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The Dirichlet problem for $-\Delta\varphi = \mathrm{e}^{-\varphi}$ in an infinite sector. Application to plasma equilibria.

Olivier Goubet*, Simon Labrunie**

Abstract

We consider here a nonlinear elliptic equation in an unbounded sectorial domain of the plane. We prove the existence of a minimal solution to this equation and study its properties. We infer from this analysis some asymptotics for the stationary solution of an equation arising in plasma physics.

Keywords: Nonlinear elliptic equations, unbounded domains, plasma physics
2010 MSC: 35J25, 35J91, 35B40, 35B65, 82D10

1. Setting of the problem

Solving elliptic PDE in unbounded domains of $\mathbb{R}^n$ such as half-spaces occurs naturally when using some blow-up argument to analyze the properties of a particular solution of a PDE in a bounded domain. We refer for instance to [1] where the analysis of the properties of the solution u_ε to

$$\begin{aligned} \varepsilon\Delta u_\varepsilon + f(u_\varepsilon) = 0, \\ u_\varepsilon = 0 \text{ on the boundary} \end{aligned} \tag{1.1}$$

in a neighborhood of a point of the boundary, leads naturally to the study of an elliptic PDE in the half-space.

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The nonlinear elliptic PDE

$$\Delta u + f(u) = 0, \tag{1.2}$$

has been widely studied in domains as half-spaces or cylindrical unbounded domains. We refer here to the articles [4, 5, 6, 7] which have been instrumental for any later results concerning the symmetry and monotonicity properties of solutions. In the literature, there are various results concerning the properties of *bounded* solutions to these equations, mainly using consequences of the maximum principle as the moving plane method or the sliding method (see [11, 10],...) We also refer to [9] where the properties of solutions in a quarter-space have been studied using tools from infinite-dimensional dynamical systems.

Let us now describe the equation we are interested in. Consider Ω a sectorial domain of $\mathbb{R}^2$ defined in polar coordinates as:

$$\Omega = \{\boldsymbol{x}(r,\theta) \in \mathbb{R}^2; |\theta| < \theta_0 \leq \pi\}.$$

We shall sometimes denote this set as $\Omega[\theta_0]$ when we need to specify the opening. We are interested in the *non-negative* solutions to the problem:

$$\begin{aligned} -\Delta\varphi &= \mathrm{e}^{-\varphi} \quad \text{in } \Omega, \\ \varphi &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

Our motivation here comes from the article [13], where the authors study stationary solutions to the Vlasov–Poisson system in a polygon and link them to those of a non-linear elliptic equation. The singular limit of the latter while some scaling parameter converges towards 0 leads to (1.3). Our aim in analyzing this equation is to provide more insight on the solutions to the original Vlasov–Poisson equation.

More specifically, we shall look for two types of solutions to (1.3). *Local variational solutions* satisfy $\varphi \in H^1(\mathcal{O})$ for any bounded open set $\mathcal{O} \subset \Omega$. By the Trudinger inequality, this implies $\int_{\mathcal{O}} \exp(\varphi^2) < +\infty$, and thus $\mathrm{e}^{-\varphi} \in L^2(\Omega)$. In this case the Dirichlet condition holds in the sense of the usual trace theory:

$$\int_\Omega \nabla\varphi \cdot \nabla v = \int_\Omega \mathrm{e}^{-\varphi} v, \quad \forall v \in H_0^1(\Omega) \text{ with bounded support.} \tag{1.4}$$

Very weak solutions are such that $\varphi \in L^2(\mathcal{O})$ and $\mathrm{e}^{-\varphi} \in H^{-1}(\mathcal{O})$:

$$-\int_\Omega \varphi\, \Delta v = \int_\Omega \mathrm{e}^{-\varphi}\, v, \qquad \forall v \in H^2 \cap H^1_0(\Omega) \text{ with bounded support.} \quad (1.5)$$

The trace is defined in a very weak sense on any bounded subset of each side of $\partial\Omega$, by an immediate generalization of [12]. Anyway, as we are interested in non-negative solutions, there automatically holds $\mathrm{e}^{-\varphi} \in L^\infty(\mathcal{O}) \subset H^{-1}(\mathcal{O})$.

For both types of solutions, there obviously holds:

Lemma 1.1. *Let φ be a solution to* (1.3) *on the sector Ω. For any isometry $\mathcal{T}$ of $\mathbb{R}^2$ (translation, rotation, reflection), the function $\varphi(\mathcal{T}\boldsymbol{x})$ is a solution on the sector $\mathcal{T}^{-1}(\Omega)$.*

Since 0 is a subsolution to the problem (1.3), the existence of a solution is equivalent to that of a non-negative supersolution to the problem. We will develop this in the sequel. Furthermore, any non-negative solution to (1.3) in $\Omega[\theta_0]$ is a supersolution in a smaller sector $\Omega[\theta_1]$, $\theta_1 < \theta_0$. Therefore, the existence of a solution in the split plane $\Omega[\pi]$ implies the solvability of (1.3) in any sector. By symmetry, it is enough to solve the mixed Dirichlet–Neumann problem in the upper half-plane

$$\begin{aligned} -\Delta\varphi_* = \mathrm{e}^{-\varphi_*} \quad &\text{in } \mathbb{R}^2_+ = [0 < \theta < \pi], \\ \varphi_* = 0 \text{ on } [\theta = \pi], \quad &\partial_n \varphi_* = 0 \text{ on } [\theta = 0]. \end{aligned} \quad (1.6)$$

Glueing this φ_* to its even reflection with respect to the axis $[\theta = 0]$ yields a solution to (1.3) on $\Omega[\pi]$.

Remark 1.2. *It is worth pointing out that there exists no (non-negative, very weak) solution to* (1.3) *in the whole plane.*

The article is written as follows. In a second section we construct a supersolution to the equation in the split plane. For this purpose we use a constructive method which relies on complex analysis. In a third section we discuss some properties, such as monotonicity, symmetry or regularity, of the *minimal* positive solution; this minimal solution is relevant for the Physics of the original problem. In a fourth section we prove the non-uniqueness of solutions and list some of their properties. Eventually, we discuss the application to the stationary Vlasov–Poisson system in a last section. In this last section, we also show the link between this asymptotic and the boundary blow-up solutions for $\Delta u = \mathrm{e}^u$, see [2, 3, 16, 15, 8]. Boundary blow-up (or large) solutions were introduced in the seminal articles [14, 17].

2. Construction of a solution in the split plane

We begin with a construction inherited from complex analysis. It is worth pointing out that this construction method works for any sectorial domain.

Proposition 2.1. *There exists a solution φ_* to equation* (1.6) *in the split plane.*

Let $z_2 = \Phi(z_1)$ or $z_1 = \Psi(z_2)$ be a conformal mapping between the complex z_1 and z_2-planes, and let $D_1,\ D_2$ be two domains conformally mapped to one another. Suppose we are given two functions $w_1,\ w_2$ on D_1 and D_2 respectively, which are transformed into each other by the formulas:

$$w_2(z_2) = \log|\Psi'(z_2)| + w_1(\Psi(z_2)); \tag{2.1}$$

$$w_1(z_1) = \log|\Phi'(z_1)| + w_2(\Phi(z_1)). \tag{2.2}$$

Using $\Delta = 4\,\partial_z\partial_{\bar z}$ and the fact that the logarithm of the modulus of an analytic function is harmonic, one easily checks the following lemma.

Lemma 2.2. *Let $\Delta_i,\ i = 1,\ 2$ be the Laplacian in the z_i-plane, and let $w_1,\ w_2$ be related by* (2.1) *or* (2.2)*. Then, w_1 satisfies $\Delta_1 w_1 = 4\mathrm{e}^{2w_1}$ in D_1 if, and only if, w_2 satisfies $\Delta_2 w_2 = 4\mathrm{e}^{2w_2}$ in D_2.*

In this section, we denote (x_i, y_i) and (r_i, θ_i) the Cartesian and polar coordinates in the z_i-plane. We choose D_1 as the upper half-plane $[0 < \theta_1 < \pi] = [y_1 > 0]$, and we introduce the mixed Dirichlet–Neumann problem

$$\Delta_1 w_1 = 4\mathrm{e}^{2w_1} \text{ in } D_1, \quad w_1 = 0 \text{ on } \Gamma_1^D, \quad \partial_{n_1} w_1 = 0 \text{ on } \Gamma_1^N, \tag{2.3}$$

where $\Gamma_1^D = [\theta_1 = \pi] = \{(x_1, 0) : x_1 < 0\}$, and $\Gamma_1^N = [\theta_1 = 0] = \{(x_1, 0) : x_1 > 0\}$. This is obviously related to (1.6) (set $\varphi_*(\boldsymbol{x}) = -2w_1(\boldsymbol{x}/\sqrt{8})$).

Consider now the conformal mapping

$$\Phi(z_1) = \frac{1}{z_1 + \mathrm{i}}, \quad \Psi(z_2) = \frac{1}{z_2} - \mathrm{i}.$$

The half-plane is mapped by Φ onto the disk D_2 centered at $\frac{1}{2\mathrm{i}}$ and of radius $\frac{1}{2}$; the negative and positive real half-axes Γ_1^D and Γ_1^N are respectively mapped to the left and right half-circles Γ_2^D and Γ_2^N of this disk.

By Lemma 2.2, the function w_2 defined by (2.1) satisfies $\Delta_2 w_2 = 4\mathrm{e}^{2w_2}$ in D_2. What about the boundary conditions? On the Dirichlet half-circle Γ_2^D, one has

$$w_2(z_2) = \log|\Psi'(z_2)| = -\log|z_2|^2 \geq 0; \tag{2.4}$$

it is worth observing that this function is non-negative and singular at $z_2 = 0$ (corresponding to $z_1 = \infty$) only. On the Neumann half-circle Γ_2^N, we compute as follows. In polar coordinates, we have:

$$w_2(r_2\,\mathrm{e}^{\mathrm{i}\theta_2}) = -2\log r_2 + w_1\left(\frac{\cos\theta_2}{r_2} - \mathrm{i}\left(1 + \frac{\sin\theta_2}{r_2}\right)\right). \tag{2.5}$$

Parametrizing the boundary as $z_2 = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\vartheta} - \mathrm{i}\right)$, we observe that:

$$r_2^2 = \frac{1-\sin\vartheta}{2}, \quad r_2\cos\theta_2 = \frac{1}{2}\cos\vartheta, \quad r_2\sin\theta_2 = \frac{-1+\sin\vartheta}{2};$$
$$2\theta_2 = \vartheta - \frac{\pi}{2}, \quad (\cos\vartheta, \sin\vartheta) = (-\sin 2\theta_2, \cos 2\theta_2).$$

Recalling that $\partial_{y_1} w_1 = -\partial_{n_1} w_1 = 0$ on Γ_1^N, we deduce:

$$\begin{aligned}
\partial_{x_2} w_2 &= -\frac{2\cos\theta_2}{r_2} - \frac{\cos 2\theta_2}{r_2^2}\partial_{x_1} w_1 \quad \text{on } \Gamma_2^N, \\
\partial_{y_2} w_2 &= -\frac{2\sin\theta_2}{r_2} - \frac{\sin 2\theta_2}{r_2^2}\partial_{x_1} w_1 \quad \text{on } \Gamma_2^N.
\end{aligned}$$

Therefore

$$\partial_{n_2} w_2 = \frac{2\sin\theta_2}{r_2} = -2 \quad \text{on } \Gamma_2^N. \tag{2.6}$$

Next, we proceed to a truncation on the boundary condition, setting $g_2^k = \min(k, -\log|z_2|^2)$, i.e., we solve the mixed Dirichlet–Neumann problem

$$\Delta_2 w_2^k = 4\mathrm{e}^{2w_2^k} \text{ in } D_2, \quad w_2^k = g_2^k \text{ on } \Gamma_2^D, \quad \partial_{n_2} w_2^k = -2 \text{ on } \Gamma_2^N. \tag{2.7}$$

The unique variational solution minimizes the strictly convex l.s.c. functional

$$\int_{D_2}\left\{\frac{1}{2}|\nabla w|^2 + 2\,\mathrm{e}^{2w}\right\} + \int_{\Gamma_2^N} 2\,w$$

on the affine space $V(g_2^k) := \{w \in H^1(D_2) : w = g_2^k \text{ on } \Gamma_2^D\}$. We remark that $w_2^k \in C(\overline{D_2})$: as argued in Section 1, $\Delta_2 w_2^k = \mathrm{e}^{2w_2^k} \in L^2(D_2)$ by the Trudinger inequality; as the boundary data are smooth enough (actually, $g_2^k \in H^{3/2-\epsilon}(\Gamma_2^D)$), one has $w_2^k \in H^{3/2-\epsilon}(D_2) \subset C(\overline{D_2})$; the regularity is limited by the change of boundary conditions [12].

We now state that $u_2(z_2) = -\log|z_2|^2$ is a supersolution to (2.7).

Lemma 2.3. *For any z_2 in D_2, $w_2^k(z_2) \le -\log|z_2|^2$.*

Proof. Computing as above, we see that u_2 solves the problem

$$\Delta_2 u_2 = 0 \text{ in } D_2, \quad u_2 = -\log r_2^2 \text{ on } \Gamma_2^D, \quad \partial_{n_2} u_2 = -2 \text{ on } \Gamma_2^N.$$

It is not a variational solution, but it is smooth where it is bounded. Moreover, $u_2 \ge 0$ in D_2. As a consequence, there holds:

$$\int_{D_2} \nabla u_2 \cdot \nabla v + \int_{\Gamma_2^N} 2\, v = 0,$$

for all $v \in H^1(D_2)$ such that $v = 0$ on Γ_2^D and u_2 is bounded above on $\operatorname{supp} v$. On the other hand, the variational formulation of (2.7) reads

$$\int_{D_2} \nabla w_2^k \cdot \nabla v + \int_{D_2} 4\mathrm{e}^{2w_2^k}\, v = -2 \int_{\Gamma_2^N} v. \tag{2.8}$$

Therefore

$$\int_{D_2} \Big\{ \nabla(w_2^k - u_2) \cdot \nabla v + 4\mathrm{e}^{2w_2^k}\, v \Big\} = 0,$$

for any admissible v. The function $v = (w_2^k - u_2)_+$ is admissible: it vanishes both on Γ_2^D and where $u_2 \ge \max w_2^k$, hence u_2 is bounded above on the support of v. One finds as usual $(w_2^k - u_2)_+ = 0$, i.e., $w_2^k \le u_2$. □

Obviously, w_2^K also is a supersolution to (2.7) for $K > k$; therefore,

$$w_2^0 \le w_2^k \le w_2^K \le u_2, \qquad \text{for } 0 \le k \le K.$$

We deduce that, for all $z_2 \in \overline{D_2} \setminus \{0\}$, $w_2^k(z_2)$ converges to a limit $w_2(z_2) \le u_2(z_2)$. By the monotone convergence theorem, $w_2^k \to w_2$ and $\mathrm{e}^{2w_2^k} \to \mathrm{e}^{2w_2}$ in $L^p(\mathcal{K})$, for any compact $\mathcal{K} \subset \overline{D_2} \setminus \{0\}$ and $p < \infty$. This implies $\Delta_2 w_2 = 4\mathrm{e}^{2w_2}$ in the sense of distributions in D_2. Furthermore, one can consider boundary conditions in a very weak sense [12]. Summing up, w_2 solves the problem:

$$\Delta_2 w_2 = 4\mathrm{e}^{2w_2} \text{ in } D_2, \quad w_2 = -\log|z_2|^2 \text{ on } \Gamma_2^D, \quad \partial_{n_2} w_2 = -2 \text{ on } \Gamma_2^N.$$

As w_2^0 is bounded on $\overline{D_2}$, the limit is bounded below: $w_2 \ge m$ on $\overline{D_2}$. Using the conformal mapping Φ, the function w_1 defined by (2.2) is a solution to (2.3) which satisfies the bounds

$$m + \log|\Phi'(z_1)| = m - \log(x_1^2 + (1+y_1)^2) \le w_1(z_1) \le 0 \quad \text{on } \overline{D_1}. \tag{2.9}$$

3. The minimal solution

From the previous section, we have constructed a solution φ_* to (1.6), which satisfies the bounds (cf. (2.9)):

$$0 \leq \varphi_*(\boldsymbol{x}) \leq 2\,\log(1 + \tfrac{\sqrt{2}}{2}\,r\,\sin\theta + \tfrac{1}{8}\,r^2) - 2\,m \tag{3.1}$$

Thus, φ_*, extended by reflection to $\Omega[\pi]$, is bounded on any bounded subset of $\Omega[\pi]$. We will provide a better version of this upper bound in the sequel.

Also, notice that the conditions, valid for any bounded subset $\mathcal{O} \subset \Omega[\pi]$

$$\varphi_* \in L^\infty(\mathcal{O}), \quad \Delta\varphi_* = \mathrm{e}^{-\varphi_*} \in L^\infty(\mathcal{O}), \quad \varphi_* = 0 \text{ on } \partial\Omega[\pi]$$

actually imply $\varphi_* \in H^1(\mathcal{O})$ (see [12] or Proposition 3.5 below);therefore φ_* is a local variational solution to (1.3) in $\Omega[\pi]$.

3.1. A limiting process

Consider the truncated domain

$$\Omega_R = \{\boldsymbol{x}(r,\theta) \in \mathbb{R}^2; |\theta| < \theta_0 \leq \pi,\ 0 < r < R\}.$$

The proof of the following result is standard.

Lemma 3.1. *There exists a unique variational solution u_R to the problem*

$$\begin{aligned} -\Delta u_R &= \mathrm{e}^{-u_R} \quad \textit{in } \Omega_R, \\ u_R &= 0 \quad \textit{on } \partial\Omega_R. \end{aligned} \tag{3.2}$$

This solution is positive and is symmetric with respect to $\theta \mapsto -\theta$.

Let us observe that, for $R' \geq R$, both $u_{R'}$ and φ_* are supersolutions to (3.2). Therefore, for any $\boldsymbol{x}$ in Ω_R

$$u_R(\boldsymbol{x}) \leq u_{R'}(\boldsymbol{x}) \leq \varphi_*(\boldsymbol{x}).$$

Passing to the limit while R goes to the infinity, setting $u(\boldsymbol{x}) = \sup_R u_R(\boldsymbol{x})$ we construct this way a solution u to the original problem. This solution is symmetric with respect to $\theta = 0$. Moreover, u is the *minimal* solution, in the sense that any (non-negative) local variational solution φ to (1.3) satisfies

$u \leq \varphi$; ([1]) therefore, u is unique. Similarly, any non-negative supersolution $\overline{\varphi}$ of local variational regularity satisfies $u \leq \overline{\varphi}$. From (3.1), we deduce that there exists $C \geq 0$ such that

$$u(\boldsymbol{x}) \leq C + 4\log(1 + |\boldsymbol{x}|), \quad \forall \boldsymbol{x} \in \Omega. \tag{3.3}$$

Remark 3.2. *We can compute the minimal solution in the half-plane* $\Omega[\pi/2]$*, which reduces to a 1D problem on the half-line, that is* $2\log(1 + \frac{x_1}{\sqrt{2}})$*. Since the minimal solution* $u_{[\theta_0]}$ *in* $\Omega[\theta_0]$ *is a supersolution if* $\theta_0 \geq \pi/2$*, then* $u_{[\theta_0]}$ *cannot be bounded by above in that case.*

3.2. Regularity

We introduce the following function spaces on a domain $\mathcal{O}$:

$$\begin{aligned} \Phi_p(\mathcal{O}) &:= \left\{ w \in H_0^1(\mathcal{O}) : \Delta w \in L^p(\mathcal{O}) \right\}, \\ N(\mathcal{O}) &:= \text{ orthogonal of } \Delta \left[H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \right] \text{ within } L^2(\mathcal{O}) \\ &= \left\{ p \in L^2(\mathcal{O}) : \Delta p = 0 \text{ in } \mathcal{O} \text{ and } p = 0 \text{ on each side of } \partial\mathcal{O} \right\}. \end{aligned}$$

In the first two lines, $\mathcal{O}$ is a bounded Lipschitz domain. The second characterisation of $N(\mathcal{O})$, proved in [12] for polygons, can be extended to curvilinear polygons[2] such as Ω_R, whose boundary is composed of smooth sides that meet at corners. To express the regularity of solutions and give asymptotic expansions, we shall generally make use of the parameter:

$$\alpha = \pi/(2\theta_0).$$

We introduce the well-known primal and dual harmonic singularities in Ω:

$$S(r,\theta) := r^\alpha \cos(\alpha\theta), \quad S^*(r,\theta) := \tfrac{1}{\pi}\, r^{-\alpha} \cos(\alpha\theta), \tag{3.4}$$

From [12], we know the following facts.

Proposition 3.3. *Let* χ *be a fixed cutoff function equal to* 1 *for* $|\boldsymbol{x}| \leq 1$ *and* 0 *for* $|\boldsymbol{x}| \geq 2$*; for any* $B > 0$ *we write* $\chi_B(r) := \chi(r/B)$*.*

[1] This is also true for very weak solutions, as a consequence of Proposition 4.2 below.

[2] A *curvilinear polygon* is an open set $\mathcal{O} \subset \mathbb{R}^2$ such that for any $\boldsymbol{x}_0 \in \overline{\mathcal{O}}$ and η sufficiently small, $\mathcal{O} \cap B(\boldsymbol{x}_0, \eta)$ is C^2-diffeomorphic, either to $\mathbb{R}^2$, or to a sector $\Omega[\theta_{\boldsymbol{x}_0}]$, with $0 < \theta_{\boldsymbol{x}_0} < \pi$. This definition includes both smooth domains and straight polygons, but excludes cusps and cracks.

- *If the angle at the tip of the sector is salient or flat, then:*

$$N(\Omega_R) = \{0\}, \quad \Phi_2(\Omega_R) = H^2 \cap H_0^1(\Omega_R), \quad \Phi_p(\Omega_R) \subset W^{2,p}(\Omega_R),$$

  for some $p > 2$.

- *If the sector is reentrant, the space $N(\Omega_R)$ is one-dimensional, spanned by*

$$P_s^R(r,\theta) = \chi_{R/2}(r)\, S^*(r,\theta) + \tilde{P}^R, \quad \textit{with} \quad \tilde{P}^R \in H_0^1(\Omega_R).$$

  The spaces $\Phi_p(\Omega_R)$ for $p \geq 2$ are embedded into $u \in H^s(\Omega_R) \subset C(\overline{\Omega_R})$, for all $s < 1+\alpha$; any $w \in \Phi_2$ admits the regular-singular decomposition:

$$w = \lambda[w]\, \chi_{R/2}\, S + \tilde{w}, \quad \textit{with} \quad \tilde{w} \in H^2 \cap H_0^1(\Omega_R),$$

  where $\lambda[w]$ is called the singularity coefficient *of w.*

Remark 3.4. *Actually, P_s^R can be computed exactly in Ω_R:*

$$P_s^R(r,\theta) = \frac{1}{\pi}\left(r^{-\alpha} - \left(\frac{r}{R^2}\right)^{\alpha}\right)\cos(\alpha\theta). \tag{3.5}$$

As a first step, we prove that the minimal solution is a local variational solution.

Proposition 3.5. *Let R be an arbitrary positive number. The minimal solution u has the following regularity in Ω_R.*

1. *If the sector is salient of flat ($\theta_0 \leq \pi/2$), $u \in C^1(\overline{\Omega_R})$.*
2. *It the sector is reentrant ($\theta_0 > \pi/2$), $u \in H^s(\Omega_R) \subset C(\overline{\Omega_R})$, for all $s < 1+\alpha$; and it admits the expansion $u(r,\theta) = \Lambda\, S(r,\theta) + \tilde{u}(r,\theta)$, with $\tilde{u} \in H^2(\Omega_R)$ and $\Lambda > 0$.*

Proof. It follows from (3.3) that u is bounded on Ω_R for any R. Fix $B > 0$, and introduce $v \in H_0^1(\Omega_{2B})$ which solves

$$\Delta v = f := \Delta(\chi_B\, u) = -\chi_B\, \mathrm{e}^{-u} + 2\,\nabla\chi_B \cdot \nabla u + (\Delta\chi_B)u \in H^{-1}(\Omega_{2B}), \tag{3.6}$$

and set $P := v - \chi_B\, u$. There holds $P \in L^2(\Omega_{2B})$, $\Delta P = 0$ and $P = 0$ on each side of $\partial\Omega_{2B}$; so $P \in N(\Omega_{2B})$. Therefore, $P = 0$ and $\chi_B\, u = v \in H_0^1(\Omega_{2B})$ in the salient case. In a reentrant sector, we have $P = A\, P_s^R$. On the other hand, a Sobolev embedding gives $v \in L^p(\Omega_{2B})$ for all $p < +\infty$; as $\chi_B\, u \in L^\infty(\Omega_{2B})$ by (3.3), one has $P \in L^p(\Omega_{2B})$ for all $p < +\infty$, which is only possible if $A = 0$, i.e., $\chi_B\, u = v \in H_0^1(\Omega_{2B})$ again.

Thus, we have in all cases $u \in H^1(\Omega_B)$; computing as in (3.6), we find $\Delta(\chi_{B/2}\, u) \in L^2(\Omega_B)$, i.e., $\chi_{B/2}\, u \in \Phi_2(\Omega_B)$. Invoking Proposition 3.3 again, this leads to:

- $\chi_{B/2}\, u \in H^2(\Omega_B) \subset W^{1,p}(\Omega_B)$ for all $p < +\infty$, in the salient case. Bootstrapping again, we deduce $\Delta(\chi_{B/2}\, u) \in L^p(\Omega_B)$, and $\chi_{B/2}\, u \in \Phi_p(\Omega_B) \subset W^{2,p}(\Omega_B)$ for some $p > 2$. In other words, $u \in W^{2,p}(\Omega_R) \subset C^1(\overline{\Omega_R})$ as B is arbitrary.

- in the reentrant case, we get the expansion

$$(\chi_{B/2}\, u)(r,\theta) = \Lambda\, \chi_{B/2}(r)\, S(r,\theta) + \tilde{v}(r,\theta), \quad \text{with} \quad \tilde{v} \in H^2 \cap H^1_0(\Omega_B),$$

  hence the decomposition of u as B is arbitrary.

To prove $\Lambda > 0$, we return to the solution u_R on Ω_R. Let Λ_R be its singularity coefficient; it controls the dominant behaviour of u_R near the corner, $u_R \sim \Lambda_R\, S(r,\theta)$ as $r \to 0$, uniformly in θ [13]. It is given by the formula [12]:

$$\Lambda_R = \int_{\Omega_R} (-\Delta u_R)\, P_s^R = \int_{\Omega_R} \mathrm{e}^{-u_R}\, P_s^R.$$

From (3.5), we see that $P_s^R > 0$ in Ω_R.([3]) One infers that $\Lambda_R > 0$. Yet, we have seen that $u \geq u_R$, thus $\Lambda \geq \Lambda_R > 0$. □

Remark 3.6. *Actually, there holds* $\Lambda = \lim_{R\to+\infty} \Lambda_R$. *In the fixed domain* Ω_{2B}, *the function* $v_R := \chi_B\, u_R \in H^1_0(\Omega_{2B})$ *solves*

$$\Delta v_R = f_R := -\chi_B\, \mathrm{e}^{-u_R} + 2\, \nabla\chi_B \cdot \nabla u_R + (\Delta\chi_B) u_R \in L^2(\Omega_{2B}).$$

By the monotone convergence theorem, we know that $u_R \to u$ *and* $\mathrm{e}^{-u_R} \to \mathrm{e}^{-u}$ *in* $L^2(\Omega_{2B})$. *Hence* $f_R \to f$ *in* $H^{-1}(\Omega_{2B})$ *and* $v_R \to v$ *in* $H^1_0(\Omega_{2B})$. *In particular,* $u_R \to u$ *in* $H^1(\Omega_B)$, *and (as in Proposition 3.5),* $\chi_{B/2}\, u_R \to \chi_{B/2}\, u \in \Phi_2(\Omega_B)$. *As the singularity coefficient is a continuous linear form on* Φ_2, *one infers* $\Lambda_R \to \Lambda$.

[3]This property is probably true for any curvilinear polygon. Anyway, it is easily seen that the dual singular function is strictly positive on some subdomain, which is enough.

3.3. Monotonicity

We can now prove a monotonicity property. Standard results in the literature assert that *bounded* solutions to this type of PDE have some monotonicity properties. Our minimal solution is not bounded, see Remark 3.2.

Proposition 3.7. *Let* $\boldsymbol{u}[\theta]$ *be the vector* $(\cos\theta, \sin\theta)$*. For any* θ *such that* $|\theta| \leq \theta_0$*,* $t \geq 0$ *and* $\boldsymbol{x} \in \Omega$*, there holds* $u(\boldsymbol{x}) \leq u(\boldsymbol{x} + t\,\boldsymbol{u}[\theta])$*.*

Remark 3.8. *This implies that* $\boldsymbol{x} \cdot \nabla u(\boldsymbol{x}) \geq 0$ *and* $\frac{\partial u}{\partial x_1} \geq 0$*.*

Proof. We use Proposition 1.1 with $\mathcal{T}$ being the translation $\mathcal{T}\boldsymbol{x} = \boldsymbol{x} + t\,\boldsymbol{u}[\theta]$. The function $u(\mathcal{T}\boldsymbol{x})$ is a super-solution in Ω, as $\Omega \subset \mathcal{T}^{-1}(\Omega)$, so $u(\mathcal{T}\boldsymbol{x}) \geq u(\boldsymbol{x})$ and the result follows promptly. □

Lemma 3.9. *The minimal solution satisfies*

$$r\,\partial_r u(r,\theta) = \boldsymbol{x} \cdot \nabla u(\boldsymbol{x}) \leq 2. \tag{3.7}$$

Proof. We observe that for any $\lambda \in (0,1)$, $u(\lambda\boldsymbol{x}) - 2\log\lambda$ is a supersolution to the equation (1.3). Thus:

$$u(\boldsymbol{x}) \leq u(\lambda\boldsymbol{x}) - 2\log\lambda. \tag{3.8}$$

Dividing the equation (3.8) by $1-\lambda$ and letting $\lambda \to 1$ yields the result. □

Remark 3.10. *Setting* $\lambda = |\boldsymbol{x}|^{-1}$ *in* (3.8)*, we find a slightly improved version of* (3.3)*:*

$$u(\boldsymbol{x}) \leq \sup_{|\boldsymbol{y}|=1} u(\boldsymbol{y}) + 2\log|\boldsymbol{x}| = \sup_{|\boldsymbol{y}|\leq 1} u(\boldsymbol{y}) + 2\log|\boldsymbol{x}|, \quad \forall |\boldsymbol{x}| \geq 1. \tag{3.9}$$

The equality of the two upper bounds follows from Proposition 3.7.

3.4. Symmetry

Proposition 3.11. *Consider* u_R *the solution of* (3.2)*. Then* $\theta \mapsto u_R(r,\theta)$ *achieves its maximum at* $\theta = 0$*.*

Corollary 3.12. *The minimal solution* u *of* (1.3) *satisfies that* $\theta \mapsto u(r,\theta)$ *achieves its maximum at* $\theta = 0$*.*

Proof. Consider the half-domain $\Omega_R^+ := [0 < \theta < \theta_0 \text{ and } r < R]$. The function $v := \partial_\theta u_R$ is solution to

$$-\Delta v + \mathrm{e}^{-u_R} v = 0, \tag{3.10}$$

and $v \leq 0$ on the boundary. Actually, $v = 0$ on the lower ray $[\theta = 0]$ and on the arc of circle $[r = R]$, while $v \leq 0$ on the upper ray $[\theta = \theta_0]$. By the maximum principle,[4] $v < 0$ in Ω_R^+. The results follows promptly. □

Remark 3.13. *We reckon that the map $\theta \mapsto u(r, \theta)$ is concave, but we do not have a proof of this fact.*

4. General properties of solutions

4.1. Non-uniqueness

For any $\mu = (\mu_-, \mu_+) \in (\mathbb{R}^+)^2$, the function $H_\mu := \mu_- S^* + \mu_+ S$ is non-negative and harmonic in Ω, and vanishes on the boundary: it is a subsolution to (1.3). We shall construct a solution in the form $\varphi_\mu = H_\mu + v^\mu$. If such a solution exists, then v^μ solves

$$-\Delta v^\mu = \mathrm{e}^{-H_\mu}\, \mathrm{e}^{-v^\mu} \quad \text{in } \Omega, \quad v^\mu = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

The corresponding problem in the truncated domain

$$-\Delta v_R^\mu = \mathrm{e}^{-H_\mu}\, \mathrm{e}^{-v_R^\mu} \quad \text{in } \Omega_R, \quad v_R^\mu = 0 \quad \text{on } \partial\Omega_R,$$

is well-posed, and $v_{R'}^\mu$ is a supersolution when $R' > R$. The minimal solution u to (1.3) also is a supersolution, so $v_R^\mu \leq v_{R'}^\mu \leq u$, and passing to the limit we obtain a solution to (4.1). Thus we have constructed a solution $\varphi_\mu = H_\mu + v^\mu$ to (1.3), which is bounded as

$$\mu_- S^* + \mu_+ S \leq \varphi_\mu \leq \mu_- S^* + \mu_+ S + u.$$

[4] To check that $v \in H^1(\Omega_R^+)$: if the sector Ω is salient or flat ($\theta_0 \leq \frac{\pi}{2}$), write: $v = \boldsymbol{x}^\perp \cdot \nabla u_R$, with $\boldsymbol{x}^\perp := (-x_2, x_1)$. As $u_R \in H^2(\Omega_R)$ in this case, one deduces $v \in H^1(\Omega_R)$. If the sector is reentrant ($\theta_0 > \frac{\pi}{2}$), write $u_R = \tilde{u}_R + \Lambda_R S$, with $\tilde{u}_R \in H^2(\Omega_R)$. Then:

$$v = \partial_\theta \tilde{u}_R + \Lambda_R\, \partial_\theta S = \boldsymbol{x}^\perp \cdot \nabla \tilde{u}_R + \Lambda_R\, (-\alpha\, r^\alpha\, \sin(\alpha\theta))$$

The second term belongs to $H^s(\Omega_R)$ for $s < 1 + \alpha$, and the first is treated as above.

Because of (3.3), there holds

$$\begin{aligned}
&\text{if } \mu_- > 0, \qquad \varphi_\mu(r,\theta) \sim \tfrac{1}{\pi}\,\mu_-\, r^{-\alpha}\,\cos(\alpha\theta) \quad \text{as } r \to 0, \quad \theta \neq \pm\theta_0 \text{ fixed;}\\
&\text{if } \mu_+ > 0, \qquad \varphi_\mu(r,\theta) \sim \mu_+\, r^{\alpha}\,\cos(\alpha\theta) \quad \text{as } r \to +\infty, \quad \theta \neq \pm\theta_0 \text{ fixed.}
\end{aligned}$$

Thus, solutions corresponding to different μ are distinct. They are local variational solutions if $\mu_- = 0$, and very weak solutions if $\mu_- > 0$ and the sector is reentrant ($\alpha < 1$). In a flat or salient sector ($\alpha \geq 1$) the solutions corresponding to $\mu_- > 0$ are not L^2 in a neighbourhood of the origin, thus they do not qualify as very weak solutions.

Remark 4.1. *One may wonder if there exists a solution to* (1.3) *that is not non-negative. Using the maximum principle in unbounded domains of* $\mathbb{R}^2$ *(see [6]) we can prove that any solution of* (1.3) *that is bounded from below is non-negative.*

4.2. Local regularity near the tip of the sector

Using the tools of Proposition 3.5 (localization, bootstrapping, and regularity theory for the linear Poisson–Dirichlet problem), one obtains the following results.

Proposition 4.2. *There holds:*

1. *Any solution to* (1.3) *belongs to* $C^\infty(\mathcal{K})$*, for any compact subset* $\mathcal{K} \subset \overline{\Omega}$ *such that the origin* $0 \notin \mathcal{K}$*.*
2. *If* Ω *is salient of flat* ($\theta_0 \leq \pi/2$)*, any very weak solution* φ *is actually a local variational solution, and all solutions satisfy* $\varphi \in W^{2,p}(\Omega_R) \subset C^1(\overline{\Omega_R})$ *for some* $p > 2$ *and all finite* R*.*
3. *It* Ω *is reentrant* ($\theta_0 > \pi/2$)*, then:*
   - *any local variational solution satisfies* $\varphi \in H^s(\Omega_R) \subset C(\overline{\Omega_R})$*, for all* $s < 1+\alpha$*, and admits the expansion* $\varphi(r,\theta) = \lambda\, S(r,\theta) + \tilde{\varphi}(r,\theta)$*, with* $\tilde{\varphi} \in H^2(\Omega_R)$*, and* $\lambda \geq \Lambda$ *if* $\varphi \geq 0$*;*
   - *any very weak solution admits the expansions*

$$\begin{aligned}
\varphi(r,\theta) &= \lambda_*\, S^*(r,\theta) + \hat{\varphi}(r,\theta), \quad \hat{\varphi} \in H^1(\Omega_R)\,; && (4.2)\\
&= \lambda_*\, S^*(r,\theta) + \lambda\, S(r,\theta) + \tilde{\varphi}(r,\theta), \quad \tilde{\varphi} \in H^2(\Omega_R). && (4.3)
\end{aligned}$$

for all finite R*. The coefficient* λ_* *is non-negative.*

Proof. We only prove the last claim; the others are similar. By definition, a variational solution belongs to $H^1(\Omega_R)$ for all finite R; computing as in (3.6) we see that $\chi_B\,\varphi \in \Phi_2(\Omega_{2B})$, hence the decomposition. As any solution $\varphi \geq 0$ is larger than the minimal solution u, its singularity coefficient λ is larger than that of u, namely Λ.

If φ only is a very weak solution, one finds $\Delta(\chi_B\,\varphi) \in H^{-1}(\Omega_{2B})$. Introducing $v \in H^1_0(\Omega_{2B})$ such that $\Delta v = \Delta(\chi_B\,\varphi)$ and $P := \chi_B\,\varphi - v$, one finds $P \in N(\Omega_{2B})$; thus, there is $\lambda_* \in \mathbb{R}$ such that $P = \lambda_*\,P_s^{2B} = \lambda_*\,(\chi_B\,S^* + \tilde{P}^{2B})$ and $\chi_B\,\varphi = \lambda_*\,\chi_B\,S^* + \varphi_B$, with $\varphi_B \in H^1_0(\Omega_{2B})$. As B is arbitrary, we deduce that for any R

$$\varphi(r,\theta) = \lambda_*\,S^*(r,\theta) + \hat{\varphi}(r,\theta), \quad \hat{\varphi} \in H^1(\Omega_R).$$

As $\Delta\hat{\varphi} = \Delta\varphi$, one finds $\chi_B\,\hat{\varphi} \in \Phi_2(\Omega_{2B})$, hence the second decomposition of φ. The condition $\lambda_* \geq 0$ is implied by the assumption $\mathrm{e}^{-\varphi} \in H^{-1}(\Omega_R)$. □

4.3. Unboundedness

We know from Remark 3.2 that the minimal solution in a flat or reentrant sector is unbounded, hence all non-negative solutions are unbounded. Actually, the same holds in a salient sector.

Proposition 4.3. *Let $\Omega[\theta_0]$ be a salient sector: $\theta_0 < \pi/2$. Any non-negative solution to* (1.3) *satisfies:*

$$\varphi(r,\theta) \geq \log(1 + \tfrac{1}{8}\,r^2\,\sin^2(\theta_0 - |\theta|)). \tag{4.4}$$

Proof. Let $\boldsymbol{x}_0(r,\theta) \in \Omega$. The radius of the largest disk D centred at $\boldsymbol{x}_0$ and contained in Ω is $R = r\,\sin(\theta_0 - |\theta|)$. We have $\varphi \in H^1(D)$ by Proposition 4.2. Introduce the variational solution $\underline{\varphi}$ to the problem

$$-\Delta\underline{\varphi} = \mathrm{e}^{-\underline{\varphi}} \text{ in } D, \quad \underline{\varphi} = 0. \tag{4.5}$$

This solution can be computed explicitly:

$$\underline{\varphi}(\boldsymbol{x}) = \log\left(\frac{(A^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2)^2}{8\,A^2}\right), \quad \text{with:} \quad A = \sqrt{2} + \sqrt{2 + R^2}.$$

On the other hand, φ is a supersolution to (4.5), and $\varphi \geq \underline{\varphi}$ on D. In particular

$$\varphi(\boldsymbol{x}_0) \geq \underline{\varphi}(\boldsymbol{x}_0) = \log\left(R^2 + 4 + \sqrt{8R^2 + 16}\right) - \log 8 \geq \log(1 + R^2/8),$$

which is (4.4). □

Remark 4.4. *The same holds in a reentrant sector, with* $R = r\,\sin(\min(\theta_0 - |\theta|, \frac{\pi}{2}))$.

5. Application to plasma equilibria

In [13] the authors study the properties of the solution ϕ to the problem

$$-\Delta\phi = \kappa\,\exp(\phi_{\rm e} - \phi) := \rho \text{ in } \Upsilon_1, \quad \text{with:} \quad \int_{\Upsilon_1} \rho = M \tag{5.1}$$

in a bounded polygonal (or curvilinear polygonal) domain Υ_1 of $\mathbb{R}^2$. The *mass* parameter M is a data of the problem; the normalization factor κ is an unknown which ensures that the constraint $\int_{\Upsilon_1} \rho = M$ is satisfied. The external potential $\phi_{\rm e}$ is fixed; it belongs to $L^\infty(\Upsilon_1)$.

We shall need the following assumptions.

1. Eq. (5.1) is supplemented with a homogeneous Dirichlet boundary condition.
2. If the domain Υ_1 has reentrant corners, it is contained in its local tangent cone at any of them.

The second condition is automatically satisfied when Υ_1 is a straight polygon with only one reentrant corner: then, it can be described as $\Upsilon_1 = \Upsilon_1^C \setminus (\mathbb{R}^2 \setminus \Omega) \subset \Omega$, where Υ_1^C is the convex envelope of Υ_1, and Ω is the tangent cone at the reentrant corner.

We say that the domain Υ_1 is *regular* if it has no reentrant corner, i.e., it is smooth where it is not convex and convex where it is not smooth. Otherwise, it is *singular*. To simplify the exposition, we shall only consider singular domains with one reentrant corner of opening $2\theta_0 = \pi/\alpha,\ 1/2 < \alpha < 1$, and such that the two sides that meet there are locally straight. In that case, the decomposition of ϕ with respect to regularity writes as in §3.2:

$$\phi = \tilde\phi + \lambda\,\chi(r)\,r^\alpha\,\cos(\alpha\theta), \quad \tilde\phi \in H^2 \cap H_0^1(\Upsilon_1), \tag{5.2}$$

assuming that the reentrant corner is located at 0 and the axes are suitably chosen. Notice, however, that these conditions are not essential as long as assumption (2) above is satisfied. If the sides are not locally straight, the expression of the singular part is modified; the case of multiple reentrant corners is treated by localization.

We shall generally discuss the cases of regular and singular domains together; statements about singularity coefficients are void for a regular domain. The goal of this section is to study the behaviour of the coefficients κ and λ as $M \to +\infty$. In [13] it was proved that:

$$\frac{M}{\kappa} \to 0 \quad \text{and} \quad \frac{\lambda}{\kappa} \to 0 \quad \text{and} \quad \lambda \to +\infty. \tag{5.3}$$

We want to refine the estimates (5.3) by obtaining explicit growth rates.

5.1. Getting rid of the external potential

For a fixed external potential ϕ_{e}, the parameters M and κ are strictly increasing functions of each other [13]. Thus, the problem (5.1) can be parametrized by κ, even though M is the significant variable. For fixed κ and ϕ_{e}, we provisionally denote $M[\kappa;\phi_{\mathrm{e}}] = \int_{\Upsilon_1} \kappa \exp(\phi_{\mathrm{e}} - \phi)$, where ϕ solves (5.1), and $\lambda[\kappa;\phi_{\mathrm{e}}]$ the singularity coefficient of ϕ.

Let $\phi_{\mathrm{e}}^{\min}$, $\phi_{\mathrm{e}}^{\max}$ be the lower and upper bounds of ϕ_{e}. Obviously, for a given κ there holds:

$$M[\kappa\, \mathrm{e}^{\phi_{\mathrm{e}}^{\min}};0] = M[\kappa;\phi_{\mathrm{e}}^{\min}] \le M[\kappa;\phi_{\mathrm{e}}] \le M[\kappa;\phi_{\mathrm{e}}^{\max}] = M[\kappa\, \mathrm{e}^{\phi_{\mathrm{e}}^{\max}};0]. \tag{5.4}$$

Similarly, we know [13] that for two given external potentials $(\phi_{\mathrm{e}}^1, \phi_{\mathrm{e}}^2)$ and normalization factors (κ^1, κ^2) the corresponding solutions (ϕ^1, ϕ^2) and their singularity coefficients satisfy

$$\phi_{\mathrm{e}}^1 + \log \kappa^1 \ge \phi_{\mathrm{e}}^2 + \log \kappa^2 \text{ in } \Upsilon_1 \;\Rightarrow\; \phi^1 \ge \phi^2 \text{ in } \Upsilon_1 \;\Rightarrow\; \lambda^1 \ge \lambda^2.$$

So we deduce:

$$\lambda[\kappa\, \mathrm{e}^{\phi_{\mathrm{e}}^{\min}};0] = \lambda[\kappa;\phi_{\mathrm{e}}^{\min}] \le \lambda[\kappa;\phi_{\mathrm{e}}] \le \lambda[\kappa;\phi_{\mathrm{e}}^{\max}] = \lambda[\kappa\, \mathrm{e}^{\phi_{\mathrm{e}}^{\max}};0]. \tag{5.5}$$

Therefore, we shall mostly concentrate on the case $\phi_{\mathrm{e}} = 0$. Using the physical scaling $\kappa = \epsilon^{-2}$, with $\epsilon \to 0$, we shall denote by ϕ_ϵ the solution to (5.1) with $\phi_{\mathrm{e}} = 0$, i.e.:

$$-\Delta\phi_\epsilon = \epsilon^{-2}\mathrm{e}^{-\phi_\epsilon} \text{ in } \Upsilon_1, \quad \phi_\epsilon = 0 \text{ on } \partial\Upsilon_1\,; \tag{5.6}$$

the corresponding mass will be written M_ϵ; and λ_ϵ is the singularity coefficient of ϕ_ϵ.

5.2. Limit of ϕ_ϵ and λ_ϵ

Now we assume that the domain Υ_1 is singular, and we set the origin at the reentrant corner. First, we prove by some blow-up argument:

Proposition 5.1. *For any $\boldsymbol{\xi}$ in the unbounded sectorial domain Ω, the rescaled function $v_\epsilon(\boldsymbol{\xi}) = \phi_\epsilon(\epsilon\boldsymbol{\xi})$, defined for ϵ small enough, converges towards $u(\boldsymbol{\xi})$, where u is the minimal solution to* (1.3) *in Ω.*

Proof. Let Γ_1 be the union of the two sides that meet at the reentrant corner 0, and R the radius of the circle centered at 0 that is tangent to $\partial\Upsilon_1 \setminus \Gamma_1$; in other words we consider R to be the maximum of r such that $\Omega_r = \Omega[\theta_0] \cap \{|\boldsymbol{x}| < r\}$ is included in Υ_1.

We now perform the blow-up argument by a dilation of factor $\frac{1}{\epsilon}$. We obtain that v_ϵ is the solution to the following problem in the dilated domain $\Upsilon_{\frac{1}{\epsilon}}$:

$$-\Delta v_\epsilon = \mathrm{e}^{-v_\epsilon} \text{ in } \Upsilon_{\frac{1}{\epsilon}}, \quad v_\epsilon = 0 \text{ on } \partial\Upsilon_{\frac{1}{\epsilon}}\,, \tag{5.7}$$

while $u_{[\theta_0]}$ is a supersolution — by assumption (2), the dilated domain is contained in $\Omega[\theta_0]$. Furthermore, we have $\Omega_{\frac{R}{\epsilon}} \subset \Upsilon_{\frac{1}{\epsilon}}$; therefore, v_ϵ is a supersolution to the same problem set in $\Omega_{\frac{R}{\epsilon}}$. Summarizing, we have for any $\boldsymbol{\xi}$ in $\Omega_{\frac{R}{\epsilon}}$:

$$u_{\frac{R}{\epsilon}}(\boldsymbol{\xi}) \le \phi_\epsilon(\epsilon\boldsymbol{\xi}) = v_\epsilon(\boldsymbol{\xi}) \le u_{[\theta_0]}(\boldsymbol{\xi}), \tag{5.8}$$

and the result follows promptly from the results of Subsection 3.1. □

Denoting the polar coordinates as (r, θ) for $\boldsymbol{x}$, $(\rho = \epsilon^{-1} r, \theta)$ for $\boldsymbol{\xi}$, the regular-singular decomposition of v_ϵ writes:

$$v_\epsilon(\rho, \theta) = \Lambda_\epsilon\, \rho^\alpha \cos(\alpha\theta) + \tilde{u}_\epsilon(\rho, \theta), \quad \tilde{u}_\epsilon \in H^2(\Upsilon_{\frac{1}{\epsilon}})\,; \tag{5.9}$$

obviously, the singularity coefficient Λ_ϵ is related to that of ϕ_ϵ as: $\Lambda_\epsilon = \epsilon^\alpha\, \lambda_\epsilon$.

On the other hand, Eq. (5.8) implies that Λ_ϵ is bounded between the singularity coefficients of $u_{\frac{R}{\epsilon}}$ and u. By Remark 3.6, one deduces $\Lambda_\epsilon \to \Lambda$. In other words, when $\phi_{\mathrm{e}} = 0$, there holds:

$$\lambda_\epsilon \sim \Lambda\, \epsilon^{-\alpha} \quad \text{as } \epsilon \to 0, \quad \text{i.e.,} \quad \lambda[\kappa; 0] \sim \Lambda\, \kappa^{\alpha/2} \quad \text{as } \kappa \to +\infty. \tag{5.10}$$

5.3. Limit of M_ϵ

We now state and prove a result that complements the numerical evidence in [13].

Proposition 5.2. *Let $|\partial\Upsilon_1|$ be the perimeter of Υ_1. When $\epsilon \to 0$, there holds:*

$$M_\epsilon \sim \sqrt{2}|\partial\Upsilon_1|\epsilon^{-1}. \tag{5.11}$$

Proof. Consider ϕ_ϵ solution to (5.6). The main idea is to write $M_\epsilon = \int_{\Upsilon_1} -\Delta\phi_\epsilon = -\int_{\partial\Upsilon_1} \partial_n\phi_\epsilon$, and to derive an equivalent to $\partial_n\phi_\epsilon$ as $\epsilon \to 0$. Unfortunately, in a non-smooth domain this equivalent is not uniform along $\partial\Upsilon_1$, and cannot be straightforwardly integrated on this boundary; thus some technicalities are needed to overcome the difficulty.

Introduce the function $w_\epsilon(\boldsymbol{x}) := -2\log\epsilon - \phi_\epsilon(\boldsymbol{x})$, which solves:

$$\begin{aligned} \Delta w_\epsilon &= \mathrm{e}^{w_\epsilon} \text{ in } \Upsilon_1, \\ w_\epsilon &= -2\log\epsilon \text{ on } \partial\Upsilon_1. \end{aligned} \tag{5.12}$$

We know (see [13]) that w_ϵ converges to the boundary blow-up solution (or large solution) to $\Delta w = \mathrm{e}^w$ in Υ_1.

For almost every point $\boldsymbol{x}_0$ on the boundary of Υ_1 (except the corners), we have the interior and exterior sphere condition: there is a (small) ball B that is included in Υ_1 (resp. $\mathbb{R}^2 \setminus \overline{\Upsilon_1}$) and tangent at $\boldsymbol{x}_0$.

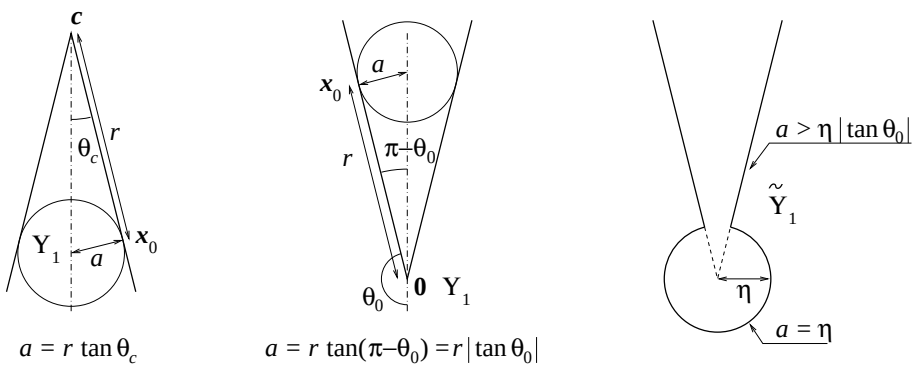

Figure 1: *Left*: The interior sphere condition near a salient corner. *Middle*: The exterior sphere condition near a reentrant corner. *Right*: The mollified/rounded domain $\tilde{\Upsilon}_1$.

We begin with a lower bound for M_ϵ. Fix $\eta > 0$ small enough. We consider $\partial\Upsilon_{1,\eta}$ the set of $\boldsymbol{x}_0 \in \partial\Upsilon_1$ such that there exists a ball B *of radius* η, included in Υ_1 and tangent at $\boldsymbol{x}_0$.

If Υ_1 is a straight polygon and $2\theta_c \in (0,\pi)$ is the angle at the corner $\boldsymbol{c}$, the maximum radius of an interior sphere tangent at a nearby point $\boldsymbol{x}_0$ is $|\boldsymbol{x}_0 - \boldsymbol{c}| \tan\theta_c$; see Figure 1 left. Elsewhere, this radius is bounded below by a constant. We deduce that there exists a constant K such that $|\partial\Upsilon_{1,\eta}| \geq |\partial\Upsilon_1| - K\eta$. This extends to a curvilinear polygon by diffeomorphism; anyway, $|\partial\Upsilon_1| - |\partial\Upsilon_{1,\eta}| \to 0$ as $\eta \to 0$.

Fix $\boldsymbol{x}_0 \in \partial\Upsilon_{1,\eta}$, and let B be the ball defined above. Solving the equation

$$\begin{aligned} \Delta\overline{v} &= \mathrm{e}^{\overline{v}} \text{ in } \ B, \\ \overline{v} &= -2\log\epsilon \text{ on } \partial B, \end{aligned} \tag{5.13}$$

we easily find that w_ϵ is a subsolution to (5.13), so $w_\epsilon(\boldsymbol{x}) \leq \overline{v}(\boldsymbol{x})$ for any $\boldsymbol{x}$ in B, and then $\partial_n w_\epsilon(\boldsymbol{x}_0) \geq \partial_n \overline{v}(\boldsymbol{x}_0)$.

We now compute the normal derivative of the solution to $\Delta v = \mathrm{e}^v$ in a given ball. Up to a translation assume that the ball is centered at the origin. The solution is radially symmetric, i.e. we solve

$$(rv'(r))' = r\mathrm{e}^{v(r)}. \tag{5.14}$$

for $r < \eta$; and $\partial_n v = v'(\eta) \geq 0$. Multiplying this equation by $rv'(r)$ and integrating between 0 and η, we find:

$$\frac{\eta^2(v'(\eta))^2}{2} = \eta^2\mathrm{e}^{-2\log\varepsilon} - 2\int_0^\eta r\mathrm{e}^{v(r)}\,dr = \frac{\eta^2}{\epsilon^2} - 2\eta v'(\eta), \tag{5.15}$$

using once more $r\mathrm{e}^{v(r)} = (rv'(r))'$ and the fact that $v = -2\log\epsilon$ on the boundary. We infer from this equality:

$$v'(\eta) \geq \frac{2\eta\epsilon^{-2}}{\sqrt{2\eta^2\epsilon^{-2}} + 2}. \tag{5.16}$$

We then have

$$M_\epsilon \geq \int_{\partial\Upsilon_{1,\eta}} \partial_n w_\epsilon(\boldsymbol{x}_0) \geq \frac{2\eta\epsilon^{-2}}{\sqrt{2\eta^2\epsilon^{-2}} + 2}\,(|\partial\Upsilon_1| - K\eta). \tag{5.17}$$

Therefore

$$\liminf(\epsilon M_\epsilon) \geq \sqrt{2}(|\partial\Upsilon_1| - K\eta). \tag{5.18}$$

We then let η go to zero to obtain the lower bound.

We now proceed to the upper bound. If $\boldsymbol{x}_0$ is not a corner, we have another small ball B_1 included in $\mathbb{R}^2 \setminus \overline{\Upsilon_1}$ that is tangent at $\boldsymbol{x}_0$. Introduce another ball B_2, with the same center as B_1, and large enough to have $\Upsilon_1 \subset B_2$. Then consider the annulus $N = B_2 \setminus B_1$ that contains Υ_1, and solve the boundary-value problem:

$$\begin{aligned} \Delta \underline{v} &= \mathrm{e}^{\underline{v}} \text{ in } N, \\ \underline{v} &= -2\log\epsilon \text{ on } \partial N. \end{aligned} \tag{5.19}$$

The solution $\underline{v} \leq -2\log\epsilon$, thus it appears as a subsolution to (5.12), and $\underline{v} \leq w_\epsilon$ in Υ_1; at $\boldsymbol{x}_0$, we have $\partial_n \underline{v} \geq \partial_n w_\epsilon$.

The rest of the proof amounts to compute the normal derivative of the solution to $\Delta v = \mathrm{e}^v$ in an annulus of radii $a < b$. Once again, the solution is radially symmetric, i.e. we solve (5.14) for $a < r < b$. In the case of (5.19), we know that there exists $c \in (a, b)$ such that $v'(c) = 0$, and that $v'(a) \leq 0$. Multiplying (5.14) by $rv'(r)$ and integrating between $r = a$ and c, we have:

$$-(av'(a))^2 = 2\int_a^c r^2\big(\mathrm{e}^{v(r)}\big)'\,dr = 2c^2\mathrm{e}^{v(c)} - 2a^2\mathrm{e}^{-2\log\epsilon} + 4av'(a), \tag{5.20}$$

as in (5.15). We thus obtain $|v'(a)| \leq \frac{2}{a} + \sqrt{\frac{4}{a^2} + \frac{2}{\epsilon^2}} \leq \frac{4}{a} + \frac{\sqrt{2}}{\epsilon}$. This estimate provides us with a sharp bound of $\partial_n w$, but it blows up near the reentrant corner in the singular case: in this case $a \to 0$. From Figure 1 middle, we see that the maximum radius of an exterior sphere tangent at $\boldsymbol{x}_0$ is $|\boldsymbol{x}_0|/|\tan\theta_0|$, if the reentrant corner is located at 0.

We overcome this difficulty as follows. For $\eta > 0$, consider a mollified/rounded domain $\tilde{\Upsilon}_1 = \Upsilon_1 \setminus B(0, \eta)$, see Figure 1 right. The maximum radius of the exterior sphere is equal to η on the rounded part of the boundary. As shown above, it is at least $\eta/|\tan\theta_0|$ on the remaining part of the sides that meet at the reentrant corner; elsewhere, it is bounded below by a constant.

Then introduce $\tilde{w}_\epsilon$ solution to

$$\begin{aligned} \Delta \tilde{w}_\epsilon &= \mathrm{e}^{\tilde{w}_\epsilon} \text{ in } \tilde{\Upsilon}_1, \\ \tilde{w}_\epsilon &= -2\log\epsilon \text{ on } \partial\tilde{\Upsilon}_1, \end{aligned} \tag{5.21}$$

extended to $-2\log\epsilon$ outside $\tilde{\Upsilon}_1$. As w_ϵ is a subsolution to this problem, there holds $w_\epsilon \leq \tilde{w}_\epsilon$ on $\tilde{\Upsilon}_1$, and also on $\Upsilon_1 \setminus \tilde{\Upsilon}_1$. Setting $\tilde{M}_\epsilon = \int_{\tilde{\Upsilon}_1} \mathrm{e}^{\tilde{w}_\epsilon}$, we then

have:

$$M_\epsilon = \int_{\Upsilon_1} \mathrm{e}^{w_\epsilon} \leq \tilde{M}_\epsilon + \int_{\Upsilon_1 \setminus \tilde{\Upsilon}_1} \frac{1}{\epsilon^2}. \tag{5.22}$$

On the one hand, since each point of $\partial\tilde{\Upsilon}_1$ satisfies the exterior sphere condition with a ball of radius proportional to η (for η small enough), we have:

$$\tilde{M}_\epsilon = \int_{\partial\tilde{\Upsilon}_1} \partial_n \tilde{w}_\epsilon \leq \left(\frac{4}{c\eta} + \frac{\sqrt{2}}{\epsilon} \right) |\partial\tilde{\Upsilon}_1|. \tag{5.23}$$

On the other hand $\int_{\Upsilon_1 \setminus \tilde{\Upsilon}_1} \frac{1}{\epsilon^2} \leq \frac{K\eta^2}{\epsilon^2}$ and $|\partial\tilde{\Upsilon}_1| \leq |\partial\Upsilon_1| + K'\eta$. Choosing $\eta = \epsilon^{\frac{2}{3}}$ and gathering these estimates, we obtain

$$\epsilon M_\epsilon \leq \epsilon \left(\frac{4}{c\epsilon^{\frac{2}{3}}} + \frac{\sqrt{2}}{\epsilon} \right) \left(|\partial\Upsilon_1| + K'\epsilon^{\frac{2}{3}} \right) + K\epsilon^{\frac{1}{3}}. \tag{5.24}$$

Therefore it follows straightforwardly that $\limsup(\epsilon M_\epsilon) \leq \sqrt{2}\,|\partial\Upsilon_1|$. This completes the proof of the proposition. □

5.4. Conclusions

From Proposition 5.2 we know that, when $\phi_\mathrm{e} \equiv 0$:

$$M[\kappa; 0] \sim \sqrt{2\kappa}\,|\partial\Upsilon_1| \quad \text{as } \kappa \to +\infty, \tag{5.25}$$

i.e., there are constants $C_1,\ C_2$ such that (cf. (5.10)):

$$\kappa \sim C_1\, M^2 \quad \text{and} \quad \lambda \sim C_2\, M^\alpha \quad \text{as } M \to +\infty. \tag{5.26}$$

When $\phi_\mathrm{e} \neq 0$, it follows from §5.1 that

$$C_1\, M^2 \leq \kappa \leq C_1'\, M^2 \quad \text{and} \quad C_2\, M^\alpha \leq \lambda \leq C_2'\, M^\alpha \quad \text{as } M \to +\infty. \tag{5.27}$$

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