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A new closure concept preserving graph Hamiltonicity and based on neighborhood equivalence

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A B S T R A C T

A graph is Hamiltonian if it contains a cycle which goes through all vertices exactly once. Determining if a graph is Hamiltonian is known as an NP-complete problem, and no satisfactory characterization for these graphs has been found.

In 1976, Bondy and Chvátal introduced a way to get round the Hamiltonicity problem complexity by using a closure of the graph. This closure is a supergraph of G which is Hamiltonian iff G is. In particular, if the closure is the complete graph, then G is Hamiltonian. Since this seminal work, several closure concepts preserving Hamiltonicity have been introduced. In particular, in 1997, Ryjáček defined a closure concept for claw-free graphs based on local completion.

Following a different approach, in 1974, Goodman and Hedetniemi gave a sufficient condition for Hamiltonicity based on the existence of a clique covering of the graph. This condition was recently generalized using the notion of Eulerian clique covering. In this context, closure concepts based on local completion are interesting since the closure of a graph contains more simplicial vertices than the graph itself, making the search for a clique covering easier.

In this article, we introduce a new closure concept based on local completion which preserves the Hamiltonicity for every graph. Note that, moreover, the closure may be claw free even when the graph is not.

1. Introduction

A graph is Hamiltonian if it contains a cycle which goes through all vertices exactly once. Determining if a graph is Hamiltonian is known as an NP-complete problem, and no satisfactory characterization for these graphs has been found. A huge body of literature exists on the subject, surveyed for instance in [7,8].

In [1], Bondy and Chvátal introduced a way to get round the Hamiltonicity problem complexity by using a closure of the graph. This closure is obtained in polynomial time by repeatedly adding edges between pairs of nonadjacent vertices whose degree sum is greater than or equal to the order of the graph, as long as such a pair exists. The closure is then proved to be Hamiltonian iff the graph is. In particular, if the closure is a complete graph, then the graph is Hamiltonian.

Since this seminal article, several closure concepts preserving Hamiltonicity have been introduced (for a survey on the topic, see for instance [6]). In particular, in [9], Ryjáček defined a closure concept for claw-free graphs based on local

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completion. The local completion is repeatedly performed on every *eligible* vertex, as long as such a vertex exists. Note that a strengthening of the closure concept of [9] was introduced in [4].

Following a different approach, in [5], Goodman and Hedetniemi gave a sufficient condition for Hamiltonicity based on the existence of a clique covering of the graph. This condition was recently generalized in [2,3] using the notion of *Eulerian* clique covering. It was also shown in [10] that there exists a Eulerian clique covering of a graph iff there exists a *normal* one, where a clique covering is normal if it contains the closed neighborhood of every simplicial vertex of the graph. In this context, closure concepts based on local completion are interesting since, then, the closure of a graph contains more simplicial vertices than the graph itself, making the search for a normal clique covering easier. For instance, a closure in the sense of [9] has at most one *normal* Eulerian clique covering.

In this article, we introduce a new closure concept based on local completion which preserves the Hamiltonicity for every graph. The closure is defined using the notion of *neighborhood equivalence*, as defined in [2,3]. First, we introduce some notation, we give some preliminary results on paths, and remind the reader of the definitions of local completion and neighborhood equivalence. Secondly, we introduce the notion of a *neighborhood-equivalence eligible* (*N-eligible*) vertex, and we show that, for every graph G , there exists an *N-eligible* free graph $cl_N(G)$, called the *neighborhood-equivalence closure* (*N-closure*) of G , such that the circumference of G is equal to the circumference of $cl_N(G)$. This implies in particular that G is Hamiltonian iff $cl_N(G)$ is. Finally, we give an example of a non-claw-free graph G such that $cl_N(G)$ is claw free and such that the closure of $cl_N(G)$ in the sense of [9] is a complete graph.

2. Preliminaries

2.1. General notation

In what follows, $|X|$ denotes the size or cardinal of the set X , and $X \setminus Y = \{x \in X : x \notin Y\}$. We also define $\mathbb{P}(X)$ as the set of all pairs $\{x, y\} \subseteq X$ such that x and y are distinct.

We always suppose a graph to be undirected, simple and finite. Thus a graph G is a pair (V, E) , where V is the vertex set of G , and E is a subset of $\mathbb{P}(V)$. To simplify notation, a pair $\{x, y\} \in E$ is simply written xy . If $X \subseteq V$, $E(X) = \{xy \in E : x, y \in X\}$.

The *open neighborhood* of $x \in V$ is the set $N(x) = \{y : xy \in E\}$, and its *closed neighborhood* is the set $N[x] = N(x) \cup \{x\}$.

The *closed neighborhood* of $X \subseteq V$ is the set $N[X] = \bigcup_{x \in X} N[x]$. The *open neighborhood* of $X \subseteq V$ is the set $N(X) = N[X] \setminus X$. Notice that the definition of $N(X)$ is coherent with the fact that $N(x) = N[x] \setminus \{x\}$. In particular, we have $N(\{x\}) = N(x)$. Notice also that $N(X)$ could not be defined as $\bigcup_{x \in X} N(x)$. Indeed, if $x, y \in X$ and $xy \in E$, then $x, y \in \bigcup_{x \in X} N(x)$, since $y \in N(x)$ and $x \in N(y)$, while $N(X) \cap X$ is always empty.

A walk in G is a sequence of vertices $w = x_0 \cdots x_k$ such that $x_i x_{i+1} \in E$, for every $i = 0, \dots, k-1$. The integer k is the length of w , x_0, x_k are its endpoints, x_0 its starting point and x_k its ending point. In particular, w is a walk of length 0 with starting and ending point x . A walk is *closed* if $k \neq 0$ and $x_0 = x_k$. A closed walk is a *cycle* if it contains no repetition of a vertex except for the endpoint. We denote by $c(G)$ the circumference of G , that is, the length of the longest cycle in G . A cycle is a *Hamilton* cycle if it contains every vertex of the graph. A graph is *Hamiltonian* if it contains a Hamilton cycle. If $w = x_0 \cdots x_k$ is a walk, then $V(w) = \{x_0, \dots, x_k\}$ and $E(w) = \{x_i x_{i+1} : i \in \{0, \dots, k-1\}\}$.

A *path* is a walk containing no repetition of a vertex. If $P = x_0 \cdots x_k$ is a path, then $x_i \xrightarrow{P} x_j$, where $0 \leq i \leq j \leq k$, denotes the subpath of P with endpoints x_i, x_j , and $x_j \xleftarrow{P} x_i$ the reverse path $x_j x_{j-1} \cdots x_i$. For $i \in \{0, \dots, k-1\}$, x_i^+ is the successor of x_i in P ; that is, $x_i^+ = x_{i+1}$. For $i \in \{1, \dots, k\}$, x_i^- is the predecessor of x_i in P ; that is, $x_i^- = x_{i-1}$.

For every path P , we define the *neighborhood* of x in P as the set $P(x)$ which contains, when defined, the predecessor and the successor of x . That is, if $P = x$, then $P(x) = \emptyset$; and if $k \neq 0$ and $P = x_0 \cdots x_k$, then $P(x_0) = \{x_0^+\}$, $P(x_k) = \{x_k^-\}$ and $P(x_i) = \{x_i^-, x_i^+\}$, for every $i \in \{1, \dots, k-1\}$. Notice that clearly $P(x) \subseteq N(x) \cap V(P)$ but, since $P(x)$ contains only the immediate predecessor and successor of x in P , we may have $P(x) \subsetneq N(x) \cap V(P)$.

For every $X \subseteq V(P)$, we let $P(X) = \bigcup_{x \in X} P(x)$. Notice that, in contrast to $N(X) \cap X$, $P(X) \cap X$ may not be empty. Finally, notice that, if P' is a subpath of P , then $P'(x) \subseteq P(x)$, for every $x \in V(P')$.

A set $X \subseteq V$ is a *clique* of G if $xy \in E$, for all distinct $x, y \in X$. A vertex is *simplicial* if $N[x]$ is a clique of G . We denote by S the set of simplicial vertices of G and by NS the set of non-simplicial vertices of G .

A graph G is connected if all distinct vertices x, y are connected by a walk, and complete if $E = \mathbb{P}(V)$. From now on, G is always supposed to be connected.

2.2. Preliminaries on paths

Definition 1. A path P with endpoints in $Y \subseteq V$ is *YX-quasi-alternating* if X, Y are disjoint subsets of $V(P)$ and $P(X) \subseteq X \cup Y$.

In other words, in a *YX-quasi-alternating* path P , the predecessor and successor of every $x \in X$ must be in $X \cup Y$.

Lemma 2. If P is *YX-quasi-alternating* and $|X| \geq |Y|$, then $E(P) \cap E(X) \neq \emptyset$.

Proof. We show the result by contradiction. So suppose that the conditions of the lemma are satisfied and that there exists no edge $xx' \in E(P)$ such that $x, x' \in X$. Hence, we have $P(X) \cap X = \emptyset$, and so $P(X) \subseteq Y$, since $P(X) \subseteq X \cup Y$.

Let $P = z_0 \cdots z_k$. We show now by induction on $i \in \{0, \dots, k\}$ that, for $P_i = z_0 \cdots z_i$, $X_i = X \cap V(P_i)$ and $Y_i = Y \cap V(P_i)$, we have either $z_i \notin X$ and $|Y_i| > |X_i|$, or $z_i \in X$ and $|Y_i| \geq |X_i|$. The case $i = 0$ is trivial, since the starting point of P is in Y and X, Y are disjoint, by definition of a YX -quasi-alternating path. Suppose now that the result is true for $i \in \{0, \dots, k-1\}$. We show the result for $i+1$ using two cases, where $z = z_{i+1}$.

1. $z \in X$, and so $X_{i+1} = X_i \cup \{z\}$ and $Y_{i+1} = Y_i$. Since a path does not contain any repetition of vertex, we have in particular $|X_{i+1}| = |X_i| + 1$. Now, since $ziz \notin E(X)$, we have $z_i \notin X$. Hence, by the induction hypothesis, $|Y_{i+1}| = |Y_i| > |X_i|$, and so $|Y_{i+1}| \geq |X_{i+1}|$.
2. $z \notin X$, and so $X_{i+1} = X_i$. First, if $z_i \in X$, we have $|Y_i| \geq |X_i|$ by the induction hypothesis. Moreover, since $P(X) \subseteq Y$, $z \in Y$, and so $Y_{i+1} = Y_i \cup \{z\}$. Hence, $|Y_{i+1}| = |Y_i| + 1 > |Y_i| \geq |X_i| = |X_{i+1}|$. Second, if $z_i \notin X$, we have $|Y_i| > |X_i|$ and so $|Y_{i+1}| \geq |Y_i| > |X_i| = |X_{i+1}|$.

Hence, since $P = P_k$ and $z_k \in Y$, we have $|Y| > |X|$, contradicting $|X| \geq |Y|$. \square

2.3. Neighborhood equivalence and local completion

Definition 3. The *neighborhood-equivalence* relation \equiv on V is defined, for all vertices x, y , by $x \equiv y$ iff $N[x] = N[y]$. We let $\bar{x} = \{y \in V : x \equiv y\}$.

Obviously, \equiv is an equivalence relation inducing a partition on V . Moreover, for every $x \in V$, $x \equiv y \Rightarrow xy \in E$, and so \bar{x} is a subset of $N[x]$ and a clique in G . Also, clearly $N[\bar{x}] = N[x]$, $N(\bar{x}) \subseteq N(x)$. Moreover, $N(\bar{x}) = N(x)$ if and only if $\bar{x} = \{x\}$. Finally, $x \equiv z$ implies that $N(\bar{x}) = N(\bar{z})$.

Definition 4. Let G be a graph, $x \in V$ and $\nu \subseteq V$.

- $B_x = \{yz \notin E : y, z \in N(\bar{x})\}$ and $B_\nu = \bigcup_{x \in \nu} B_x$.
- The *local completion* of G at ν is the graph $G_\nu = (V, E_\nu)$, where $E_\nu = E \cup B_\nu$.

Obviously, $E \cap B_\nu = \emptyset$. Moreover, if $y, z \in N[x]$, then the hypothesis $yz \notin E$ implies that $y, z \notin \bar{x}$ (indeed, if $y \in \bar{x}$, we get $z \in N[x] = N[y]$). Hence, it is easy to check that, for every vertex x , $B_x = \{yz \notin E : y, z \in N[x]\}$. The following result comes straightforwardly from this remark and Definition 4.

Fact 5. G is a spanning subgraph of G_ν , for every $\nu \subseteq V$. Moreover, for all $x \in \nu$ and $z \in \bar{x}$, $N[z]$ is a clique in G_ν .

In what follows, we denote respectively by $N_\nu[z]$, \bar{z}^ν , \equiv_ν and S_ν (respectively NS_ν) the neighborhood, the neighborhood-equivalence class of z , the neighborhood-equivalence relation in G_ν and the set of simplicial vertices (respectively non-simplicial vertices) of G_ν .

3. Neighborhood-equivalence closure of a graph

We define below *neighborhood-equivalence eligibility* (N -eligibility) as well as the *neighborhood-equivalence closure* (N -closure) of a graph. We fix a graph G with set of all N -eligible vertices ν and local completion G_ν .

3.1. N -eligibility

Definition 6. A vertex x of G is *neighborhood-equivalence eligible* (N -eligible) if $x \in NS$ and $|\bar{x}| \geq |N(\bar{x})|$.

Lemma 7. If ν is the set of N -eligible vertices of G , then we have $\nu = \bigcup_{x \in \nu} \bar{x}$ and $\nu \cap \bigcup_{x \in \nu} N(\bar{x}) = \emptyset$.

Proof. It is immediate that $\nu \subseteq \bigcup_{x \in \nu} \bar{x}$, and it is easy to check, from the definitions of \bar{x} and N -eligibility, that, if $x \in \nu$, then $\bar{x} \subseteq \nu$. This shows the first part of the lemma. We now show the second part by contradiction. So suppose that $x, y \in \nu$ such that $y \in N(\bar{x})$. Notice first that $N[\bar{x}] = N[x]$ and $N[\bar{y}] = N[y]$. Let $Y = N[y] \setminus N[x]$, $Z = (N[y] \cap N[x]) \setminus (\bar{x} \cup \bar{y})$ and let $|\bar{x}| = m$, $|\bar{y}| = n$, $|Y| = l$ and $|Z| = k$. Since $xy \in E$, it is easy to check that $x, y \in N[x] \cap N[y]$, and so $\bar{x}, \bar{y} \subseteq N[x] \cap N[y]$. We also have $\bar{y} \cap \bar{x} = \emptyset$, since $y \notin \bar{x}$, and so $\bar{y} \subseteq N(\bar{x})$ and $\bar{x} \subseteq N(\bar{y})$. Hence, $\{\bar{x}, \bar{y}, Z\}$ is a partition of $N[x] \cap N[y]$. So, $\{\bar{x}, \bar{y}, Z, Y\}$ is a partition of $N[y]$ and $\{\bar{x}, Z, Y\}$ is a partition of $N(\bar{y})$. Hence, $|N(\bar{y})| = m + k + l$ and, since y is N -eligible, we have $n \geq m + k + l$. Since, moreover, x is N -eligible, we have $m \geq |N(\bar{x})|$. Now, since $\bar{y} \cup Z \subseteq N(\bar{x})$, we have $|N(\bar{x})| \geq n + k$, and hence $m \geq n + k \geq n \geq m + k + l$. Hence, $k = l = 0$ and $m = n$. This implies that $Z = \emptyset = Y$, and so $N[y] = \bar{x} \cup \bar{y}$. Notice now that, as shown above, $y \in N(\bar{x})$ implies that $\bar{x} \subseteq N(\bar{y})$, and so $x \in N(\bar{y})$. So, by swapping x and y in the proof above, we also get $N[x] = \bar{x} \cup \bar{y}$. This implies that $N[x] = N[y]$, contradicting $y \notin \bar{x}$. \square

Corollary 8. $N_\nu[z] = N[z]$, for every $z \in \nu$ and every $z \notin \bigcup_{x \in \nu} N[x]$.

Proof. Since G is a subgraph of G_v , $N[z] \subseteq N_v[z]$, for every $z \in V$. Now, if $N_v[z] \neq N[z]$, there exists $y \in N_v[z] \setminus N[z]$. So, by definition of B_v , there exists $x \in v$ such that $zy \in B_x$. We have $zy \notin E$ and $z, y \in N(\bar{x})$. Hence, $z \in \cup_{x \in v} N[x]$ and, since $v \cap N(\bar{x}) = \emptyset$ (Lemma 7), $z \notin v$. This shows the corollary by contraposition. \square

Lemma 9. $v \subseteq S_v \setminus S$ and $S \subseteq S_v$.

Proof. Since, for every $z \in v$, $N[z]$ is a clique of G_v (Fact 5) and $N_v[z] = N[z]$ (Corollary 8), we have immediately that $v \subseteq S_v$, and so $v \subseteq S_v \setminus S$, since $v \subseteq NS$. Suppose now that $s \in S$; we must show that $yz \in E_v$, for all distinct $z, y \in N_v[s]$. If $yz \in E$, then the result is immediate, since $E \subseteq E_v$. Hence, we can suppose that $yz \notin E$, and so there is at least one vertex among y, z which is not in $N[s]$ (otherwise, we would have $yz \in E$ from $s \in S$). So suppose that, for instance, $y \in N_v[s] \setminus N[s]$. There exists $x \in v$ such that $sy \in B_x$, and so $s, y \in N(\bar{x})$. Since $N(\bar{x}) \subseteq N[x]$, we have in particular $s \in N[x]$ and $x \in N[s]$. If, now, $z \in N[s]$, we get $xz \in E$ by simplicity of s , and so $y, z \in N[x]$. Since $N[x]$ is a clique of G_v (Fact 5), we conclude that $yz \in E_v$. Finally, if $z \notin N[s]$, there exists $u \in v$ such that $sz \in B_u$ and $s, z \in N(\bar{u})$. Since $s \in S$, we have $xu \in E$, and so $u \in N[x] = N[\bar{x}]$. Since, by Lemma 7, $u \notin N(\bar{x})$, we have $u \in \bar{x}$. This implies, in particular, that $z, y \in N[x]$, and so $yz \in E_v$ by Fact 5. \square

3.2. N -closure of a graph

Theorem 11 below proves the existence of the N -closure of G , that is, of a N -eligible-free supergraph of G with the same circumference as G . It is essentially a corollary of the following proposition.

Proposition 10. $c(G) = c(G_v)$.

Proof. Since every cycle in G is also a cycle in G_v , clearly $c(G) \leq c(G_v)$. It remains to prove $c(G_v) \leq c(G)$. To do so, it is sufficient to show that every cycle C of G_v of maximal length can be transformed into a cycle \mathbb{C} of G such that $V(\mathbb{C}) = V(C)$. The proof is by induction on $n = |E_v(C) \cap B_v|$. If $n = 0$, then C is a cycle in G and, taking $\mathbb{C} = C$, the result is immediate. Suppose now that C contains $n + 1$ edges of B_v , and let $yz \in E_v(C) \cap B_v$. There exists $x \in v$ such that $yz \in B_x = \{yz \notin E : y, z \in N(\bar{x})\}$. Let $P = y \vec{C} z$; that is, up to a rotation, P is the path obtained from C by removing yz . Notice that $V(P) = V(C)$, $E_v(P) = E_v(C) \setminus \{yz\}$ (and so P contains n edges of B_v), and $E(P) = E(C)$. If we suppose now that there exists $u \in \bar{x} \setminus V(C)$, then, since $y, z \in N[x] = N[u] \subseteq N_v[u]$, $zuy \vec{C} z$ would be a cycle in G_v strictly longer than C , contradicting the maximality of C . Hence, $\bar{x} \subseteq V(P)$. Let now $Y = N(\bar{x}) \cap V(P)$. By definition, we have $y, z \in Y$ and $\bar{x} \cap Y = \emptyset$. Moreover, suppose that $v \in P(\bar{x})$. There exist $u \in \bar{x}$ such that $v \in P(u)$ and so, in particular, $v \in N[u] = N[x] = N[\bar{x}] = N(\bar{x}) \cup \bar{x}$. Since $v \in V(P)$ and $\bar{x} \subseteq V(P)$, $v \in (N(\bar{x}) \cup \bar{x}) \cap V(P) = (N(\bar{x}) \cap V(P)) \cup (\bar{x} \cap V(P)) = Y \cup \bar{x}$. This proves that $P(\bar{x}) \subseteq \bar{x} \cup Y$. Hence, P is a $Y\bar{x}$ -quasi-alternating path in G_v . Since $Y \subseteq N(\bar{x})$, and x is N -eligible, we get $|\bar{x}| \geq |N(\bar{x})| \geq |Y|$. So, by Lemma 2, there exists an edge $uv \in E_v(\bar{x}) \cap E_v(P)$. Without loss of generality, we can suppose that u appears before v in P , and so, since $y, z \in N[x] = N[u] = N[v]$, $C' = y \vec{P} u \overset{\leftarrow}{P} v y$ is a cycle in G_v . Moreover, since $uz, vy \in E$ and $|E_v(P) \cap B_v| = n$, clearly C' contains at most n edges of B_v . Obviously, $V(C') = V(P) = V(C)$, and so C' is of maximal length in G_v . Hence, by the induction hypothesis, there exists a cycle \mathbb{C} of G such that $V(\mathbb{C}) = V(C')$, and so $V(\mathbb{C}) = V(C)$. This proves the result. \square

Theorem 11. For every graph G there exists a graph $cl_N(G)$ which contains no N -eligible vertex, such that G is a spanning subgraph of $cl_N(G)$ and such that $c(G) = c(cl_N(G))$.

Proof. The proof is done by induction on the number n of non-simplicial vertices of the graph. If $NS = \emptyset$ or $v = \emptyset$, then $G = cl_N(G)$ is the graph we are looking for. Otherwise, since $v \neq \emptyset$, we have $S \subsetneq S_v$ by Lemma 9. Hence, $NS_v \subsetneq NS$ and so, by the induction hypothesis, there exists a graph $cl_N(G_v)$ which contains no N -eligible vertex, such that G_v is a spanning subgraph of $cl_N(G_v)$ and such that $c(G_v) = c(cl_N(G_v))$. Since G is a spanning subgraph of G_v (Fact 5), we let $cl_N(G) = cl_N(G_v)$, and we conclude easily by Proposition 10. \square

Corollary 12. G is Hamiltonian iff $cl_N(G)$ is.

The construction used to obtain the graph $cl_N(G)$ is deterministic; there is no choice involved, and so the N -closure $cl_N(G)$ of G is unique. We show below that in fact G_v does not contain any N -eligible vertex, and so $cl_N(G) = G_v$.

Lemma 13. For every $z \in V$, $\bar{z} \subseteq \bar{z}^v$.

Proof. Let $z, y \in V$ such that $N[z] = N[y]$. Note first that $z \in v$ iff $y \in v$, by Lemma 7, and so, if $z \in v$, then we get $N_v[z] = N[z] = N[y] = N_v[y]$ (Corollary 8). Suppose now the contrary; that is, $z, y \notin v$. We show by contradiction that the hypothesis $N_v[z] \neq N_v[y]$ leads to a contradiction. Indeed, suppose that either $N_v[z] \not\subseteq N_v[y]$ or $N_v[y] \not\subseteq N_v[z]$. We show the result for the first case. The other case can be symmetrically dealt with. Let $u \in N_v[z] \setminus N_v[y]$. Since $N[z] = N[y] \subseteq N_v[y]$, we have $u \in N_v[z] \setminus N[z]$. Hence, there exists $x \in v$ such that $zu \in B_x$; that is, $zu \notin E$ and $z, u \in N(\bar{x})$. Hence, $z, u \in N[x]$, and so $x \in N[z] = N[y]$. So, we have $u, y \in N[x]$, which is a clique of G_v (Fact 5). Hence, $uy \in E_v$, contradicting the hypothesis $u \notin N_v[y]$. \square

Lemma 14. G_v does not contain any N -eligible vertex, and so $G_v = cl_N(G)$.

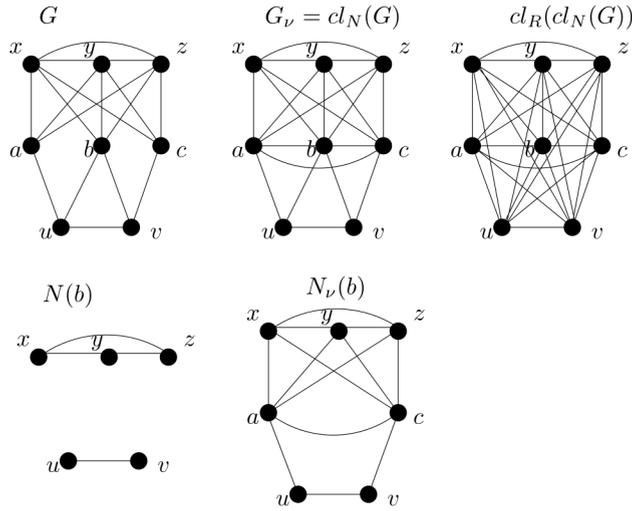


Fig. 1. The complete Ryjáček closure of the N -closure of a graph G .

Proof. Let $z \in V$. We must show that z is not N -eligible in G_v . If $z \in S_v$, then the result is immediate by Definition 6. So we can suppose that $z \in NS_v$, and so $z \notin v$, since $v \subseteq S_v$ by Lemma 9. Hence, since z is not N -eligible in G , we have $|\bar{z}| < |N(\bar{z})|$, and we must show that $|\bar{z}^v| < |N_v(\bar{z}^v)|$. We now present two cases.

1. Suppose first that $z \notin \cup_{x \in v} N[x]$. We have $N[z] = N_v[z]$ by Corollary 8 and $\bar{z} \subseteq \bar{z}^v$ by Lemma 13. Suppose now that there exists $u \in \bar{z}^v \setminus \bar{z}$; that is, $N_v[z] = N_v[u]$ but $N[z] \neq N[u]$. Hence, $N[z] = N_v[z] = N_v[u]$, and either $N[z] \not\subseteq N[u]$ or $N[u] \not\subseteq N[z]$. The second case is impossible, since $N[u] \subseteq N_v[u] = N[z]$ and so there exists $y \in N[z] \setminus N[u] = N_v[u] \setminus N[u]$. So there exists $x \in v$ such that $xy \in E$. Thus $xy \notin E$ and $u, y \in N(\bar{x})$. But $u \in N[x]$ implies that $x \in N[u] \subseteq N_v[u] = N[z]$. So $z \in N[x]$, contradicting $z \notin \cup_{x \in v} N[x]$. So, by contradiction, $\bar{z}^v = \bar{z}$, and so $N[\bar{z}^v] \setminus \bar{z} = N_v[\bar{z}] \setminus \bar{z}^v$. Hence, we get finally $|\bar{z}^v| = |\bar{z}| < |N(\bar{z})| = |N_v(\bar{z}^v)|$.
2. Suppose now that there exists $x \in v$ such that $z \in N[x]$. Note first that, since $N[x]$ is a clique in G_v (Fact 5), we have $N[x] \subseteq N_v[z]$. If $N_v[z] \subseteq N[x]$, then $z \in S_v$, contradicting $z \in NS_v$. So we can suppose that $N_v[z] \not\subseteq N[x]$, and so $Y = N_v[z] \setminus N[x] \neq \emptyset$. Now let $y \in Y$ and $u \in \bar{x}$. We have $N_v[u] = N[u] = N[x]$ (by Corollary 8 and the definition of \bar{x}). Hence, $y \notin N_v[u]$, and so $u \notin N_v[y]$. Since, moreover, $y \in N_v[z]$, we get $u \neq_v z$. Since $u \in N[x] \subseteq N_v[z]$, we also get $y \neq_v z$. So, we have proved that $\bar{z}^v \cap \bar{x} = \emptyset = \bar{z}^v \cap Y$, and so $Y \cup \bar{x} \subseteq N_v(\bar{z}^v)$. We have $Y \cap \bar{x} = \emptyset$, by the definition of Y , and so $|\bar{x}| < 1 + |\bar{x}| \leq |Y| + |\bar{x}| = |Y \cup \bar{x}| \leq |N_v(\bar{z}^v)|$. Now, if $v \in \bar{z}^v$, we have $x \in N_v[z] = N_v[v]$, and so $v \in N_v[x] = N[x]$. Hence, $\bar{z}^v \subseteq N[x]$ and, since $\bar{z}^v \cap \bar{x} = \emptyset$, $\bar{z}^v \subseteq N(\bar{x})$. Hence, finally, $|\bar{z}^v| \leq |N(\bar{x})| \leq |\bar{x}| < |N_v(\bar{z}^v)|$. \square

We give now an example of a graph G which is not claw free but whose N -closure $cl_N(G)$ is. Hence, by the main result of [9], there exists a closure $cl_R(cl_N(G))$ of $cl_N(G)$ which contains no eligible vertex and which is Hamiltonian iff $cl_N(G)$ is. Since, moreover, $cl_R(cl_N(G))$ is complete and thus Hamiltonian, both $cl_N(G)$ and G are (Corollary 12). We recall that a vertex is eligible in [9] if the subgraph induced by its (open) neighborhood is connected.

Example 15. The graph G of Fig. 1 contains three N -eligible vertices x, y, z , and it is not claw free. For instance, the subgraph induced by $\{x, a, b, c\}$ is a claw. Note that, moreover, $N(b)$ does not induce a connected subgraph of G . The graph G_v , where $v = \{x, y, z\}$, contains no N -eligible vertex, and is the N -closure $cl_N(G)$ of G . It is claw free, and b is eligible in this graph since the subgraph induced by $N_v(b)$ is connected.

4. Conclusion and future work

In this article, we have introduced a new closure concept which preserves the Hamiltonicity for every graph. This concept, the N -closure, is based on local completion and neighborhood equivalence. Fig. 1 gives an example of how the N -closure operation can be combined with the closure operation of [9] to prove the Hamiltonicity of a graph. A systematic study of the power of such combinations has still to be done. The time complexity of building the N -closure seems to be polynomial, but a formal proof should be provided. Finally, several generalizations of the N -closure seem possible; one of them has already been studied.

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