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# Proper Generalized Decomposition for the dynamic analysis of uncertain structures

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**ABSTRACT:** The paper deals with the random frequency response of uncertain structure in the low frequency band and with a parametric vision of uncertainties. The dynamic analysis of uncertain structures requires fine spatial and stochastic approximations to well describe the non smooth random response with respect to the input parameters for the studied excitation frequency can be a resonance or anti-resonance depending depending on the values taken by the input parameters. In order to ease the solution of the stochastic partial differential equations in the context of spectral stochastic methods, we use the Proper Generalized Decomposition method which relies on the *a priori* construction of a separated representation of the solution and which only requires the solution of relatively small deterministic or stochastic problems. Two strategies are considered for multidimensional dynamic problems. The first one introduces a full separated representation associated with the tensor product space of the solution. The second one consists in first applying a two dimensional separated representation (deterministic/stochastic) and to consider a second level of separation by exploiting the tensor product structure of stochastic approximation spaces.

**KEY WORDS:** Dynamics; Vibration; Uncertainty quantification; Spectral stochastic methods; Proper Generalized Decomposition; Tensor product approximation; Separated representation.

## 1 INTRODUCTION

The introduction of uncertainties in the physical models is essential for the robust design of structures, especially in dynamics for which the random response happens to be highly irregular with respect to the uncertain parameters. In a parametric stochastic framework, the random frequency response  $\mathbf{u} : I \times \Xi \mapsto \mathbb{C}^n$  on a frequency band  $I$  is solution of a system of random equations:

$$(-\omega^2 \mathbf{M}(\xi) + i\omega \mathbf{C}(\omega, \xi) + \mathbf{K}(\xi))\mathbf{u}(\omega, \xi) = \mathbf{f}(\omega, \xi) \quad (1)$$

where  $\omega$  denotes the frequency in  $I$  and  $\xi$  is a set of random variables which are the classical random parameters of the continuous mechanical model that define the probability space. Galerkin type spectral methods define the approximate solution using a weak formulation of problem (1) in which the solution is searched under the form of a development  $\mathbf{u}(\omega, \xi) = \sum_{\alpha=1}^P H_{\alpha}(\xi)\mathbf{u}_{\alpha}(\omega)$  where  $H_{\alpha}$  form a basis of the stochastic approximation space. While spectral approaches are attractive because they give accurate and explicit predictions of the response in terms of the random variables (see [1], [2], [3], [4], [5], [6]), they also are time and memory consuming since they require the solution of large systems of equations. This matter is especially encountered in dynamic analyses for which the numerical models need to be refined at the stochastic level because of the irregularity of the random response with respect to  $\xi$ .

We propose to use a model reduction method based on Proper Generalized Decomposition (PGD) for an efficient resolution of problem (1). PGD methods consist in building *a priori* separated variables representations of the solution of models

defined in tensor product spaces (see [7], [8], [9], [10]). The particular case where the deterministic and random variables are separated,  $\mathbf{u}(\xi) \approx \mathbf{u}_{\mathbf{m}}(\xi) = \sum_{i=1}^{\mathbf{m}} \mathbf{w}_i \lambda_i(\xi)$ , have been introduced in [11][12], where definitions and algorithms were proposed for generating quasi-optimal representations of the solution on a reduced basis of deterministic vectors and of stochastic functions ( $m \ll P$ ). It has been extended to the complex framework in [13] in order to solve the stochastic problem (1) on the frequency band  $I$ . The deterministic and stochastic functions are defined progressively and a global basis of deterministic modes valid on the whole frequency band  $I$  can be easily constructed. The drastic reduction of computational costs allows for the use of refined discretization and even for adaptive strategies at the stochastic level to control the accuracy of the representation of the irregular random response with respect to the random variables. For higher stochastic dimensions, a multidimensionnal PGD method for the representation of the solution in separated variables at the stochastic level is proposed in [14] so to circumvent the so-called curse of dimensionality and which consists in introducing an *a priori* representation of the stochastic functions. The strategy is studied here for dynamic analyses.

The paper is articulated as follows. The stochastic dynamic problem is first presented in section 2. Sections 3 and 4 present two strategies for the separated representation in order to deal with multidimensional problems. The first one relies on the full exploitation of the tensor structure of the solution where the PGD is used for the construction of the separated representation of the solution under the form  $\mathbf{u}(\xi) \approx \mathbf{u}_{\mathbf{z}}(\xi) = \sum_{i=1}^{\mathbf{z}} \mathbf{w}_i \varphi_i^1(\xi^1) \dots \varphi_i^r(\xi^r)$ , where  $\xi^1, \dots, \xi^r$  are  $r$  independent sets of random variables. The second

strategy consists in performing a hierarchic PGD. The idea is to perform a quasi-optimal deterministic-stochastic separation  $\mathbf{u}(\xi) \approx \sum_{i=1}^m \mathbf{w}_i \lambda_i(\xi)$  (based on Galerkin orthogonality criteria), where the stochastic functions  $\lambda_i$  are approximated with a second level of accurate separated representations. Finally these methodologies are illustrated in section 5.

## 2 STOCHASTIC DYNAMICAL PROBLEM

### 2.1 Modeling of the uncertainties and dynamic problem

In the parametric vision of uncertainties, the uncertainties are supposed to be represented with a finite number of random parameters  $\xi = (\xi^1, \dots, \xi^r)$  associated with the probability space  $(\Xi, \mathcal{B}, \mathcal{P}_\xi)$ ,  $\Xi \subset \mathbb{R}^r$  being the range of  $\xi$ ,  $\mathcal{B}$  the associated  $\sigma$ -algebra and  $P_\xi$  the probability measure of  $\xi$ .

We consider a forced vibration problem at frequency  $\omega$  and a finite element approximation is used at the spatial level. The stochastic problem then reduces to the resolution of a system of random equations: find the random frequency response  $\mathbf{u} : \Xi \rightarrow \mathbb{C}^n$  such that we have  $P_\xi$  almost surely

$$Re(\mathbf{v}^H \mathbf{A}(\xi) \mathbf{u}(\xi)) = Re(\mathbf{v}^H \mathbf{b}(\xi)) \quad \forall \mathbf{v} \in \mathbb{C}^n \otimes \mathcal{S} \quad (2)$$

where  $Re$  denotes the real part and  $\cdot^H$  denotes the conjugate transpose.  $\mathbf{A}(\xi) = -\omega^2 \mathbf{M}(\xi) + i\omega \mathbf{C}(\xi) + \mathbf{K}(\xi)$  is the dynamic stiffness matrix where  $\mathbf{M}$ ,  $\mathbf{K}$  et  $\mathbf{C}$  designate respectively the mass, stiffness and damping matrices. These matrices are random and can be frequency dependent.  $\mathbf{b}(\xi)$  is the excitation vector at frequency  $\omega$ .

### 2.2 Spectral stochastic methods

In the context of the spectral stochastic methods, an approximation of the solution is searched under the form  $\mathbf{u}(\xi) = \sum_{\alpha=1}^P \mathbf{u}_\alpha H_\alpha(\xi) \in \mathbb{C}^n \otimes \mathcal{S}_P$ .  $\{H_\alpha(\xi)\}_{\alpha=1}^P$  is a basis of an approximation space  $\mathcal{S}_P \subset \mathcal{S}$  and  $\mathbf{u}_\alpha \in \mathbb{C}^n$  are the coefficients of  $\mathbf{u}$  on this basis.

The Galerkin type spectral methods use a weak formulation of the dynamic problem (2) and the approximation  $\mathbf{u} \in \mathbb{C}^n \otimes \mathcal{S}_P$  is then defined by:

$$\mathbf{u} \in \mathbb{C}^n \otimes \mathcal{S}_P, \quad \langle \mathbf{v}, \mathbf{A}\mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{b} \rangle \quad \forall \mathbf{v} \in \mathbb{C}^n \otimes \mathcal{S}_P \quad (3)$$

with

$$\langle \mathbf{v}, \mathbf{A}\mathbf{u} \rangle = \mathbb{E}(Re(\mathbf{v}^H \mathbf{A}\mathbf{u})) \quad (4)$$

$$\langle \mathbf{v}, \mathbf{b} \rangle = \mathbb{E}(Re(\mathbf{v}^H \mathbf{b})) \quad (5)$$

where  $\mathbb{E}$  denotes the mathematical expectation defined by  $\mathbb{E}(\mathbf{v}) = \int_{\Xi} \mathbf{v}(y) dP_\xi(y)$ . The determination of an approximation of the frequency response at a given frequency requires the solution of the discretized problem (3) which is a system of equations in the complex field of size  $n \times P$ . Its solution may reveal important for dynamic analyses. In fact, at the spatial level, dynamic problems usually need a fine spatial approximation space which means an important number of degree of freedom  $n$ . Furthermore, the frequency response can be highly irregular with respect to the random variables. An accurate description of the non smooth behavior then requires the use of very fine approximation spaces at the stochastic level (large  $P$ ). The use of standard Krylov type iterative

algorithms for the solution of (3) can consequently become prohibitive, especially when dealing with high dimensional stochastic problems involving a large number of parameters  $\xi$ .

## 3 PROPER GENERALIZED DECOMPOSITION

Here, we propose a model reduction method based on Proper Generalized Decomposition (PGD) for the solution of problem (3) in the case of multidimensional problems where the uncertainties are represented with  $r$  independant sets of random parameters  $\xi = (\xi^1, \dots, \xi^r)$ . PGD methods exploit the tensor structure of the solution  $\mathbf{u} \in \mathbb{C}^n \otimes \mathcal{S}_P$  with  $\mathcal{S}_P = \otimes_{j=1}^r \mathcal{S}_{P_j}^j$  and consist in constructing an *a priori* separated representation of the solution. The idea is thus to search the solution  $\mathbf{u}(\xi) \in \mathbb{C}^n \otimes \mathcal{S}_P$  of problem (3) at a given frequency  $\omega$  under the form

$$\mathbf{u}(\xi) \approx \mathbf{u}_Z(\xi) = \sum_{i=1}^Z \mathbf{w}_i \ell_i^1(\xi^1) \dots \ell_i^r(\xi^r) \quad (6)$$

where  $\mathbf{w}_i \in \mathbb{C}^n$  are deterministic vectors and  $\ell_i^j \in \mathcal{S}_{P_j}^j$  are stochastic functions. This decomposition can be seen as a generalized spectral expansion

$$\mathbf{u}(\xi) \approx \mathbf{u}_Z(\xi) = \sum_{i=1}^Z \mathbf{w}_i H_i(\xi) \quad (7)$$

where the  $H_i(\xi) = \ell_i^1(\xi^1) \dots \ell_i^r(\xi^r)$  are not known *a priori* but are constructed in an optimal way.

The tensor product representation enables to handle multidimensional stochastic problems, The Proper Generalized Decomposition method is used for an optimal construction of the separated representation. Several definitions have been proposed in [14], sole the so-called progressive PGD is presented here.

The progressive PGD consists in determining a rank-one element  $\mathbf{w}_Z \ell_Z^1(\xi^1) \dots \ell_Z^r(\xi^r)$  which is added to a previously computed decomposition  $\mathbf{u}_{Z-1}$  to obtain the rank- $Z$  approximation:  $\mathbf{u}_Z = \mathbf{u}_{Z-1} + \mathbf{w}_Z \ell_Z^1 \dots \ell_Z^r$ .

### 3.1 Progressive Minimal Residual PGD

Bilinear form  $\langle \mathbf{v}, \mathbf{A}\mathbf{u} \rangle$  being non symmetric for dynamic problems, problem (3) is reformulated as an optimization problem with a minimal residual formulation. The new rank one element  $\mathbf{w}_Z \ell_Z^1 \dots \ell_Z^r$  of the progressive Minimal Residual PGD is defined by the optimization problem

$$\begin{aligned} & \left\| \mathbf{b} - \mathbf{A}(\mathbf{u}_{Z-1} + \mathbf{w}_Z \ell_Z^1 \dots \ell_Z^r) \right\|^2 = \\ & \min_{\mathbf{w} \in \mathbb{C}^n, \ell^j \in \mathcal{S}_{P_j}^j} \left\| \mathbf{b} - \mathbf{A}(\mathbf{u}_{Z-1} + \mathbf{w} \ell^1 \dots \ell^r) \right\|^2 \end{aligned} \quad (8)$$

where  $\|\cdot\|$  is the norm associated with the inner product  $\langle \cdot, \cdot \rangle$  defined in equations (4) and (5). Denoting  $\mathbf{r} = \mathbf{b} - \mathbf{A}\mathbf{u}_{Z-1}$ , the new rank one element  $\mathbf{w}_Z \ell_Z^1 \dots \ell_Z^r$  must then verify the  $r+1$  following coupled equations

$$\begin{aligned} & \langle \tilde{\mathbf{w}} \ell_Z^1 \dots \ell_Z^r, \mathbf{A}^H \mathbf{A} \mathbf{w}_Z \ell_Z^1 \dots \ell_Z^r \rangle = \langle \tilde{\mathbf{w}} \ell_Z^1 \dots \ell_Z^r, \mathbf{A}^H \mathbf{r} \rangle, \forall \tilde{\mathbf{w}} \in \mathbb{C}^n \\ & \langle \mathbf{w}_Z \tilde{\ell}^1 \dots \tilde{\ell}^r, \mathbf{A}^H \mathbf{A} \mathbf{w}_Z \ell_Z^1 \dots \ell_Z^r \rangle = \langle \mathbf{w}_Z \tilde{\ell}^1 \dots \tilde{\ell}^r, \mathbf{A}^H \mathbf{r} \rangle, \forall \tilde{\ell}^1 \in \mathcal{S}_{P_1}^1 \\ & \dots \\ & \langle \mathbf{w}_Z \ell_Z^1 \dots \tilde{\ell}^r, \mathbf{A}^H \mathbf{A} \mathbf{w}_Z \ell_Z^1 \dots \ell_Z^r \rangle = \langle \mathbf{w}_Z \ell_Z^1 \dots \tilde{\ell}^r, \mathbf{A}^H \mathbf{r} \rangle, \forall \tilde{\ell}^r \in \mathcal{S}_{P_r}^r \end{aligned}$$

These equations are necessary conditions of optimality of problem (26) and the optimal element must verify these  $r + 1$  equations simultaneously. An alternated minimization algorithm can be used to construct the rank one element  $\mathbf{w}_Z \ell_Z^1 \cdots \ell_Z^r$  solution of (26).

Though robust for non symmetric problems, PGD based on Minimal Residual formulation may lead to a poor convergence rate of the decomposition. Besides, the efficiency of PGD methods relies on the separated representations of the operator  $\mathbf{A}^H \mathbf{A}$  and right hand  $\mathbf{A}^H \mathbf{b}$ , the ranks of which are much higher than the ranks of  $\mathbf{A}$  and  $\mathbf{b}$ .

### 3.2 Progressive Galerkin PGD

For the reasons mentioned above, we introduce the PGD based on the Galerkin formulation. In the progressive Galerkin PGD, an optimal rank one element  $\mathbf{w}_Z \ell_Z^1 \cdots \ell_Z^r$  is searched such that it verifies the  $r + 1$  following Galerkin orthogonality criteria

$$\begin{aligned} \langle \tilde{\mathbf{w}} \ell_Z^1 \cdots \ell_Z^r, \mathbf{A} \mathbf{w}_Z \ell_Z^1 \cdots \ell_Z^r \rangle &= \langle \tilde{\mathbf{w}} \ell_Z^1 \cdots \ell_Z^r, \mathbf{r} \rangle, \quad \forall \tilde{\mathbf{w}} \in \mathbb{C}^n \\ \langle \mathbf{w}_Z \tilde{\ell}^1 \cdots \ell_Z^r, \mathbf{A} \mathbf{w}_Z \ell_Z^1 \cdots \ell_Z^r \rangle &= \langle \mathbf{w}_Z \tilde{\ell}^1 \cdots \ell_Z^r, \mathbf{r} \rangle, \quad \forall \tilde{\ell}^1 \in \mathcal{S}_{P_1}^1 \\ \dots \\ \langle \mathbf{w}_Z \ell_Z^1 \cdots \tilde{\ell}^r, \mathbf{A} \mathbf{w}_Z \ell_Z^1 \cdots \ell_Z^r \rangle &= \langle \mathbf{w}_Z \ell_Z^1 \cdots \tilde{\ell}^r, \mathbf{r} \rangle, \quad \forall \tilde{\ell}^r \in \mathcal{S}_{P_r}^r \end{aligned}$$

A construction of the separated representation of the solution based on Galerkin orthogonality criteria should be preferred when convergent although monotone convergence is not guaranteed for non symmetric problems.

**Remark** In order to improve the convergence of the decomposition, update steps can be performed along the deterministic dimension and the  $r$  stochastic dimensions, or along some of them only. The idea can be resumed from that explained in 4.2.1 and the authors refer to [14] for more details.

It is to be noted that in the particular case of symmetric operator  $\mathbf{A}$ , the progressive PGD can be interpreted as a generalization of the multidimensional Singular Valued Decomposition.

## 4 GENERALIZED SPECTRAL DECOMPOSITION

Here is presented an alternative to the previous definition of the separated representation (7) which consists in a hierarchic PGD of the solution. The idea is to introduce a first level of separation in two dimensions, deterministic and stochastic, of the form

$$\mathbf{u}(\xi) \approx \mathbf{u}_m(\xi) = \sum_{i=1}^m \mathbf{w}_i \lambda_i(\xi) \quad (9)$$

which is constructed with the PGD method. In [11] and [12] different definitions and algorithms were proposed for generating quasi-optimal representations of the solution on a reduced basis of deterministic vectors and of stochastic functions ( $m \ll n$  and  $m \ll P$ ) in two dimensional problems. In [12], the author shows that in the case of a symmetric operator  $\mathbf{A}$  the PGD in the deterministic-stochastic separation framework can be interpreted as a generalization of the Karhunen Loève decomposition and was then introduced as the Generalized Spectral Decomposition (GSD).

In practice, the two dimensional (deterministic-stochastic) PGD based on the Galerkin formulation leads to good

convergence properties in many cases and has notably proved robust for dynamic analyses in [13].

In order to deal with multidimensional stochastic problems, a second tensor product approximation is introduced at the stochastic level where the stochastic functions are themselves searched under the separated form

$$\lambda_m(\xi) = \sum_{k=1}^s \lambda_k^1(\xi^1) \cdots \lambda_k^1(\xi^r) \quad (10)$$

and defined using the progressive Galerkin or Minimal Residual definition of the PGD. Here only the Minimal Residual PGD for the definition of the separated form of the stochastic functions will be considered.

### 4.1 GSD method for the deterministic-stochastic separated representation

In the progressive Galerkin GSD, a new optimal rank one element  $\mathbf{w}_m \lambda_m$  should verify the two following Galerkin orthogonality criteria

$$\langle \tilde{\mathbf{w}} \lambda_m, \mathbf{A} \mathbf{w}_m \lambda_m \rangle = \langle \tilde{\mathbf{w}} \lambda_m, \mathbf{b}^m \rangle, \quad \forall \tilde{\mathbf{w}} \in \mathbb{C}^n \quad (11)$$

$$\langle \mathbf{w}_m \tilde{\lambda}, \mathbf{A} \mathbf{w}_m \lambda_m \rangle = \langle \mathbf{w}_m \tilde{\lambda}, \mathbf{b}^m \rangle, \quad \forall \tilde{\lambda} \in \mathcal{S}_P \quad (12)$$

where  $\mathbf{b}^m = \mathbf{b} - \mathbf{A} \mathbf{u}_{m-1}$ . The optimal couple  $(\mathbf{w}_m, \lambda_m)$  verifies simultaneously  $\mathbf{w}_m = \mathbf{F}_m(\lambda_m)$  and  $\lambda_m = f_m(\mathbf{w}_m)$  where  $\mathbf{F}_m : \mathcal{S}_P \rightarrow \mathbb{C}^n$  and  $f_m : \mathbb{C}^n \rightarrow \mathcal{S}_P$  are the mappings defined by equations (11) and (12).

Although not based on an optimality criterion, in many cases this definition based on Galerkin projection proves efficient to capture good rank-one approximations.

An alternated direction algorithm for solving (11) and (12) is used to find the approximation of an optimal couple  $(\mathbf{w}_m, \lambda_m)$ . Starting from an initial random function  $\lambda^0 \in \mathcal{S}_P$ ,  $\mathbf{w}^{k+1} \in \mathbb{C}^n$  and  $\lambda^{k+1} \in \mathcal{S}_P$  with  $k \geq 0$  are successively constructed with the definitions

$$\mathbf{w}^{k+1} = \mathbf{F}_m(\lambda^k) \quad \text{and} \quad \lambda^{k+1} = f_m(\mathbf{w}^{k+1}) \quad (13)$$

In practice only few iterations are sufficient to obtain a good estimation of the optimal couple  $(\mathbf{w}_m, \lambda_m)$ .

In [12], the author shows that problem (26) can be seen as successive pseudo eigen problems where  $\mathbf{w}_m$  (respectively  $\lambda_m$ ) is the dominant eigen function of the operator  $T_m = \mathbf{F}_m \circ f_m$  (respectively  $T_m^* = f_m \circ \mathbf{F}_m$ ). It is then shown that the above algorithm for the progressive construction of the decomposition corresponds to a power-type algorithm with deflation to solve the pseudo-eigen problem which defines the couples  $(\mathbf{w}_i, \lambda_i)$  of the separated representation (7).

#### 4.1.1 Application of mapping $F_m$

For a fixed stochastic function  $\lambda$ , the calculation of  $\mathbf{w} = F_m(\lambda)$  appears as a simple deterministic problem of size  $n$  ( $2n$  in the real field)

$$Re(\tilde{\mathbf{w}}^H \mathbf{A}_\lambda \mathbf{w}) = Re(\tilde{\mathbf{w}}^H \mathbf{b}_\lambda^m) \quad \forall \tilde{\mathbf{w}} \in \mathbb{C}^n \quad (14)$$

where  $\mathbf{A}_\lambda = \mathbb{E}(\mathbf{A} \lambda \lambda)$  and  $\mathbf{b}_\lambda^m = \mathbb{E}(\mathbf{b}^m \lambda)$ . Considering the real and imaginary parts of the quantities, equation (14) can be

written under the form

$$\begin{bmatrix} \mathbb{E}(\mathbf{A}_R \lambda^2) & -\mathbb{E}(\mathbf{A}_I \lambda^2) \\ \mathbb{E}(\mathbf{A}_I \lambda^2) & \mathbb{E}(\mathbf{A}_R \lambda^2) \end{bmatrix} \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix} = \begin{bmatrix} \mathbb{E}(\mathbf{r}_R^m \lambda) \\ \mathbb{E}(\mathbf{r}_I^m \lambda) \end{bmatrix} \quad (15)$$

where  $\cdot_R$  and  $\cdot_I$  denote the real and imaginary parts of the quantities and which express, for the operator,  $\mathbf{A}_R = \mathbf{K} - \omega^2 \mathbf{M}$  and  $\mathbf{A}_I = \omega \mathbf{C}$ .

#### 4.1.2 Application of mapping $f_m$

For a fixed deterministic function  $\mathbf{w}$ , the calculation of  $\lambda = f_m(\mathbf{w})$  appears as a stochastic algebraic equation approximated in  $\mathcal{S}_P$  (problem of size  $P$ )

$$\mathbb{E}(\tilde{\lambda} A_w \lambda) = \mathbb{E}(\tilde{\lambda} b_w^m) \quad \forall \tilde{\lambda} \in \mathcal{S}_P \quad (16)$$

with

$$A_w(\xi) = \text{Re}(\mathbf{w}^H \mathbf{A}(\xi) \mathbf{w}) = \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix}^T \begin{bmatrix} \mathbf{A}_R & -\mathbf{A}_I \\ \mathbf{A}_I & \mathbf{A}_R \end{bmatrix} \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix} \quad (17)$$

$$b_w^m(\xi) = \text{Re}(\mathbf{w}^H \mathbf{b}^m(\xi)) = \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix}^T \begin{bmatrix} \mathbf{b}_R^m \\ \mathbf{b}_I^m \end{bmatrix} \quad (18)$$

#### 4.2 PGD method for the separated representation of the stochastic functions

At this stage, the tensor product structure of the stochastic approximation space can be exploited when applying mapping  $f_m$ . We thus propose to use the PGD method for the solution of equation (16) which can be equivalently written

$$\langle \tilde{\lambda}, A_w \lambda \rangle = \langle \tilde{\lambda}, b_w^m \rangle \quad \forall \tilde{\lambda} \in \mathcal{S}_P \quad (19)$$

We introduce a tensor product approximation of the stochastic function  $\lambda(\xi) \in \otimes_{k=1}^r \mathcal{S}_{P_k}^k$

$$\lambda(\xi) \approx \sum_{j=1}^s \lambda_j^1(\xi^1) \cdots \lambda_j^r(\xi^r) \quad (20)$$

A rank  $(s-1)$  separated representation  $\lambda_{s-1}$  being computed, the progressive Minimal Residual PGD defines a new rank one element with the optimization problem

$$\begin{aligned} & \left\| b_w^m - A_w(\lambda_{s-1} + \lambda_s^1 \cdots \lambda_s^r) \right\|^2 = \\ & \min_{\lambda^k \in \mathcal{S}_{P_k}^k} \left\| b_w^m - A_w(\lambda_{s-1} + \lambda^1 \cdots \lambda^r) \right\|^2 \end{aligned} \quad (21)$$

Necessary conditions of optimality are

$$\begin{aligned} & \langle \tilde{\lambda}^1 \cdots \lambda^r, A_w A_w \lambda^1 \cdots \lambda^r \rangle = \langle \tilde{\lambda}^1 \cdots \lambda^r, A_w b_w^m \rangle, \quad \forall \tilde{\lambda}^1 \in \mathcal{S}_{P_1}^1 \\ & \dots \\ & \langle \lambda^1 \cdots \tilde{\lambda}^r, A_w A_w \lambda^1 \cdots \lambda^r \rangle = \langle \lambda^1 \cdots \tilde{\lambda}^r, A_w b_w^m \rangle, \quad \forall \tilde{\lambda}^r \in \mathcal{S}_{P_r}^r \end{aligned}$$

##### 4.2.1 Updated progressive PGD

In order to improve the convergence of the progressive PGD, an update along the stochastic dimension is performed [12], [10]. Let  $\mathbf{W} = (\mathbf{w}_1 \cdots \mathbf{w}_m)$  and  $\Lambda = (\lambda_1 \cdots \lambda_m)^T$  denote the

bases of deterministic and stochastic functions respectively. An approximation of the solution can be written

$$\mathbf{u}_m(\xi) = \mathbf{W} \Lambda(\xi) \quad (22)$$

Given the deterministic basis  $\mathbf{W}$  of a rank- $m$  decomposition  $\mathbf{u}_m$ , the update of the stochastic functions consists in solving

$$\Lambda \in \mathbb{R}^m \otimes \mathcal{S}_P, \quad \langle \mathbf{W} \tilde{\Lambda}, \mathbf{A} \mathbf{W} \Lambda \rangle = \langle \mathbf{W} \tilde{\Lambda}, \mathbf{b} \rangle \quad \forall \tilde{\Lambda} \in \mathbb{R}^m \otimes \mathcal{S}_P \quad (23)$$

It can be seen as a stochastic problem projected on the reduced basis  $\mathbf{W}$ ,  $\mathbf{W}^H \mathbf{A} \mathbf{W}$  and  $\mathbf{W}^H \mathbf{b}$  being the reduced operator and right hand side. Denoting  $\mathbf{B} = \text{Re}(\mathbf{W}^H \mathbf{A} \mathbf{W})$  and  $\mathbf{d} = \text{Re}(\mathbf{W}^H \mathbf{b})$ , equation (23) can be equivalently written

$$\langle \tilde{\Lambda}, \mathbf{B} \Lambda \rangle = \langle \tilde{\Lambda}, \mathbf{d} \rangle \quad \forall \tilde{\Lambda} \in \mathbb{R}^m \otimes \mathcal{S}_P \quad (24)$$

We introduce the separated representation

$$\Lambda(\xi) \approx \Lambda_Z(\xi) = \sum_{s=1}^Z \phi_s^0 \phi_s^1(\xi^1) \cdots \phi_s^r(\xi^r) \quad (25)$$

with  $\phi_k^0 \in \mathbb{R}^m$  and  $\phi_s^k \in \mathcal{S}_{P_k}^k$  for  $k = 1 \cdots r$ .

A rank- $(Z-1)$  separated representation  $\Lambda_{Z-1}$  being computed, the progressive Minimal Residual PGD defines a new rank one element  $\phi_Z^0 \phi_Z^1 \cdots \phi_Z^r$  with the optimization problem

$$\begin{aligned} & \left\| \mathbf{d} - \mathbf{B}(\Lambda_{Z-1} + \phi_Z^0 \phi_Z^1 \cdots \phi_Z^r) \right\|^2 = \\ & \min_{\phi^0 \in \mathbb{R}^m, \phi^k \in \mathcal{S}_{P_k}^k} \left\| \mathbf{d} - \mathbf{B}(\Lambda_{Z-1} + \phi^0 \phi^1 \cdots \phi^r) \right\|^2 \end{aligned} \quad (26)$$

The  $r+1$  coupled equations below are necessary conditions of optimality of the new rank one element  $\phi_Z^0 \phi_Z^1 \cdots \phi_Z^r$

$$\begin{aligned} & \langle \tilde{\phi}^0 \phi^1 \cdots \phi^r, \mathbf{B}^T \mathbf{B} \phi^0 \phi^1 \cdots \phi^r \rangle = \langle \tilde{\phi}^0 \phi^1 \cdots \phi^r, \mathbf{B}^T \mathbf{d} \rangle, \quad \forall \tilde{\phi}^0 \in \mathbb{R}^m \\ & \langle \phi^0 \tilde{\phi}^1 \cdots \phi^r, \mathbf{B}^T \mathbf{B} \phi^0 \phi^1 \cdots \phi^r \rangle = \langle \phi^0 \tilde{\phi}^1 \cdots \phi^r, \mathbf{B}^T \mathbf{d} \rangle, \quad \forall \tilde{\phi}^1 \in \mathcal{S}_{P_1}^1 \\ & \dots \\ & \langle \phi^0 \phi^1 \cdots \tilde{\phi}^r, \mathbf{B}^T \mathbf{B} \phi^0 \phi^1 \cdots \phi^r \rangle = \langle \phi^0 \phi^1 \cdots \tilde{\phi}^r, \mathbf{B}^T \mathbf{d} \rangle, \quad \forall \tilde{\phi}^r \in \mathcal{S}_{P_r}^r \end{aligned}$$

The rank-one element solution is constructed using an alternated minimization algorithm.

Finally, the PGD approximation (22) can thus be recasted under the separated form (7) by letting  $\mathbf{w}_s = \mathbf{W} \phi_s^0$  and  $\iota_s^k = \phi_s^k$ .

## 5 ILLUSTRATION

### 5.1 Description of the example, stochastic modeling and approximation

The method is applied on a two plate elastic structure represented in figure 1. Homogeneous Dirichlet conditions and a harmonic loading are imposed on parts  $\Gamma_1$  and  $\Gamma_2$  of the boundary respectively. The complementary part of the boundary is a free boundary. A spatial finite element approximation is used and the discrete deterministic model contains  $n = 5556$  degrees of freedom.

We consider a non-dimensional analysis where the mass density is chosen equal to 1 and the Young modulus  $E$  is a uniform random variable on  $[0.9, 1.1]$  and is expressed as a

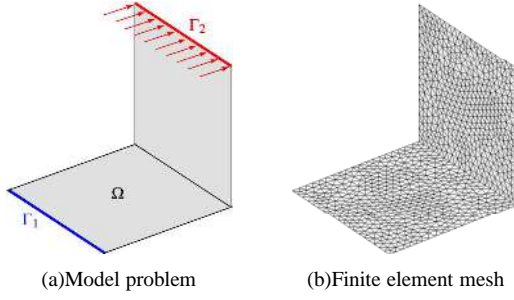


Figure 1. Mesh and boundary conditions of an elastic plate structure under harmonic bending load.

function of a uniform random variable  $\xi_1$ :  $E = 0.9 + 0.2\xi_1$  with  $\Xi_1 = (0, 1)$ . We consider proportional damping in the structure with  $\mathbf{C} = \beta\mathbf{K}$ , where the damping is a log-uniform random variable on  $(0.005, 0.01)$  expressed as  $\beta = 0.005 \cdot 2^{\xi_2}$  with  $\xi_2$  a uniform random variable on  $\Xi_2 = (0, 1)$ . The source of uncertainties is thus represented with  $r = 2$  independent uniform random variables. Figure 2 shows some realizations of the frequency response function of the out of plane displacement of the upper right node of the structure. The associated response surfaces for three excitation frequencies obtained by evaluating the semi-discretized solution  $\mathbf{u}_{ref} \in \mathbb{C}^n \otimes \mathcal{S}$  for samples of  $\xi$ , are plotted in Figure 3 and show that the dynamic behavior can significantly change when considering an excitation frequency near the resonances of the mean structure. Therefore a good description of this non-linear behavior at the stochastic level is carried out using a fine stochastic approximation. Here we introduce a finite element approximation space for  $\mathcal{S}_P = \mathcal{S}_{P_1}^1 \otimes \mathcal{S}_{P_2}^2$  where  $\mathcal{S}_{P_l}^l$  are finite elements spaces with a uniform partition of  $\Xi_l$  with 32 elements and polynomial degree 4 in the first dimension ( $l = 1$ ) and a polynomial degree 2 on 1 element in the second dimension ( $l = 2$ ), which leads to a stochastic approximation space of dimension  $P = 480$ .

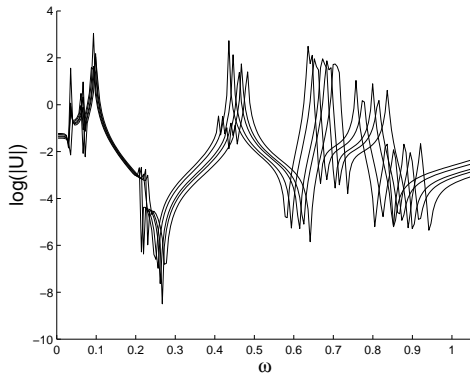


Figure 2. Samples of the random frequency response functions.

## 5.2 Multidimensional PGD

First a multidimensional progressive PGD using the strategy presented in section 3 is performed for the three selected frequencies above. An update step only among the 2 stochastic dimensions is performed. The convergence of the rank- $Z$

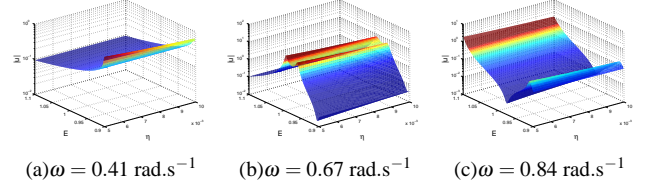


Figure 3. Reference response surface of the modulus of the out of plane displacement  $|u|$  of the upper right node of the two-plate structure for three frequencies.

Galerkin or Minimal progressive PGD is estimated with the relative error defined by

$$\varepsilon^Z = \frac{\|\mathbf{u}_{ref} - \mathbf{u}_Z\|}{\|\mathbf{u}_{ref}\|} \quad (27)$$

where the norm is estimated with Monte-Carlo simulations. The evolutions of the relative errors for the three mentioned frequencies above are plotted in Figure 4. It comes out that the Galerkin PGD converges for the sole frequency  $\omega = 0.41 \text{ rad.s}^{-1}$  far from a resonance zone and fails for the other frequencies that can be eigen frequencies of the structure for some parameter  $\xi$ . As for the Minimal residual PGD, it appears to converge but with an extremely small convergence rate. To illustrate these comments, the response surfaces of the PGD approximation of rank  $Z = 30$  are plotted in figures 5 and 6 respectively using Galerkin and the Minimal Residual definitions respectively. They are to be compared to those of figure 3.

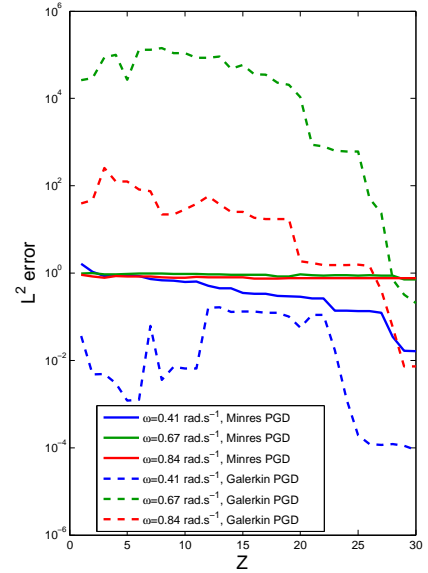


Figure 4. Evolution of the error  $\varepsilon^Z$  with  $Z$ .

In order to improve the convergence of the Minimal residual PGD, an update along the deterministic dimension could be performed but this update step reveals computationally cumbersome for it involves the solution of a system of size  $Z \times n$ .

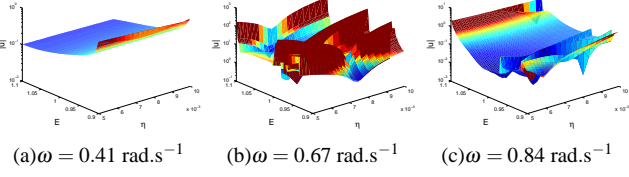


Figure 5. Response surface of the modulus of the out of plane displacement  $|u_{Z=30}|$  of the upper right node of the two-plate structure obtained with the progressive Galerkin PGD for three frequencies.

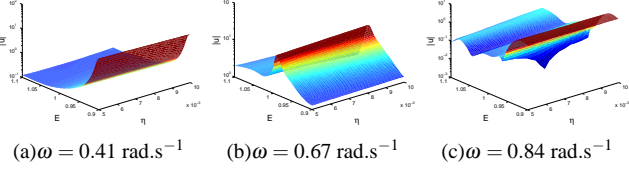


Figure 6. Response surface of the modulus of the out of plane displacement  $|u_{Z=30}|$  of the upper right node of the two-plate structure obtained with the progressive Minimal Residual PGD for three frequencies.

### 5.3 Generalized spectral decomposition

In this section we illustrate the two level separated representation of the solution presented in section 4. Given the good convergence of the Galerkin PGD for two dimensional (deterministic-stochastic) separated representation and given the non convergence of the Galerkin PGD for multidimensional separated representation, here a progressive Galerkin PGD is performed for construction of a deterministic-stochastic separated representation, and a progressive Minimal Residual PGD is performed within the stochastic dimension for an approximation of the stochastic functions  $\lambda_i$ . We introduce a first residual error to achieve the separated representation of the stochastic functions defined by equation (21)

$$\varepsilon_{res}^s(\lambda) = \frac{\|b_w - A_w \lambda_s\|}{\|b_w\|} \quad (28)$$

The PGD method for the *a priori* separated representation of the stochastic functions is here completed with a given tolerance  $\varepsilon_{res}^s(\lambda) < 10^{-2}$ . The separated representation of the updated progressive PGD uses  $Z = 30$  stochastic functions.

A second residual error is introduced to assess the convergence of the two dimensional decomposition (9) and is defined by

$$\varepsilon_{res}^m(\mathbf{u}) = \frac{\|\mathbf{b} - \mathbf{A}\mathbf{u}_m\|}{\|\mathbf{b}\|} \quad (29)$$

The evolution of  $\varepsilon_{res}^m$  with  $m$  is plotted in figure 7. The relative errors, defined as in (27), between the PGD approximation of rank  $m = 10$  and the semi-discretized solution  $\mathbf{u}_{ref} \in \mathbb{C}^n \otimes \mathcal{S}$  are  $\varepsilon^{(m=10)} = 2.710^{-5}$ ,  $4.110^{-3}$  and  $1.810^{-3}$  for the respective frequencies  $\omega = 0.41 \text{ rad.s}^{-1}$ ,  $\omega = 0.67 \text{ rad.s}^{-1}$  and  $\omega = 0.84 \text{ rad.s}^{-1}$ .

Figure 8 depicts the response surfaces of the PGD approximation  $\mathbf{u}_{m=10}$  of rank  $m = 10$  that are to be compared with those

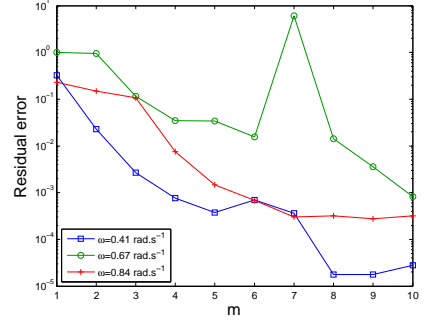


Figure 7. Evolution of the residual error  $\varepsilon_{res}^m(\mathbf{u})$  with  $m$ .

of figure 3. Figures 9 and 10 compare the reference solution  $\mathbf{u}_{ref}$  and the PGD approximation  $\mathbf{u}_{m=10}$  for the respective frequencies  $\omega = 0.67 \text{ rad.s}^{-1}$  and  $\omega = 0.84 \text{ rad.s}^{-1}$ . We observe a good concordance between the response computed with the PGD and the reference.

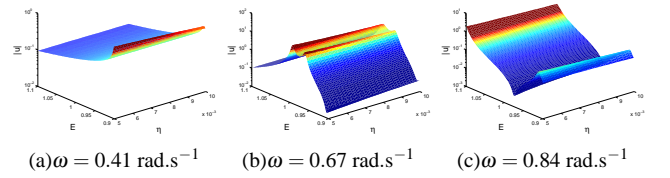


Figure 8. Response surface of the modulus of the out of plane displacement  $|u_{m=10}|$  of the upper right node of the two-plate structure obtained with the progressive hierarchic PGD for three frequencies.

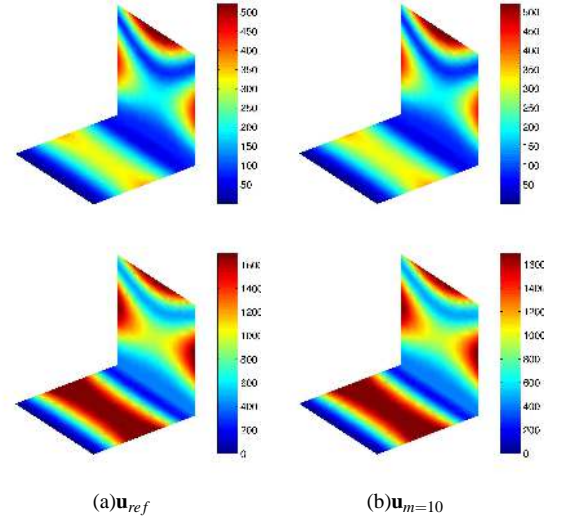


Figure 9. Samples of the solution  $|u|$  at frequency  $\omega = 0.67 \text{ rad.s}^{-1}$ . Comparison of the reference solution  $\mathbf{u}_{ref}$  (a) and the approximation  $\mathbf{u}_{m=10}$  of rank  $m = 10$  (b).

We can see that the generalized spectral decomposition converges on the whole but still may encounter some difficulties for frequencies concerned with resonances as can be observed for the rank  $m = 7$  of the excitation frequency  $\omega = 0.67 \text{ rad.s}^{-1}$ .

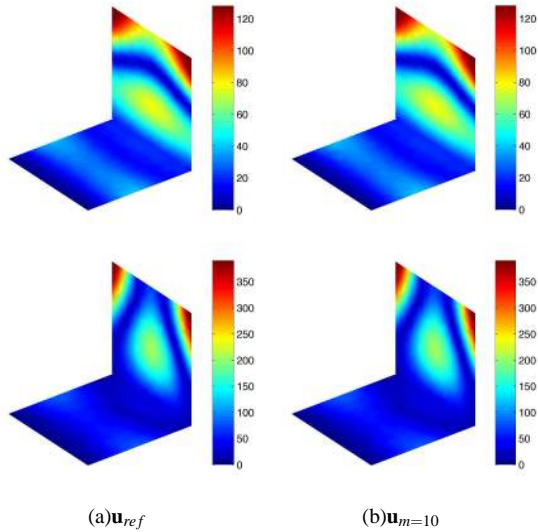


Figure 10. Samples of the solution  $|u|$  at frequency  $\omega = 0.84 \text{ rad.s}^{-1}$ . Comparison of the reference solution  $\mathbf{u}_{ref}$  (a) and the approximation  $\mathbf{u}_{(m=10)}$  of rank  $m = 10$ (b).

Furthermore, the convergence properties for the frequencies in the resonance zones seem sensitive to the initial random function  $\lambda^0$  chosen in the alternated direction algorithm for the solution of (11) and (12).

## 6 CONCLUSION

In this work, we have shown that the PGD method provides a good mean to deal with multidimensional problems and enables to automatically generate a separated representation of the frequency response at a given frequency. Although the PGD based on the Galerkin formulation has proved robust for the construction of a deterministic-stochastic separated representation, it has not revealed robust in the case of a multidimensional separated representation for dynamic analyses and the use of the PGD method based on a Minimal Residual formulation in multidimensional cases has not shown efficient either. Meanwhile, the Galerkin PGD in the context of a two dimension, deterministic and stochastic, separated representation combined with a tensor product approximation of the stochastic approximation constructed with the Minimal Residual PGD has proved convergent. However further studies need to be conducted to propose robust and more efficient methods for the frequencies in the resonance zones.

## ACKNOWLEDGMENTS

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