



PGD in Computational Mechanics: State of the Art and Prospective

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Proper Generalized Decomposition: extension of POD

Part 1 (PL): Nonlinear time dependent problems (possibly with parameters)

Part 2 (FC): High-dimensional problems, inverse problems, ...

The PGD (for PDE)

Shape functions: not a priori given

$$s_h(t, M) = \sum_{i=1}^m a_i \Psi_i(t, M) \quad \text{over } [0, T] \times \Omega$$

Ideas

Variable separation (or?)

$$\Psi_i(t, M) = \lambda_i(t) \Lambda_i(M)$$

Interest

Few shape functions: m small (computational time $\searrow \searrow \searrow$)

Question

How?

Question

How?

Residue?

$$R(s) = \int_0^T \int_{\Omega} dt d\Omega r(s)$$

$$R \geq 0$$

$$R = 0 \Rightarrow s = s_{ex}$$

$$\min_{\lambda_i, \Lambda_i, i=1\dots m} R \left(\sum_{i=1}^m \lambda_i(t) \Lambda_i(M) \right)$$



greedy algorithms

The PGD

Question

How?

Residue?

Our answer

LATIN method (time-dependent nonlinear problems)



[Ladevèze 1985]

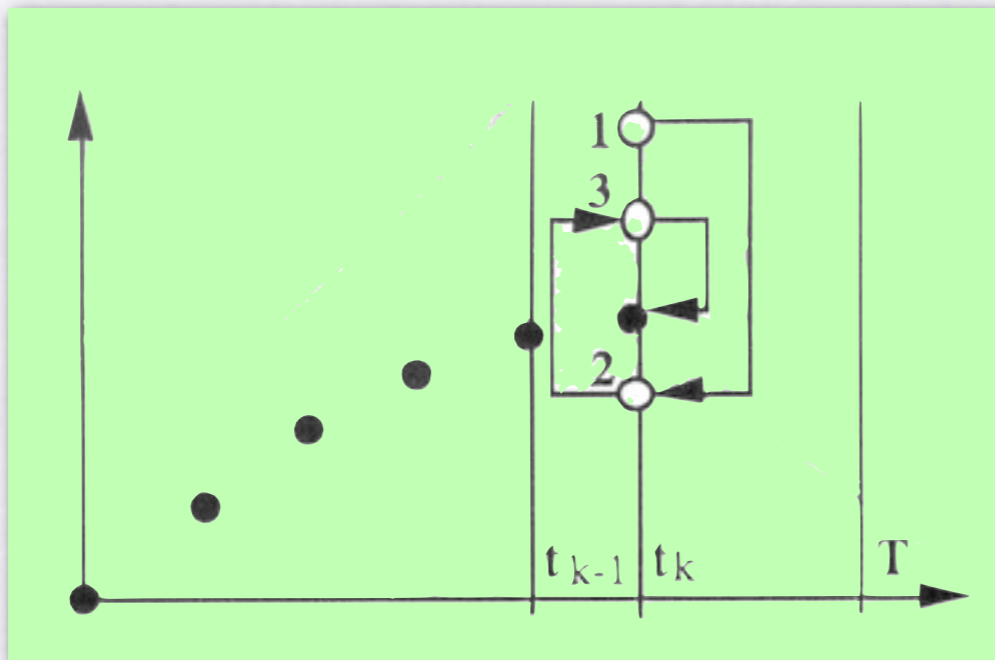
PGD: radial loading approximations

Numerous works (contact problems, mutiscale, multiphysics, large displacements, ...)

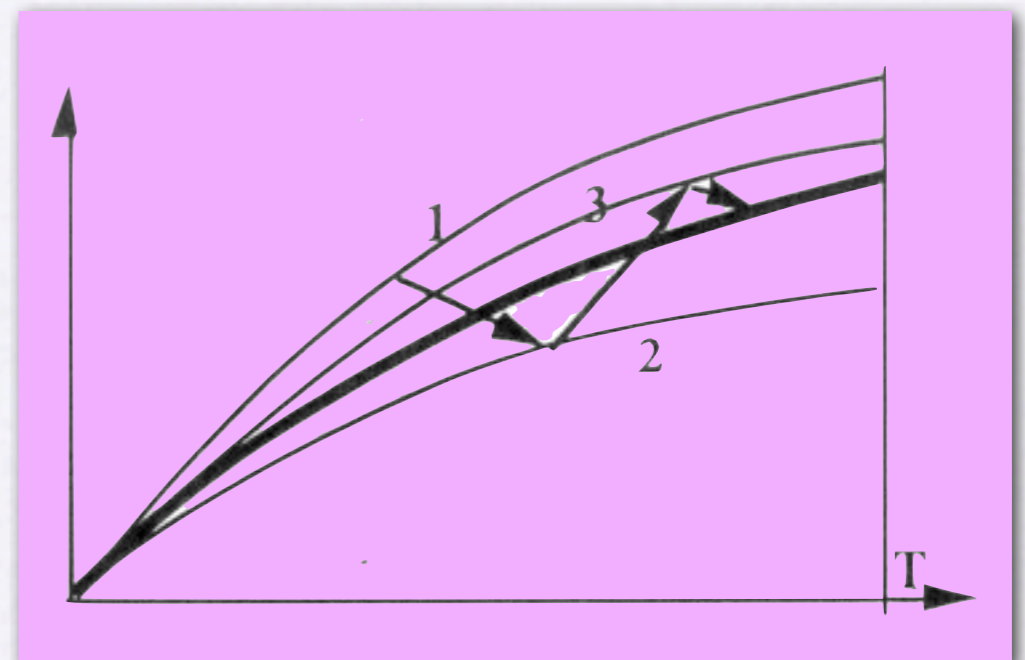
[Ladevèze 1985] and **book** [Springer NY 1999]

Theses (Cachan): Boisse, Cognard, Royer, Allix, Bussy, Liu, Champaney, Ryckelynk, Boucard, Michel, Dureisseix, Loiseau, Nouy, Néron, Guidault, Passieux

The LATIN method



Classical step-by-step methods



LATIN method

Idea: performance ↗↗ = mechanics-based



LATIN :Large Time INcrement method



Outline

- 1. The reference problem to be solved over the time-space domain**
 - remarkable properties?
- 2. The LATIN method and the last PGD version-Small displacements**
 - basic features?
- 3. Why are PGD-approximations efficient?**
- 4. Conclusion**

Reference problem to be solved over $[0, T] \times \Omega$

Classical formulation



Framework

- small perturbations
- quasi-static evolution
- isothermal



State of the structure

- defined by $\mathbf{s} = (\dot{\boldsymbol{\varepsilon}}_p, \dot{\mathbf{X}}, \boldsymbol{\sigma}, \mathbf{Y})$

- $\boldsymbol{\varepsilon}_p$ inelastic part of strain field

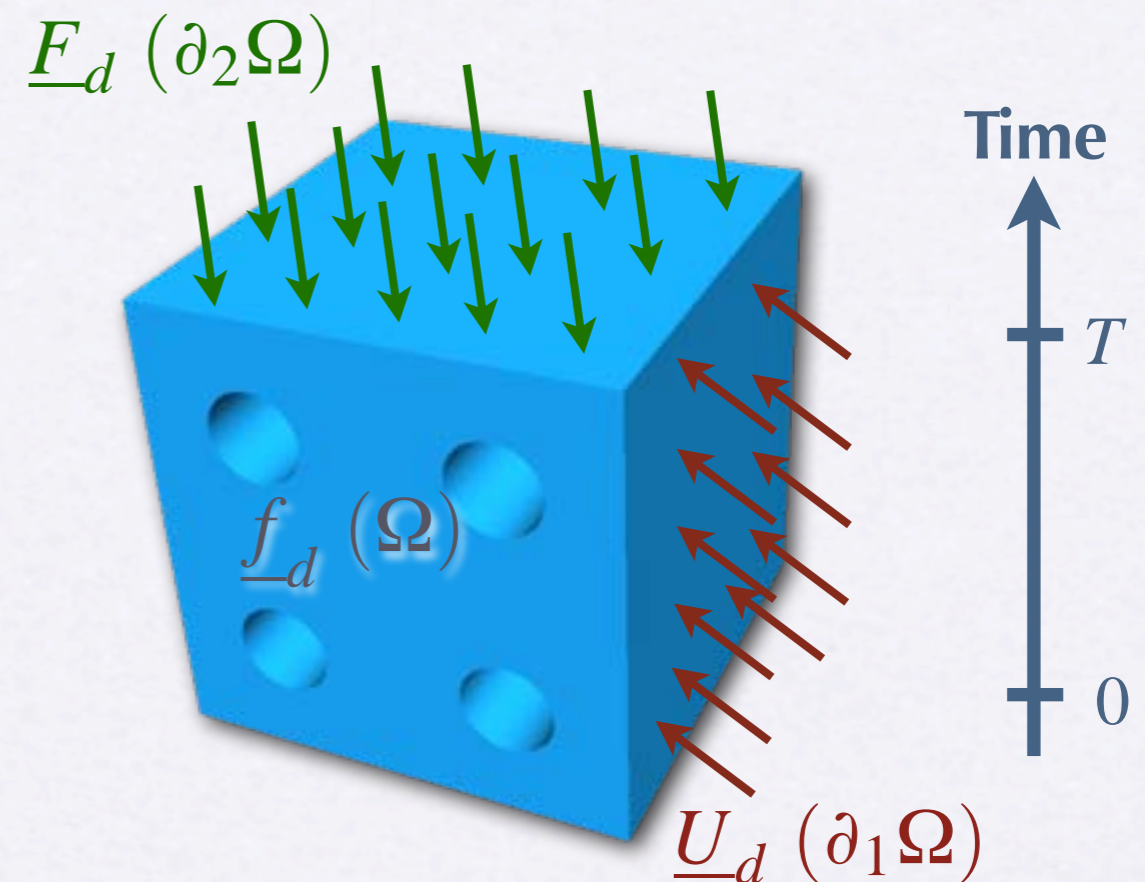
- $\boldsymbol{\varepsilon}_e$ elastic part of strain field

- $\boldsymbol{\varepsilon}$ strain field corresponding to displacement \underline{U} : $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_p + \boldsymbol{\varepsilon}_e$

- \mathbf{X} remaining internal variables

- $\boldsymbol{\sigma}$ stress field

- \mathbf{Y} variables conjugate of \mathbf{X}



Reference problem to be solved over $[0, T] \times \Omega$

Classical formulation

Find $\underline{s} = (\dot{\epsilon}^p, \dot{\mathbf{X}}, \boldsymbol{\sigma}, \mathbf{Y}) \in \mathcal{S}^{[0, T]}$
such that

Kinematic admissibility

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\underline{U}) \quad \underline{U}|_{\partial_1 \Omega} = \underline{U}_d$$

Static admissibility

$$\forall \underline{U}^* \in \mathcal{U}_0^{[0, T]},$$

$$-\int_{[0, T]} \text{Tr}[\boldsymbol{\sigma} \boldsymbol{\epsilon}^*] d\Omega dt + \int_{[0, T] \times \Omega} \underline{f}_d \cdot \underline{U}^* d\Omega dt + \int_{[0, T] \times \partial_2 \Omega} \underline{F}_d \cdot \underline{U}^* dS dt = 0$$

Constitutive relations

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_p + \boldsymbol{\epsilon}_e$$

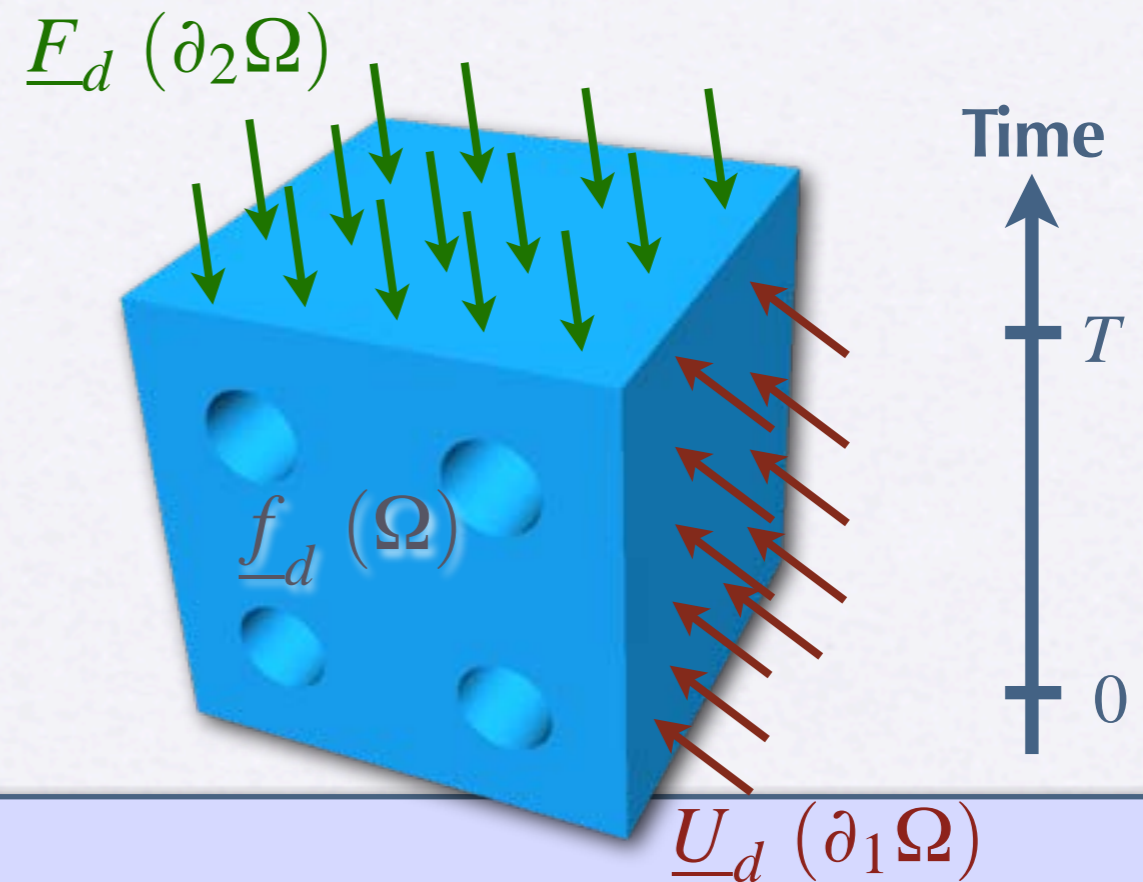
State equations

$$\boldsymbol{\sigma} = \mathbf{K} \boldsymbol{\epsilon}_e$$

$$\mathbf{Y} = \boldsymbol{\Lambda} \mathbf{X}$$

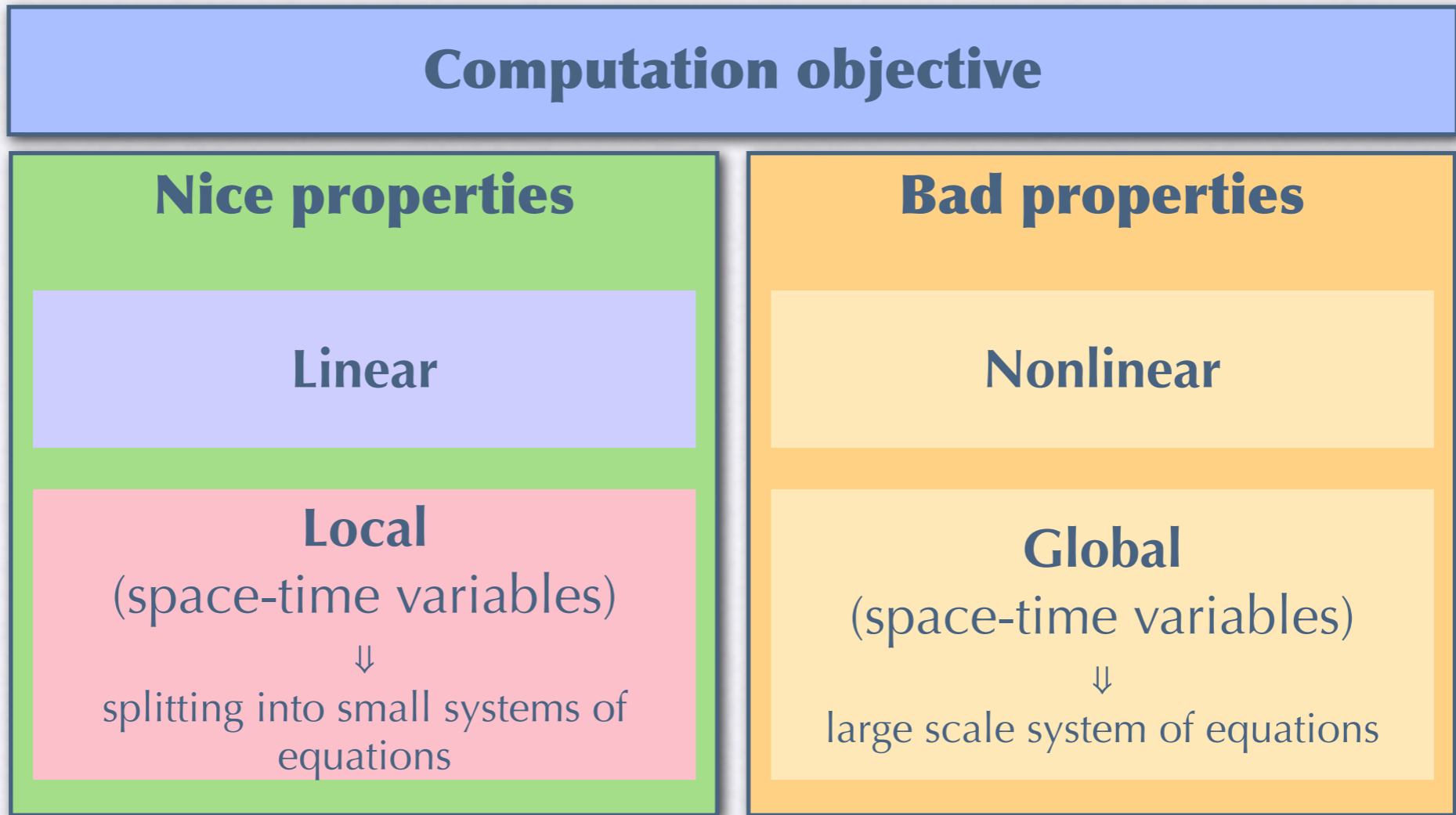
State evolution laws

$$\begin{bmatrix} \dot{\boldsymbol{\epsilon}}_p \\ -\dot{\mathbf{X}} \end{bmatrix} = \mathbf{B} \begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{Y} \end{bmatrix}$$



Reference problem to be solved over $[0, T] \times \Omega$

Remarkable properties?



Example of **local** relation : Hooke law

$$\sigma(\underline{M}) = K(\underline{M})\varepsilon(\underline{M})$$

Reference problem to be solved over $[0, T] \times \Omega$

Remarkable properties?

Find $\mathbf{s} = (\dot{\epsilon}^p, \dot{\mathbf{X}}, \boldsymbol{\sigma}, \mathbf{Y}) \in \mathcal{S}^{[0, T]}$ such that

 **Kinematic admissibility**



Linear

 **Static admissibility**



Local

 **State equations**



 **State evolution laws**





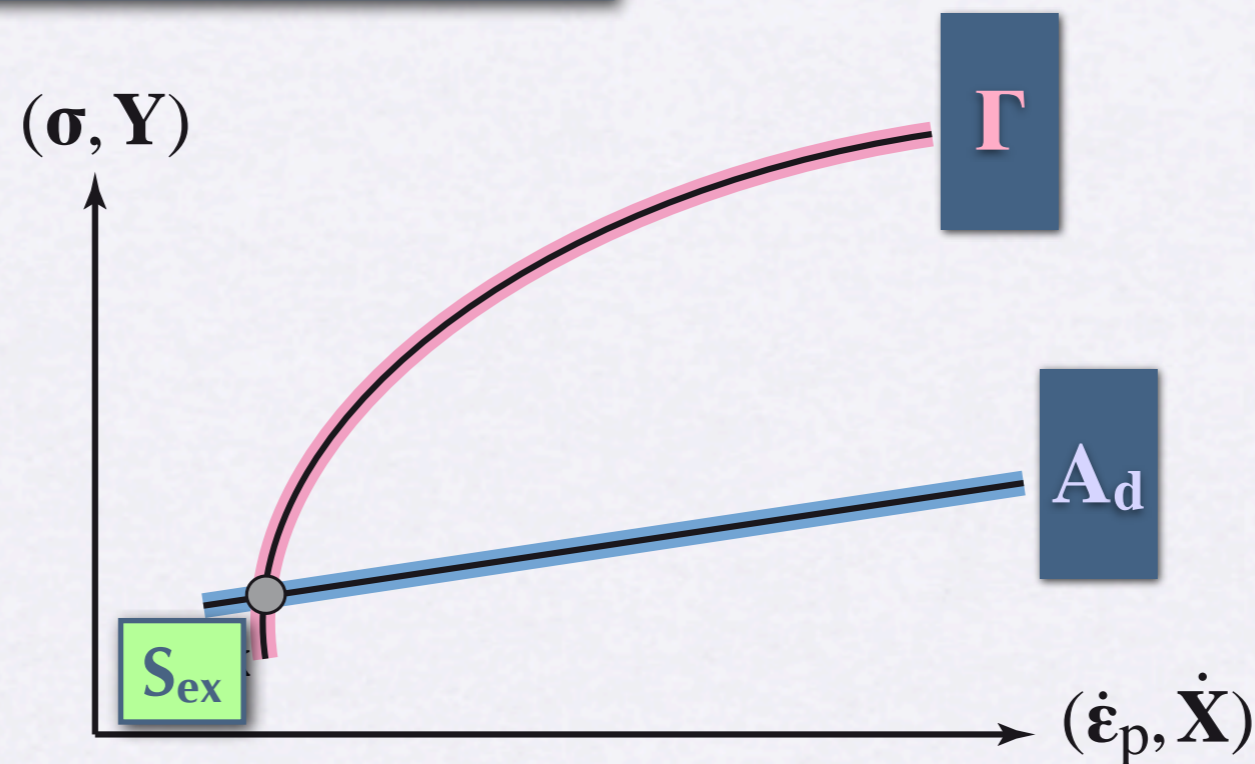
Outline

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 - basic features?
- 3. Why are PGD-approximations efficient?-Error control**
- 4. Conclusion**

The LATIN method - Principles P1 and P2

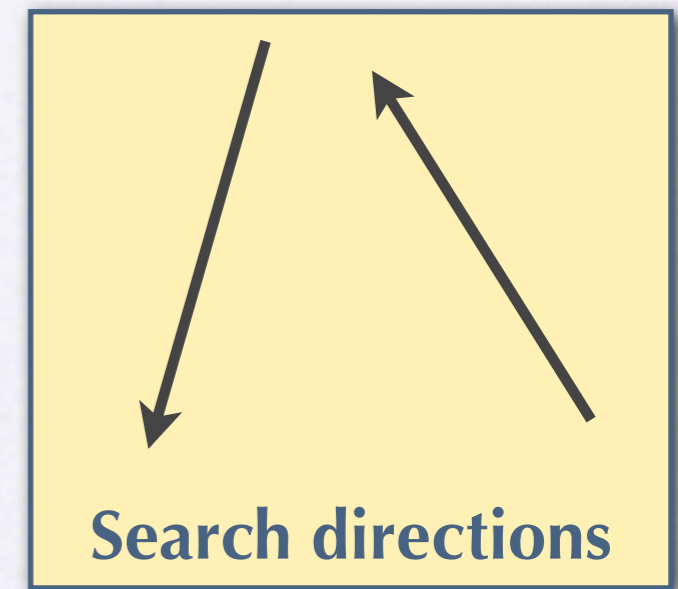
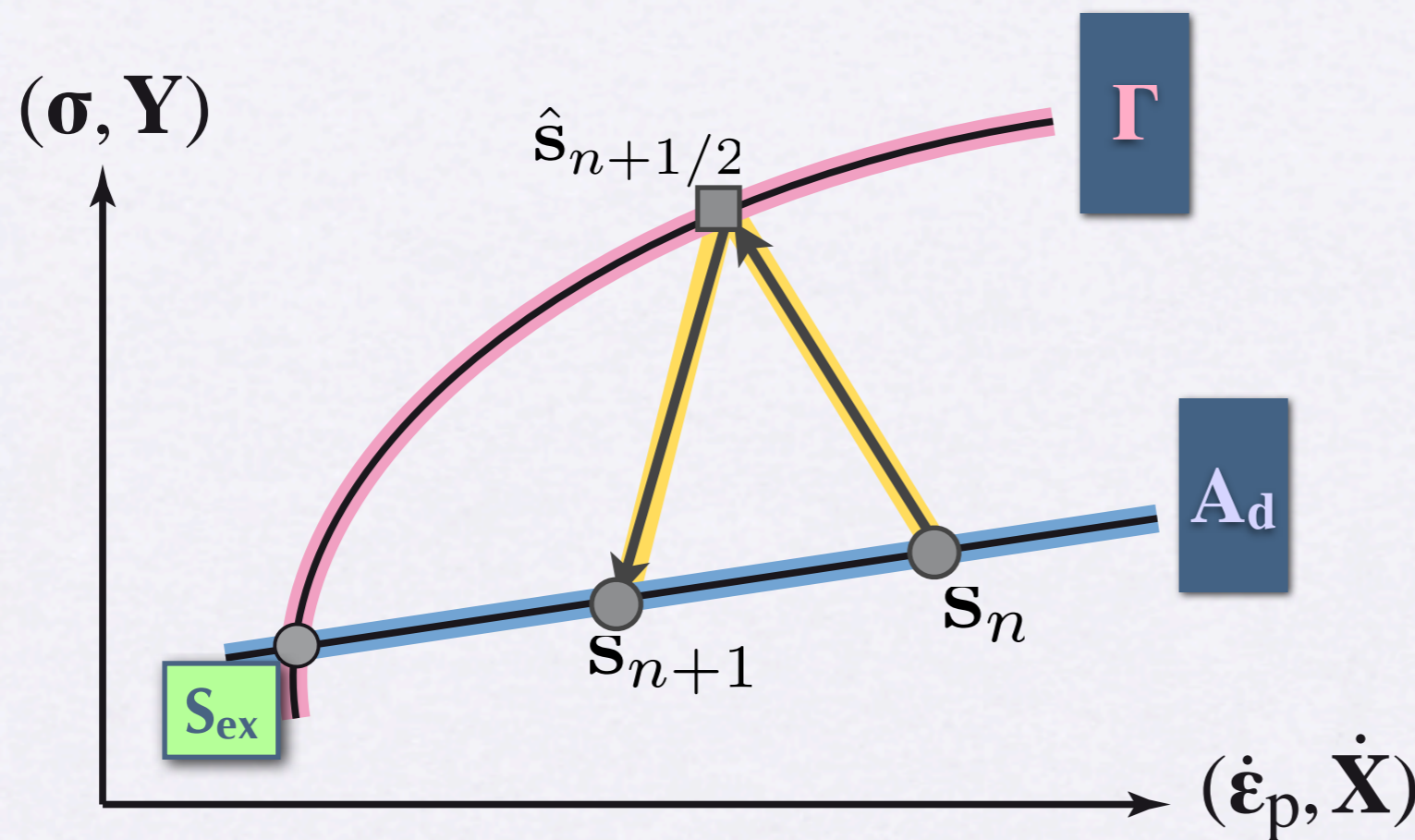
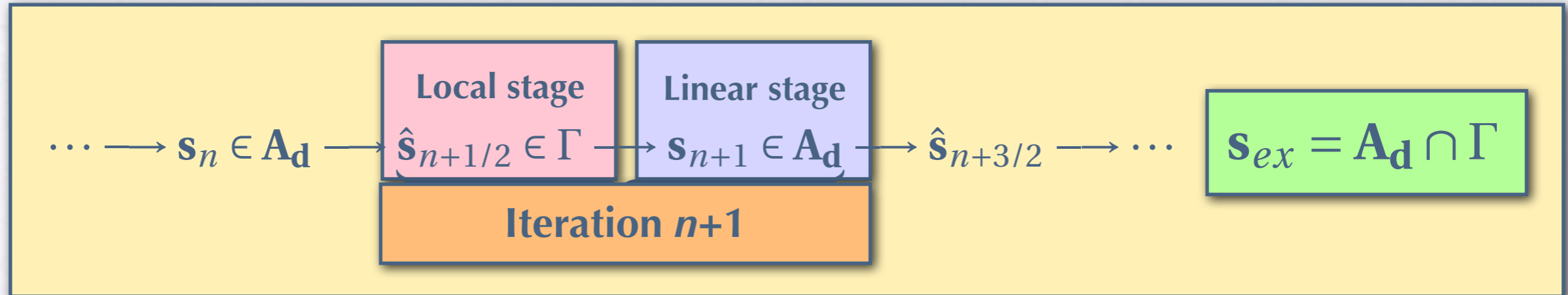
Principle P1: splitting of difficulties

Find $\mathbf{s} = (\dot{\epsilon}^p, \dot{\mathbf{X}}, \boldsymbol{\sigma}, \mathbf{Y}) \in \mathcal{S}^{[0,T]}$ such that



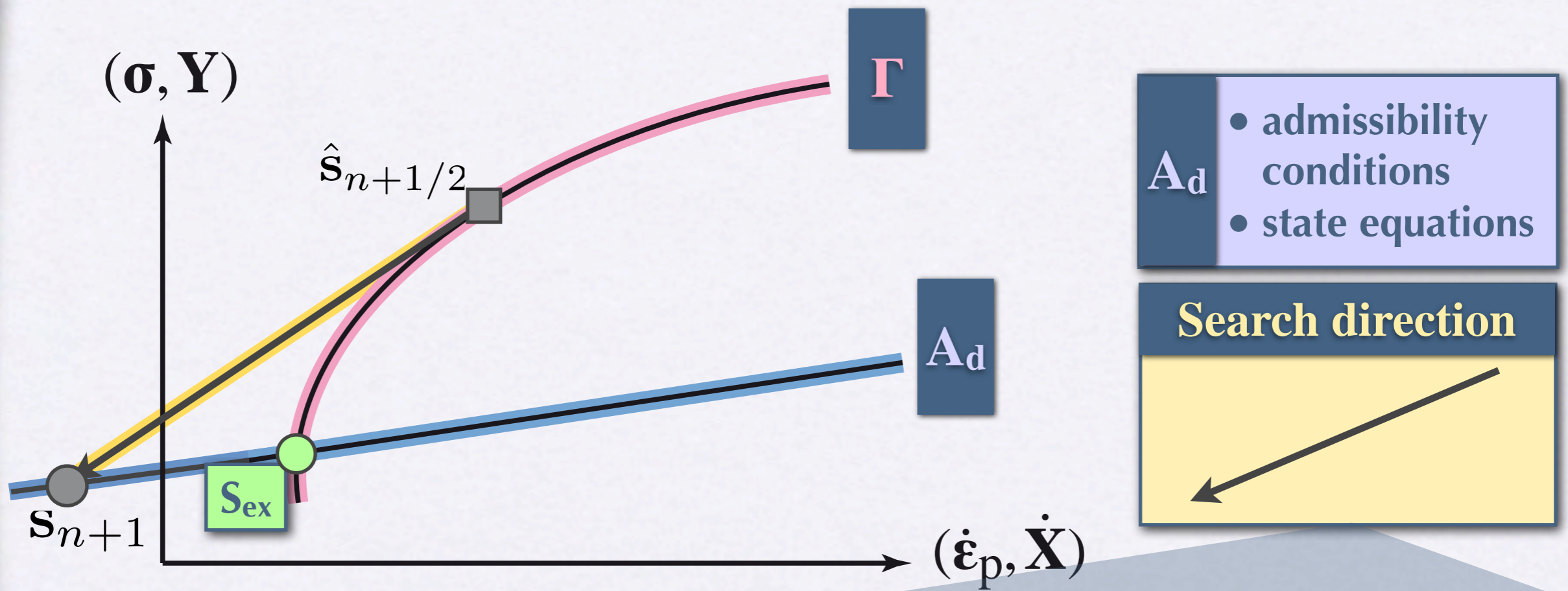
The LATIN method - Principles P1 and P2

Principle P2: iterative and alternative scheme



The LATIN method - Principles P1 and P2

Linear stage at Iteration n : $\hat{\mathbf{S}}_{n+1/2} \rightarrow \mathbf{S}_{n+1}$



Problem to be solved

Linear

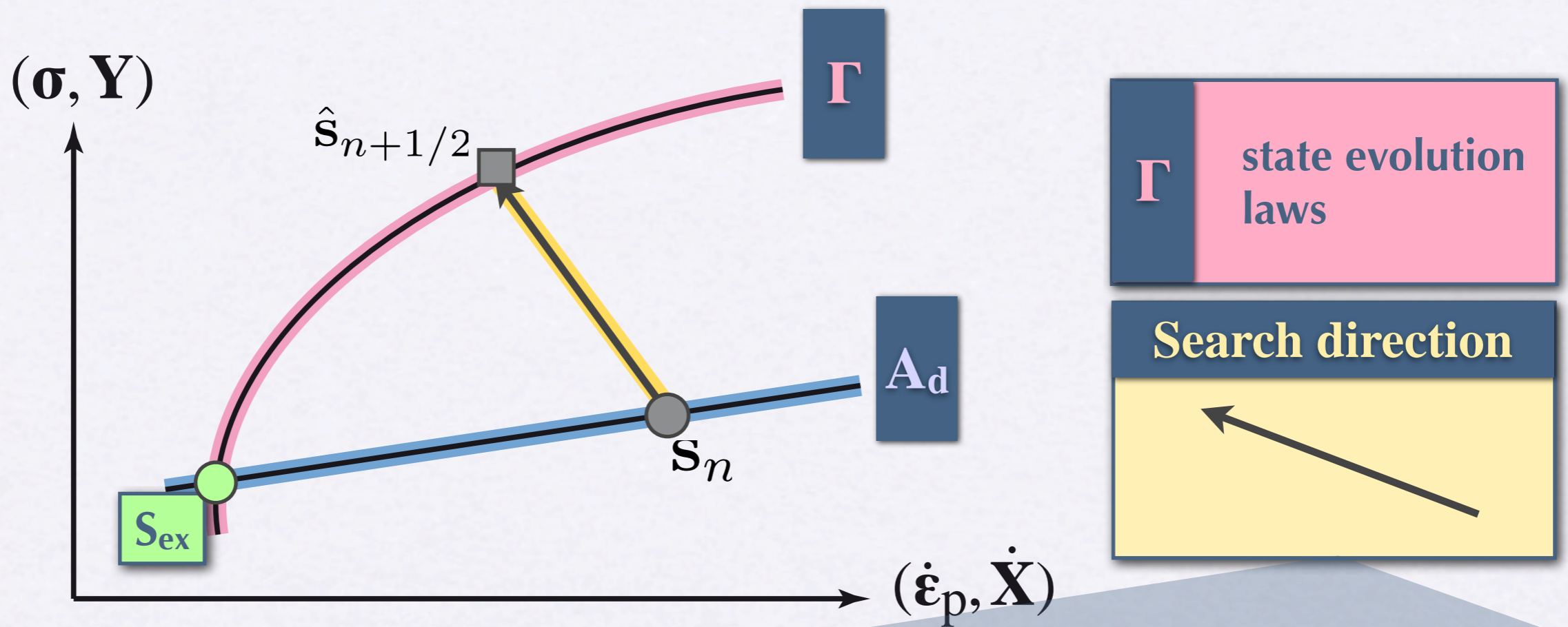
Global over $[0, T] \times \Omega$

$$\begin{bmatrix} \dot{\epsilon}_{p,n+1} - \hat{\epsilon}_{p,n+1/2} \\ -\dot{X}_{p,n+1} + \hat{X}_{p,n+1/2} \end{bmatrix} - \mathbf{H}^{-1} \begin{bmatrix} \sigma_{n+1} - \hat{\sigma}_{n+1/2} \\ \mathbf{Y}_{n+1} - \hat{\mathbf{Y}}_{n+1/2} \end{bmatrix} = 0$$

not a drawback but a great advantage!

The LATIN method - Principles P1 and P2

Local stage at Iteration n : $S_n \rightarrow \hat{S}_{n+1/2}$



Problem to be solved

Local in space-time

Nonlinear

$$\begin{bmatrix} \hat{\epsilon}_{p,n+1/2} - \epsilon_{p,n} \\ -\hat{X}_{p,n+1/2} + X_{p,n} \end{bmatrix} + H^+ \begin{bmatrix} \hat{\sigma}_{n+1/2} - \sigma_n \\ \hat{Y}_{n+1/2} - Y_n \end{bmatrix} = 0$$

very suitable for parallel computing!

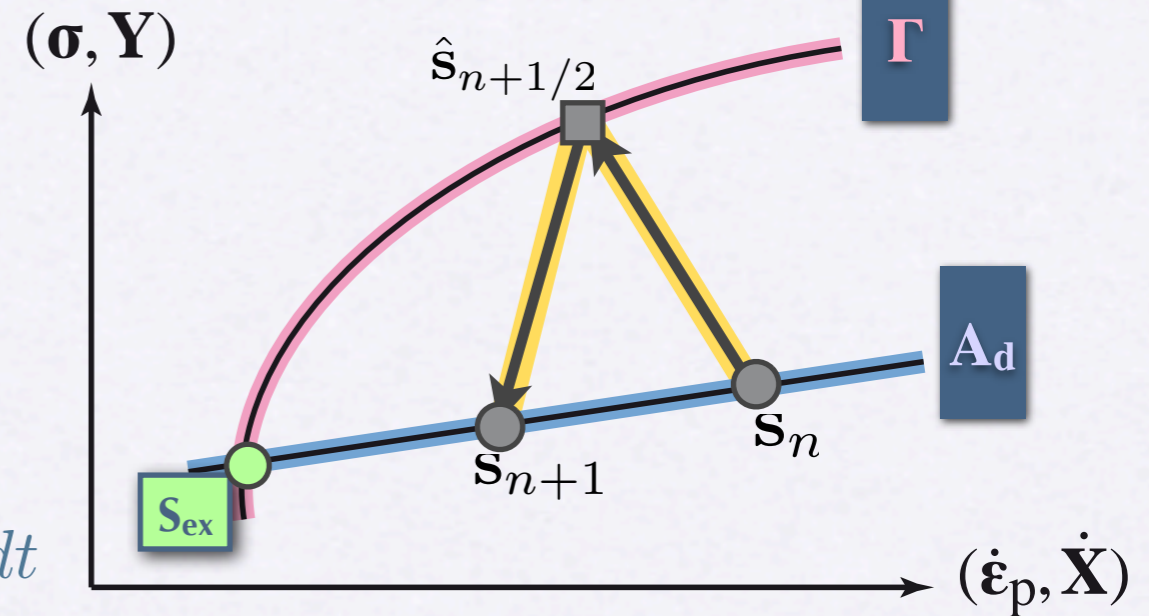
The LATIN method - Principles P1 and P2

A word about convergence

Dissipation bilinear form:

$$(\mathbf{s}, \mathbf{s}') \in S^{[0,T]} \times S^{[0,T]} \mapsto [\mathbf{s}, \mathbf{s}']_t$$

$$[\mathbf{s}, \mathbf{s}]_t \equiv \int_0^t \int_{\Omega} [\text{Tr}[\boldsymbol{\sigma} \dot{\boldsymbol{\varepsilon}}_p] - \mathbf{Y} \circ \dot{\mathbf{X}}] d\Omega dt$$



Theorem

Convergence of the LATIN method if:

- **B** monotone material operator
- Λ positive definite
- Conjugate search directions $H^+ = H^-$ positive definite

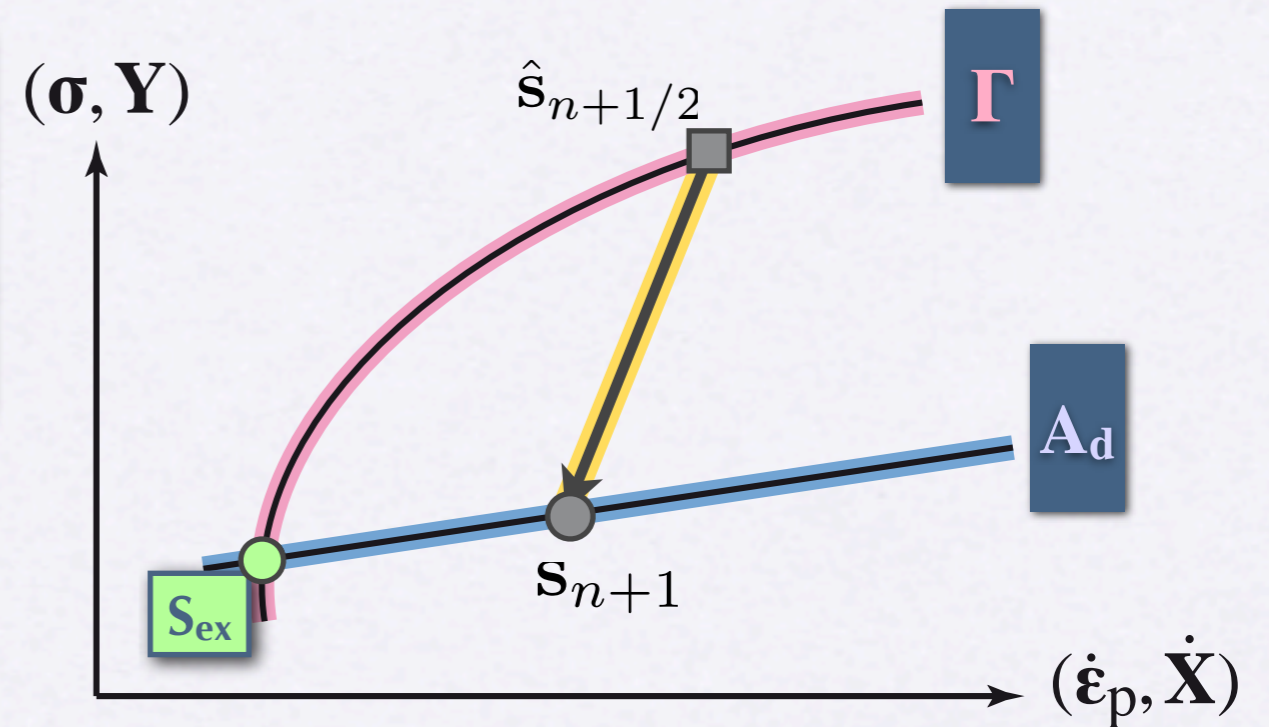
Error indicator

$$\frac{\|\hat{\mathbf{s}}_{n+1/2} - \mathbf{s}_n\|}{\frac{1}{2} \|\hat{\mathbf{s}}_{n+1/2} + \mathbf{s}_n\|}$$

The LATIN method - Principle P3 and PGD

Comeback: Linear stage at Iteration n : find S_{n+1}

**Problem defined
over $[0, T] \times \Omega$**



**Remarkable
property**

**Minimization formulation
over $[0, T] \times \Omega$**

**Scalar product defined over $[0, T] \times \Omega$
and used to introduce appropriate approximations**

The LATIN method - Principle P3 and PGD

Idea: PGD approximation for quasi-static loadings

[Ladevèze 1985], **book** [Springer NY 1999],
[Nouy, Ladevèze, cmame 2004], [Ladevèze, Néron, Passieux 2008]

Time-space approximation over $[0, T] \times \Omega$ of the unknowns of Λ_d

$$\forall (t, \underline{M}) \in [0, T] \times \Omega, \quad f(t, \underline{M}) = \sum_{i=1}^m \lambda_i(t) \Lambda_i(\underline{M})$$

- $m = 1$, well-known approximation in (visco)plasticity
- hot issue : F. Chinesta (high-dimensional problems and inverse problems)
 - A. Nouy (stochastic problems)
 - D. Néron (functional framework)
 - ...

The LATIN method - Principle P3 and PGD

The best PGD approximation of order m :

$$\tilde{f}_m(t, M)$$

Ladevèze 1996

$f \in \mathbf{F}^{[0, T]}$ defined over $[0, T] \times \Omega$, the norm being $\| \cdot \|_{\mathbf{F}^{[0, T]}}$

$$\forall (t, M) \in [0, T] \times \Omega, \quad \tilde{f}_m(t, M) = \sum_{i=1}^m \lambda_i(t) \Lambda_i(M)$$

$$\lambda_i \in \lambda, \quad \Lambda_i \in \Lambda$$

- result of an **eigenvalue problem** whose eigenfunctions are time functions λ_i

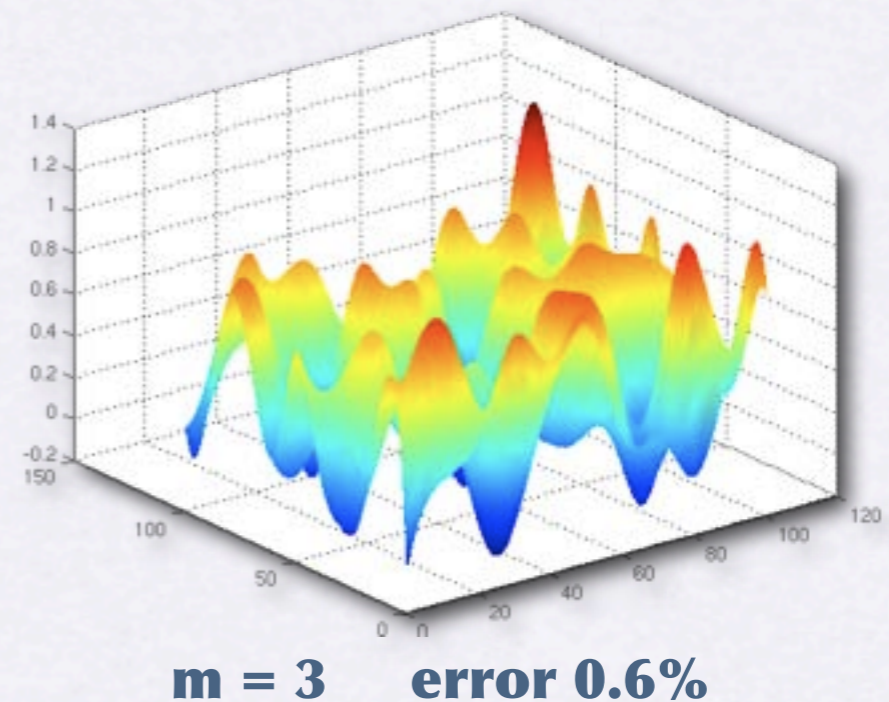
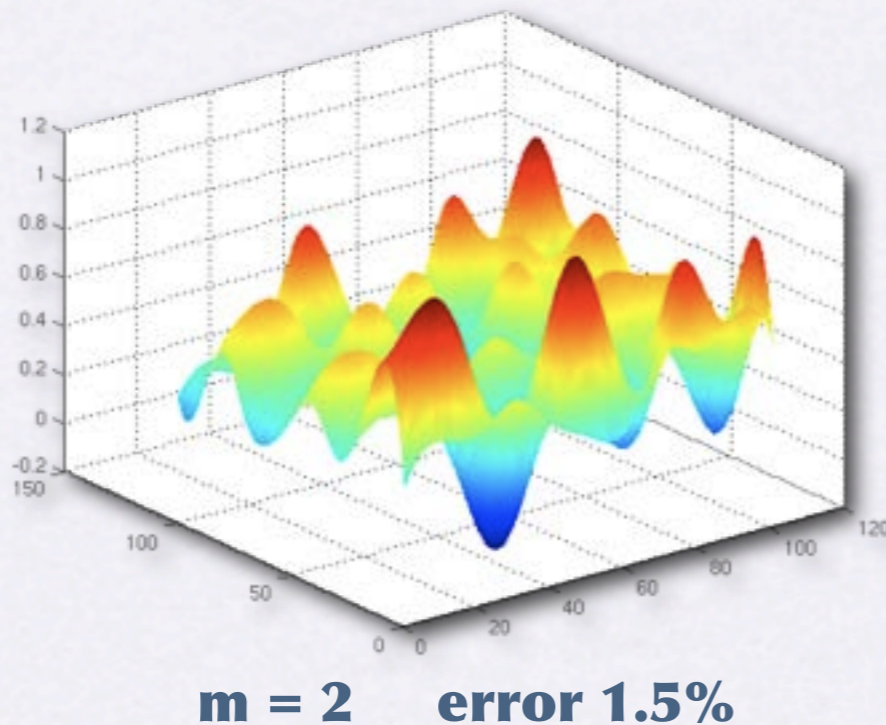
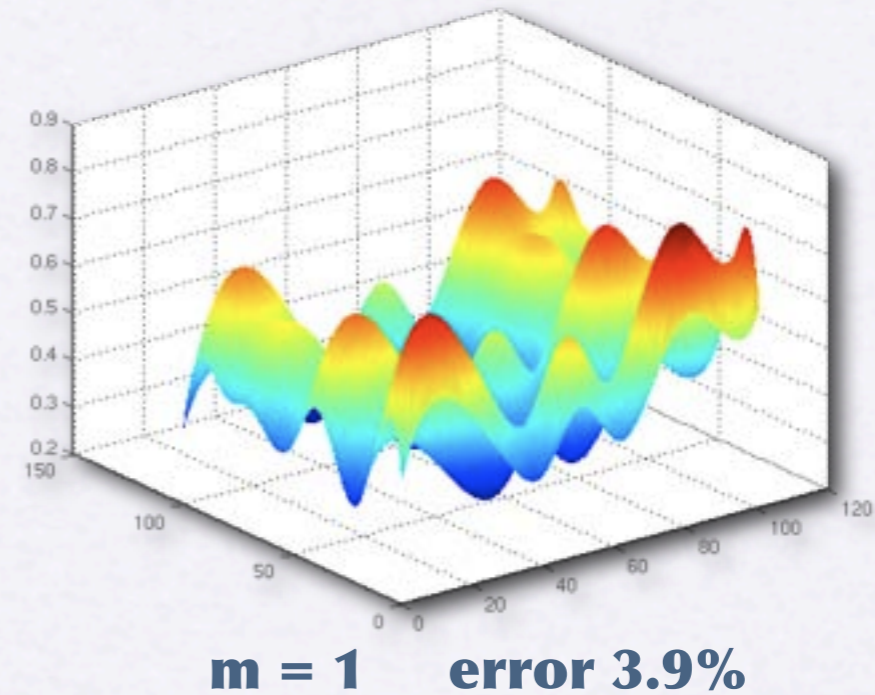
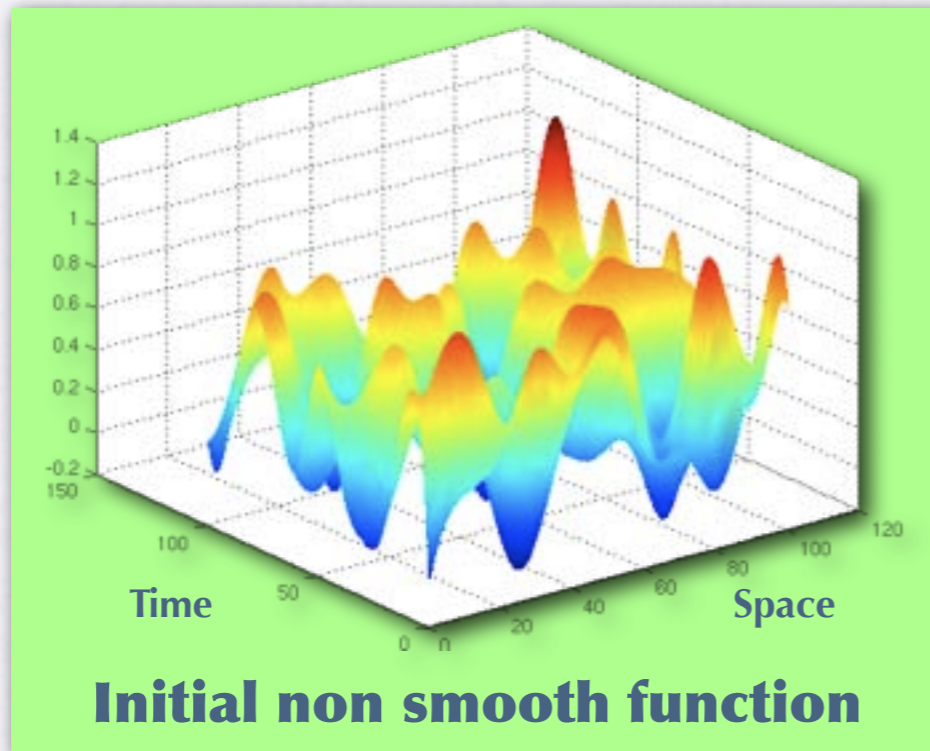
- and the corresponding space functions $\Lambda_i = \frac{\langle f, \lambda_i \rangle_{\lambda}}{\|\lambda_i\|_{\lambda}^2}$

- **Convergence result**

$$\lim_{m \rightarrow \infty} \| f - \tilde{f}_m \|_{\mathbf{F}^{[0, T]}} = 0$$

The LATIN method - Principle P3 and PGD

The best PGD time-space approximation of order m : example



The LATIN method - Principle P3 and PGD

Comeback to the linear stage at Iteration n : \mathbf{S}_{n+1} ?

$$\Delta \mathbf{s} \equiv \mathbf{S}_{n+1} - \mathbf{S}_n$$

Find $\Delta \mathbf{s} \in \mathbf{S}_{\text{Ad}}^{[0,T]}$ which minimizes:

$$\Delta \mathbf{s} \mapsto E(\Delta \mathbf{s})$$

$$\mathbf{S}_{\text{Ad}}^{[0,T]} \longrightarrow \mathbb{R}$$

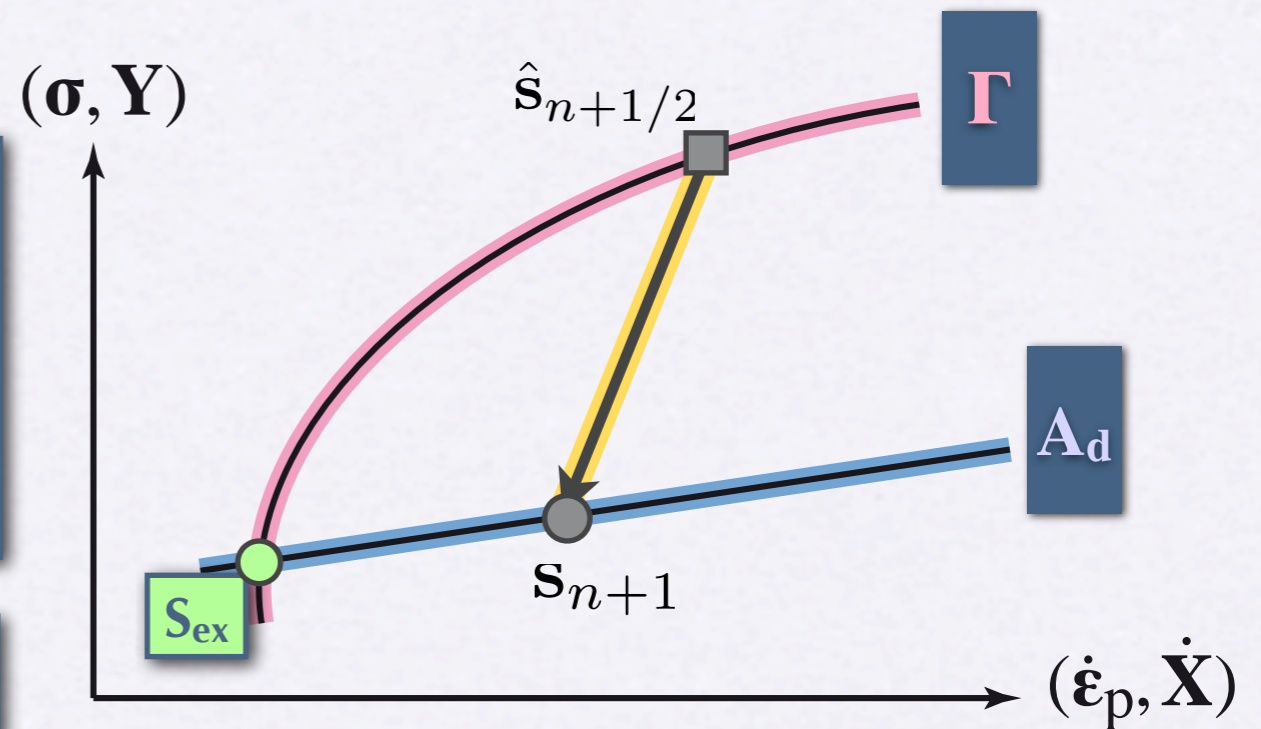
Radial time-space approximation

Subspace $\mathbf{S}_{\text{Ad},m}^{[0,T]}$:

- $\Delta \boldsymbol{\varepsilon}_p(t, \underline{M}) = \sum_{k=1}^m a^k(t) \mathbf{E}_p^k(\underline{M})$

- $\Delta \mathbf{X}(t, \underline{M}) = \sum_{k=1}^{m'} b^k(t) \mathbf{D}_p^k(\underline{M})$

- and the \mathbf{A}_d -equations



Drastic cost reduction

The LATIN method - Principle P3 and PGD

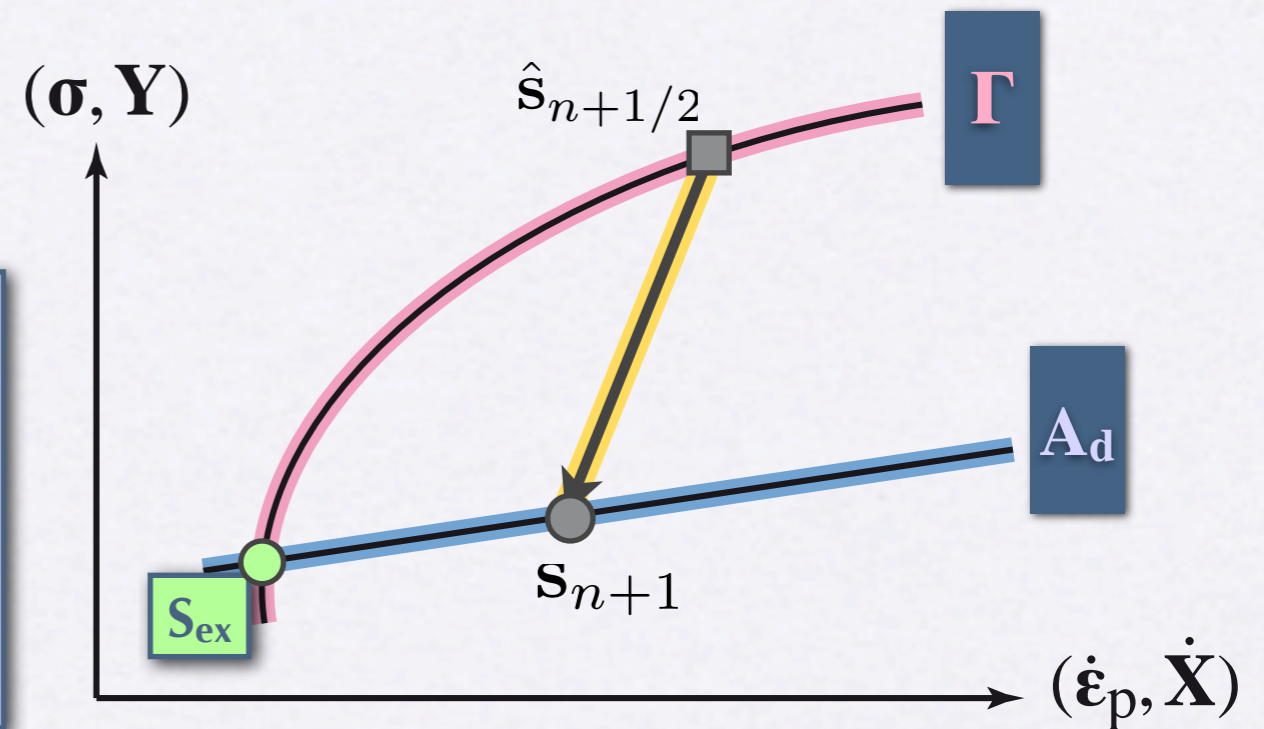
Comeback to the linear stage at Iteration n : Δs ?

$$\Delta \mathbf{s} \equiv \mathbf{s}_{n+1} - \mathbf{s}_n$$

Find $\Delta \mathbf{s} \in \mathbf{S}_{\text{Ad},m}^{[0,T]}$ which minimizes:

$$\Delta \mathbf{s} \longmapsto E(\Delta \mathbf{s})$$

$$\mathbf{S}_{\text{Ad},m}^{[0,T]} \longrightarrow \mathbb{R}$$



- practical technique (1 or 2 subiterations)
 - minimization alternatively on time functions (differential system of equations) and on space functions ("spatial" problem)
 - 1 "product " per iteration at the most
 - at each iteration : PGD space functions are kept and reused and then PGD time functions are updated
- drawback
 - computation of integrals over $[0,T] \times \Omega$: cost very high

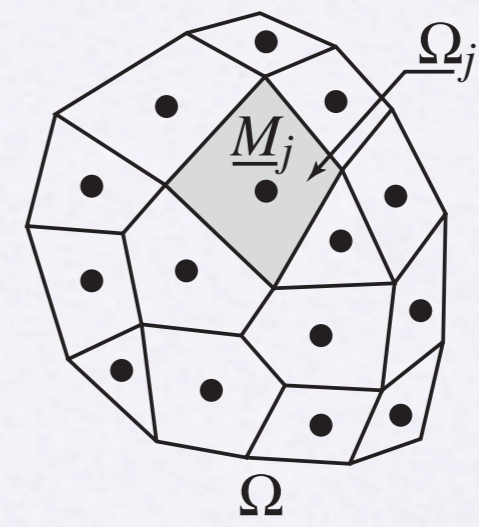
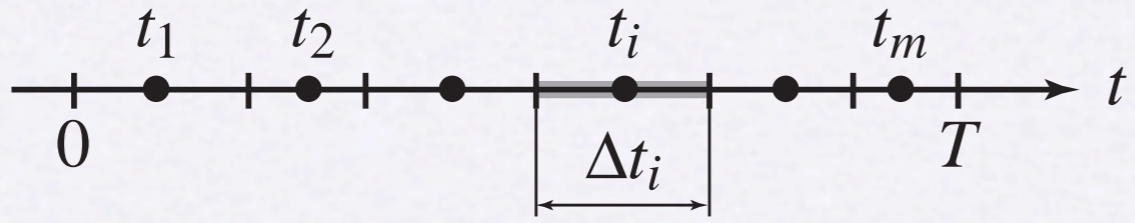
The LATIN method - PGD framework

All quantities : $\forall (t, \underline{M}) \in [0, T] \times \Omega, \quad \tilde{f}_m(t, \underline{M}) = \sum_{i=1}^m \lambda_i(t) \Lambda_i(\underline{M})$

Reference times and points

Reference times t_i

Reference points \underline{M}_j

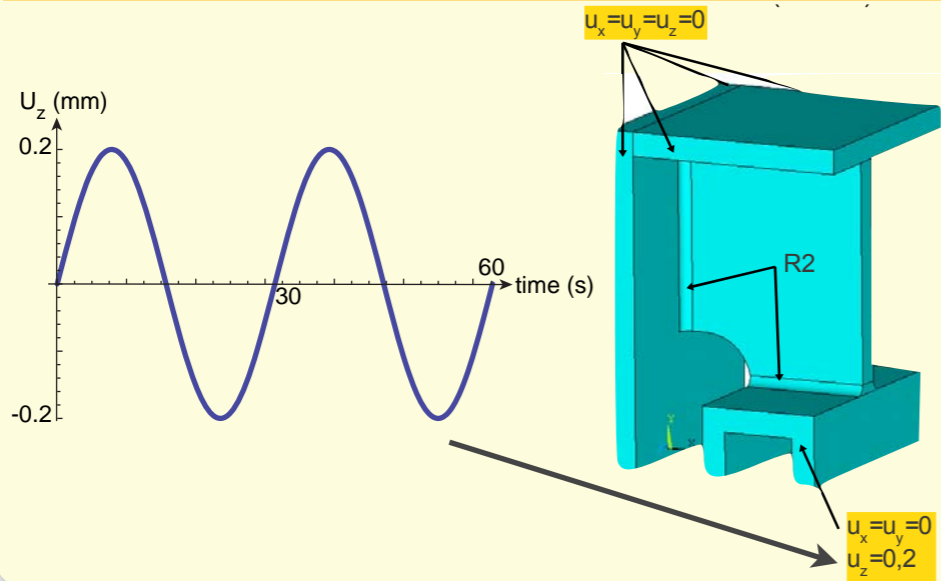


Generalized PGD components

$$f(t, \underline{M}) : \hat{f} = \begin{cases} t_i, i = 1, \dots, n & \hat{f}_i(\underline{M}) & \text{over } \Omega \\ \underline{M}_j, i = 1, \dots, m & \hat{f}^j(t) & \text{over } [0, T] \end{cases}$$

- “easy” reconstruction
- “PGD” algebra: operations done on \hat{f} quantities [Néron 11]

Problem



- visco-plastic behavior (Chaboche)
- 141753 dofs, 60 time step
- 17 pairs of time and space functions needed

Illustrations

[Relun et al 2011]

ANR APPRoFi

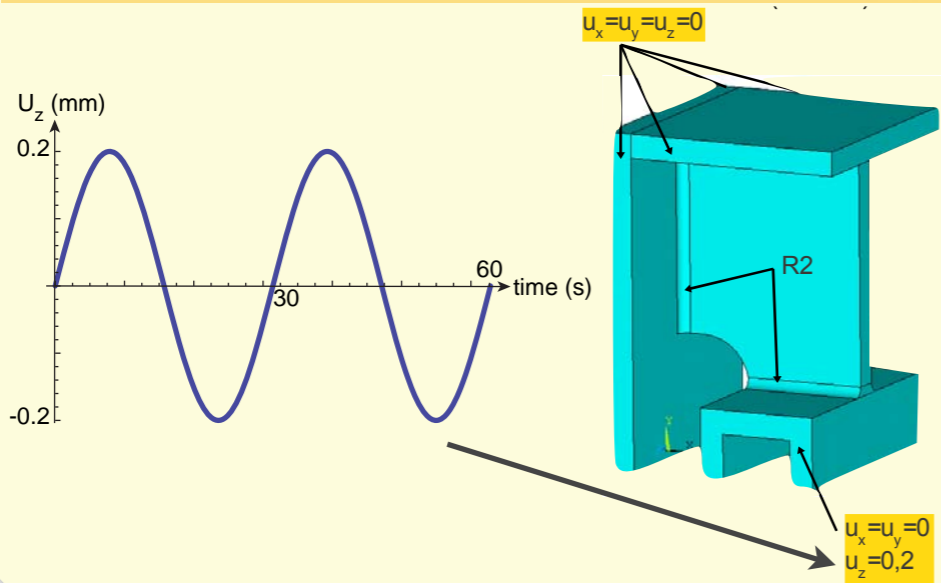


LATIN

Comparison

ABAQUS

Problem



- visco-plastic behavior (Chaboche)
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- 17 pairs of time and space functions needed

Illustrations

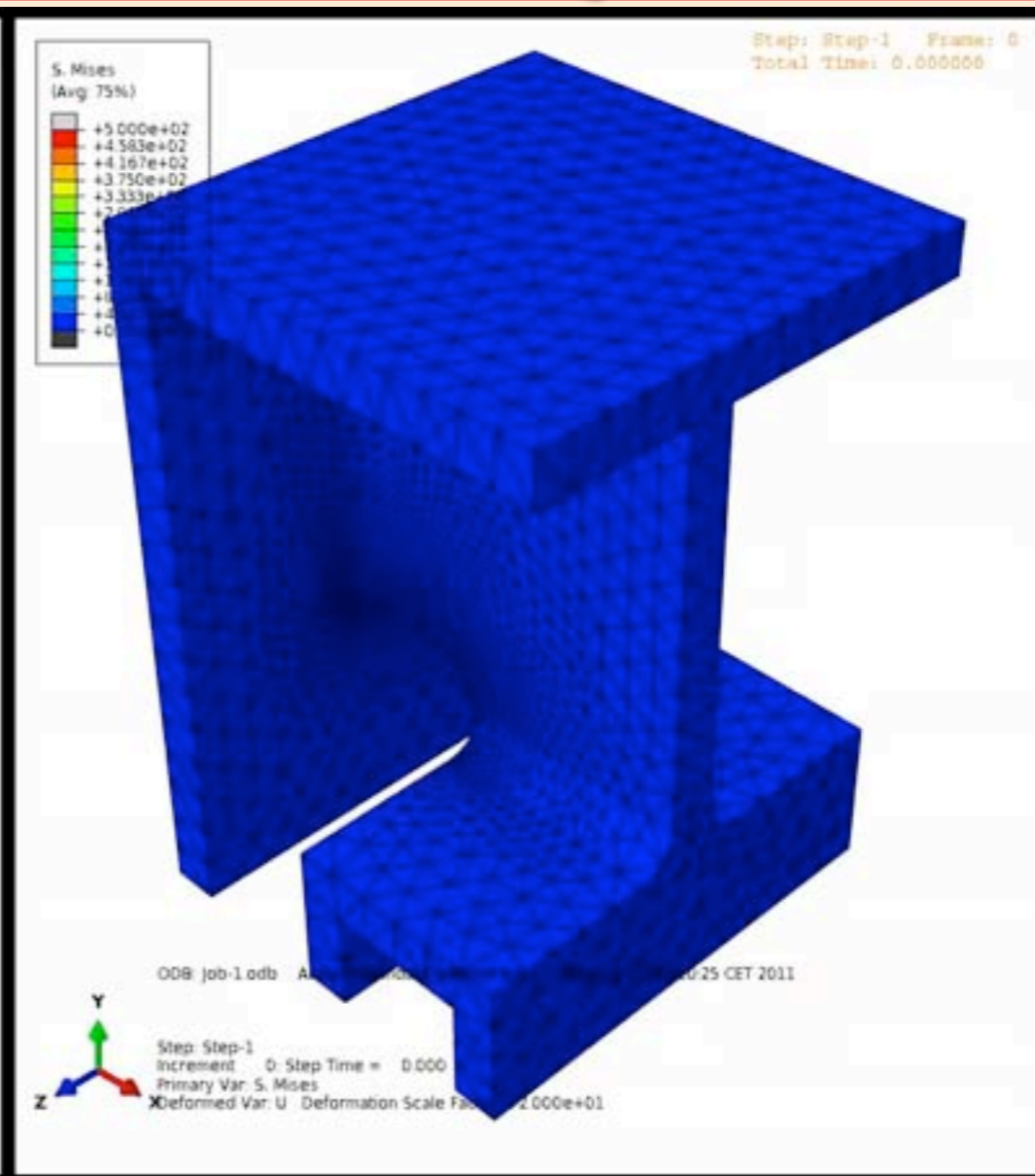
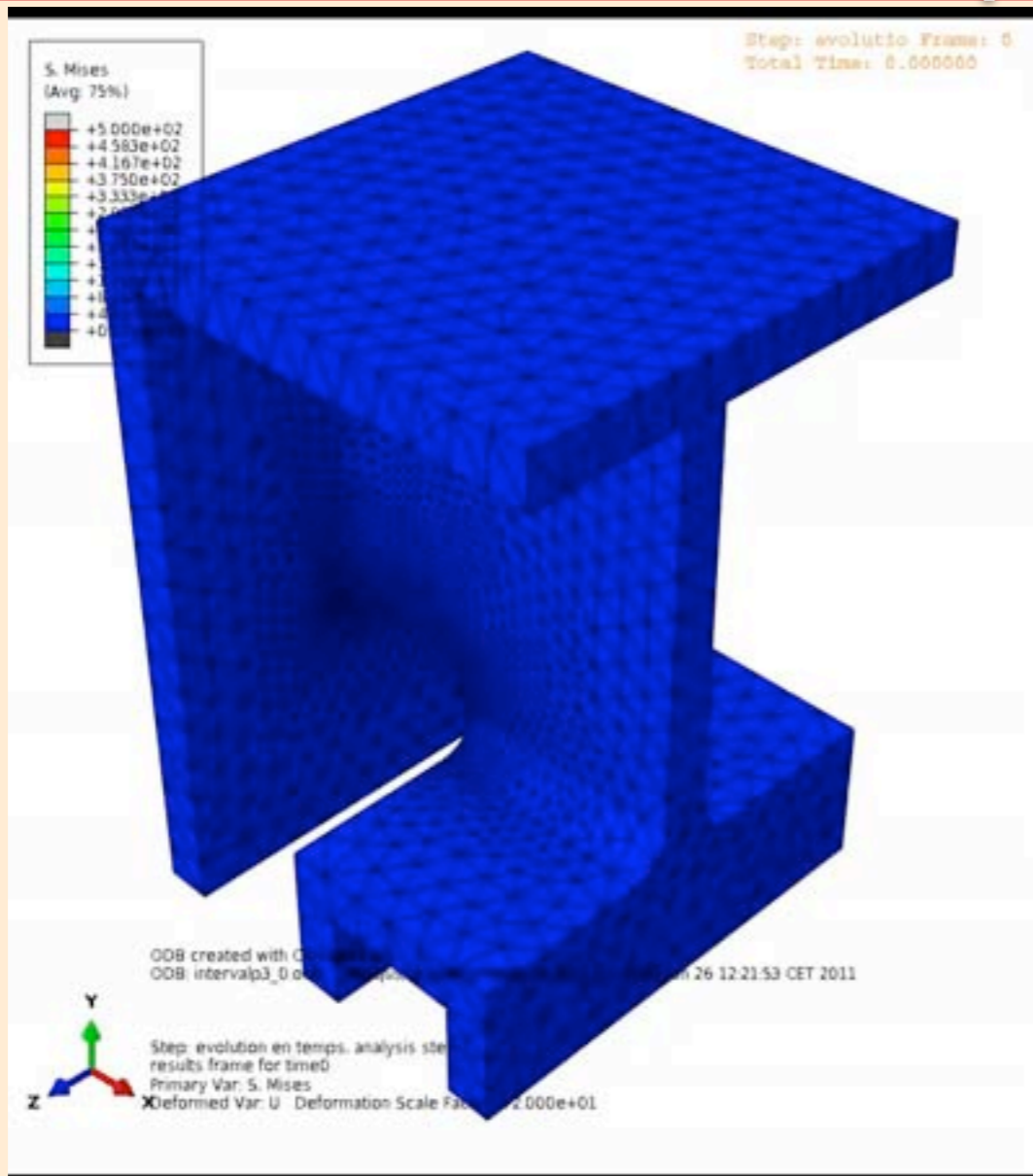
[Relun et al 2011]
ANR APPRoFi



LATIN

Comparison

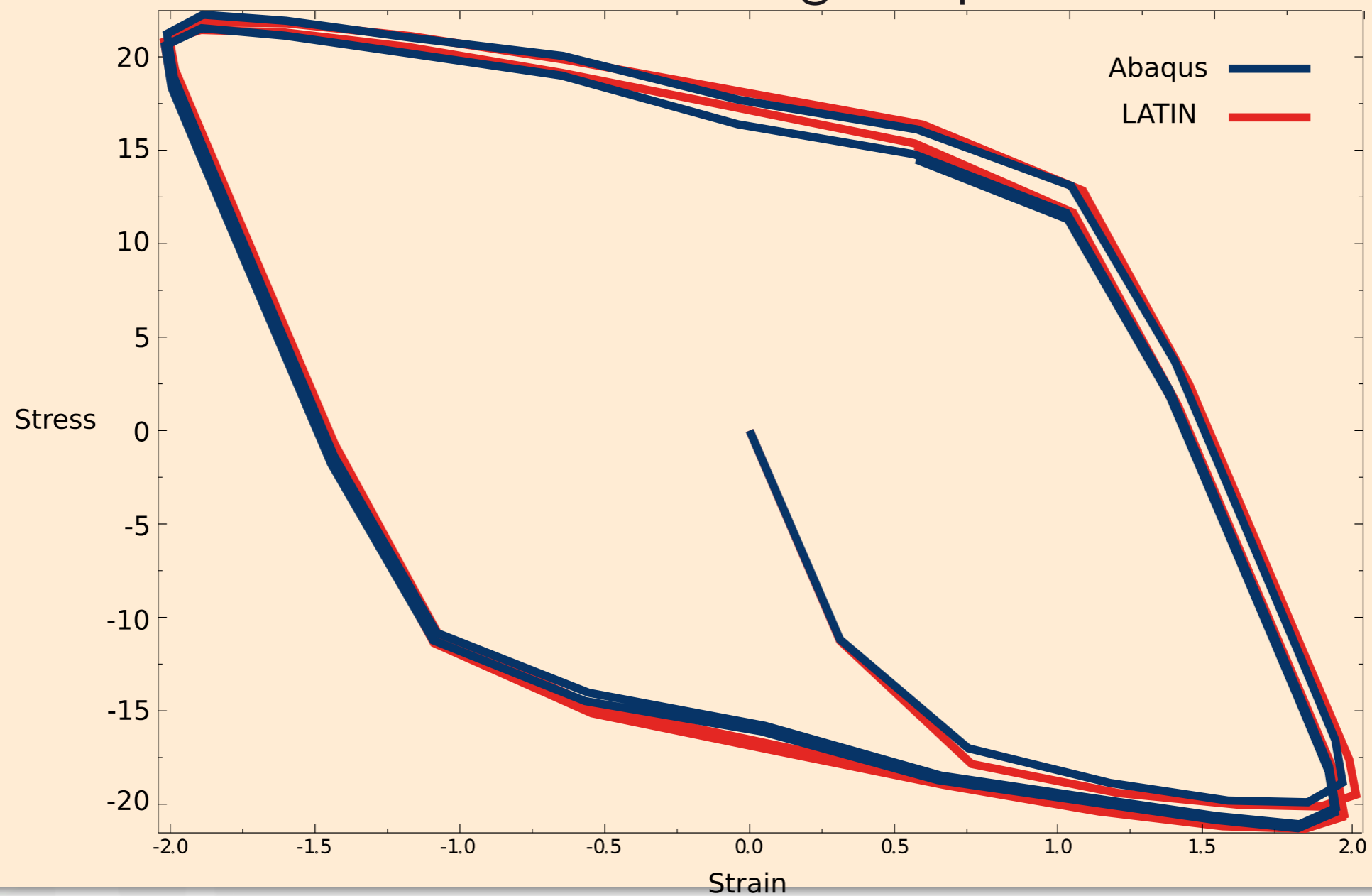
ABAQUS



Results obtained

- Comparison with ABAQUS

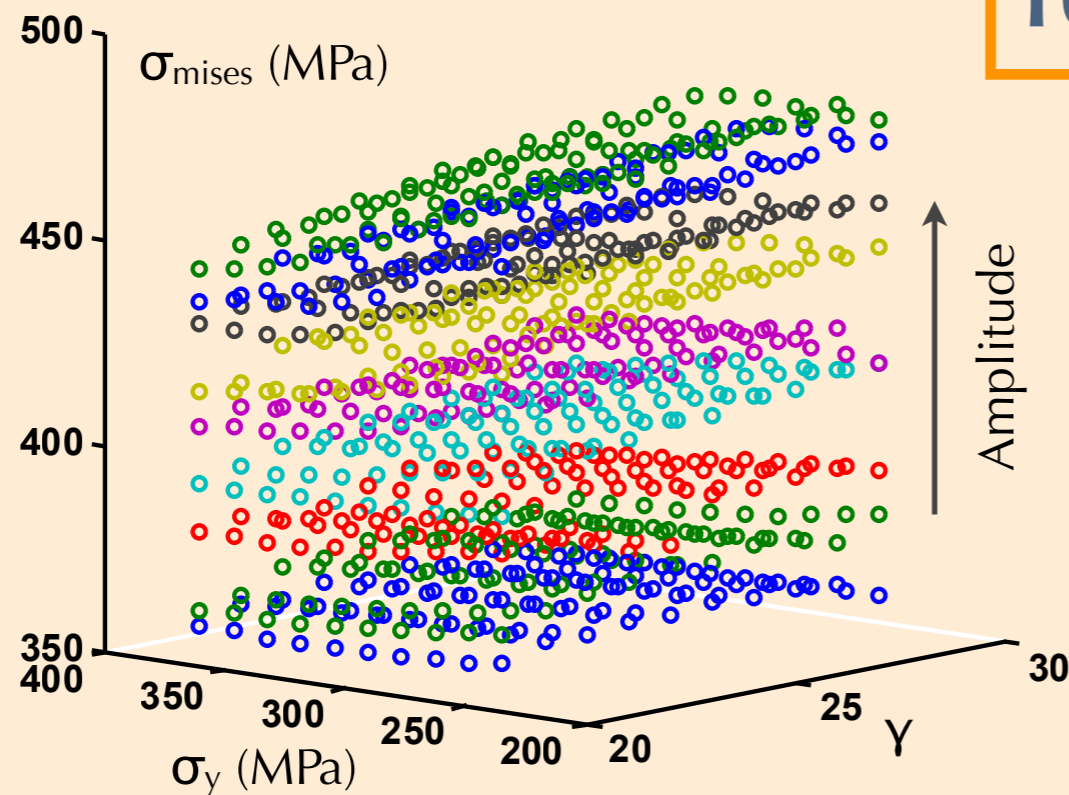
At the most loaded gauss point



Associated stochastic problem

Stochastic variables

- amplitude of the load,
- viscoplastic material constants
 - yield stress σ_y
 - kinematic hardening recall term γ



1000 parameter sets

One month with ABAQUS

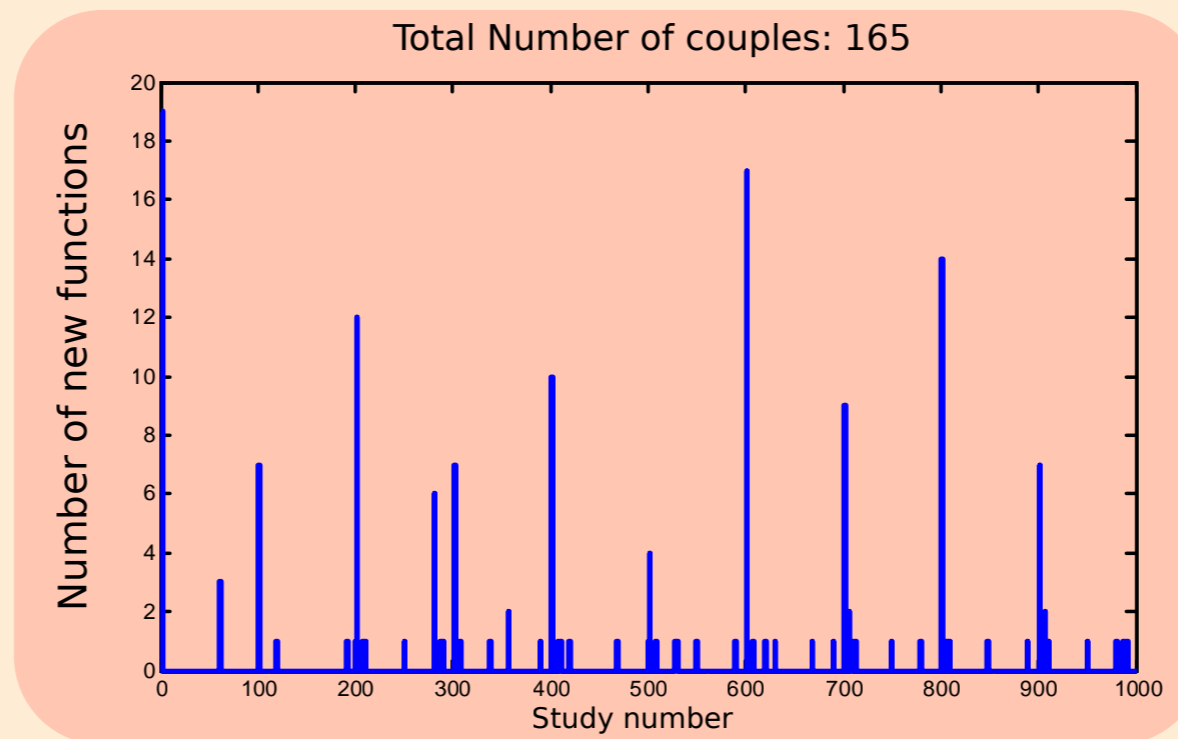
Technique and results

Incremental description of the parameters

→ parameter set $P_{i+1} = \text{parameter set } P_i + \Delta P_i$

Solution (t, M, P_{i+1})

→ initialisation with: solution (t, M, P_i)



final gain (one proc.): 40



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Why PGD-approximations are efficient ?

Heuristic arguments

📌 **Model problem over $[0,T] \times \Omega$ in quasi-statics**

$$\underline{\mathbb{A}}(\underline{s}) = \underline{F}$$

$$\textcircled{\text{H}} \quad \underline{F} = \sum_{i=1}^m f_i(t) \underline{F}_i(X)$$

usual engineering description

📌 **Case 1**

$\underline{\mathbb{A}}$ linear and constant

$$\underline{s} = \sum_{i=1}^m f_i(t) \underline{\mathcal{A}}(\underline{F}_i(X))$$

📌 **Case 2**

$\underline{\mathbb{A}}$ linear and $\underline{\mathbb{A}} = \underline{\mathbb{A}}_0 + \sum_{i=1}^n a_i(t) \underline{\mathbb{A}}_i$

$$\underline{s} = \sum_{i=1}^k s_i(t) \underline{S}_i(X)$$

Why PGD-approximations are efficient ?

Heuristic arguments

Case 3 (plasticity, viscoplasticity)

A nonlinear $\underline{F} = f(t)\underline{F}(X)$ radial loading

Theorem under the hypotheses:
- elasticity neglected
- isotropic hardening
- $R = p^n$

the solution verifies:

$$\begin{aligned}\sigma &= f(t)C(X) \\ \varepsilon_p &= g(t)\Sigma(X)\end{aligned}$$

with $\Sigma(X) = C_D(X)$ and $|g|^n |\Sigma|^n = \sup_{\tau \leq t} f_\tau |C|$

➔ (Hencky-Mises): accurate first approximation

Why PGD-approximations are efficient? A posteriori PGD verification



Our goal

- guaranteed error bound (PGD, Reduced Model,...)

[Ladevèze, Chamoin CMAME 2010]

Why PGD-approximations are efficient? A priori PGD verification

?



Conclusion

- **A tool for robust design: used to solve engineering problems not affordable by commercial codes**
 - cyclic (visco-)plasticity
 - damping virtual testing of the joints of Ariane 5
 - fracture of laminated composites at the micro scale
 - multi-parameter engineering problems
 - ...
- **Challenging questions in progress**
 - very large scale time-dependent nonlinear problems (numerous parameters, complex loading, large displacement,..)
 - dynamics problems and acoustics over the LF + MF range (frequency-space domain)
 - verification of PGD (goal oriented quantities)



Not the end
thank you

