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Kevin Langlois, Alexander Perepechko. Demazure roots and spherical varieties: the example of horizontal  $SL(2)$ -actions. 2014. hal-01011047

**HAL Id: hal-01011047**

**<https://hal.science/hal-01011047>**

Preprint submitted on 22 Jun 2014

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# DEMAZURE ROOTS AND SPHERICAL VARIETIES: THE EXAMPLE OF HORIZONTAL $\mathrm{SL}_2$ -ACTIONS

KEVIN LANGLOIS AND ALEXANDER PEREPECHKO

ABSTRACT. Let  $G$  be a connected reductive group, and let  $X$  be an affine  $G$ -spherical variety. We show that the classification of  $\mathbb{G}_a$ -actions on  $X$  normalized by  $G$  can be reduced to the description of quasi-affine homogeneous spaces under the action of a semi-direct product  $\mathbb{G}_a \rtimes G$  with the following property. The induced  $G$ -action is spherical and the complement of the open orbit is either empty or a  $G$ -orbit of codimension one. These homogeneous spaces are parametrized by a subset  $\mathrm{Rt}(X)$  of the character lattice  $\mathbb{X}(G)$  of  $G$ , which we call the set of Demazure roots of  $X$ . We give a complete description of the set  $\mathrm{Rt}(X)$  when  $G$  is a semi-direct product of  $\mathrm{SL}_2$  and an algebraic torus; we show particularly that  $\mathrm{Rt}(X)$  can be obtained explicitly as the intersection of a finite union of polyhedra in  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}(G)$  and a sublattice of  $\mathbb{X}(G)$ . We conjecture that  $\mathrm{Rt}(X)$  can be described in a similar combinatorial way for an arbitrary affine spherical variety  $X$ .

*Key words:* Demazure root, Luna–Vust theory, polyhedral divisor, spherical variety.  
MSC 2010: 14M25 14M27 14R20.

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## INTRODUCTION

In this article we are interested in the automorphism groups of spherical varieties defined over an algebraically closed field  $\mathbb{K}$  of characteristic zero. Let  $G$  be a connected reductive group. Recall that a  $G$ -spherical variety is a normal variety endowed with

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The second author is supported by the SPbGU grant for postdocs no. 6.50.22.2014. He expresses his gratitude to the Institut Fourier for the hospitality it provided during the writing of this paper.

a  $G$ -action and containing an open orbit of a Borel subgroup. The Luna–Vust theory describes such varieties with combinatorial objects defined by the geometric structure of the open  $G$ -orbit [Kno91]. Their study has been motivated by numerous important examples (see [Sat60, Dem70, KKMS73, MO73, Pau81, CP83]) coming in particular from representation theory and enumerative geometry.

To begin studying automorphisms of  $G$ -spherical varieties it is natural to consider the varieties with a relatively small automorphism group, for instance which neutral component is expected to be generated by the  $G$ -action. The following families of such spherical varieties are studied:

- (1) smooth complete toric varieties [Dem70, AG10];
- (2) flag varieties [Dem77];
- (3) spherical regular varieties [BB96];
- (4) wonderful varieties [Bri07, Pez09];

On the other side, there are families of spherical varieties with quite rich automorphism group, namely such that their automorphism group acts infinitely transitively on the regular locus. Excluding one-dimensional case, the following families are appropriate:

- (5) affine cones over flag varieties [AKZ12, Section 1];
- (6) non-degenerate affine toric varieties [AKZ12, Section 2];
- (7) equivariant  $\mathrm{SL}_2$ -embeddings considered as spherical varieties under the extension of  $\mathrm{SL}_2$  by a torus [BH08], [AFKKZ13, Section 5.2];
- (8) smooth affine  $G$ -spherical varieties under a semisimple group  $G$  [AFKKZ13, Section 5.2];

Related to this topic, our general aim is to provide a method of constructing automorphisms of an arbitrary  $G$ -spherical variety in a combinatorial way. In this article we study the classification of the  $\mathbb{G}_a$ -actions on affine  $G$ -spherical varieties that are normalized by  $G$ . Restricting to toric varieties, such actions allow studying the automorphism groups in case (1) via total coordinate spaces [Cox95, Cox14]. They also allow establishing the flexibility [AKZ12, Definition 0.1] in case (6).

In order to explain our results, we introduce the following notation. Let  $X$  be an affine  $G$ -spherical variety. We say that two  $\mathbb{G}_a$ -actions  $\varphi_1 : \mathbb{G}_a \times X \rightarrow X$  and  $\varphi_2 : \mathbb{G}_a \times X \rightarrow X$  are *equivalent*, if there exists a constant  $\lambda \in \mathbb{K} \setminus \{0\}$  such that

$$\varphi_1(\lambda\mu, x) = \varphi_2(\mu, x) \text{ for any } x \in X, \mu \in \mathbb{G}_a.$$

In particular, two equivalent actions share the same orbits. Denote the  $G$ -action on  $X$  by  $(g, x) \mapsto g \cdot x$ . A  $\mathbb{G}_a$ -action

$$\gamma : \mathbb{G}_a \times X \rightarrow X, (\lambda, x) \mapsto \lambda * x$$

is called *normalized* by  $G$  if there exists a character  $\chi : G \rightarrow \mathbb{G}_m$  such that

$$g \cdot (\lambda * (g^{-1} \cdot x)) = (\lambda\chi^{-1}(g)) * x$$

for all  $x \in X$ ,  $g \in G$ , and  $\lambda \in \mathbb{G}_a$ . The  $\mathbb{G}_a$ -action normalized by  $G$  naturally defines an action of the semidirect product  $\mathbb{G}_a \rtimes_\chi G$  on  $X$ , see 2.6.

In Theorem 4.5 we provide a correspondence in terms of Luna–Vust theory between

- equivalence classes of  $\mathbb{G}_a$ -actions on  $X$  normalized by  $G$  and
- quasi-affine homogeneous spaces under the action of a semi-direct product  $\mathbb{G}_a \rtimes_\chi G$  enjoying the following property: the induced  $G$ -action is spherical and the complement of the open orbit is either empty or a  $G$ -orbit of codimension

one. Furthermore, the character  $\chi$  satisfies a certain explicit condition imposed by  $X$ ; such characters are called *Demazure roots* of  $X$  and form the set denoted by  $\text{Rt}(X)$ . This definition is coherent with a classical one for toric varieties, see [Dem70, Section 4.5, p. 571].

The set  $\text{Rt}(X)$  is defined in a geometric way. A natural question arises whether  $\text{Rt}(X)$  is a combinatorial object. More precisely:

**Question 0.1** (see 4.11). Can  $\text{Rt}(X)$  be obtained explicitly as the intersection of a sublattice of the character lattice  $\mathbb{X}(G)$  and a finite union of polyhedra in  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}(G)$ ?

The answer to this question is affirmative if  $G$  is an algebraic torus. The simplest non-abelian connected reductive group which we can consider is the semidirect product  $G_e = \text{SL}_2 \rtimes_e \mathbb{T}$ . It is the reductive group defined via the multiplication law

$$(A, t_1) \cdot (B, t_2) = (A \cdot \varepsilon_e(t_1)(B), t_1 \cdot t_2), \text{ where } \varepsilon_e(t) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \chi^{-e}(t)b \\ \chi^e(t)c & d \end{pmatrix}.$$

We prove that the answer is still affirmative for every affine  $G_e$ -spherical variety  $X$ , see Theorem 4.18. We also provide a necessary and sufficient condition for affine  $G_e$ -spherical homogeneous spaces to admit a nontrivial normalized  $\mathbb{G}_a$ -action, see 4.14.

To obtain these results we use *AL-colored polyhedral divisors* (by Arzhantsev and Liendo) which provide a combinatorial description of compatible  $\text{SL}_2$ -actions on affine  $\mathbb{T}$ -varieties of complexity at most one, see [AL12]. This description allows us to characterize all spherical  $G_e$ -actions on affine varieties. In fact, we establish an explicit relation between AL-colorings on  $G_e$ -spherical varieties and corresponding colored cones in the Luna–Vust theory, see Theorem 3.7 and Propositions 3.13, 3.18.

Let us outline the structure of this article. In Section 1 we recall elements of the Luna–Vust theory for the embeddings of spherical homogeneous spaces. In Section 2 we briefly explain the Altmann–Hausen construction of polyhedral divisors (see [FZ03], [AH06], [Tim08]) for  $\mathbb{T}$ -varieties of complexity one and provide several known results concerning normalized  $\mathbb{G}_a$ -actions (from [FZ05], [Lie10a], [Lie10b], [AL12], [Lan13, Chapter 5], and [LL]). In Section 3 we establish the connection between AL-colored polyhedral divisors and colored cones. In Section 4 we deduce Theorem 4.5, we pursue Question 4.11, in particular we answer it for affine  $G_e$ -spherical varieties in Theorem 4.18, and we study the existence of normalized  $\mathbb{G}_a$ -actions in Theorem 4.14.

**Convention 0.2.** Throughout this paper  $\mathbb{K}$  is an algebraically closed field of characteristic zero. By a variety we mean an integral separated scheme of finite type over  $\mathbb{K}$ . If  $X$  is a variety then  $\mathbb{K}[X]$  denotes the coordinate ring of  $X$ , and  $\mathbb{K}(X)$  denotes its field of rational functions. A point of  $X$  is assumed to be a  $\mathbb{K}$ -rational point. Furthermore, all algebraic group actions are in particular morphisms of varieties. For an algebraic group  $G$  acting on  $X$  the natural action on  $\mathbb{K}[X]$  (resp.  $\mathbb{K}(X)$ ) is defined by the formula

$$(g \cdot f)(x) = f(g^{-1} \cdot x),$$

where  $g \in G$ ,  $f \in \mathbb{K}[X]$  (resp.  $f \in \mathbb{K}(X)$ ), and  $x \in X$ .

## 1. PRELIMINARIES ON LUNA–VUST THEORY

In this section, we recall some notation from the Luna–Vust theory for the embeddings of spherical homogeneous spaces. We are mainly interested in the case where

the embeddings are affine. We refer the reader to [LV83, Kno91, Pez10, Tim11] for the general theory. Let  $G$  be a connected reductive linear algebraic group and consider a Borel subgroup  $B \subset G$ . We start by introducing classical definitions.

**Definition 1.1.** A variety with a  $G$ -action is called *spherical* (or  *$G$ -spherical* when one needs to emphasize the acting group) if it is normal and contains an open  $B$ -orbit.

To define the notion of embeddings of a spherical homogeneous space, we introduce the scheme of geometric localities (see [LV83, Section 1]).

**1.2.** Let  $G/H$  be a spherical homogeneous space. A subalgebra of  $\mathbb{K}(G/H)$  is *affine* if it is finitely generated and if  $\mathbb{K}(G/H)$  is equal to its field of fractions. A *geometric locality* is a local algebra obtained by the localization of an affine subalgebra of  $\mathbb{K}(G/H)$  and having residue field equal to  $\mathbb{K}$ . The set  $\text{Sch}(G/H)$  of geometric localities is naturally endowed with a structure of  $\mathbb{K}$ -scheme, where the spectrum of each affine subalgebra of  $\mathbb{K}(G/H)$  is an open subset. The group  $G$  acts on  $\text{Sch}(G/H)$  as an abstract group; this action is induced by the  $G$ -action on  $\mathbb{K}(G/H)$ . We denote by  $\text{Emb}(G/H)$  the maximal  $G$ -stable normal open subset which the  $G$ -action is regular (see [LV83, Propositions 1.2 and 1.4]). An *embedding* of  $G/H$  is a  $G$ -stable separated noetherian open subset of  $\text{Emb}(G/H)$ . If  $G$  is an algebraic torus, then an embedding of  $G$  is called a *toric variety*.

The toric varieties are described by objects from the convex geometry called fans, e.g. see [KKMS73, MO73, Oda88, Ful93]. The Luna–Vust theory is a generalization of this description for embeddings of homogeneous spaces. In the sequel, we denote by  $X$  an affine embedding of a spherical homogeneous space  $G/H$ . We choose  $x \in X$  so that  $B.x$  is the open  $B$ -orbit.

**Definition 1.3.** Let  $K$  be an algebraic group acting on a variety  $Z$ . A  *$K$ -divisor* of  $Z$  is a  $K$ -stable prime divisor on  $Z$ . In our situation, a  $B$ -divisor of the spherical variety  $X$  that is not  $G$ -stable is called a *color*.

Identifying  $G.x$  with  $G/H$  we may view a color as the closure in  $X$  of a  $B$ -divisor of  $G/H$ . Thus, colors are determined by the homogeneous space  $G/H$ .

**1.4.** The algebra of rational functions  $\mathbb{K}(X) = \mathbb{K}(G/H)$  is endowed with a natural linear  $B$ -action. Consider the lattice  $L$  of  $B$ -weights of  $\mathbb{K}(X)$  for this action. For a  $B$ -eigenvector  $f$  in  $\mathbb{K}(X)$  we denote by  $\chi_f$  the corresponding weight. Since the quotient of two eigenvectors of the same weight is  $B$ -invariant and therefore constant on the open orbit  $B.x$ , the  $B$ -eigenspaces in  $\mathbb{K}(X)$  are of dimension one, hence the weight  $\chi_f$  defines  $f$  up to a scalar. Summarizing, we have the exact sequence

$$1 \rightarrow \mathbb{K}^* \rightarrow \mathbb{K}(X)^{(B)} \rightarrow L \rightarrow 0,$$

where  $\mathbb{K}(X)^{(B)}$  denotes the multiplicative subgroup of  $B$ -eigenvalues of  $\mathbb{K}(X)^*$ . The epimorphism  $\mathbb{K}(X)^{(B)} \rightarrow L$  is given by  $f \mapsto \chi_f$ .

**1.5.** Recall that a *discrete valuation* on  $\mathbb{K}(X)$  is a map  $v : \mathbb{K}(X)^* \rightarrow \mathbb{Q}$  such that

- $v(f + g) \geq \min\{v(f), v(g)\}$  for all  $f, g \in \mathbb{K}(X)^*$  satisfying  $f + g \in \mathbb{K}(X)^*$ ;
- $v$  is a group morphism from  $(\mathbb{K}(X)^*, \times)$  to  $(\mathbb{Q}, +)$ ;
- the subgroup  $\mathbb{K}^*$  is contained in the kernel of the morphism  $v$ .

Let  $V = \text{Hom}(L, \mathbb{Z})$  be the dual lattice of  $L$  and let  $V_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} V$  be the associated  $\mathbb{Q}$ -vector space. For every discrete valuation  $v$  of  $\mathbb{K}(X)$  we associate a vector  $\varrho(v) \in V_{\mathbb{Q}}$  by letting

$$\varrho(v)(\chi_f) = \langle \chi_f, \varrho(v) \rangle = v(f) \text{ for all } f \in \mathbb{K}(X)^{(B)}.$$

Since  $X$  is normal, each prime divisor  $D \subset X$  naturally defines a discrete valuation  $v_D$  on  $\mathbb{K}(X)$ . So, we may identify any subset of  $B$ -divisors with the corresponding set of discrete valuations, which we denote by the same letter.

**1.6.** A discrete valuation  $v$  on  $\mathbb{K}(X)$  is said to be *G-invariant* if every fiber of the map  $v$  is  $G$ -stable, for the natural linear  $G$ -action on  $\mathbb{K}(X)$ . It is well known that the restriction of  $\varrho$  to the subset of discrete  $G$ -invariant valuations is injective [LV83, Section 7.4]. Furthermore, the image of this restriction is a polyhedral cone  $\mathcal{V}$  of  $V_{\mathbb{Q}}$  (see [BP87, Corollary 3.2]), called the *valuation cone* of  $G/H$ . Thus, we will say that  $V$  is the *valuation lattice*. Recall that the spherical variety  $X$  is said to be *horospherical* if the isotropy group of a general point contains a maximal unipotent subgroup of  $G$ . It is well-known that  $X$  is horospherical if and only  $\mathcal{V} = V_{\mathbb{Q}}$  (see [Kno91, Corollary 6.2]).

Let us introduce the classical notion of colored cone [LV83, p. 230]. In this definition, an element of  $L$  is seen as a linear form on  $V_{\mathbb{Q}}$ .

**Definition 1.7.** A *colored cone* of the homogeneous space  $G/H$  is a pair  $(\mathcal{C}, \mathcal{F})$ , where

- $\mathcal{F}$  is a subset of colors of  $G/H$  such that  $0 \notin \varrho(\mathcal{F})$ ;
- $\mathcal{C} \subset V_{\mathbb{Q}}$  is a strongly convex polyhedral cone generated by the union of  $\varrho(\mathcal{F})$  and of a finite subset of  $\mathcal{V}$ ;
- the intersection of the relative interior of  $\mathcal{C}$  with the cone  $\mathcal{V}$  is non-empty.

Moreover, a colored cone  $(\mathcal{C}, \mathcal{F})$  will be called *affine* if there exists a lattice vector  $m \in L$  satisfying the following conditions:

- $m$  is non-positive on  $\mathcal{V}$ ;
- $m$  is zero on  $\mathcal{C}$ ;
- $m$  is positive on all colors of  $G/H$  not belonging to  $\mathcal{F}$ .

There is a natural way to build an affine colored cone of  $G/H$  from the embedding  $X$ , see [Kno91, Section 2]. In the next paragraph, we explain this construction.

**1.8.** Note that the open orbit  $B.x$  is affine (see [Tim11, Theorem 3.5]), and so its complement in  $X$  is a union of  $B$ -divisors. So,  $X$  has a finite number of  $B$ -divisors. We denote by  $\mathcal{P}$  the set of  $G$ -divisors of  $X$  identified with a subset of  $V_{\mathbb{Q}}$ . Moreover, since  $X$  is affine,  $X$  has a unique closed  $G$ -orbit  $Y$ , which is contained in the closure of all the  $G$ -orbits of  $X$ . We denote by  $\mathcal{F}_Y$  the set of colors of  $X$  containing  $Y$ . The *associated colored cone* of  $X$  is the pair  $(\mathcal{C}, \mathcal{F}_Y)$ , where  $\mathcal{C} \subset V_{\mathbb{Q}}$  is the cone generated by the union of  $\mathcal{P}$  and of the subset  $\varrho(\mathcal{F}_Y)$ . According to [Kno91, Section 2] and [Kno91, Theorem 6.7] the pair  $(\mathcal{C}, \mathcal{F}_Y)$  is in fact an affine colored cone of  $G/H$ . Note that the embedding  $X$  is called *toroidal* if  $\mathcal{F}_Y = \emptyset$ .

Conversely, to a colored cone  $(\mathcal{C}, \mathcal{F})$  of  $G/H$  one can associate an embedding.

**1.9.** Let  $(\mathcal{C}, \mathcal{F})$  be a colored cone of  $G/H$ . For a discrete valuation  $v$  of  $\mathbb{K}(G/H)$  we write  $\mathcal{O}_v$  the corresponding local algebra. Let  $X_B$  be the open  $B$ -orbit in  $G/H$ .

Considering a finite subset  $\mathcal{P} \subset \mathcal{V}$  such that  $\mathcal{P} \cup \varrho(\mathcal{F})$  generates the cone  $\mathcal{C}$ , we define the algebra

$$A(\mathcal{P}, \mathcal{F}) = \mathbb{K}[X_B] \cap \bigcap_{v \in \mathcal{P}} \mathcal{O}_v \cap \bigcap_{D \in \mathcal{F}} \mathcal{O}_{v_D}.$$

By [LV83, Section 8],  $A(\mathcal{P}, \mathcal{F})$  is a normal affine subalgebra of  $\mathbb{K}(G/H)$ . The variety  $X_0 = \text{Spec } A(\mathcal{P}, \mathcal{F})$  does not depend on the choice of the set  $\mathcal{P}$ . Furthermore,  $X_0$  is an open subset of  $\text{Emb}(G/H)$  (compare with [LV83, Proposition 8.10]). The associated spherical embedding of  $(\mathcal{C}, \mathcal{F})$  is the subscheme  $X = G \cdot X_0$  [LV83, Proposition 1.5]. This construction is also explained in [Tim11, Chapter 3, Section 13].

More precisely, we have the following classical result (see [Kno91, Theorem 6.7]).

**Theorem 1.10.** *The map defined by sending an affine embedding  $X$  of  $G/H$  to its associated colored cone  $(\mathcal{C}, \mathcal{F}_X)$  is a bijection between*

- *the set of affine embeddings of  $G/H$  and*
- *the set of affine colored cones of  $G/H$ .*

The next well-known lemma will be useful afterwards (see the proof of [Kno91, 6.7]).

**Lemma 1.11.** *Let  $X$  be an affine embedding of  $G/H$  with associated colored cone  $(\mathcal{C}, \mathcal{F}_X)$ . Let  $\mathcal{F}_0$  be the set of colors of  $G/H$ . Then the cone  $\Gamma$  dual to the cone generated by the  $B$ -weights of  $\mathbb{K}[X]$  is the polyhedral cone in  $V_{\mathbb{Q}}$  generated by the subset  $\mathcal{C} \cup \varrho(\mathcal{F}_0)$ . In addition,  $\Gamma$  is strongly convex.*

*Proof.* Let  $U$  be the unipotent part of  $B$ . Since  $X$  is affine, the field of fractions of  $\mathbb{K}[X]^U$  is equal to  $\mathbb{K}(X)^U$  (see [Tim11, Lemma D.7]). Consequently, every  $B$ -eigenvector of  $\mathbb{K}(X)$  is the quotient of two eigenvectors of  $\mathbb{K}[X]$ . This shows that  $\Gamma$  is a strongly convex cone. The rest of the proof is straightforward.  $\square$

## 2. PRELIMINARIES ON POLYHEDRAL DIVISORS AND NORMALIZED $\mathbb{G}_a$ -ACTIONS

In this section, we recall some basic facts from the theory of polyhedral divisors for the case of complexity one [FZ03, AH06, Tim08]. We present also a brief survey of known results concerning the classification of normalized  $\mathbb{G}_a$ -actions on affine  $\mathbb{T}$ -varieties of complexity one, see [FZ05], [Lie10a], [Lie10b], [AL12], [Lan13, Chapter 5], and [LL].

### 2.1. Polyhedral divisors and affine $\mathbb{T}$ -varieties of complexity one.

We denote by  $\mathbb{T}$  an algebraic torus of dimension  $n$ . We fix a lattice  $M$  isomorphic to the character group of  $\mathbb{T}$  via  $m \mapsto \chi^m$ .

**Definition 2.1.** The *complexity* of a  $\mathbb{T}$ -action on a variety  $X$  is the transcendence degree over  $\mathbb{K}$  of the field extension  $\mathbb{K}(X)^{\mathbb{T}}$ . By a result of Rosenlicht [Ros63], the complexity is also the codimension of a general orbit in  $X$ . In particular, if the action is faithful, then the complexity is equal to  $\dim X - \dim \mathbb{T}$ . A  *$\mathbb{T}$ -variety* is a normal variety endowed with a faithful  $\mathbb{T}$ -action.

Note that having a  $\mathbb{T}$ -action on an affine variety  $X = \text{Spec } A$  is equivalent to endowing  $A$  with an  $M$ -grading. If  $\mathbb{T}$  acts on  $X$  and  $m \in M$ , then we define the  $m$ -th graded component of  $A$  by letting

$$A_m = \{f \in A \mid t \cdot f = \chi^m(t)f \text{ for all } t \in \mathbb{T}\}.$$

In [AH06], a combinatorial description of  $M$ -graded algebras corresponding to affine  $\mathbb{T}$ -varieties is given in terms of polyhedral divisors. We recall this construction in the setting of the complexity one. We also introduce other notation from convex geometry which will be useful in the sequel.

**2.2.** Let  $N = \text{Hom}(M, \mathbb{Z})$  be the dual lattice of  $M$ . Denote by  $M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$  and  $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$  the associated vector spaces. For a polyhedral cone  $\tau \subset N_{\mathbb{Q}}$  the symbols  $\tau^{\vee}$ ,  $\text{lin.}(\tau)$ , and  $\text{rel.int.}(\tau)$  stand respectively for the dual cone in  $M_{\mathbb{Q}}$ , the linear part  $(-\tau) \cap \tau$ , and the relative interior of  $\tau$ . Considering  $v \in N_{\mathbb{Q}}$  we may write  $v^{\perp} = \{m \in M_{\mathbb{Q}} \mid v(m) = 0\}$  for its orthogonal space and denote  $\mu(v) = \inf\{r \in \mathbb{Z}_{>0} \mid rv \in N\}$ .

For every polyhedron  $\Delta$  of  $N_{\mathbb{Q}}$  we denote by  $\Delta(r)$  the set of its  $r$ -dimensional faces. Let  $\sigma \subset N_{\mathbb{Q}}$  be a strongly convex polyhedral cone. A subset  $\Delta \subset N_{\mathbb{Q}}$  is called a  $\sigma$ -polyhedron if  $\Delta$  is a Minkowski sum  $Q + \sigma$  for a polytope<sup>1</sup>  $Q \subset N_{\mathbb{Q}}$ .

A  $\sigma$ -polyhedral divisor over a curve  $C$  is a formal sum  $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ , where each  $\Delta_z$  is a  $\sigma$ -polyhedron, with the condition that  $\Delta_z = \sigma$  for all but finitely many  $z \in C$ . The finite subset  $\{z \in C \mid \Delta_z \neq \sigma\}$  is called the *support* of  $\mathfrak{D}$ . The *degree* of  $\mathfrak{D}$ , denoted by  $\text{deg } \mathfrak{D}$ , is the Minkowski sum of all the coefficients of  $\mathfrak{D}$ , which is a  $\sigma$ -polyhedron (see [AH06, 2.12]). For a vector  $m \in M_{\mathbb{Q}}$  we let

$$\mathfrak{D}(m) = \sum_{z \in C} \min_{v \in \Delta_z(0)} \{v(m)\} \cdot z,$$

which is a  $\mathbb{Q}$ -divisor on  $C$ . If  $C$  is a normal curve, then the associated  $M$ -graded algebra of  $\mathfrak{D}$  is

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma^{\vee} \cap M} H^0(C, \mathcal{O}_C([\mathfrak{D}(m)])) \chi^m.$$

The multiplication on  $A[C, \mathfrak{D}]$  is defined on homogeneous elements via the maps

$$H^0(C, \mathcal{O}_C([\mathfrak{D}(m)])) \chi^m \times H^0(C, \mathcal{O}_C([\mathfrak{D}(m')])) \chi^{m'} \rightarrow H^0(C, \mathcal{O}_C([\mathfrak{D}(m+m')])) \chi^{m+m'}$$

sending  $(f_1, f_2)$  to  $f_1 \cdot f_2$ . The polyhedral divisor  $\mathfrak{D}$  is called *linear* on a polyhedral cone  $\omega \subset M_{\mathbb{Q}}$  if

$$\mathfrak{D}(m+m') = \mathfrak{D}(m) + \mathfrak{D}(m') \text{ for all } m, m' \in \omega.$$

We denote by  $\Lambda(\mathfrak{D})$  the coarsest quasifan of  $\sigma^{\vee}$  where  $\mathfrak{D}$  is linear on each cone (see [AH06, Section 1] for details).

**Definition 2.3.** A  $\sigma$ -polyhedral divisor  $\mathfrak{D}$  on a normal curve  $C$  is said to be *proper* if  $\mathfrak{D}(m)$  is semi-ample for any  $m \in \sigma^{\vee}$  and big for any  $m \in \text{rel.int.}(\sigma^{\vee})$ . This condition is automatically fulfilled in case of an affine curve  $C$ . Otherwise, if  $C$  is projective, then the properness condition can be split in two parts:

- $\text{deg } \mathfrak{D} \not\subseteq \sigma$ ;
- for any  $m \in \sigma^{\vee}$  such that  $\min_{v \in \text{deg } \mathfrak{D}} \{v(m)\} = 0$ ,  $m$  belongs to the boundary of  $\sigma^{\vee}$  and  $\mathfrak{D}(m)$  is a nonzero  $\mathbb{Q}$ -principal divisor.

In this paper we deal with proper polyhedral divisors over  $\mathbb{A}^1$  or  $\mathbb{P}^1$ . In this case, we have a simple characterization of the properness: a  $\sigma$ -polyhedral divisor over  $\mathbb{P}^1$  is proper if and only if  $\text{deg } \mathfrak{D} \not\subseteq \sigma$ .

<sup>1</sup>We recall that a polytope in a  $\mathbb{Q}$ -vector space is the convex hull of a non-empty finite subset.



**2.4.** If  $\mathfrak{D}$  is a proper polyhedral divisor over a normal curve  $C$ , then  $X = \text{Spec } A[C, \mathfrak{D}]$  is an affine  $\mathbb{T}$ -variety of complexity one. The cone  $\sigma^\vee$  is the weight cone of the  $M$ -graded algebra  $\mathbb{K}[X]$ . The curve  $C$  is a rational quotient of  $X$ , i.e.  $\mathbb{K}(X)^\mathbb{T} = \mathbb{K}(C)$ . Conversely, every affine  $\mathbb{T}$ -variety of complexity one is obtained in this way [AH06, Theorem 3.1]. For the uniqueness of such representation, see [AH06, Section 8].

Let  $\mathfrak{D}$  be a proper  $\sigma$ -polyhedral divisor over a normal curve  $C$ . Consider the associated algebra  $A = A[C, \mathfrak{D}]$  and let  $X = \text{Spec } A$ . In the following paragraph we explain a classification of  $\mathbb{T}$ -divisors in terms of the pair  $(C, \mathfrak{D})$  corresponding to  $X$ . See [AH06, Section 7], [Tim08], and [PS11] for further details.

**2.5.** Let  $\sigma(1)^*$  be the following subset of  $\sigma(1)$ . If  $C$  is affine, then we let  $\sigma(1)^* = \sigma(1)$ . Otherwise,  $C$  is projective and  $\sigma(1)^*$  is the set of 1-dimensional faces of  $\sigma$  that do not meet  $\text{deg } \mathfrak{D}$ . For every element  $\rho \in \sigma(1)$  we denote by the same letter  $\rho$  its extremal ray. There are two types of  $\mathbb{T}$ -divisors in the variety  $X$ .

- *Horizontal* divisors denoted by  $D_\rho$  for each  $\rho \in \sigma(1)^*$ . The corresponding  $M$ -graded ideal of such divisor  $D_\rho$  is

$$I(D_\rho) = \bigoplus_{m \in (\sigma^\vee \setminus \rho^\perp) \cap M} A_m \chi^m, \text{ where } A_m = H^0(C, \mathcal{O}_C([\mathfrak{D}(m)])).$$

A  $\mathbb{T}$ -divisor  $D$  is horizontal if and only if the induced  $\mathbb{T}$ -action on  $D$  is of complexity one.

- *Vertical* divisors denoted by  $D_{(z,v)}$ , where  $z \in C$  and  $v \in \Delta_z(0)$  is a vertex of  $\Delta_z$ . The associated  $M$ -graded ideal is

$$I(D_{(z,v)}) = \bigoplus_{m \in \sigma^\vee \cap M} (A_m \cap \{f \in \mathbb{K}(C) \mid \text{ord}_z f > -v(m)\}) \chi^m.$$

A  $\mathbb{T}$ -divisor  $D$  is vertical if and only if the induced  $\mathbb{T}$ -action on  $D$  is of complexity zero.

Moreover, if  $f\chi^m \in A$  is homogeneous of degree  $m$ , then the corresponding principal divisor is given by the formula

$$\text{div}(f\chi^m) = \text{div}(f\chi^m)_{\text{hor}} + \text{div}(f\chi^m)_{\text{ver}}$$

where

$$\begin{aligned} \text{div}(f\chi^m)_{\text{hor}} &= \sum_{\rho \in \sigma(1)^*} \rho(m) \cdot D_\rho, \\ \text{div}(f\chi^m)_{\text{ver}} &= \sum_{z \in C} \sum_{v \in \Delta_z(0)} \mu(v) (v(m) + \text{ord}_z f) \cdot D_{(z,v)}, \end{aligned}$$

see [PS11, Proposition 3.14]. The divisor  $\text{div}(f\chi^m)_{\text{hor}}$  (resp.  $\text{div}(f\chi^m)_{\text{ver}}$ ) will be called the *horizontal part* (resp. *vertical part*) of  $\text{div}(f\chi^m)$ .

## 2.2. $\mathbb{G}_a$ -actions on affine $\mathbb{T}$ -varieties of complexity one.

In this subsection, we review results in [Lie10a] concerning the classification of normalized additive group actions on affine  $\mathbb{T}$ -varieties of complexity one.

**2.6.** Let  $Z$  be a variety. Let

$$K \times Z \rightarrow Z, (g, x) \mapsto g \cdot x$$

be an action of an algebraic group  $K$  on  $Z$ . An additive group action

$$\gamma : \mathbb{G}_a \times Z \rightarrow Z, (\lambda, x) \mapsto \lambda * x$$

is called *normalized* by  $K$  if there exists a character  $\chi : K \rightarrow \mathbb{G}_m$  such that

$$g \cdot (\lambda * (g^{-1} \cdot x)) = (\lambda \chi^{-1}(g)) * x$$

for all  $x \in Z$ ,  $g \in K$ , and  $\lambda \in \mathbb{G}_a$ . The character  $\chi$  will be called a *degree* of  $\gamma$ . Note that if  $\gamma$  is non-trivial, then  $\chi$  is uniquely determined by  $\gamma$ .

Consider the algebraic group  $\mathbb{G}_a \rtimes_\chi K$ , where the underlying variety is  $\mathbb{G}_a \times K$  and the group law is given by

$$(\lambda_1, g_1) \cdot (\lambda_2, g_2) = (\lambda_1 + \chi^{-1}(g_1)\lambda_2, g_1 \cdot g_2)$$

for  $\lambda_i \in \mathbb{G}_a$  and  $g_i \in K$ ; this algebraic group is the semi-direct product of  $\mathbb{G}_a$  and  $K$  associated to the character  $\chi$ . Moreover, a  $K$ -action on  $Z$  and an additive group action normalized by  $K$  of degree  $\chi$  naturally generate the action of  $\mathbb{G}_a \rtimes_\chi K$  on  $Z$  via the formula

$$(\lambda, g) \cdot x = \lambda * (g \cdot x),$$

where  $(\lambda, g) \in \mathbb{G}_a \rtimes_\chi K$  and  $x \in X$ . If  $K$  is the torus  $\mathbb{T}$  and  $e \in M$  is lattice vector, then we denote  $\mathbb{G}_a \rtimes_e \mathbb{T} = \mathbb{G}_a \rtimes_{\chi^e} \mathbb{T}$ .

The following definition is introduced in [FZ05] to describe normal quasi-homogeneous affine surfaces and extended in [Lie10a, Section 1.2] to varieties with a  $\mathbb{T}$ -action.

**Definition 2.7.** Let  $Z$  be a variety endowed with a  $\mathbb{T}$ -action. An additive group action on  $Z$  normalized by  $\mathbb{T}$  is called *vertical* if a general  $\mathbb{G}_a$ -orbit of  $Z$  is contained in the closure of a  $\mathbb{T}$ -orbit. Otherwise, the action is called *horizontal*.

**2.8.** For an affine variety  $Z$ , there is a well-known one-to-one correspondence between the  $\mathbb{G}_a$ -actions on  $Z$  and the locally nilpotent derivations on  $\mathbb{K}[Z]$  (called LNDs for brevity). Indeed, let  $\mathbb{K}[\mathbb{G}_a] = \mathbb{K}[\lambda]$  for a variable  $\lambda$  over  $\mathbb{K}$ . Then, given an LND  $\partial$  on  $\mathbb{K}[Z]$ , the morphism

$$\mathbb{K}[Z] \rightarrow \mathbb{K}[Z] \otimes_{\mathbb{K}} \mathbb{K}[\lambda], \quad f \mapsto \sum_{i \geq 0} \frac{\partial^i(f)}{i!} \otimes \lambda^i,$$

defines a  $\mathbb{G}_a$ -action  $\gamma_\partial$  on  $Z$ , and vice-versa. See [Fre06] for details concerning the theory of LNDs. Now assume that  $Z$  is endowed with a  $\mathbb{T}$ -action. It is also known that  $\mathbb{G}_a$ -actions on  $Z$  normalized by  $\mathbb{T}$  of degree  $\chi^e$  correspond to homogeneous LNDs of degree  $e$  with respect to the  $M$ -grading on  $\mathbb{K}[Z]$  (see the proof of [FZ05, Lemma 2.2]).

For the next paragraphs, we keep the same notation as above for the  $\sigma$ -polyhedral divisor  $\mathfrak{D}$ . We let  $X = \text{Spec } A[C, \mathfrak{D}]$  be the associated affine  $\mathbb{T}$ -variety of complexity one.

**2.9.** Let  $\partial$  be a homogeneous LND on  $\mathbb{K}[X]$ . We have the following characterization.

- $\gamma_\partial$  is vertical if and only if  $\mathbb{K}(C) \subset \mathbb{K}(X)^\partial$ , where  $\mathbb{K}(X)^\partial \subset \mathbb{K}(X)$  is the field of invariants.
- $\gamma_\partial$  is horizontal if and only if  $\mathbb{K}(C) \cap \mathbb{K}(X)^\partial = \mathbb{K}$ . In other words, the algebraic group  $\mathbb{G}_a \rtimes_e \mathbb{T}$  acts on  $X$  with an open orbit.

We call the corresponding LNDs *vertical* and *horizontal* respectively. The set of non-zero vertical (resp. horizontal) LNDs on  $\mathbb{K}[X]$  is denoted by  $\text{LND}_{\text{ver}}(X)$  (resp.  $\text{LND}_{\text{hor}}(X)$ ).

The following paragraph presents the classical description of vertical homogeneous LNDs in complexity one. See [Lie10b] for the general case.

**2.10.** Recall from [Dem70, Section 4.5, p. 571] that a vector  $e \in M$  is a *Demazure root* of the cone  $\sigma$  with distinguished ray  $\rho_e \in \sigma(1)$  if  $\rho_e(e) = -1$  and  $\rho(e) \geq 0$  for all  $\rho \in \sigma(1) \setminus \{\rho_e\}$ . We denote by  $\text{Rt}(\sigma)$  the set of all roots of  $\sigma$ . If  $e \in \text{Rt}(\sigma)$ , then we let

$$\Phi_e^* = H^0(C, \mathcal{O}_C([\mathcal{D}(e)])) \setminus \{0\}.$$

Furthermore,  $\Phi_e^* \neq \emptyset$  if and only if  $\rho_e \in \sigma(1)^*$ , see [Lie10a, Corollary 3.13]. Assume that  $\Phi_e^*$  has an element  $\varphi$ . We define a homogeneous LND  $\partial_{e,\varphi}$  of vertical type on the algebra  $A = A[C, \mathcal{D}]$  by the formula

$$\partial_{e,\varphi}(f\chi^m) = \rho_e(m)\varphi f\chi^{m+e},$$

where  $f\chi^m \in A$  is an homogeneous element of degree  $m$ , see [Lie10a, Lemma 3.6]. More precisely, let

$$\text{Rt}^*(\sigma) = \{e \in \text{Rt}(\sigma) \mid \rho_e \in \sigma(1)^*\}.$$

Considering the set

$$\Phi = \{(e, \varphi) \mid e \in \text{Rt}^*(\sigma), \varphi \in \Phi_e^*\},$$

the map  $\Phi \rightarrow \text{LND}_{\text{ver}}(A)$ ,  $(e, \varphi) \mapsto \partial_{e,\varphi}$  is a bijection [Lie10a, Theorem 3.8].

In the sequel, we consider the case of horizontal homogeneous LNDs in complexity one. Note that if  $X = \text{Spec } A[C, \mathcal{D}]$  admits a normalized  $\mathbb{G}_a$ -action of horizontal type, then either  $C = \mathbb{A}^1$  or  $C = \mathbb{P}^1$  (see [Lie10a, 3.16]). The next paragraph introduces combinatorial objects that describe homogeneous LNDs of horizontal type on  $A = A[C, \mathcal{D}]$  (see [Lie10a, Section 3.2], [AL12, Section 1.4], [Lan13, Section 5.6], and [LL]). Note that our terminology will be slightly different.

**2.11.** Let  $\mathcal{D}$  be a proper  $\sigma$ -polyhedral divisor over  $\mathbb{A}^1$ . An (*affine*) *AL-coloring*<sup>2</sup> on the polyhedral divisor  $\mathcal{D}$  is a triple  $\mathcal{D}_\bullet = (\mathcal{D}, \{v_z^\bullet \mid z \in \mathbb{A}^1\}, z_0)$ , where  $z_0 \in \mathbb{A}^1$ , satisfying the following conditions:

- each  $v_z^\bullet$  is a vertex of  $\Delta_z$ ;
- $v_{\text{deg}}^\bullet := \sum_{z \in \mathbb{A}^1} v_z^\bullet$  is a vertex of the polyhedron  $\text{deg } \mathcal{D} = \sum_{z \in \mathbb{A}^1} \Delta_z$ ;
- $v_z^\bullet \in N$  for any  $z \in \mathbb{A}^1 \setminus \{z_0\}$ .

The elements  $v_z^\bullet$  for  $z$  in the support of  $\mathcal{D}$  are called the *AL-colors* of  $\mathcal{D}_\bullet$ . The point  $z_0$  is called the *marked point* of  $\mathcal{D}_\bullet$ . The *support* of  $\mathcal{D}_\bullet$  is defined as the union of the support of  $\mathcal{D}$  and of the marked point  $z_0$ . The polyhedron

$$\omega_\bullet = \text{Cone}(\text{deg } \mathcal{D} - v_{\text{deg}}^\bullet)$$

is called the *associated cone* of  $\mathcal{D}_\bullet$ .

Now let  $\mathcal{D}$  be a proper  $\sigma$ -polyhedral divisor over  $\mathbb{P}^1$ . A (*projective*) *AL-coloring* on the polyhedral divisor  $\mathcal{D}$  is a quadruple  $\mathcal{D}_\bullet = (\mathcal{D}, \{v_z^\bullet \mid z \in \mathbb{A}^1\}, z_0, z_\infty)$ , where  $z_\infty \in \mathbb{P}^1$ ,  $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{z_\infty\}$ , and the triple  $\mathcal{D}_\bullet^{\text{aff}} = (\mathcal{D}|_{\mathbb{A}^1}, \{v_z^\bullet \mid z \in \mathbb{A}^1\}, z_0)$  is an affine AL-coloring. We define the AL-colors, the marked point, the associated cone, and the support of  $\mathcal{D}_\bullet$  as those of  $\mathcal{D}_\bullet^{\text{aff}}$ . We also call  $z_\infty$  the *point at the infinity* of  $\mathcal{D}_\bullet$ .

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<sup>2</sup>named by I. Arzhantsev and A. Liendo. The sign  $\bullet$  will be substituted by  $-$  and  $+$  later on, when we consider different AL-colorings of a polyhedral divisor.

*Remark 2.12.* The definition of  $\omega_\bullet$  is motivated by the following relation:

$$(1) \quad \mathfrak{D}(m)|_{\mathbb{A}^1} = \sum_{z \in \mathbb{A}^1} v_z^\bullet(m) \cdot z \text{ if and only if } m \in \omega_\bullet^\vee.$$

**2.13.** Since we will consider several colorings on a same polyhedral divisor, it is convenient to represent them by a weighted graph. Let  $\{z_0, \dots, z_r\} \subset \mathbb{A}^1$  be the support of  $\mathfrak{D}_\bullet$ , where  $z_0$  is the marked point. A representation of  $\mathfrak{D}_\bullet$  will be drawn as

$$\cdots \text{---} \underset{z_0}{\overset{v_{z_0}^\bullet}{\circ}} \text{---} \underset{z_1}{\overset{v_{z_1}^\bullet}{\circ}} \text{---} \cdots \text{---} \underset{z_r}{\overset{v_{z_r}^\bullet}{\circ}} \text{---} \cdots,$$

where  $v_{z_0}^\bullet, \dots, v_{z_r}^\bullet$  are the corresponding AL-colors of  $\mathfrak{D}_\bullet$ .

Let us introduce other objects attached to a coloring of  $\mathfrak{D}$ . In the following definition, we keep the same notation as above for the AL-coloring  $\mathfrak{D}_\bullet$ .

**Definition 2.14.** Let  $\tilde{\omega}_\bullet \subset (N \oplus \mathbb{Z})_{\mathbb{Q}}$  be the cone generated by  $(\omega_\bullet, 0)$  and  $(v_{z_0}^\bullet, 1)$  if  $C = \mathbb{A}^1$ , and by  $(\omega_\bullet, 0)$ ,  $(v_{z_0}^\bullet, 1)$ , and  $(\Delta_{z_\infty} + v_{\deg}^\bullet - v_{z_0}^\bullet + \omega_\bullet, -1)$  if  $C = \mathbb{P}^1$ . The polyhedral cone  $\tilde{\omega}_\bullet$  will be called the *domain* of  $\mathfrak{D}_\bullet$ . Let  $e \in M$ . A pair  $(\mathfrak{D}_\bullet, e)$  is said to be *coherent* if the following conditions are satisfied.

- Let  $s = -\frac{1}{d(e)} - v_{z_0}^\bullet(e)$ , where  $d(e) = \inf\{d \in \mathbb{Z}_{>0} \mid dv_{z_0}^\bullet(e) \in \mathbb{Z}\}$ . The vector  $(e, s)$  is a Demazure root of  $\tilde{\omega}_\bullet$ ;
- $v(e) \geq 1 + v_z^\bullet(e)$  for any  $z \in \mathbb{A}^1 \setminus \{z_0\}$  and for any  $v \in \Delta_z(0) \setminus \{v_z^\bullet\}$ ;
- $v(e) \geq -s$  for any  $v \in \Delta_{z_0} \setminus \{v_{z_0}^\bullet\}$ ;
- If  $C = \mathbb{P}^1$ , then  $v(e) \geq -\frac{1}{d(e)} - v_{\deg}^\bullet(e)$  for any  $v \in \Delta_{z_\infty}(0)$ .

The next theorem provides a description of normalized  $\mathbb{G}_a$ -actions of horizontal type on the  $\mathbb{T}$ -variety  $X = \text{Spec } A[C, \mathfrak{D}]$  in terms of coherent pairs [AL12, Theorem 1.10]. We fix a variable  $t$  over  $\mathbb{K}$  such that  $z_0 = \{t = 0\}$  and  $z_\infty = \{t = \infty\}$  (if exists), in particular  $\mathbb{K}[\mathbb{A}^1] = \mathbb{K}[t]$ .

**Theorem 2.15.** *Let  $\mathfrak{D}$  be a proper  $\sigma$ -polyhedral divisor over  $C = \mathbb{A}^1$  or  $C = \mathbb{P}^1$  and let  $A = A[C, \mathfrak{D}]$ .*

*i) Consider a triplet  $(\lambda, \mathfrak{D}_\bullet, e)$ , where  $\lambda \in \mathbb{K}^*$ ,  $\mathfrak{D}_\bullet$  is an AL-coloring of  $\mathfrak{D}$ , and  $(\mathfrak{D}_\bullet, e)$  is a coherent pair. Assume without loss of generality that  $v_z^\bullet = 0$  for all  $z \in \mathbb{A}^1 \setminus \{0\}$ . Each such triplet  $(\lambda, \mathfrak{D}_\bullet, e)$  defines a horizontal LND  $\partial$  on  $A$  via the formula*

$$\partial(t^r \chi^m) = \lambda \cdot d(e) \cdot (v_0^\bullet(m) + r) \cdot t^{r+s} \chi^{m+e} \text{ for all } (m, r) \in M \oplus \mathbb{Z},$$

*where  $s = -\frac{1}{d(e)} - v_0^\bullet(e)$ . The kernel of the LND  $\partial$  corresponding to the triplet  $(\lambda, \mathfrak{D}_\bullet, e)$  is*

$$\ker \partial = \bigoplus_{m \in \omega_\bullet^\vee \cap L} \mathbb{K} \varphi_m \chi^m,$$

*where  $L = \{m \in M \mid v_0^\bullet(m) \in \mathbb{Z}\}$ ,  $\varphi_m \in \mathbb{K}(C)^*$  such that  $(\text{div}(\varphi_m) + \mathfrak{D}(m))|_{\mathbb{A}^1} = 0$ , and  $\omega_\bullet$  is the associated cone of  $\mathfrak{D}_\bullet$ . Moreover,  $\omega_\bullet^\vee$  is a maximal cone of  $\Lambda(\mathfrak{D}|_{\mathbb{A}^1})$  (see 2.2).*

*ii) All horizontal LNDs on  $A$  are obtained in this way.*

*Remark 2.16.* More generally, if  $z_0$  is arbitrary,  $z_\infty = \infty$ , and  $\mathfrak{D}_\bullet$  be represented by

$$\cdots \text{---} \underset{z_0}{\overset{v_{z_0}^\bullet}{\circ}} \text{---} \underset{z_1}{\overset{v_{z_1}^\bullet}{\circ}} \text{---} \cdots \text{---} \underset{z_k}{\overset{v_{z_k}^\bullet}{\circ}} \text{---} \cdots$$

then the LND  $\partial_\bullet$  is given by the formula

$$\partial_\bullet((t - z_0)^r \xi_m \chi^m) = d(e)(v_{z_0}^\bullet(m) + r)(t - z_0)^{r-s} \xi_{m+e} \chi^{m+e},$$

where

$$\xi_m = \prod_{i=1}^k (t - z_i)^{-v_{z_i}^\bullet(m)} \text{ and } s = -\frac{1}{d(e)} - v_{z_0}^\bullet(e).$$

### 2.3. Horizontal $\mathrm{SL}_2$ -actions.

In this subsection, we recall generalities on compatible  $\mathrm{SL}_2$ -actions (see [AL12, Section 2]) of horizontal type for affine  $\mathbb{T}$ -varieties of complexity one.

**2.17.** We make the convention that  $\mathrm{SL}_2$  is the algebraic group of  $2 \times 2$  matrices over  $\mathbb{K}$  with determinant one. For every  $e \in M$  we define the reductive group  $G_e = \mathrm{SL}_2 \rtimes_e \mathbb{T}$  as follows. The underlying variety is  $\mathrm{SL}_2 \times \mathbb{T}$  and the multiplication law is given by the relation

$$(A, t_1) \cdot (B, t_2) = (A \cdot \varepsilon_e(t_1)(B), t_1 \cdot t_2),$$

where  $(A, t_1), (B, t_2) \in G_e$  and the function  $\varepsilon_e$  satisfies

$$\varepsilon_e(t) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \chi^{-e}(t)b \\ \chi^e(t)c & d \end{pmatrix}$$

for all  $t \in \mathbb{T}$  and  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$ . The group  $G_e$  is the semi-direct product of  $\mathrm{SL}_2$  and  $\mathbb{T}$  via the map  $\varepsilon_e$ .

**2.18.** In the sequel, we let  $T \subset \mathrm{SL}_2$  be the maximal torus which consists of diagonal matrices. We also consider the root subgroups

$$U_- = \left\{ \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \mid \lambda \in \mathbb{K} \right\} \text{ and } U_+ = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{K} \right\}$$

with respect to  $T$ . We make also the convention that  $\mathrm{SL}_2 \subset G_e$  via the natural isomorphism  $\mathrm{SL}_2 \cong \mathrm{SL}_2 \times \{1\}$ . We denote by  $B_- \subset G_e$  (resp.  $B_+ \subset G_e$ ) the Borel subgroup generated by  $U_-$  (resp.  $U_+$ ) and  $T \times \mathbb{T}$ .

In the following lemma we describe the subgroups of  $G_e$ .

**Lemma 2.19.** I. *The group  $G_e$  possesses a non-trivial center*

$$(2) \quad Z(G_e) = \left\{ \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, t \right) \mid \chi^e(t) = a^2, a, t \in \mathbb{K}^* \right\}.$$

II. *Any closed subgroup of  $G_e$  up to conjugation is of one of the following types:*

- (i) “center extension”, i.e. a subgroup  $H \subset G_e$  such that the neutral component  $H^\circ$  is contained in the center  $Z(G)$ .
- (ii) quasitorus  $Q \subset T \times \mathbb{T}$  such that  $Q^\circ \not\subset Z(G_e)$ ;
- (iii) “normalizing subgroup”  $N(Q) = Q \cup \tau Q$  such that  $Q^\circ \not\subset Z(G_e)$ , where  $\tau = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s \right) \in G_e$  satisfies the condition  $\tau^2 \in Q$ ;
- (iv) semidirect product  $U_+ \rtimes Q$ , where the quasitorus  $Q$  is a subgroup of  $T \times \mathbb{T}$ ;
- (v) semidirect product  $\mathrm{SL}_2 \rtimes Q'$ , where the quasitorus  $Q'$  is a subgroup of  $\mathbb{T}$ .

III. *The spherical subgroups of  $G_e$  are those with a non-central neutral component, i.e. of types (ii)–(v).*

*Proof.* Let  $(A, t)$  be an arbitrary element in  $G_e$ . Conjugating by  $(B, t^{-1}) \in G_e$ , we obtain

$$(3) \quad (B \cdot \varepsilon_e(t^{-1})(A \cdot \varepsilon_e(t)(B^{-1})), t) = (B \cdot \varepsilon_e(t^{-1})(A) \cdot B^{-1}, t).$$

It is easy to see that up to conjugation we may assume  $A$  to be upper (or lower) triangular for any  $(A, t)$ , and diagonal for a semisimple element  $(A, t) \in G_e$ . In particular, if  $(A, t)$  is central, then  $A$  is diagonal.

Let us study the center  $Z(G)$ . Consider the conjugation of a diagonal element  $z = \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, b \right) \in G$  by an arbitrary  $g = \left( \begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right) \in G$ :

$$(4) \quad g \cdot z \cdot g^{-1} = \left( \begin{pmatrix} axw - a^{-1}yz\chi^e(b), & (a^{-1} - a\chi^e(b^{-1}))xy \\ (a - a^{-1}\chi^e(b))wz, & a^{-1}xw - ayz\chi^e(b^{-1}) \end{pmatrix}, b \right).$$

So,  $z \in Z(G)$  if and only if  $a^2 = \chi^e(b)$ . So, the first statement of the lemma is proved.

Let  $H \subset G_e$  be a closed subgroup. Assume that all its elements are semisimple. Then the connected component is diagonalizable, so up to conjugation  $H^\circ \subset T \times \mathbb{T}$ .

If  $H^\circ \subset Z(G)$ , then our group is of type (i), i.e. a ‘‘center extension’’. Otherwise, there exists  $h = \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, b \right) \in H^\circ$  such that  $h \notin Z(G_e)$ . Since  $H^\circ$  is normal in  $H$ , for any  $g = \left( \begin{pmatrix} x & y \\ z & w \end{pmatrix}, t \right) \in H$  the conjugation  $g \cdot h \cdot g^{-1}$  lies in  $H^\circ$  and in particular is diagonal. By (4),  $xy = wz = 0$ . This is possible if  $g$  is either of form  $\left( \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, * \right)$  or of form  $\left( \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, * \right)$ . Since  $Q = H \cap (T \times \mathbb{T})$  is a quasitorus, either  $H = Q$  or  $H = Q \cup \tau Q$  for some  $\tau = \left( \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}, b \right)$ . In the latter case,  $\tau^2 \in Q$ . In addition, if we conjugate  $H$  by a diagonal element, then  $Q$  is preserved and  $\tau$  can be reduced to form  $\tau = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, b \right)$ .

Now assume there is  $h \in H$  that is not semisimple. By (3), we may assume up to conjugation that  $h$  is upper-triangular. Consider the Jordan decomposition  $h = h_s h_u$  into the semisimple part  $h_s$  and the unipotent part  $h_u$ . Since  $H$  is closed,  $h_u \in H$  and  $H$  contains  $U_+$ .

Let  $H$  be contained in the Borel subgroup  $\left\{ \left( \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, * \right) \right\}$ . Then any element of  $H$  is decomposed into an element of  $U$  and an element of  $T \times \mathbb{T}$ . So,  $H = U \rtimes Q$ , where  $Q \subset T \times \mathbb{T}$  is a quasitorus.

Assume now that there exists  $h \in H$  that is not contained in the Borel subgroup. Then by direct computation we obtain that the subset

$$U_+ \cdot h \cdot U_+ \cdot h^{-1} \cdot U_+ \subset \mathrm{SL}_2$$

is three-dimensional and dense in  $\mathrm{SL}_2$ . Thus,  $H = \mathrm{SL}_2 \rtimes Q$ ,  $Q \subset \mathbb{T}$ .

Finally, let us describe spherical subgroups. A subgroup  $H \subset G_e$  is spherical if and only if the neutral component  $H^\circ$  is spherical. The connected subgroup  $H^\circ$  is spherical if and only if  $B \cdot H^\circ$  is dense in  $G_e$  for some Borel subgroup  $B$ . Since  $B \subset G_e$  is of codimension one, the latter condition is equivalent to  $H^\circ \not\subset Z$ . So, a subgroup  $H \subset G_e$  is spherical if and only if  $H^\circ \not\subset Z(G_e)$ .  $\square$

*Remark 2.20.* In fact, normal closed subgroups of  $G_e$  are exhausted by:

- (1) central quasitori  $Q \subset Z(G_e)$ ;
- (2) semidirect products  $\mathrm{SL}_2 \rtimes Q'$ , where  $Q' \subset \mathbb{T}$ .

**2.21.** Let us describe a  $G_e$ -action on an affine variety  $Z$ . This is equivalent to having (see [CD03, Theorem 3])

- an  $M$ -grading on  $\mathbb{K}[Z]$ , and

- three derivations  $\partial_-, \delta, \partial_+$  on  $\mathbb{K}[Z]$  satisfying

$$[\delta, \partial_{\pm}] = \pm 2\partial_{\pm} \text{ and}$$

$$[\partial_+, \partial_-] = \partial_+ \circ \partial_- - \partial_- \circ \partial_+ = \delta,$$

where  $\partial_{\pm}$  is a homogeneous LND of degree  $\pm e$  and  $\delta$  is a semi-simple derivation inducing a downgrading of the  $M$ -grading.

Note that  $\partial_-, \partial_+$ , and  $\delta$  are given by the induced actions of  $U_-, U_+$ , and  $T$  respectively, whereas the  $M$ -grading corresponds to the induced  $\mathbb{T}$ -action. Assume that the  $\mathbb{T}$ -action on  $Z$  is of complexity at least one. Then the variety  $Z$  is  $G_e$ -spherical if and only if  $\partial_-$  and  $\partial_+$  are both of horizontal type.

**2.22.** By [AL12, Corollary 2.3 (ii)], in case  $e = 0$  we may consider the bigger torus and fall into  $e \neq 0$ . So, the classification of affine  $G_e$ -spherical varieties falls into the following categories:

- (1)  $e \neq 0$  and  $Z$  is not toric for the action of a larger torus than  $\mathbb{T}$ .
- (2)  $e \neq 0$  and  $Z$  is toric for the  $\mathbb{T}$ -action.

**Definition 2.23.** An affine  $G_e$ -spherical variety  $Z$  satisfying (1) (resp. (2)) is called of *type I* (resp. of *type II*).

The following theorem gives a complete description of affine  $G_e$ -spherical varieties of type I in terms of polyhedral divisors (see [AL12, Theorem 2.18]). Here  $t$  is a local parameter on  $\mathbb{P}^1$  and  $0, 1, \infty \in \mathbb{P}^1$  are chosen accordingly.

**Theorem 2.24.** *Let  $X$  be an affine variety with a  $G_e$ -action such that the induced  $\mathbb{T}$ -action is faithful. Then  $X$  is  $G_e$ -spherical of type I if and only if  $\mathbb{K}[X]$  is  $G_e$ -isomorphic to a rational  $G_e$ -algebra  $A = A[C, \mathfrak{D}]$  enjoying the following properties.*

- (a)  $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$  is a proper  $\sigma$ -polyhedral divisor over  $C = \mathbb{A}^1$  or  $C = \mathbb{P}^1$  supported in at most three points  $0, 1, \infty$ .
- (b) The horizontal LNDs  $\partial_-, \partial_+$  are given by the coherent pairs<sup>3</sup>  $(\mathfrak{D}_-, -e), (\mathfrak{D}_+, e)$ , where the marked point of  $\mathfrak{D}_{\pm}$  is 0 and the point at the infinity of  $\mathfrak{D}_{\pm}$  is  $\infty$ .
- (c)  $\mathfrak{D}, \mathfrak{D}_-,$  and  $\mathfrak{D}_+$  are defined by the following conditions separated in two cases.
  - Either (reflexive case)  $\Delta_0 = \text{Conv}(0, v_0) + \sigma, \Delta_1 = \text{Conv}(0, v_1) + \sigma$ , where  $v_0, v_1 \in N \setminus \sigma, v_0(e) = 1, v_1(e) = -1, \Delta_{\infty}(0) \subset e^{\perp}$ , and<sup>4</sup>  $(\pm e, -\frac{1}{2} \mp \frac{1}{2}) \in \text{Rt}(\tilde{\omega}_{\pm})$ . Representations of  $\mathfrak{D}_-, \mathfrak{D}_+$  are given respectively by

$$\begin{array}{ccccccc} & v_0 & 0 & & & 0 & v_1 \\ \cdots & \text{---} & \text{---} & \cdots & \cdots & \text{---} & \text{---} & \cdots \\ & \circ & \circ & & & \circ & \circ & \\ & 0 & 1 & & & 0 & 1 & \end{array}$$

- Or (skew case)  $\Delta_0 = v_0 + \sigma, \Delta_1 = \text{Conv}(0, v_1) + \sigma$ , where  $2v_0, v_1 \in N \setminus \sigma, 2v_0(e) = 1, v_1(e) = -1, \Delta_{\infty}(0) \subset e^{\perp}$ , and  $(\pm e, -\frac{1}{2} \mp \frac{1}{2}) \in \text{Rt}(\tilde{\omega}_{\pm})$ . Representations of  $\mathfrak{D}_-, \mathfrak{D}_+$  are given respectively by

$$\begin{array}{ccccccc} & v_0 & 0 & & & v_0 & v_1 \\ \cdots & \text{---} & \text{---} & \cdots & \cdots & \text{---} & \text{---} & \cdots \\ & \circ & \circ & & & \circ & \circ & \\ & 0 & 1 & & & 0 & 1 & \end{array}$$

<sup>3</sup>By the abuse of notation we allow a coherent pair  $(\mathfrak{D}_{\bullet}, e)$  to denote a triplet  $(1, \mathfrak{D}_{\bullet}, e)$ , cf. Theorem 2.15.

<sup>4</sup>The polyhedral cone  $\tilde{\omega}_{\pm}$  is the domain of  $\mathfrak{D}_{\pm}$  (see 2.14).

*Remark 2.25.* The LND  $\partial_{\pm}$  corresponding to the coherent pair  $(\mathfrak{D}_{\pm}, \pm e)$  in Theorem 2.24 is defined via the formulae (see the comment before [AL12, Lemma 2.16] and the proof of [AL12, Lemma 2.17])

either (reflexive case)

$$\begin{aligned} \partial_{\pm}(t^r \xi_m^{\pm} \chi^m) &= \left[ \left( \frac{1}{2} \mp \frac{1}{2} \right) v_0(m) + r \right] \cdot t^{r - \frac{1}{2} \mp \frac{1}{2}} \xi_{m \pm e}^{\pm} \chi^{m \pm e} \quad \text{for all } (m, r) \in M \oplus \mathbb{Z}, \\ \delta(t^r \chi^m) &= (v_0 - v_1)(m) \cdot t^r \chi^m \quad \text{for all } (m, r) \in M \oplus \mathbb{Z}, \end{aligned}$$

where  $\xi_m^{\pm} = (t - 1)^{-\left(\frac{1}{2} \pm \frac{1}{2}\right)v_1(m)}$ ;

or (skew case)

$$\begin{aligned} \partial_{\pm}(t^r \xi_m^{\pm} \chi^m) &= 2(v_0(m) + r) \cdot t^{r - \frac{1}{2} \mp \frac{1}{2}} \xi_{m \pm e}^{\pm} \chi^{m \pm e} \quad \text{for all } (m, r) \in M \oplus \mathbb{Z}, \\ \delta(t^r \chi^m) &= -2v_1(m) \cdot t^r \chi^m \quad \text{for all } (m, r) \in M \oplus \mathbb{Z}, \end{aligned}$$

where  $\xi_m^{\pm}$  is again  $(t - 1)^{-\left(\frac{1}{2} \pm \frac{1}{2}\right)v_1(m)}$ .

*Remark 2.26.* The formulas for  $\partial_{\pm}$  can also be expressed for an arbitrary  $Q \in \mathbb{K}(t)$ . In reflexive case they are:

$$\begin{aligned} \partial_{-}(Q\chi^m) &= (v_0(m)Q + tQ')\chi^{m-e}, \\ \partial_{+}(Q\chi^m) &= (v_1(m)Q + (t-1)Q')\chi^{m+e}, \end{aligned}$$

where  $Q' = \frac{dQ}{dt}$ . In skew case they are:

$$\begin{aligned} \partial_{-}(Q\chi^m) &= 2(v_0(m)Q + tQ')\chi^{m-e}, \\ \partial_{+}(Q\chi^m) &= 2\left(\left(v_0(m)\left(1 - \frac{1}{t}\right) + v_1(m)\right)Q + (t-1)Q'\right)\chi^{m+e}. \end{aligned}$$

### 3. COLORED CONES VERSUS COLORED POLYHEDRAL DIVISORS

Let  $G_e$  be the previously introduced reductive group  $\mathrm{SL}_2 \rtimes_e \mathbb{T}$ . The purpose of this section is to express the approach of Arzhantsev–Liendo in terms of the Luna–Vust theory applied to affine  $G_e$ -spherical varieties. Sections 3.1 and 3.2 are devoted to the type I and Section 3.3 to the type II.

#### 3.1. Classification of the type I.

Let  $X$  be an affine  $G_e$ -spherical variety of type I. Suppose that  $\mathbb{K}[X] = A[C, \mathfrak{D}]$  admits a presentation as in 2.24 (either skew or reflexive case). We identify the open  $G_e$ -orbit of  $X$  with a homogeneous space  $G_e/H$  so that  $X$  is its embedding.

*Remark 3.1.* Since  $X$  is not horospherical [AL12, Proposition 3.8], Lemma 2.19 implies that  $H$  is reductive. By the Matsushima criterion,  $G_e/H$  is affine.

**3.2.** Denote by

$$\begin{array}{ccccccc} & v_0^- & v_1^- & & & v_0^+ & v_1^+ \\ \cdots & \text{---} & \text{---} & \cdots & \cdots & \text{---} & \text{---} & \cdots \\ & \circ & \circ & & & \circ & \circ & \\ & 0 & 1 & & & 0 & 1 & \end{array}$$

the representations of  $\mathfrak{D}_{-}$  and  $\mathfrak{D}_{+}$  respectively. For instance, in the reflexive case,  $v_0^- = v_0$ ,  $v_1^- = 0$ ,  $v_0^+ = 0$ , and  $v_1^+ = v_1$ . For every  $m \in M$  we let

$$\varphi_m^{\pm} = (t - 1)^{-v_1^{\pm}(m)} t^{-\lfloor v_0^{\pm}(m) \rfloor}.$$



Then we have

$$(5) \quad \ker \partial_{\pm} = \bigoplus_{m \in \omega_{\pm}^{\vee} \cap L} \mathbb{K} \varphi_m^{\pm} \chi^m,$$

where  $\omega_{\pm}$  is the associated cone of  $\mathfrak{D}_{\pm}$  and

$$L = \{m \in M \mid v_0^+(m) \in \mathbb{Z}\} = \{m \in M \mid v_0^-(m) \in \mathbb{Z}\}.$$

In the following lemma we use the notation from 2.18.

**Lemma 3.3.** *Let  $A_M^{\pm} = \bigoplus_{m \in M} \mathbb{K}[t] \cdot \varphi_m^{\pm} \chi^m$ . Then the following assertions hold.*

- (i) *The lattice of  $B_{\pm}$ -weights of the  $B_{\pm}$ -algebra  $\mathbb{K}(X)$  is  $L$ .*
- (ii) *The variety  $X_M^{\pm} = \text{Spec } A_M^{\pm}$  is a principal open subset of  $X$  which coincides with the open  $B_{\pm}$ -orbit.*

*More precisely, the supports of all divisors  $\text{div}(\varphi_m^{\pm} \chi^m)$  for  $m \in \text{rel.int.}(\omega_{\pm}^{\vee}) \cap L$  are equal and form the complement  $X \setminus X_M^{\pm}$ .*

*Proof.* (i) The first assertion results from the fact that  $B_{\pm}$ -eigenvectors in  $\mathbb{K}(X)$  are exactly quotients of non-zero homogeneous elements of  $\ker \partial_{\pm}$ .

(ii) By Remark 2.12, we can add a principal polyhedral divisor to  $\mathfrak{D}$  so that for any  $m \in \text{rel.int.}(\omega_{\pm}^{\vee}) \cap L$  the divisor  $\mathfrak{D}(m)|_{\mathbb{A}^1}$  is supported in at most the point 0 and that  $\varphi_m^{\pm} = t^{-[v_0^{\pm}(m)]}$ . Let us fix  $m \in \text{rel.int.}(\omega_{\pm}^{\vee}) \cap L$  and denote  $\alpha_{\pm} = \varphi_m^{\pm} \chi^m$ . By [AHS08, Proposition 3.3], the localization  $\mathbb{K}[X]_{\alpha_{\pm}}$  is equal to  $A_M^{\pm}$ . Since  $\alpha_{\pm} \in \ker \partial_{\pm}$ , the derivation  $\partial_{\pm}$  extends to an LND on  $A_M^{\pm}$  written by the same letter  $\partial_{\pm}$ . Hence,  $X_M^{\pm}$  is a  $B_{\pm}$ -stable principal open subset of  $X$ .

Let us deduce that  $X_M^{\pm}$  is the open  $B_{\pm}$ -orbit. Denote by  $E_M^{\pm}$  the normalization of the algebra  $A_M^{\pm}[\sqrt[d]{\varphi_{\pm de}^{\pm} \chi^{\pm e}}]$ , where  $d = d(e)$ . Then the inclusion  $A_M^{\pm} \subset E_M^{\pm}$  yields a finite surjective map

$$\pi : Z_M^{\pm} = \text{Spec } E_M^{\pm} \rightarrow X_M^{\pm}.$$

By [Lan13, Corollary 5.3.6], the derivation  $\partial_{\pm}$  admits a unique extension to  $E_M^{\pm}$ , hence there is a natural  $B_{\pm}$ -action on  $Z_M^{\pm}$  and  $\pi$  is  $B_{\pm}$ -equivariant. Denote  $\zeta = \sqrt[d]{t}$ . The map

$$\gamma : E_M^{\pm} \rightarrow \mathbb{K}[M][[\zeta]] = \bigoplus_{m \in M} \mathbb{K}[\zeta] \chi^m, \quad \zeta^l \chi^m \mapsto \zeta^{l+dv_0^{\pm}(m)} \chi^m$$

is an isomorphism of  $M$ -graded algebras. Consider the extension of  $\partial_{\pm}$  to  $E_M^{\pm}$  denoted by  $\tilde{\partial}_{\pm}$  and introduce  $\tilde{\partial}'_{\pm} = \gamma \circ \tilde{\partial}_{\pm} \circ \gamma^{-1}$ , which is a homogeneous LND on  $\mathbb{K}[M][[\zeta]]$ . Then the induced  $B_{\pm}$ -action on  $\text{Spec } \mathbb{K}[M][[\zeta]] \cong \mathbb{T} \times \mathbb{A}^1$  is transitive, and so is the  $B_{\pm}$ -action on  $Z_M^{\pm}$ . Since  $\pi$  is  $B_{\pm}$ -equivariant and surjective, the  $B_{\pm}$ -action on  $X_M^{\pm}$  is also transitive.  $\square$

Now we give an explicit description of the  $B_{\pm}$ -divisors on  $X$  and compute their images in the vector space  $V_{\mathbb{Q}}$  (see 1.4, 1.5), where  $V = \text{Hom}(L, \mathbb{Z})$  is the valuation lattice. Note that  $V_{\mathbb{Q}}$  is identified with  $N_{\mathbb{Q}}$ .

**Proposition 3.4.** *Denote by  $\varrho_{\pm}$  the map that sends a discrete valuation  $\nu$  of  $\mathbb{K}(X)$  into the linear form  $\varrho_{\pm}(\nu) \in V_{\mathbb{Q}}$  defined as follows. For any  $B_{\pm}$ -eigenvector  $\alpha \in \mathbb{K}(X)$  of weight  $\chi_{\alpha} \in L$  we let*

$$\varrho_{\pm}(\nu)(\chi_{\alpha}) = \nu(\alpha).$$

*To simplify the notation, we also let  $\varrho(D) = \varrho(v_D)$  for a prime divisor  $D \subset X$ . Then the colors of  $X$  are given in the following table in accordance with notation 2.5.*



by the formula

$$\operatorname{div}(\varphi_m^+ \chi^m) = \sum_{\rho \in \sigma(1)^*} \varrho_+(D_\rho)(m) \cdot D_\rho + \sum_{z \in \{0,1,\infty\}} \sum_{v \in \Delta_z(0)} \varrho_+(D_{(z,v)})(m) \cdot D_{(z,v)}.$$

□

*Remark 3.5.* Let us consider the subalgebra  $A^h = A[\mathbb{A}^1, \mathfrak{D}_{G_e/H}] \subset \mathbb{K}(t)[M]$  defined by the polyhedral divisor

$$\mathfrak{D}_{G_e/H} = \Delta(G_e/H)_0 \cdot \{0\} + \Delta(G_e/H)_1 \cdot \{1\}$$

with the conditions  $\Delta(G_e/H)_0 = \operatorname{Conv}(0, v_0)$ ,  $\Delta(G_e/H)_1 = \operatorname{Conv}(0, v_1)$  in the reflexive case, and  $\Delta(G_e/H)_0 = \{v_0\}$ ,  $\Delta(G_e/H)_1 = \operatorname{Conv}(0, v_1)$  otherwise.

Then  $\mathbb{K}[X] \subset A^h$ , and  $A^h$  is  $G_e$ -stable as a subalgebra of  $\mathbb{K}(X)$ . Therefore,  $X^h = \operatorname{Spec} A^h$  is an embedding of  $G_e/H$ , see 1.2. Since  $X^h$  does not include  $G_e$ -divisors by Proposition 3.4, and  $G_e/H$  is affine by Remark 3.1,  $X^h = G_e/H$ . So, the polyhedral divisor  $\mathfrak{D}_{G_e/H}$  characterizes the open orbit.

The closed  $G_e$ -orbit  $Y \subset X$  is described as follows.

**Lemma 3.6.** *The vanishing ideal of  $Y$  is*

$$I = \bigoplus_{m \in (\sigma^\vee \setminus \operatorname{lin}(\sigma^\vee)) \cap M} A_m \chi^m, \text{ where } A_m = H^0(C, \mathcal{O}_C([\mathfrak{D}(m)])).$$

*Proof.* Let  $R = \mathbb{K}[X]/I$  and  $Z = \operatorname{Spec} R$ . We will prove that  $Z = Y$ .

*Case  $C = \mathbb{A}^1$ .* Since  $I$  is generated by  $\bigcup_{\rho \in \sigma(1)} I(D_\rho)$ , the ideal  $I$  is  $G_e$ -stable and  $R$  is a rational  $G_e$ -algebra. Let us describe  $R$  in terms of polyhedral divisors. Consider the projection  $\pi : N \rightarrow \tilde{N}$ , where  $\tilde{N} = N/(\operatorname{lin}(\sigma^\vee)^\perp \cap N)$ . The lattice  $\tilde{N}$  is the dual of  $\operatorname{lin}(\sigma^\vee) \cap M$ . Let  $\tilde{\mathfrak{D}}$  be the polyhedral divisor with coefficients in  $\tilde{N}_\mathbb{Q} = \mathbb{Q} \otimes_{\mathbb{Z}} \tilde{N}$  given by

$$\tilde{\mathfrak{D}} = \operatorname{Conv}(0, \pi(v_0)) \cdot \{0\} + \operatorname{Conv}(0, \pi(v_1)) \cdot \{1\},$$

in the reflexive case, and by

$$\tilde{\mathfrak{D}} = \{\pi(v_0)\} \cdot \{0\} + \operatorname{Conv}(0, \pi(v_1)) \cdot \{1\}$$

in the skew case. Note that  $e \in \operatorname{lin}(\sigma^\vee) \setminus \{0\}$  (see [Lie10a, Lemma 3.1]) and  $\pi(v_0), \pi(v_1)$  are nonzero. We have a  $G_e$ -equivariant isomorphism  $R \cong A[C, \tilde{\mathfrak{D}}]$ . The induced homogeneous LNDs on  $A[C, \tilde{\mathfrak{D}}]$  are horizontal and represented by the graphs

$$\begin{array}{ccccccc} & \pi(v_0^-) & \pi(v_1^-) & & & \pi(v_0^+) & \pi(v_1^+) \\ \cdots & \text{---} & \text{---} & \cdots & \cdots & \text{---} & \text{---} & \cdots \\ & 0 & 1 & & & 0 & 1 & \end{array}$$

In particular,  $Z$  is an affine spherical variety of type I which is homogeneous under  $G_e$ , according to Remark 3.5. So,  $Z$  is the closed  $G_e$ -orbit.

*Case  $C = \mathbb{P}^1$ .* By properness of  $\mathfrak{D}$ , every homogeneous element of degree  $m \in \operatorname{lin}(\sigma^\vee) \cap M$  is invertible and so belongs to  $\ker \partial_- \cap \ker \partial_+$ . Hence  $I$  is  $G_e$ -stable. Furthermore, the  $\mathbb{T}$ -action on  $Z$  is transitive. This allows us to conclude. □

The following theorem provides the Luna–Vust description of the affine  $G_e$ -spherical variety  $X$  in terms of its AL-colorings  $\mathfrak{D}_-, \mathfrak{D}_+$ .

**Theorem 3.7.** *Let  $X$  be an affine  $G_e$ -spherical variety admitting a presentation as in Theorem 2.24. Then the following statements hold.*

- (i) *If  $C = \mathbb{A}^1$  then  $X$  is a toroidal spherical variety with colored cone  $(\sigma, \emptyset)$ .*
- (ii) *If  $C = \mathbb{P}^1$  then the colored cone of  $X$  is given by the following table.*

	<i>Reflexive case</i>	<i>Skew case</i>
$B_-$ -action	$(\omega_-, \{D_{(0,0)}, D_{(1,v_1)}\})$	$(\omega_-, D_{(1,v_1)})$
$B_+$ -action	$(\omega_+, \{D_{(0,v_0)}, D_{(1,0)}\})$	$(\omega_+, D_{(1,0)})$

*Proof.* By Proposition 3.4 and Lemma 3.6, the closed orbit is contained in all colors of  $X$  in the case  $C = \mathbb{P}^1$  and is not contained in any colors in the case  $C = \mathbb{A}^1$ . In the former case, one concludes by Lemma 1.11.  $\square$

### 3.2. Horizontal and vertical $G_e$ -invariant valuations.

As before,  $X$  is an affine embedding of type I of a spherical homogeneous space  $G_e/H$  with the associated colored cone  $(\mathcal{C}, \mathcal{F}_Y)$ . In this subsection, we provide the representation of  $\mathbb{K}[X]$  by a polyhedral divisor and AL-colorings in terms of  $(\mathcal{C}, \mathcal{F}_Y)$ . For this, we examine the valuation cone of  $G_e/H$ .

**3.8.** Without loss of generality, we may suppose that  $\mathbb{K}[G_e/H]$  is equal to the  $G_e$ -algebra  $A[\mathbb{A}^1, \mathfrak{D}_{G_e/H}]$  defined in Remark 3.5. Thus, a variety  $Z$  is an affine embedding of  $G_e/H$  if and only if  $\mathbb{K}[Z]$  is a  $G_e$ -stable normal affine subalgebra of  $A[\mathbb{A}^1, \mathfrak{D}_{G_e/H}]$ .

Inspired by [Tim08], we introduce the following terminology.

**Definition 3.9.** A non-trivial  $G_e$ -invariant valuation  $v$  on  $\mathbb{K}(G_e/H)$  is called *horizontal* (or *central*) if  $v$  is trivial on the subfield  $\mathbb{K}(t)$ . Otherwise,  $v$  is said to be *vertical*. We denote by  $\mathcal{V}_{\text{hor}}$  (resp.  $\mathcal{V}_{\text{vert}}$ ) the subset of horizontal (resp. vertical) valuations in the valuation cone  $\mathcal{V}$  of  $G_e/H$ .

We make the convention that  $\mathcal{V}_{\text{hor}} \cap \mathcal{V}_{\text{vert}} = \{0\}$ , i.e. the trivial valuation is both horizontal and vertical. For the next result, we emphasize that vectors in  $N_{\mathbb{Q}}$  are regarded as primitive with respect to the valuation lattice  $V = \text{Hom}(L, \mathbb{Z})$ .

**Lemma 3.10.** *Consider the  $B_{\pm}$ -action on  $G_e/H$ . Then  $\mathcal{V} = (\mathbb{Q}_{\geq 0}(\mp e))^{\vee}$ . Further, denote*

$$S = \text{Cone}(-\rho_{\pm}(\mathcal{F}_0)),$$

where  $\mathcal{F}_0$  is the set of colors of  $G_e/H$ . Then  $\mathcal{V}_{\text{hor}} \cap (\mathcal{V} \setminus S) = e^{\perp} \setminus \{0\}$ .

*Proof.* Let us show that  $\mathcal{V} = \text{Cone}(\mp e)^{\vee}$  with respect to the  $B_{\pm}$ -action. For any primitive  $\rho \in e^{\perp} \setminus \{0\}$ , let us define the affine embedding  $Z = \text{Spec } A[\mathbb{A}^1, \mathfrak{D}_{\rho}]$  of  $G_e/H$  via

$$\mathfrak{D}_{\rho} = \Delta_0^{\rho} \cdot \{0\} + \Delta_1^{\rho} \cdot \{1\},$$

where  $\Delta_0^{\rho}$  and  $\Delta_1^{\rho}$  are as in Theorem 2.24 with  $\sigma = \text{Cone}(\rho)$ . Then by Proposition 3.4,  $Z$  has a unique horizontal  $G_e$ -divisor  $D_{\rho}$ , hence  $\rho = \varrho_{\pm}(D_{\rho})$  is contained in  $\mathcal{V}$ . This yields  $e^{\perp} \subset \mathcal{V}_{\text{hor}}$ . Using again Proposition 3.4, we may construct an affine embedding of  $G_e/H$  having a vertical  $G_e$ -divisor  $D_{(v,\infty)}$  such that  $\varrho_{\pm}(D_{(v,\infty)}) \in \{v \in N_{\mathbb{Q}} \mid \mp e(v) > 0\}$ . Hence

$$\text{Cone}((\mp e)^{\vee}) = \text{Cone}(e^{\perp} \cup \varrho_{\pm}(D_{(v,\infty)})) \subset \mathcal{V}.$$

Since  $G_e/H$  is not horospherical, by [Kno91, Corollary 6.2] we obtain  $\mathcal{V} = \text{Cone}(\mp e)^\vee$ .

Take a primitive vector  $v \in \mathcal{V} \setminus S$  and consider the colored cone  $(\mathcal{C}, \mathcal{F}_Y)$  of  $G_e/H$  given by the following conditions:

- (1)  $\mathcal{F}_Y$  is the set of all colors of  $G_e/H$ ;
- (2)  $\mathcal{C} = \mathbb{Q}_{\geq 0}v + \sum_{D \in \mathcal{F}_Y} \mathbb{Q}_{\geq 0}\varrho(D)$ .

We note that  $\mathcal{C}$  is strongly convex, since  $v \notin S$ . Assume further that  $v \notin e^\perp$ . In this case  $(\mathcal{C}, \mathcal{F}_Y)$  is an affine colored cone. Indeed,  $\varrho(\mathcal{F}_Y) \cap \mathcal{V} = \emptyset$  by Proposition 3.4, and we set  $m = 0$  in Definition 1.7. Let  $Z'$  be the embedding corresponding to  $(\mathcal{C}, \mathcal{F}_Y)$ . Then  $Z'$  is not toroidal and so admits a presentation  $\mathbb{K}[Z'] = A[\mathbb{P}^1, \mathfrak{D}']$  as in Theorem 2.24. By 3.4, the only  $G_e$ -divisor  $D'$  on  $Z'$  is vertical and  $\varrho(D') = v$ . Thus,  $v \in \mathcal{V}_{\text{vert}}$ . The assertion follows.  $\square$

*Remark 3.11.* The primitive vectors in  $\mathcal{V} \setminus S$  are exactly the images of the  $G_e$ -divisors of all affine embeddings of  $G_e/H$ .

**Definition 3.12.** Given a vector  $v \in N_{\mathbb{Q}} \setminus e^\perp$ , we introduce the following function  $\beta_v : N_{\mathbb{Q}} \rightarrow \mathbb{Q}$ . Consider the linear projection  $\pi_v : N_{\mathbb{Q}} \rightarrow \mathbb{Q} \cdot v$  along  $e^\perp$ . Then given  $x \in N_{\mathbb{Q}}$ ,  $\beta_v(x)$  is the unique rational number such that  $\pi_v(x) = \beta_v(x)v$ . We say that  $\pi_v$  is the projection defining  $\beta_v$ .

This notation is used in the following proposition that introduces the explicit construction of the AL-colorings corresponding to an affine embedding of  $G_e/H$  of type I. This is the inverse of Theorem 3.7.

**Proposition 3.13.** *Let  $X$  be an affine  $G_e$ -spherical variety of type I, whose open  $G_e$ -orbit is identified with the homogeneous space  $G_e/H$ . Let  $\beta_-$  be  $\beta_{v_0}$  and  $\beta_+$  be either  $\beta_{v_1}$  in the reflexive case or  $\beta_{v_0+v_1}$  in the skew case (cf. latter table in Proposition 3.4). Denote*

$$\mathcal{C}_{\pm}^{\text{vert}} = \left\{ \frac{v}{\beta_{\pm}(v)} \mid v \in \mathcal{C}(1), v(e) > 0 \right\}.$$

Let  $\sigma \subset N_{\mathbb{Q}}$  be the strongly convex polyhedral cone equal to either

$$\sigma = \text{Cone}(\mathcal{C}(1) \cap e^\perp, \mathcal{C}_{\pm}^{\text{vert}}, \mathcal{C}_{\pm}^{\text{vert}} \pm v_0, \mathcal{C}_{\pm}^{\text{vert}} \mp v_1, \mathcal{C}_{\pm}^{\text{vert}} \pm v_0 \mp v_1)$$

in the reflexive case or

$$\sigma = \text{Cone}(\mathcal{C}(1) \cap e^\perp, \mathcal{C}_{\pm}^{\text{vert}}, \mathcal{C}_{\pm}^{\text{vert}} \mp v_1)$$

in the skew case. We have  $\mathbb{K}[X] = A[C, \mathfrak{D}]$ , where  $\mathfrak{D}$  is a  $\sigma$ -polyhedral divisor over  $C = \mathbb{A}^1$ , if  $\mathcal{C} \subset e^\perp$ , and over  $C = \mathbb{P}^1$ , otherwise. The non-trivial coefficients of  $\mathfrak{D}$  are defined as follows (see 3.5).

$$\Delta_0 = \Delta(G_e/H)_0 + \sigma, \quad \Delta_1 = \Delta(G_e/H)_1 + \sigma, \quad \text{and}$$

$$\Delta_\infty = \text{Conv} \left( \left\{ \frac{v - \pi_{\pm}(v)}{\beta_{\pm}(v)} \mid v \in \mathcal{C}(1) \setminus e^\perp \right\} \right) + \sigma \text{ if } C = \mathbb{P}^1,$$

where  $\pi_{\pm}$  is the projection defining  $\beta_{\pm}$ .

*Proof.* First, we note by [Kno91, Lemma 2.4] that  $\mathcal{C}(1) \cap \mathcal{V}$  consists exactly of images in  $N_{\mathbb{Q}}$  of the  $G_e$ -divisors of  $X$ , and elements of  $\mathcal{C}(1) \cap e^\perp$  (resp.  $\mathcal{C}(1) \setminus e^\perp$ ) correspond to horizontal (resp. vertical)  $G_e$ -divisors.

The  $G_e$ -algebra  $\mathbb{K}[X] = A[C, \mathfrak{D}]$  admits a presentation as in Theorem 2.24, where  $\mathfrak{D}$  is a  $\sigma'$ -polyhedral divisor for some strongly convex polyhedral cone  $\sigma' \subset N_{\mathbb{Q}}$ . Using

Proposition 3.4 and Theorem 3.7, we remark that  $\sigma \subset N_{\mathbb{Q}}$  is generated by  $\sigma'(1)$  if  $C = \mathbb{A}^1$  and by  $\sigma'(1)^*$  and  $\deg \mathcal{D}$  if  $C = \mathbb{P}^1$ . Hence  $\sigma = \sigma'$ . Clearly, we have  $\Delta_0 = \Delta(G/H)_0 + \sigma$  and  $\Delta_1 = \Delta(G/H)_1 + \sigma$ . For the case  $C = \mathbb{P}^1$ , the description of  $\Delta_{\infty}$  follows again from 3.4.  $\square$

### 3.3. Classification of the type II.

In this subsection, we explain the connection between Demazure roots and affine colored cones for the type II. Let  $e$  be a nonzero vector of  $M$ . Note by [AL12, Proposition 3.8] an affine  $G_e$ -variety  $X$  is not of type I if and only if  $X$  is horospherical. The following result shows that an affine  $G_e$ -spherical  $X$  that is not of type I can be assumed to be of type II for a larger reductive group.

**Lemma 3.14.** *Let  $\Omega$  be a quasi-affine  $G_e$ -horospherical homogeneous space. Assume that the induced  $\mathbb{T}$ -action on  $\Omega$  is of complexity one. Then there exists an open subset  $\tilde{\Omega}$  of the affine closure  $\bar{\Omega} = \text{Spec } \mathbb{K}[\Omega]$ , which is a homogeneous space under a reductive group  $\tilde{G}_{\tilde{e}} = \text{SL}_2 \rtimes_{\tilde{e}} \tilde{\mathbb{T}}$  containing  $G_e$  with the following properties.*

- (a) *The torus  $\mathbb{T}$  is a subtorus of  $\tilde{\mathbb{T}}$ .*
- (b) *The  $\tilde{\mathbb{T}}$ -action on  $\tilde{\Omega}$  is of complexity 0.*
- (c)  *$\tilde{\Omega}$  is an embedding of the  $G_e$ -spherical homogeneous space  $\Omega$ .*

Moreover, the inclusion map  $\Omega \rightarrow \tilde{\Omega}$  naturally identifies the affine  $G_e$ -embeddings of  $\Omega$  with the affine  $\tilde{G}_{\tilde{e}}$ -embeddings of  $\tilde{\Omega}$ .

*Proof.* First of all,  $\mathbb{K}[\Omega]$  is finitely generated, so  $\bar{\Omega} = \text{Spec } \mathbb{K}[\Omega]$  is an affine variety and  $\text{codim}_{\bar{\Omega}}(\bar{\Omega} \setminus \Omega) \geq 2$ . Since the  $\text{SL}_2$ -action induced by  $G_e$  on  $\bar{\Omega}$  is horospherical and horizontal, [AL12, Proposition 3.8] implies the existence of the  $\tilde{G}_{\tilde{e}}$ -homogeneous space  $\tilde{\Omega} \subset \bar{\Omega}$  that is a  $G_e$ -equivariant embedding of  $\Omega$ .

Moreover, let  $X$  be an affine  $G_e$ -embedding of  $\Omega$ . The  $\tilde{G}'_{\tilde{e}}$ -action on  $X$  is constructed in [AL12, Proposition 3.8] by extending the torus  $\mathbb{T} \subset G_e$  by the torus  $S$  commuting with  $G_e$ , see [AL12, Corollary 3.4]. The torus  $S$  corresponds to the grading defined by the decomposition into isotypic components with respect to the  $\text{SL}_2$ -action. Hence it does not depend on the embedding  $X$ . Therefore,  $\tilde{G}'_{\tilde{e}} = \tilde{G}_{\tilde{e}}$  and  $\tilde{G}_{\tilde{e}}$ -action on  $X$  is the extension of  $\tilde{G}_{\tilde{e}}$ -action on  $\tilde{\Omega}$ .  $\square$

For the next paragraphs, we let  $X$  be an affine  $G_e$ -spherical variety of type II and assume that  $\mathbb{T}$ -action on  $X$  is faithful.

**3.15.** Since the torus  $\mathbb{T}$  acts on  $X$  with an open orbit, we may represent the  $M$ -graded algebra  $\mathbb{K}[X]$  as the semigroup algebra

$$\mathbb{K}[\sigma^{\vee} \cap M] = \bigoplus_{m \in \sigma^{\vee} \cap M} \mathbb{K}\chi^m,$$

where  $\sigma \subset N_{\mathbb{Q}}$  is a strongly convex polyhedral cone. For any  $m \in \sigma^{\vee} \cap M$  the character  $\chi^m : \mathbb{T} \rightarrow \mathbb{G}_m$  is identified with a regular function on  $X$ . Recall that the set of *semi-simple roots* [Oda88, Section 3.4] of  $\sigma$  is

$$\text{Rt}_{\text{ss}}(\sigma) = \text{Rt}(\sigma) \cap (-\text{Rt}(\sigma)).$$

By [AL12, Theorem 2.7], we have  $e \in \text{Rt}_{\text{ss}}(\sigma)$ . Denoting by  $\rho_{\pm}$  the distinguished ray of  $\pm e$ , the  $\mathfrak{sl}_2$ -triplet on  $\mathbb{K}[X]$  corresponding to the  $G_e$ -action on  $X$  is  $\{\partial_-, \delta, \partial_+\}$ , where  $\partial_{\pm}(\chi^m) = \langle m, \rho_{\pm} \rangle \chi^{m \pm e}$  for all  $m \in \sigma^{\vee} \cap M$ , and  $\delta = [\partial_+, \partial_-]$ .

**Lemma 3.16.** *The following assertions hold.*

- (i) *The lattice  $L_{\pm}$  of  $B_{\pm}$ -weights of  $\mathbb{K}(X)$  is the sublattice of  $M$  generated by  $\tau_{\pm} \cap M$ , where  $\tau_{\pm} \subset \sigma^{\vee}$  is the dual facet of the ray  $\rho_{\pm} \subset \sigma$ .*
- (ii) *The open  $B_{\pm}$ -orbit on  $X$  is  $X_{B_{\pm}} = \text{Spec } \mathbb{K}[\rho_{\pm}^{\vee} \cap M]$ .*
- (iii) *The open  $G_e$ -orbit on  $X$  is the toric  $\mathbb{T}$ -variety  $X_{\Sigma}$  associated to the fan  $\Sigma = \{\rho_-, \{0\}, \rho_+\}$ .*

*Proof.* (i) This follows from the equality  $\ker \partial_{\pm} = \mathbb{K}[\tau_{\pm} \cap M]$ .

(ii)  $X_{B_{\pm}} \cong \mathbb{A}^1 \times \mathbb{G}_m^{n-1}$ , where  $n = \dim \mathbb{T}$ , is an open  $B_{\pm}$ -stable subset of  $X$ . It consists of two  $\mathbb{T}$ -orbits, namely  $\{0\} \times \mathbb{G}_m^{n-1}$  and  $\mathbb{G}_m \times \mathbb{G}_m^{n-1} \cong \mathbb{T}$ . Since a  $U_{\pm}$ -orbit, which is an affine line, cannot be contained in neither  $\mathbb{T}$ -orbit, one concludes.

(iii) By the above step, the open  $B_{\pm}$ -orbit corresponds to the fan  $\{\{0\}, \rho_{\pm}\}$ , hence the open  $G_e$ -orbit contains  $X_{\Sigma}$ . So it remains to show that  $X_{\Sigma}$  is  $\text{SL}_2$ -stable. The coordinate ring of  $X_{\Sigma}$  is  $\mathbb{K}[\Lambda \cap M]$ , where  $\Lambda = \rho_-^{\vee} \cap \rho_+^{\vee} \subset M_{\mathbb{Q}}$ . Since  $\mathbb{K}[X_{\Sigma}]$  is  $\text{SL}_2$ -stable, it remains to show that the complement  $\text{Spec}(\mathbb{K}[\Lambda \cap M]) \setminus X_{\Sigma}$  is also  $\text{SL}_2$ -stable. The ideal of the complement

$$I = \bigoplus_{m \in (\Lambda \setminus \text{lin}(\Lambda)) \cap M} \mathbb{K} \chi^m \subset \mathbb{K}[\Lambda^{\vee} \cap M]$$

is  $\partial_{\pm}$ -stable, which proves the assertion.  $\square$

*Remark 3.17.* Given an affine horospherical embedding  $X$ , the associated colored cone is  $(\mathcal{C}, \mathcal{F})$ , where  $\mathcal{F}$  is the set of all colors of  $X$ , see Definition 1.7. In particular, for any affine embedding of  $X_{\Sigma}$  the associated colored cone contains exactly one color.

The main result of this subsection is the following proposition. It can also be obtained from [AKP13, Theorem 1.1].

**Proposition 3.18.** *Let  $X$  be an affine  $G_e$ -spherical variety of type II. Assume that the induced  $\mathbb{T}$ -action on  $X$  is faithful.*

- (i) *Assume that  $X$  is the toric  $\mathbb{T}$ -variety corresponding to the strongly convex cone  $\sigma \subset N_{\mathbb{Q}}$ . Let  $\omega_{\pm}$  be the image of the cone*

$$\sum_{\rho \in \sigma(1) \setminus \{\rho_{\pm}\}} \mathbb{Q}_{\geq 0} \cdot \rho \subset \sigma$$

*under the projection  $N_{\mathbb{Q}} \rightarrow (V_{\pm})_{\mathbb{Q}} = \text{Hom}(L_{\pm}, \mathbb{Q}) = (N/\langle \rho_{\pm} \rangle)_{\mathbb{Q}}$ , and  $D_{\rho_{\pm}}$  be the  $\mathbb{T}$ -divisor on  $X$  corresponding to the ray  $\rho_{\pm}$ . Then the associated colored cone of  $X$  in  $(V_{\pm})_{\mathbb{Q}}$  with respect to the  $B_{\pm}$ -action is  $(\omega_{\pm}, D_{\rho_{\pm}})$ .*

- (ii) *Conversely, assume that  $X$  is an affine embedding of the  $G_e$ -homogeneous space  $X_{\Sigma}$  (see 3.16) given by a colored cone  $(\mathcal{C}, D_{\pm})$ , where  $D_{\pm}$  is the color of  $X$  for the  $B_{\pm}$ -action. Then  $X$  is equal to the toric  $\mathbb{T}$ -variety  $\text{Spec } \mathbb{K}[\mathcal{C}^{\vee} \cap \rho_{\pm}^{\vee} \cap M]$ , where  $\rho_{\pm} \in N$  is the primitive vector corresponding to the  $\mathbb{T}$ -divisor  $D_{\pm}$ .*

*Proof.* (i) By Lemma 3.16, the  $G_e$ -divisors of  $X$  correspond to elements of  $\sigma(1) \setminus \{\rho_-, \rho_+\}$  and the image of the unique color (with respect to  $B_{\pm}$ ) is  $\rho_{\pm}$ .

Assertion (ii) follows directly from (i).  $\square$

#### 4. DEMAZURE ROOTS AND AFFINE SPHERICAL VARIETIES

The existence of normalized  $\mathbb{G}_a$ -actions on an affine spherical variety imposes restrictive conditions on its geometric structure. We study some of these conditions in the

framework of the Luna–Vust theory. In particular, we define the notion of Demazure roots of an affine spherical embedding  $X$  (see 4.3), which describes the normalized  $\mathbb{G}_a$ -actions on  $X$ . Section 4.2 is devoted to the example of affine  $G_e$ -spherical varieties, where  $G_e$  is the reductive group  $\mathrm{SL}_2 \rtimes_e \mathbb{T}$ .

#### 4.1. General case.

We denote by  $G$  a connected reductive group and fix a Borel subgroup  $B \subset G$ . We also let  $U$  be the unipotent radical of  $B$ . We begin with a preliminary lemma.

**Lemma 4.1.** *Let  $Z$  be a quasi-affine variety endowed with a  $G$ -action. Let  $\partial$  be an LND on  $\mathbb{K}[Z]$ . Then the  $\mathbb{G}_a$ -action corresponding to  $\partial$  is normalized of degree  $\chi$  if and only if*

$$g \cdot \partial(f) = \chi(g) \cdot \partial(g \cdot f)$$

for all  $f \in \mathbb{K}[Z]$  and  $g \in G$ . If the latter condition holds, then  $\partial$  sends  $B$ -eigenvectors into  $B$ -eigenvectors. Furthermore, if  $Z$  contains an open  $B$ -orbit and  $\partial$  is nonzero, then  $\partial$  is uniquely determined by the character  $\chi$  up to the multiplication by a nonzero constant.

*Proof.* See [FKZ13, Section 1.1] for the existence and properties of the LND  $\partial$ . The first claims are obtained similarly to the proof of [FZ05, Lemma 2.2]. Moreover, for any  $b \in B$  and  $B$ -eigenvector  $f \in \mathbb{K}[Z]$  with  $B$ -weight  $\chi_f$ , one has

$$b \cdot \partial(f) = \chi(b) \cdot \partial(b \cdot f) = (\chi \cdot \chi_f)(b) \cdot \partial(f).$$

Hence,  $\partial$  sends  $B$ -eigenvectors into  $B$ -eigenvectors.

Assume that  $Z$  contains an open  $B$ -orbit. Passing to the normalization, we may suppose that  $Z$  is normal. Since the semi-simple  $G$ -module  $\mathbb{K}[Z]$  has no multiplicities, the subalgebra  $\mathbb{K}[Z]^U$  is a  $\partial$ -stable normal finitely generated semi-group algebra. Furthermore, the restriction  $\partial'$  of  $\partial$  on  $\mathbb{K}[Z]^U$  is a homogeneous LND. Since every simple submodule of  $\mathbb{K}[X]$  is generated by  $G \cdot f$  for some  $B$ -weight  $f \in \mathbb{K}[X]$ ,  $\partial$  is uniquely determined by  $\partial'$ . One concludes by [Lie10a, Corollary 2.8].  $\square$

In the sequel, we fix an affine embedding  $X$  of a spherical homogeneous space  $G/H$  given by a colored cone  $(\mathcal{C}, \mathcal{F}_Y)$ . We denote by  $\mathcal{F}_0$  the set of colors of  $X$  and by  $\Gamma$  the cone generated by  $\varrho(\mathcal{F}_0) \cup \mathcal{C}$ . We recall that  $\Gamma^\vee$  is the cone of  $B$ -weights of  $\mathbb{K}[X]$  (see Lemma 1.11).

**4.2.** An embedding  $Z$  of  $G/H$  is called *elementary* if  $Z$  has two orbits so that the complement of the open orbit is a divisor. In particular, every elementary embedding is smooth. There is a natural bijection between non-trivial normalized discrete  $G$ -invariant valuations on  $\mathbb{K}(G/H)$  and elementary embeddings of  $G/H$  (see [BP87, 2.2]).

Let  $v \in \Gamma(1)$ . We say that  $v$  *corresponds to a  $G$ -divisor* of  $X$  if there exists a  $G$ -divisor  $D \subset X$  such that  $v = \varrho(D)$ . In the sequel, we make the following convention. If  $v$  corresponds to a  $G$ -divisor of  $X$ , then  $X_v$  denotes the associated elementary embedding of  $G/H$  given by the colored cone  $(\mathbb{Q}_{\geq 0} \cdot v, \emptyset)$ . Otherwise,  $v \in \mathbb{Q}_{\geq 0} \cdot \varrho(D_0)$  for some  $D_0 \in \mathcal{F}_0$  (see [Kno91, Lemma 2.4]) and we let  $X_v = G/H$ .

Let us introduce the main definition of this subsection.

**Definition 4.3.** A character  $\chi$  of  $G$  is called a *Demazure root* of  $X$  if  $\chi \in \mathrm{Rt}(\Gamma)$  and the following holds. The embedding  $X_v$ , where  $v = \rho_\chi$  is the distinguished ray of  $\chi$



(see 2.10), admits a normalized  $\mathbb{G}_a$ -action of degree  $\chi$  and is homogeneous under the induced  $\mathbb{G}_a \rtimes_\chi G$ -action.

We denote by  $\text{Rt}(X)$  the set of Demazure roots of  $X$ . A root  $\chi \in \text{Rt}(X)$  is said to be *exterior* if  $\rho_\chi$  corresponds to a  $G$ -divisor. Otherwise,  $\chi$  is called *interior*. We denote by  $\text{Rt}_{\text{ext}}(X)$  (resp.  $\text{Rt}_{\text{int}}(X)$ ) the set of exterior (resp. interior) roots of  $X$ . Clearly, we have

$$\text{Rt}_{\text{ext}}(X) \cap \text{Rt}_{\text{int}}(X) = \emptyset \text{ and } \text{Rt}(X) = \text{Rt}_{\text{ext}}(X) \cup \text{Rt}_{\text{int}}(X).$$

**4.4.** Two nonzero LNDs  $\partial, \partial'$  on  $\mathbb{K}[X]$  are called equivalent if  $\partial = \lambda \cdot \partial'$  for some  $\lambda \in \mathbb{K}^*$ . We denote by  $\text{LND}_G(X)$  the set of equivalence classes of nonzero LNDs on  $\mathbb{K}[X]$  corresponding to normalized  $\mathbb{G}_a$ -actions. Since  $\deg \partial = \deg \partial'$  for equivalent  $\partial$  and  $\partial'$ , the degree is well defined on  $\text{LND}_G(X)$ . It also coincides with the degree of the homogeneous LND on  $\mathbb{K}[X]^U$  induced by any  $\partial' \in [\partial]$ .

The following theorem allows to reduce the classification of normalized  $\mathbb{G}_a$ -actions on  $X$  to the case when  $X$  is either quasi-affine homogeneous or elementary, and the induced action of  $\mathbb{G}_a \rtimes_\chi G$  on  $X$  is transitive.

**Theorem 4.5.** *The map*

$$(6) \quad \iota : \text{LND}_G(X) \rightarrow \text{Rt}(X), \quad [\partial] \mapsto \deg \partial$$

*is well defined and bijective. Furthermore, let  $\partial \in \text{LND}_G(X)$  and  $\chi = \deg \partial \in \text{Rt}(X)$  have the distinguished ray  $v$ . Then the following hold.*

- (i) *The open orbit of the induced  $\mathbb{G}_a \rtimes_\chi G$ -orbit on  $X$  is  $X_v$ .*
- (ii) *We have  $-v \in \mathcal{V}$ .*
- (iii) *The kernel  $\ker \partial$  is a finitely generated subalgebra.*

*Proof.* Consider  $[\partial] \in \text{LND}_G(X)$ . Let us show that  $\chi = \deg \partial \in \text{Rt}(X)$ . By Lemmas 1.11, 4.1, and [Lie10a, Theorem 2.7], we have  $\chi \in \text{Rt}(\Gamma)$ . Choose a  $B$ -eigenvector  $f \in \mathbb{K}[X]$  such that  $\chi_f \in \text{rel.int.}(\Gamma \cap \rho_\chi^\perp) \cap L$ , and let  $B.x$  be the open  $B$ -orbit of  $X$ . Since  $f \in \ker \partial$ , the subset  $X_f = X \setminus \text{div}(f)$  is stable under the  $\mathbb{G}_a$ -action induced by  $\partial$ .

Assume first that  $v = \rho_\chi \in \Gamma(1)$  does not correspond to a  $G$ -divisor of  $X$ . Then we may consider the non-empty subset of colors

$$\mathcal{F}' = \{D' \in \mathcal{F}_0 \mid \varrho(D') = v\}.$$

Using the relation

$$\text{div } f = \sum_{D \in \mathcal{F}_0 \cup \mathcal{P}} \langle \chi_f, \varrho(D) \rangle \cdot D,$$

where  $\mathcal{P}$  is the set of  $G$ -divisors of  $X$ , we obtain the equality

$$X_f = X \setminus \bigcup_{\mathcal{P} \cup \mathcal{F}_0 \setminus \mathcal{F}'} D.$$

The complement to  $(G/H) \cap X_f$  in  $X_f$  is of codimension at least two, hence the subset

$$Z = \mathbb{G}_a \star (X_f \setminus (G/H))$$

is  $\mathbb{G}_a$ -stable and of codimension at least one. Thus,  $G/H = G \cdot (X_f \setminus Z)$  is also  $\mathbb{G}_a$ -stable. Hence  $\chi \in \text{Rt}_{\text{int}}(X)$  and (i) is proven for this case.

Now assume that  $v = \rho_\chi \in \Gamma(1)$  corresponds to a  $G$ -divisor  $D_v$  of  $X$ . Then

$$X_f = X \setminus \bigcup_{(\mathcal{P} \cup \mathcal{F}_0) \setminus \{D_v\}} D.$$

So, the open subset  $G \cdot X_f$  is  $\mathbb{G}_a$ -stable and its unique  $G$ -divisor is  $D_v$ .

Let us prove by contradiction that  $\mathbb{G}_a \star D_v$  is dense in  $X$ . If  $\mathbb{G}_a \star D_v$  is not dense, then  $G/H$  is  $\mathbb{G}_a$ -stable. Remark that the dual cone  $\Gamma_0$  of the  $B$ -weights of  $\mathbb{K}[G/H]$  is the subcone of  $\Gamma$  generated by  $\varrho(\mathcal{F}_0)$ . By considering the induced LND on  $\mathbb{K}[G/H]^U$  (see 4.1), which has same degree  $\chi$ , we have  $\chi \in \text{Rt}(\Gamma_0)$ . Let  $v'$  be the distinguished ray of  $\chi$  with respect to  $\Gamma_0$ . The inclusion

$$\mathbb{K}[X]^U \cap \mathbb{K}(X)^\partial \subset \mathbb{K}[G/H]^U \cap \mathbb{K}(X)^\partial$$

yields the inclusion of facets  $v^\perp \cap \Gamma^\vee \subset v'^\perp \cap \Gamma_0^\vee$ . Hence  $v^\perp = v'^\perp$ . Since  $v, v'$  are primitive vectors and  $\Gamma$  is strongly convex, we have  $v = v' \in \Gamma(1) \cap \Gamma_0(1)$ . Therefore  $v$  does not correspond to a  $G$ -divisor of  $X$ , a contradiction. Hence  $\mathbb{G}_a \star D_v$  is dense in  $X$ .

Let  $G.y$  be the orbit which closure in  $G \cdot X_f$  is  $D_v$ . Such an orbit  $G.y$  exists due to the fact that  $X$  has a finite number of orbits. Then the subset

$$Z_0 = G \cdot X_f \setminus (G/H \cup G.y)$$

is closed of codimension at least two. So,  $Z_1 = \overline{\mathbb{G}_a \star Z_0}$  is a  $\mathbb{G}_a \rtimes_\chi G$ -stable subset of codimension at least one. Hence  $Z_1 \cap G/H = \emptyset$ . If  $Z_1 \cap G.y \neq \emptyset$ , then  $Z_1$  contains  $D_v$ , yielding a contradiction. Therefore,  $Z_1 = Z_0$  and  $X_v$  is  $\mathbb{G}_a$ -stable. Since  $D_v$  is not  $\mathbb{G}_a$ -stable,  $\mathbb{G}_a \rtimes_\chi G$  acts homogeneously on  $X_v$  and  $\chi \in \text{Rt}_{\text{ext}}(X)$ . This proves (i) and shows that  $\iota$  in (6) is well defined. Note that the injectivity of  $\iota$  follows from Lemma 4.1 and the surjectivity is a consequence of (i).

(ii) We define the discrete  $G$ -invariant valuation  $w$  on  $\mathbb{K}(G/H)$  by letting

$$w(f) = -\max\{i \in \mathbb{N} \mid \partial^i(f) \neq 0\}$$

for every nonzero  $f \in \mathbb{K}[X_v]$ . Further, if  $f$  is a  $B$ -eigenvector, then  $w(f) = \langle \chi_f, -v \rangle$ . Hence  $w = -v \in \text{lin}.\mathcal{V}$ .

(iii) This follows from the fact that the  $(\ker \partial)^U$  is finitely generated (see [Kur03, Theorem 1.2] and [Lie10a, Corollary 2.11]) and that  $\ker \partial = G \cdot (\ker \partial)^U$ .  $\square$

By the previous result we may introduce the following definition.

**Definition 4.6.** A non-trivial normalized  $\mathbb{G}_a$ -action of degree  $\chi$  on  $X$  is said to be *exterior* (resp. *interior*) if  $\chi \in \text{Rt}_{\text{ext}}(X)$  (resp.  $\chi \in \text{Rt}_{\text{int}}(X)$ )

As a direct consequence of Theorem 4.5, we have the following results.

**Corollary 4.7.** *If  $G$  is semi-simple, then  $\text{Rt}(X) = \emptyset$ . If  $\mathcal{V}$  is strongly convex, then  $\text{Rt}_{\text{ext}}(X) = \emptyset$ .*

*Proof.* This follows from the fact that  $\text{Rt}(X) \subset \text{Rt}(\Gamma)$  and from the part (ii) of Theorem 4.5.  $\square$

**Corollary 4.8.** *If  $X$  has three orbits including a fixed point  $y$ , then  $\text{Rt}_{\text{ext}}(X) = \emptyset$ .<sup>5</sup>*

*Proof.* Take  $\chi \in \text{Rt}_{\text{ext}}(X)$ . By Theorem 4.5,  $X$  admits a normalized  $\mathbb{G}_a$ -action on  $X$  that induces a transitive  $\mathbb{G}_a \rtimes_\chi G$ -action on  $X_v$ , where  $v \in \mathcal{C}(1) \cap \mathcal{V}$ . Therefore,  $X \setminus X_v$  is  $\mathbb{G}_a$ -stable and  $y$  is the unique fixed point for the  $\mathbb{G}_a$ -action. This yields a contradiction (see [Bia73]).  $\square$

<sup>5</sup>We are thankful for valuable suggestions to Hanspeter Kraft and other participants of the seminar “Algebra and Geometry” at the Mathematisches Institut, Basel Universität.

**Example 4.9.** Consider the spherical action of  $G = \mathrm{GL}_2 \times \mathrm{GL}_2$  on the variety  $X = \mathrm{M}_{2 \times 2}$  of  $2 \times 2$  matrices via the relation  $(P, Q) \cdot M = PMQ^{-1}$  for all  $(P, Q) \in G$  and  $M \in X$ . Then  $X$  has three orbits:

$$G \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad G \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the first is the fixed point. Hence by Corollary 4.8, every exterior normalized  $\mathbb{G}_a$ -action on  $X$  is trivial.

In the following paragraph, we recall the construction of the odd symplectic Grassmannian (cf [Mi07]). This construction provides examples of non-trivial exterior normalized  $\mathbb{G}_a$ -actions in the projective situation.

**Example 4.10.** Let  $E$  be a  $\mathbb{K}$ -vector space of dimension  $2r + 1$ . Consider a skew form  $\omega \in \wedge E^*$  of rank  $2r$ , where  $E^*$  is the dual vector space of  $E$ . The closed subgroup of  $\mathrm{GL}(E)$  preserving the form  $\omega$  is called the *odd symplectic group*  $\mathrm{Sp}_{2r+1}$ . Up to an appropriate basis of  $E$ , this algebraic group is isomorphic to a semi-direct product  $\mathbb{G}_a^{2r} \rtimes (\mathrm{Sp}_{2r} \times \mathbb{G}_m)$ , where  $\mathrm{Sp}_{2r} \subset \mathrm{GL}_{2r}$  is the classical symplectic group. Recall that a linear subspace  $F \subset E$  is said to be *isotropic* (with respect to  $\omega$ ) if  $\omega(v_1, v_2) = 0$  for any  $v_1, v_2 \in F$ . To specify the latter property we will write  $\omega(F) = 0$ . Denote by  $\mathrm{Gr}_d(E)$  the Grassmannian of  $d$ -dimensional linear subspaces of  $E$ . Then the *odd symplectic Grassmannian* is defined as the closed irreducible subset

$$X = \{F \in \mathrm{Gr}_d(E) \mid \omega(F) = 0\}.$$

Assume that  $d \leq r$ . The algebraic group  $\mathrm{Sp}_{2r+1}$  acts naturally on  $X$ . Let  $G = \mathrm{Sp}_{2r} \times \mathbb{G}_m \subset \mathrm{Sp}_{2r+1}$ , then the variety  $X$  is  $G$ -spherical. By [Mi07, Proposition 4.3], the variety has an open  $\mathrm{Sp}_{2r+1}$ -orbit  $\mathcal{O}$ , which decomposes in two  $G$ -orbits. Hence the induced  $\mathbb{G}_a^{2r}$ -action on  $X$  is non-trivial. Furthermore,  $\mathcal{O}$  is the total space of a line bundle over the symplectic Grassmannian  $Z$  of  $d$ -linear subspaces of  $\mathbb{K}^{2r}$ , whose fibers coincide with the  $\mathbb{G}_a^{2r}$ -orbits and also with the closures of the  $\mathbb{G}_m$ -orbits in  $\mathcal{O}$ . The closed  $G$ -orbit in  $\mathcal{O}$  corresponds to the zero section and is therefore isomorphic to  $Z$ . In particular, every one-parameter unipotent subgroup  $H \subset \mathbb{G}_a^{2r}$  yields a vertical  $\mathbb{G}_a$ -action on  $\mathcal{O}$  normalized by  $\mathbb{G}_m$ .

The set  $\mathrm{Rt}(X)$  of Demazure roots of an affine spherical variety  $X$  is defined in geometric terms. A natural question is to know whether  $\mathrm{Rt}(X)$  is described in combinatorial terms. More precisely, we propose the following problem.

**Question 4.11.** For any affine spherical variety  $X$ , is the set  $\mathrm{Rt}(X)$  equal to the intersection of a finite union of polyhedra in  $L_{\mathbb{Q}}$  with the lattice  $L$ ?

#### 4.2. Description for the reductive group $G_e$ .

Let  $X$  be an affine  $G_e$ -spherical variety with associated colored cone  $(\mathcal{C}, \mathcal{F})$ . We assume that  $e \neq 0$ . The aim of this subsection is to provide an explicit description of the set  $\mathrm{Rt}(X)$  of the Demazure roots of  $X$  in terms of the pair  $(\mathcal{C}, \mathcal{F})$ , see Theorem 4.18.

**4.12.** Let  $\partial$  be the LND on  $\mathbb{K}[X]$  corresponding to a non-trivial normalized  $\mathbb{G}_a$ -action of degree  $\chi^\theta$ , where  $\theta \in M$  is nonzero. Equivalently,  $\partial$  is nonzero, homogeneous of degree  $\theta$  with respect to the  $M$ -grading of  $\mathbb{K}[X]$ , and satisfies  $[\partial, \partial_-] = [\partial, \partial_+] = 0$ , where  $\partial_\pm$  is the LND corresponding to the  $U_\pm$ -action on  $X$ .

Applying Lemma 4.1 to the Borel subgroups  $B_-$  and  $B_+$  of  $G_e$ , we immediately obtain the following result.

**Corollary 4.13.** *Assume that  $X$  is of type I. Without loss of generality, we may suppose that  $\mathbb{K}[X]$  admits a presentation as in 2.24. Then in the notation of Section 3 and 4.12, we have  $\theta \in \text{Rt}(\omega_-) \cap \text{Rt}(\omega_+)$ , and there exists a facet  $\varsigma_{\pm} \subset \omega_{\pm}^{\vee}$  such that*

$$\ker \partial_{\pm} \cap \ker \partial = \bigoplus_{m \in \varsigma_{\pm} \cap L} \mathbb{K} \varphi_m^{\pm} \chi^m.$$

*Proof.* By Lemma 4.1, the restriction of  $\partial$  to  $\ker \partial_{\pm} = \mathbb{K}[X]^{U_{\pm}}$  is well defined, non-trivial, and homogeneous. Since  $\ker \partial_{\pm}$  is a semigroup algebra, we conclude by [Lie10a, Lemma 2.6].  $\square$

The following theorem gives examples of affine spherical homogeneous spaces admitting a non-trivial normalized  $\mathbb{G}_a$ -action.

**Theorem 4.14.** *Let  $X$  be a  $G_e$ -spherical homogeneous space of type I. Consider the representation  $\mathbb{K}[X] = A[\mathbb{A}^1, \mathfrak{D}]$  as in Remark 3.5, where  $\mathfrak{D} = \mathfrak{D}_{G_e/H}$ . Then  $X$  admits a normalized  $\mathbb{G}_a$ -action if and only if  $X$  is reflexive and  $v_0, v_1$  are linearly independent. Under this condition, all normalized  $\mathbb{G}_a$ -actions are horizontal, and we have*

$$\text{Rt}(X) = \text{Rt}_{\text{int}}(X) = \{\theta \in L \mid v_0(\theta) = v_1(\theta) = 1 \text{ or } v_0(\theta) = v_1(\theta) = -1\}.$$

*Proof.* Let  $\partial$  be the LND corresponding to a non-trivial normalized  $\mathbb{G}_a$ -action on  $X$ . Denote by  $\theta$  the degree of  $\partial$ , and by  $\varpi$  the weight cone of  $\ker \partial$ . By [Lie10a, Lemma 3.1],  $\partial$  is horizontal with respect to the  $M$ -grading of  $\mathbb{K}[X]$ .

Assume that  $X$  is skew. Then the quasifan  $\Lambda(\mathfrak{D})$  (see 2.2) has exactly two maximal elements. By Theorem 2.15, they are exactly  $\omega_-^{\vee}$  and  $\omega_+^{\vee}$ , hence  $\varpi$  coincides with one of them. But this contradicts Corollary 4.13. Thus, we may suppose that  $X$  is reflexive and  $\Lambda(\mathfrak{D})$  has at least three maximal cones. The latter condition is equivalent to  $v_0, v_1$  being linearly independent. In fact, in this case  $\Lambda(\mathfrak{D})$  has four maximal cones, and  $\omega_-$ ,  $\omega_+$ ,  $\varpi$  are among them.

By Theorem 2.15, the LND  $\partial$  can be given by a triple  $(\lambda, \mathfrak{D}_{\bullet}, \theta)$ , where we let  $\lambda = 1$  for simplicity and where the colored polyhedral divisor is given by

$$\mathfrak{D}_{\bullet} = (\mathfrak{D}, \{\mu_z \mid z \in \mathbb{A}^1\}, z_0).$$

Since  $\omega_-$  and  $\omega_+$  correspond to diagrams

$$\begin{array}{ccc} \cdots & \overset{v_0}{\circ} & \overset{0}{\circ} & \cdots & \cdots & \overset{0}{\circ} & \overset{v_1}{\circ} & \cdots \\ & \text{---} & \text{---} & & \text{---} & \text{---} & \text{---} & \\ & \underset{0}{\circ} & \underset{1}{\circ} & & \underset{0}{\circ} & \underset{1}{\circ} & & \end{array}$$

the two possible cases remaining for  $\varpi$  are

$$\begin{array}{ccc} \cdots & \overset{0}{\circ} & \overset{0}{\circ} & \cdots & \cdots & \overset{v_0}{\circ} & \overset{v_1}{\circ} & \cdots \\ & \text{---} & \text{---} & & \text{---} & \text{---} & \text{---} & \\ & \underset{0}{\circ} & \underset{1}{\circ} & & \underset{0}{\circ} & \underset{1}{\circ} & & \end{array}$$

where the marked point  $z_0$  is not yet known.

*Case 1.* The coloration  $\mathfrak{D}_{\bullet}$  is represented by one of diagrams

$$\begin{array}{ccccccc} \cdots & \overset{0}{\circ} & \overset{0}{\circ} & \cdots & \cdots & \overset{0}{\circ} & \overset{0}{\circ} & \overset{0}{\circ} & \cdots \\ & \text{---} & \text{---} & & \text{---} & \text{---} & \text{---} & \text{---} & \\ & \underset{z_0=0}{\circ} & \underset{1}{\circ} & & \underset{0}{\circ} & \underset{z_0=1}{\circ} & & \underset{0}{\circ} & \underset{1}{\circ} & \underset{z_0}{\circ} & \cdots \end{array}$$

The LND  $\partial$  is given by the relation

$$\partial(Q\chi^m) = Q'\chi^{m+\theta} \text{ for any } Q \in \mathbb{K}(t), \chi \in M.$$

Let us study the conditions  $[\partial, \partial_+] = [\partial, \partial_-] = 0$ . Recall that

$$\begin{aligned} \partial_-(Q\chi^m) &= (v_0(m)Q + tQ')\chi^{m-e}, \text{ for any } Q \in \mathbb{K}(t), \chi \in M, \\ \partial_+(Q\chi^m) &= (v_1(m)Q + (t-1)Q')\chi^{m+e} \text{ for any } Q \in \mathbb{K}(t), \chi \in M. \end{aligned}$$

Then

$$\begin{aligned} \partial \circ \partial_-(Q\chi^m) &= (v_0(m)Q' + Q' + tQ'')\chi^{m+\theta-e}, \\ \partial_- \circ \partial(Q\chi^m) &= (v_0(m+\theta)Q' + tQ'')\chi^{m+\theta-e}, \\ \partial \circ \partial_+(Q\chi^m) &= (v_1(m)Q' + Q' + (t-1)Q'')\chi^{m+\theta+e}, \\ \partial_+ \circ \partial(Q\chi^m) &= (v_1(m+\theta)Q' + (t-1)Q'')\chi^{m+\theta+e}. \end{aligned}$$

The conditions  $[\partial, \partial_-] = 0$  and  $[\partial, \partial_+] = 0$  are therefore equivalent  $v_0(\theta) = 1$  and  $v_1(\theta) = 1$  respectively.

*Case 2.* The set of AL-colors  $\{\mu_z \mid z \in \mathbb{A}^1\}$  is  $\{v_0, v_1\}$ . Since  $\varsigma_{\pm} = \varpi \cap \omega_{\pm}^{\vee}$  is a facet that separates full-dimensional cones  $\varpi$  and  $\omega_{\pm}^{\vee}$  in  $\sigma^{\vee}$ , we have  $\varsigma_{\pm} \cap \text{rel.int.}(\sigma^{\vee}) \neq \emptyset$ . This implies that

$$\ker \partial = \bigoplus_{m \in \varpi \cap L} \mathbb{K}t^{-v_0(m)}(t-1)^{-v_1(m)},$$

and therefore  $z_0 \in \{0, 1\}$  by Remark 2.16. By applying the formula from Remark 2.16, one can see that in both cases  $z_0 = 0$  and  $z_0 = 1$  the derivation  $\partial$  is the same. namely for any  $Q \in \mathbb{K}(t), m \in M$

$$\partial(Q\chi^m) = \left( \left( \frac{v_0(m)}{t} + \frac{v_1(m)}{t-1} \right) Q + Q' \right) \gamma \chi^{m+\theta} \text{ where } \gamma = t^{-v_0(\theta)}(t-1)^{-v_1(\theta)}.$$

Let us study the conditions  $[\partial, \partial_-] = [\partial, \partial_+] = 0$ . Using  $\gamma' = -v_0(\theta)\frac{\gamma}{t} - v_1(\theta)\frac{\gamma}{t-1}$ , we have

$$\begin{aligned} \partial \circ \partial_-(t) &= \partial(t\chi^{-e}) = \frac{t}{t-1}\gamma\chi^{\theta-e}, \\ \partial_- \circ \partial(t) &= \partial_-(\gamma\chi^{\theta}) = -v_1(\theta)\frac{t}{t-1}\gamma\chi^{\theta-e}, \\ \partial \circ \partial_+(t) &= \partial((t-1)\chi^e) = \frac{t-1}{t}\gamma\chi^{\theta+e}, \\ \partial_+ \circ \partial(t) &= \partial_+(\gamma\chi^{\theta}) = -v_0(\theta)\frac{t-1}{t}\gamma\chi^{\theta+e}, \end{aligned}$$

The conditions  $\partial \circ \partial_+(t) = \partial_+ \circ \partial(t)$  and  $\partial \circ \partial_-(t) = \partial_- \circ \partial(t)$  are therefore equivalent  $v_0(\theta) = -1$  and  $v_1(\theta) = -1$  respectively. Thus,  $\gamma = t(t-1)$  and

$$\partial(Q\chi^m) = ((v_0(m)(t-1) + v_1(m)t)Q + t(t-1)Q')\chi^{m+\theta}.$$

It is easy to check that  $\partial \circ \partial_-(\chi^m) = \partial_- \circ \partial(\chi^m)$  and  $\partial \circ \partial_+(\chi^m) = \partial_+ \circ \partial(\chi^m)$  for any  $m \in M$ . Therefore,  $\partial$  commutes with  $\partial_-$  and  $\partial_+$ .

Finally, from cases 1 and 2 we obtain the condition  $v_0(\theta) = v_1(\theta) = \pm 1$ , which implies that  $\theta$  is the Demazure root of the domain of  $\mathfrak{D}_{\bullet}$ . Now the proof of the theorem is completed.  $\square$

Let us give an example of a non-trivial interior normalized  $\mathbb{G}_a$ -action.

**Example 4.15.** Assume that  $M = N = \mathbb{Z}^2$  and  $M_{\mathbb{Q}} = N_{\mathbb{Q}} = \mathbb{Q}^2$ . The pairing between  $M$  and  $N$  is given by the usual scalar product. Consider the polyhedral divisor

$$\mathfrak{D} = \text{Conv}(0, v_0) \cdot \{0\} + \text{Conv}(0, v_1) \cdot \{1\}$$

over the affine line  $\mathbb{A}^1$ , where  $v_0 = (-1, 0)$  and  $v_1 = (0, -1)$ . An easy computation shows that  $A = A[\mathbb{A}^1, \mathfrak{D}] = \mathbb{K}[u_1, u_2, u_3, u_4]$ , where  $u_1 = \chi^{(-1,0)}$ ,  $u_2 = \chi^{(0,-1)}$ ,  $u_3 = (t-1)\chi^{(0,1)}$ , and  $u_4 = t\chi^{(1,0)}$  satisfy the irreducible relation  $u_1u_4 - u_2u_3 - 1$ . Hence  $X = \text{Spec } A$  is identified with the hypersurface  $\mathbb{V}(x_1x_4 - x_2x_3 - 1) \subset \mathbb{A}^4$ . The action of the torus  $\mathbb{T} = (\mathbb{K}^*)^2$  on the coordinates is given by the formula

$$(t_1, t_2) \cdot (x_1, x_2, x_3, x_4) = (t_1 \cdot x_1, t_2 \cdot x_2, t_2^{-1} \cdot x_3, t_1^{-1} \cdot x_4),$$

where  $(t_1, t_2) \in \mathbb{T}$ . Consider the vector  $e = (-1, 1)$ . Then we have  $\langle v_0, e \rangle = 1$  and  $\langle v_1, e \rangle = -1$ . The corresponding LNDs are given by

$$\partial_- = u_2 \frac{\partial}{\partial u_1} + u_4 \frac{\partial}{\partial u_3} \quad \text{and} \quad \partial_+ = u_1 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_4}.$$

Thus, the action of the algebraic group  $G_e = \text{SL}_2 \rtimes_e \mathbb{T}$  is defined by the formula

$$g \cdot (x_1, x_2, x_3, x_4) = (dt_1x_1 + ct_2x_2, bt_1x_1 + at_2x_2, dt_2^{-1}x_3 + ct_1^{-1}x_4, bt_2^{-1}x_3 + at_1^{-1}x_4),$$

where  $g = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (t_1, t_2) \right) \in G_e$ . In particular, we have

$$g \cdot (1, 0, 0, 1) = (dt_1, bt_1, ct_1^{-1}, at_1^{-1}).$$

The isotropy subgroup of  $(1, 0, 0, 1)$  is equal to

$$H = \left\{ \left( \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix}, (t_1, t_2) \right) \mid t_1, t_2 \in \mathbb{K}^* \right\},$$

and we have a  $G_e$ -isomorphism  $X \simeq G_e/H$ . Let us describe the non-trivial normalized  $\mathbb{G}_a$ -actions on  $X$ . By Theorem 4.14, we observe that  $\text{Rt}_{\text{int}}(X) = \{(1, 1), (-1, -1)\}$ . The associated LNDs to the vectors  $(-1, -1)$ ,  $(1, 1)$  are given respectively by

$$u_3 \frac{\partial}{\partial u_1} + u_4 \frac{\partial}{\partial u_2} \quad \text{and} \quad u_1 \frac{\partial}{\partial u_3} + u_2 \frac{\partial}{\partial u_4},$$

and the associated  $\mathbb{G}_a$ -actions by

$$\lambda \star (x_1, x_2, x_3, x_4) = (x_1 + \lambda x_3, x_2 + \lambda x_4, x_3, x_4)$$

and

$$\lambda \star (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3 + \lambda x_1, x_4 + \lambda x_2)$$

for any  $\lambda \in \mathbb{G}_a$ .

Let us describe the  $\mathbb{G}_a \rtimes_{\theta} G_e$ -homogeneous space  $X = X_{\rho}$  associated to the root  $\theta \in \text{Rt}_{\text{int}}(X)$  with the distinguished ray  $\rho$ . If  $\theta = (-1, -1)$ , then we have

$$(\lambda, g) \cdot (1, 0, 0, 1) = \lambda \star (g \cdot (1, 0, 0, 1)) = (dt_1 + \lambda ct_1^{-1}, bt_1 + \lambda at_1^{-1}, ct_1^{-1}, at_1^{-1}).$$

The isotropy subgroup of  $(1, 0, 0, 1)$  for the  $\mathbb{G}_a \rtimes_{\theta} G_e$ -action is

$$H_1 = \left\{ \left( -bt_1, \begin{pmatrix} t_1 & b \\ 0 & t_1^{-1} \end{pmatrix}, (t_1, t_2) \right) \mid b \in \mathbb{K}, (t_1, t_2) \in (\mathbb{K}^*)^2 \right\}.$$

We have a  $\mathbb{G}_a \rtimes_{\theta} G_e$ -isomorphism  $X \simeq (\mathbb{G}_a \rtimes_{\theta} G_e)/H_1$ . Similarly, for  $\theta = (1, 1)$  the isotropy subgroup of  $(1, 0, 0, 1)$  is

$$H_2 = \left\{ \left( -ct_1^{-1}, \begin{pmatrix} t_1 & 0 \\ c & t_1^{-1} \end{pmatrix}, (t_1, t_2) \right) \mid c \in \mathbb{K}, (t_1, t_2) \in (\mathbb{K}^*)^2 \right\},$$

and  $X \simeq (\mathbb{G}_a \rtimes_{\theta} G_e)/H_2$ .

The next two lemmas will be used in the proof of Theorem 4.18.

**Lemma 4.16.** *Let  $X$  be an affine  $G_e$ -spherical variety of type I. Assume that the torus  $\mathbb{T}$  acts faithfully on  $X$ . Then every exterior normalized  $\mathbb{G}_a$ -action on  $X$  is vertical for the induced  $\mathbb{T}$ -action.*

*Proof.* According to Theorem 4.5, it is enough to consider the  $\mathbb{G}_a \rtimes_{\theta} G_e$ -homogeneous variety  $X_{\rho}$ , where  $\rho$  is the distinguished ray of  $\theta \in \text{Rt}_{\text{ext}}(X)$ . Note that a priori  $X_{\rho}$  is not necessarily affine. By Theorem 4.5(ii) and Lemma 3.10, we have  $\rho \in \text{lin}(\mathcal{V}) = e^{\perp}$ . This implies that  $X_{\rho}$  is affine and described by a  $\sigma$ -polyhedral divisor over  $\mathbb{A}^1$ , where  $\sigma = \mathbb{Q}_{\geq 0} \cdot \rho$ . By [Lie10a, Lemma 3.1], the equality  $\rho(\theta) = -1$  implies that the  $\mathbb{G}_a$ -action on  $X_{\rho}$  is vertical.  $\square$

**Lemma 4.17.** *Let  $\varphi \in \mathbb{K}(t)^*$ ,  $s \in \mathbb{Z}$ ,  $d \in \mathbb{Z} \setminus \{0\}$ ,  $v \in N_{\mathbb{Q}}$ ,  $\rho \in N_{\mathbb{Q}} \setminus \{0\}$ , and  $e, \theta \in M$ . Consider the derivations  $\partial, \partial_{\bullet}$  on  $\mathbb{K}(t)[M]$  defined by the relations*

$$\begin{aligned} \partial_{\bullet}(t^r \chi^m) &= d \cdot (v(m) + r) \chi^{m+e} t^{r+s}, \\ \partial(t^r \chi^m) &= \rho(m) \varphi t^r \chi^{m+\theta} \text{ for all } (m, r) \in M \oplus \mathbb{Z}. \end{aligned}$$

*Then  $[\partial, \partial_{\bullet}] = 0$  if and only if  $\rho \in e^{\perp}$ ,  $v(\theta) \in \mathbb{Z}$ , and  $\varphi = \lambda t^{-v(\theta)}$  for some  $\lambda \in \mathbb{K}^*$ .*

*Proof.* Let us compute the Lie bracket  $[\partial_{\bullet}, \partial](t^r \chi^m)$ . For all  $(m, r) \in M \oplus \mathbb{Z}$ , we have

$$\begin{aligned} \partial_{\bullet} \circ \partial(t^r \chi^m) &= d \cdot \rho(m) \left[ \frac{d\varphi}{dt} t + (v(m+\theta) + r) \varphi \right] t^{r+s} \chi^{m+e+\theta}, \text{ and} \\ \partial \circ \partial_{\bullet}(t^r \chi^m) &= d \cdot \rho(m+e) (v(m) + r) \varphi t^{r+s} \chi^{m+e+\theta}. \end{aligned}$$

Thus, the condition  $[\partial, \partial_{\bullet}](t^r \chi^m) = 0$  is equivalent to

$$\rho(m) \left( v(\theta) \varphi + t \frac{d\varphi}{dt} \right) = \rho(e) (v(m) + r) \varphi$$

for any  $r \in \mathbb{Z}$ ,  $m \in M$ . The left-hand side does not depend on  $r$ , hence  $[\partial, \partial_{\bullet}] = 0$  implies  $\rho(e)r\varphi = 0$  for all  $r \in \mathbb{Z}$  and is equivalent to

$$\begin{cases} \rho(e) = 0, \\ v(\theta) \varphi + t \frac{d\varphi}{dt} = 0. \end{cases}$$

The latter equation is equivalent to  $\varphi = \lambda t^{-v(\theta)}$  for some  $\lambda \in \mathbb{K}^*$ .  $\square$

The following result gives a description of Demazure roots of an affine  $G_e$ -spherical variety.

**Theorem 4.18.** *Let  $X$  be an affine  $G_e$ -spherical variety. Assume that  $e \neq 0$  and that  $\mathbb{T}$  acts faithfully on  $X$ . Let  $(\mathcal{C}, \mathcal{F})$  be the associated colored cone of  $X$  with respect to the  $B_{\pm}$ -action. Let  $\Gamma$  be the polyhedral cone generated by the subset  $\mathcal{C}$  and by the images of all colors of  $X$ . Then the following hold.*

(i) If  $X$  is of type I, then  $X$  is isomorphic to an embedding of  $G_e/H$ , where  $G_e/H$  is as in 3.5. In the reflexive case,

$$\text{Rt}_{\text{ext}}(X) = \{\theta \in \text{Rt}(\Gamma) \cap L \mid \rho_\theta \in \text{lin.}(\mathcal{V}), v_0(\theta) = v_1(\theta) = 0\},$$

and in the skew case

$$\text{Rt}_{\text{ext}}(X) = \{\theta \in \text{Rt}(\Gamma) \cap L \mid \rho_\theta \in \text{lin.}(\mathcal{V}), v_1(\theta) = 0\}.$$

Furthermore, if  $X$  is skew or  $v_0, v_1$  are linearly dependent, then  $\text{Rt}_{\text{int}}(X) = \emptyset$ ; otherwise, we have

$$\text{Rt}_{\text{int}}(X) = \{\theta \in \text{Rt}(\Gamma) \cap L \mid v_0(\theta) = v_1(\theta) = -1 \text{ or } v_0(\theta) = v_1(\theta) = 1\}.$$

(ii) If  $X$  is not of type I, then  $X$  is a  $\tilde{G}_{\bar{e}}$ -spherical variety of type II (see Lemma 3.14) isomorphic to an embedding of  $X_\Sigma$  as in Lemma 3.16. We have  $\text{Rt}_{\text{int}}(X) = \emptyset$  and, in the notation of 3.18,

$$\text{Rt}_{\text{ext}}(X) = \{\theta \in \text{Rt}(\Gamma) \cap L_\pm \mid \theta \in \rho_-^\perp \cap \rho_+^\perp \text{ and } \rho_\theta \in e^\perp\}.$$

*Proof.* (i) Note that the description of  $\text{Rt}_{\text{int}}(X)$  is a direct consequence of Theorem 4.14. Let us describe the exterior Demazure roots of  $X$ . Let  $\theta \in \text{Rt}_{\text{ext}}(X)$  with the distinguished root  $\rho = \rho_\theta$ . Then by Lemma 4.16, the corresponding  $\mathbb{G}_a$ -action on  $X_\rho$  is obtained from a vertical LND  $\partial$  on  $\mathbb{K}[X_\rho]$  of degree  $\theta$  such that the algebra  $\mathbb{K}[X]$  is  $\partial$ -stable, see 2.10. By Lemma 4.17, the condition  $[\partial, \partial_-] = 0$  implies that  $v_0(\theta) \in \mathbb{Z}$  and  $\partial$  is given by the formula

$$\partial(Q\chi^m) = \rho(m)t^{-v_0(\theta)}Q\chi^{m+\theta},$$

where  $Q \in \mathbb{K}(t)$  and  $m \in M$ . The other implication  $\rho \in e^\perp$  is equivalent to  $\rho \in \text{lin.}(\mathcal{V})$ .

*Reflexive case.* Using formula for  $\partial_+$  in Remark 2.26 and the equality  $\rho(e) = 0$ , we have

$$[\partial_+, \partial](Q\chi^m) = \rho(m) \left( v_1(\theta) - v_0(\theta) \frac{t-1}{t} \right) t^{-v_0(\theta)} Q\chi^{m+e+\theta} \text{ for any } Q \in \mathbb{K}(t), m \in M.$$

Therefore,  $\partial$  and  $\partial_+$  commute if and only if  $v_0(\theta) = v_1(\theta) = 0$ . We conclude by remarking that  $\partial(\mathbb{K}[X]) \subset \mathbb{K}[X]$  gives the condition  $\theta \in \text{Rt}(\Gamma)$ .

*Skew case.* Similarly,

$$[\partial_+, \partial](Q\chi^m) = 2\rho(m)v_1(\theta)t^{-v_0(\theta)}Q\chi^{m+e+\theta} \text{ for any } Q \in \mathbb{K}(t), m \in M.$$

Thus,  $\partial$  and  $\partial_+$  commute if and only if  $v_1(\theta) = 0$ . Moreover, let  $\mathcal{D}'$  be the polyhedral divisor over  $\mathbb{A}^1$  associated to  $X_\rho$ ; here we identify  $\mathbb{K}[X_\rho]$  with a subalgebra of  $\mathbb{K}[G_e/H]$ . Note that the condition on the vertical derivation  $\partial = \partial_{\theta, t^{-v_0(\theta)}}$  imposed by 2.10, namely

$$t^{-v_0(\theta)} \in \Phi_\theta^* = H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}([\mathcal{D}'(\theta)])) \setminus \{0\},$$

is always fulfilled. Again, we have the condition  $\theta \in \text{Rt}(\Gamma)$ .

(ii)  $\mathbb{K}[X]$  is the semigroup algebra  $\mathbb{K}[\sigma^\vee \cap M]$ , where  $\sigma^\vee \subset M_\mathbb{Q}$  is the weight cone for the  $\mathbb{T}$ -action. Let  $\theta \in \text{Rt}(X)$  with the distinguished root  $\rho = \rho_\theta$ . Then  $\theta \in \text{Rt}(\sigma)$  and it corresponds to the homogeneous LND  $\partial$  given by

$$\partial(\chi^m) = \rho(m)\chi^{m+\theta}.$$

The conditions  $[\partial, \partial_-] = [\partial, \partial_+] = 0$  are equivalent to (see the notation of 3.15)

$$\begin{cases} \rho_-(\theta)\rho(m) = -\rho(e)\rho_-(m) \text{ for any } m \in \sigma^\vee \cap M, \\ \rho_+(\theta)\rho(m) = \rho(e)\rho_+(m) \text{ for any } m \in \sigma^\vee \cap M. \end{cases}$$



In other words,  $\rho_-(\theta)\rho = -\rho(e)\rho_-$  and  $\rho_+(\theta)\rho = \rho(e)\rho_+$ . Moreover, these two vectors are collinear. Since  $\rho_-$  and  $\rho_+$  are not,  $\rho(e) = 0$ . Moreover,  $\rho \neq 0$  implies  $\rho_-(\theta) = \rho_+(\theta) = 0$ . But  $\rho(\theta) = -1$  by definition, hence  $\rho \notin \{\rho_-, \rho_+\}$ . So, Lemma 3.16(iii) implies that  $\rho$  is exterior and  $\text{Rt}_{\text{int}}(X) = \emptyset$ . Finally, under the conditions  $\rho(e) = \rho_-(\theta) = \rho_+(\theta) = 0$  the induced  $\mathbb{G}_a$ -action is indeed normalized by  $G_e$ .  $\square$

**Corollary 4.19.** *Let  $e \in M$  be a nonzero element. Then the answer to Question 4.11 is affirmative for the reductive group  $G_e$ .*

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