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# PGD and separated space variables representation for linear elasticity in 3D representation of plate domains

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**Abstract.** In this paper, we focus on the simulation of linear elastic behaviour of plates using a 3D approach which numerical cost only scales like a 2D one. In the case of plates, the kinematic hypothesis introduced in plate theories to go from 3D to 2D is usually unsatisfactory where one cannot rely on St Venant's principle (usually close to the plate edges).

We propose to apply the PGD (Proper Generalized Decomposition) method [1] to the simulation of the linear elastic behavior of plates. This method allows us to separately search for the in-plane and the out-of plane contributions to the 3D solution, yielding significant savings in computational cost. The method is validated on a simple case and its full potential is then presented for the simulation of the behavior of laminated composite plates.

**Keywords:** Linear elasticity, Model reduction, PGD, Plate domains, 2D+1D to 3D

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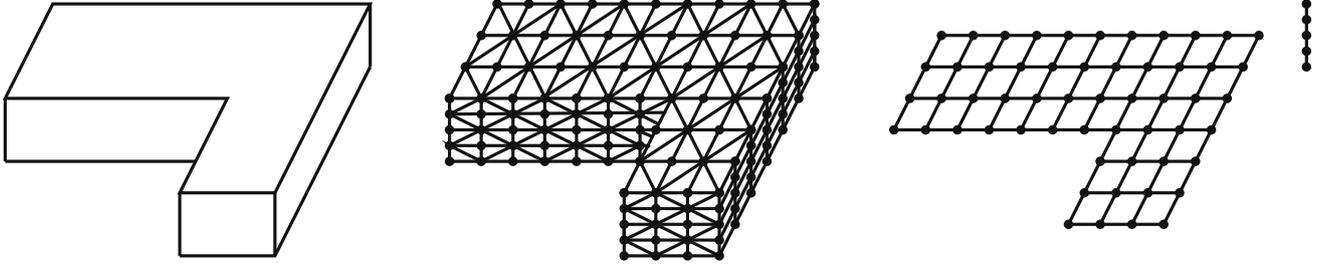
## INTRODUCTION

When computing elastic response of plates, two dimensional plate theories are usually preferred to the numerically expensive solution of the full three-dimensional elastic problem. Going from a 3D elastic problem to a 2D plate theory usually involves some kinematic hypothesis [2] on the variation of the solution through the thickness of the plate.

Despite the quality of existing plate theories, their solution close to the plate edge is usually wrong as the displacement fields are truly 3D in those regions and do not satisfy the kinematic hypothesis. Indeed, the kinematic hypothesis is a good approximation where Saint-Venant's principle is verified.

In an attempt to overcome this problem while maintaining a low computation cost, we use the Proper Generalized Decomposition (PGD) approach to seek a solution of the 3D elastic problem under the following space variable separated form:

$$\underline{u}(x, y, z) = \begin{pmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{pmatrix} = \sum_{i=1}^N \begin{pmatrix} u_{xy}^i(x, y)u_z^i(z) \\ v_{xy}^i(x, y)v_z^i(z) \\ w_{xy}^i(x, y)w_z^i(z) \end{pmatrix} = \sum_{i=1}^N U_{xy}^i(x, y)U_z^i(z), \quad (1)$$



**FIGURE 1.** Solid part, 3D FEM mesh, and 2D and 1D associated discretization for the PGD

where  $u_{xy}(x, y)$ ,  $v_{xy}(x, y)$ ,  $w_{xy}(x, y)$  are functions of the surface coordinates, and  $u_z(z)$ ,  $v_z(z)$ ,  $w_z(z)$  are functions of the thickness (Fig. 1), and  $N$  is the number of superimposed couples of functions (modes) describing the solution.

As we allow the number of modes to be greater than 1, and as the  $z$  dependence of  $U_z(z)$  modes is not fixed *a priori*, this representation of the solution is flexible enough to represent the solution of the full 3D problem. It is also straightforward to notice that with enough modes, we can describe the 3D FEM solution to an arbitrary precision.

However, even in the case of a single mode, the simultaneous determination of  $U_{xy}$  and  $U_z$  is a non-linear problem. Following [3] we use a fixed point method to compute  $U_{xy}$  and  $U_z$  in such a way that only 2D and 1D linear problems have to be solved.

In the next sections, we first present the reformulation adapted to PGD for the chosen separation of space variables of 3D linear elasticity, then we briefly describe the fixed point algorithm used in this particular case. The method is then validated by comparing the PGD solution to the solution of a simple 3D elastic problem computed with finite elements. Finally, we demonstrate the power of the method on a more complex case.

## APPLICATION TO LINEAR ELASTICITY

Let's consider a linear elasticity problem on a domain  $\Omega$ . The weak formulation associated to such a problem reads:

$$\langle \underline{\varepsilon}(\underline{\hat{u}}) : \underline{K} : \underline{\varepsilon}(\underline{u}) \rangle = \langle \underline{\hat{u}} \cdot f_d \rangle + \ll \underline{\hat{u}} \cdot F_d \gg, \forall \underline{\hat{u}} \in \hat{U}, \quad (2)$$

with  $\underline{K}$  the stiffness 6 by 6 matrix,  $\underline{\varepsilon}$  the strain, and  $F_d$  and  $f_d$  respectively volume and surface forces.

The separation of variables introduced in Eq. (1) yields the following expression for the strain:

$$\underline{\varepsilon}(\underline{u}(x, y, z)) = \sum \begin{pmatrix} u_{xy,x} \cdot u_z \\ v_{xy,y} \cdot v_z \\ w_{xy} \cdot w_{z,z} \\ u_{xy,y} \cdot u_z + v_{xy,x} \cdot v_z \\ u_{xy} \cdot u_{z,z} + w_{xy,x} \cdot w_z \\ v_{xy} \cdot v_{z,z} + w_{xy,y} \cdot w_z \end{pmatrix}. \quad (3)$$

Depending of the number of non-zero elements in the  $\underline{K}$  matrix, the development of  $\langle \underline{\varepsilon}(\underline{\hat{u}}) : \underline{K} : \underline{\varepsilon}(\underline{u}) \rangle$  gives from 21 to 41 terms. In the case of an isotropic material, we

obtain 21 terms, in the case of an orthotropic one, we obtain up to 41 terms depending on the orientation of the principal directions.

Assuming that the first  $N$  modes of the solution are known, let us look for an additional mode made of a couple of functions:  $R(x, y)$  and  $S(z)$ . The augmented solution writes:

$$\underline{u}_{N+1}(x, y, z) = \underbrace{\sum_{i=1}^N U_{xy}^{(i)}(x, y) U_z^{(i)}(z)}_{\underline{u}_N(x, y, z)} + R(x, y) S(z). \quad (4)$$

For consistency, we define  $\underline{u}_0(x, y, z) = 0$ . As proposed by Chinesta et al. [3], we introduce the following test function:

$$\hat{u}(x, y, z) = R(x, y) S^*(z) + R^*(x, y) S(z). \quad (5)$$

Injecting this test function in the weak formulation (2) leads to:

$$\int_{x,y} \int_z \left[ \underline{\varepsilon}(\hat{u}(x, y, z)) \cdot \underline{K} \cdot \underline{\varepsilon}(\underline{u}_{N+1}(x, y, z)) \right] dx dy dz = \langle \hat{u} \cdot f_d \rangle + \ll \hat{u} \cdot F_d \gg. \quad (6)$$

Because the simultaneous determination of  $R(x, y)$  and  $S(z)$  is non-linear, we alternatively look for each one with a fixed point algorithm.

Given an initial approximation  $S^0(z)$  of  $S(z)$ , all  $z$  dependent functions are known (hence  $S^*(z) = 0$ ) and Eq. (6) therefor reduces to a 2D ( $x$  and  $y$  dependent) problem where  $R(x, y)$  is the unknown field. Its solution yields  $R^1(x, y)$ , a first approximation of  $R(x, y)$ . Then by approximating  $R(x, y)$  by  $R^1(x, y)$  in Eq. (6) we similarly obtain a 1D problem ( $z$  dependent) which allows us to compute  $S^1(z)$  the next approximation of  $S(z)$ .

This fixed point loop keeps running until  $R^i(x, y)$  and  $S^i(z)$  are stabilized:

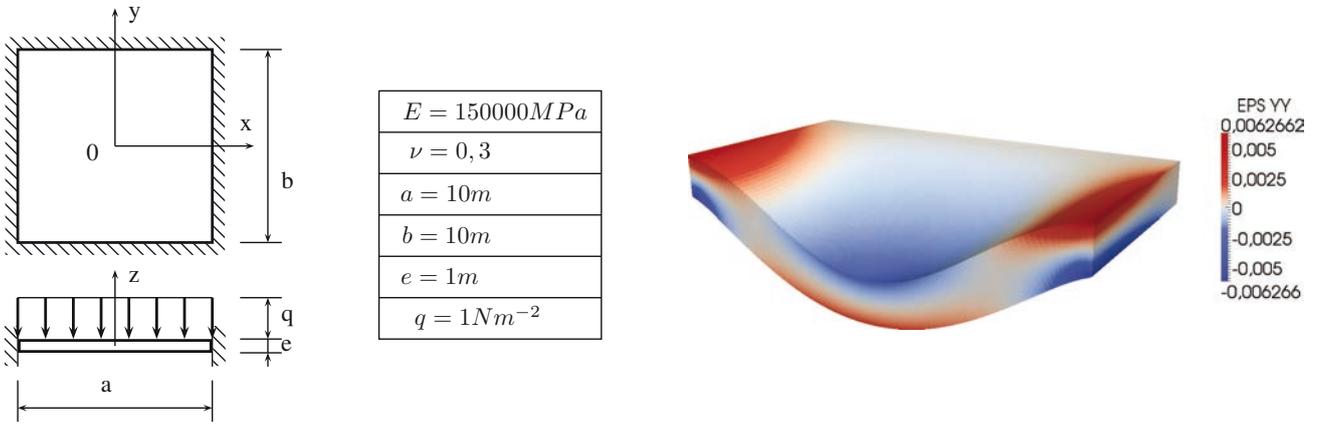
$$\int_{x,y,z} \left( R^{(j)}(x, y) \cdot S^{(j)}(z) - R^{(j-1)}(x, y) \cdot S^{(j-1)}(z) \right)^2 dx dy dz \leq \varepsilon, \quad (7)$$

$\varepsilon$  is chosen close to the machine precision in order to make  $R^{(j)}(x, y) \cdot S^{(j)}(z)$  and  $R^{(j-1)}(x, y) \cdot S^{(j-1)}(z)$  numerically indistinguishable. The solution is enriched with new modes until  $e_i/e_0 \leq \varepsilon_2$ , where  $e_j$  is the norm of the residue of Eq. (2) with  $\underline{u} = \underline{u}_j$ .  $\varepsilon$  controls the convergence of each mode in the fixed point algorithm while  $\varepsilon_2$  controls the enrichment procedure and therefor the global accuracy of the solution.

## ISOTROPIC LINEAR ELASTICITY ON RECTANGULAR PLATE

To validate the proposed solution method, we compare the classical 3D FEM solution of a square plate problem to the PGD solution with an equivalent discretization.

Let us consider a square plate with all edges built-in (Fig. 2). The applied load is a uniform pressure imposed on the upper face. For this example we used a regular 100x100x50 mesh of trilinear elements for the 3D FEM modeling while 100x100 bi-linear elements in the plane, and 50 linear elements in the thickness were used for the



**FIGURE 2.** Reference problem and half of the solution calculated with PGD

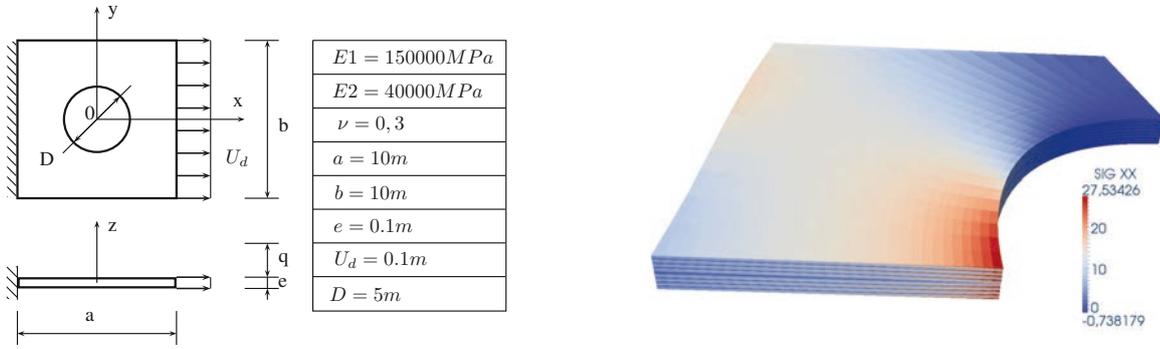


**FIGURE 3.** Relative error in strain energy density between 3D FEM and PGD based method

PGD approach. The reference 3D solution is computed with the Castem [4] software, and the PGD solution is computed with the Matlab [5] software.

The solution found with the separated space variables PGD method is presented in Fig. 2. On this simple problem, the edge effect is clearly visible, and confirms the necessity of several modes to correctly describe the solution. Furthermore, depending on geometrical and material characteristics those effects can penetrate more or less into the structure possibly making their contribution more important.

The PGD solution is composed of 9 modes and a total of 165 fixed point iterations was necessary to converge to the final solution. on Fig. 3, we illustrate the accuracy of the PGD solution by plotting the relative strain energy density error between the PGD and the FEM solutions. This error is everywhere lower than 0.3 percent, excepted on the corner elements where it reaches 0.57 percent. Thanks to the separation of the problems in a 2D and in a 1D problem, the interpolation in those two dimensions can be of a very different nature. In our case, Finite Element approximation is taken for both. Even if the interpolation is of the same type for the two problems, the discretization can be very different in term of characteristic length in the two dimensions while in case of 3D FEM, elements have to keep a shape close to the reference element. For example we can easily consider a very fine mesh in the plate thickness while keeping a coarse mesh on the plate surface.



**FIGURE 4.** load applied on the plate and quarter of the PGD solution

## APPLICATION TO A LAMINATED COMPOSITE PLATE WITH HOLE

A very large number of laminated composite plates has a constant thickness, because of the uniform number of plies in the structure. Moreover, the orthotropic direction is uniform in each layer in case of planar draping. In this case, it is easy to apply the previously presented method on laminated composite plates. Indeed, in the absence of localized additional plies, the material parameters are only function of the thickness, so those parameters can be integrated in the 1D PGD subproblems. Let us consider a square plate of with a hole in its center (See Fig 4 for details). This plate is built-in on one side, and submitted to a forced displacement on the opposite side. The plies are stacked as:  $[0,45,-45,90]_{4s}$ . The complex structure of the stress field along the plate thickness is illustrated in Fig. 4.

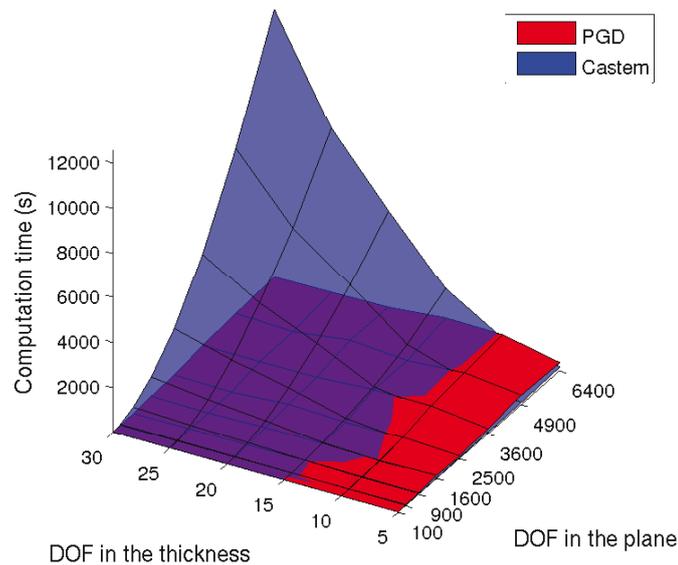
Laminated composite plates are typically the kind of problems which are interesting to solve with the PGD. Indeed, in such cases a large number of degrees of freedom (DOF) are necessary in the thickness to be able to capture the solution at the scale of a layer. The PGD method is highly appropriate in those cases because a full 3D solution with so many DOF would be prohibitive.

In Fig. 5, we show how the PGD method overcomes the classical 3D FEM method as the number of DOF increases in the plate thickness. We compare the raw computation times of 3D FEM Castem solution and of our PGD method, programmed in Matlab. In the interest of simplicity the problem solved here is the isotropic square plate studied in the previous section. Although it is difficult to compare raw timings, it is clearly visible that, as the number of DOF in the plate thickness increases, the PGD computation time increases only slightly while the FEM computation time simply blows up.

## CONCLUSION

We have shown that it is possible to solve three dimensional linear elasticity problems with the PGD method on plate domains. Naturally, the solution is separated between the thickness variable  $z$  and the plane variables  $(x,y)$ . Using a fixed point algorithm to compute the different modes of the solution, we show the convergence of the PGD solution to the finite element solution of the full 3D problem.

The main advantage of the proposed approach is that no three-dimensional problem is



**FIGURE 5.** Comparison of computation times of PGD and Castem full 3D solution of the same problem

ever solved. Only two-dimensional and one-dimensional averaged problems are solved (possibly many times), yielding significant savings in computational and memory costs with respect to the FE solution.

Although the method might not be of particular interest for homogeneous isotropic materials, it reaches its full potential when applied to laminated composite plates, where the material parameters vary only along the plate thickness. In this case a fine discretization is needed in the thickness dimension, which renders the FE solution horrendously prohibitive compared to the PGD.

## REFERENCES

1. A. Ammar, B. Mokdad, F. Chinesta, and R. Keunings, *Journal Of Non-Newtonian Fluid Mechanics* **139**, 153–176 (2006).
2. S. Timoshenko, and S. Woinowsky-Krieger, *Theory of plates and shells*, McGraw-Hill, 1959, second edition edn.
3. F. Chinesta, A. Ammar, and E. Cueto, *Archives of Computational Methods in Engineering* **In press** (2009).
4. *Commissariat à l’Energie Atomique CEA - DEN/DM2S/SEMT* (2009), URL <http://www-cast3m.cea.fr>.
5. M. Inc., *MATLAB The Language of Technical Computing: MATLAB Application Program Interface Guide* (1998).