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# Optimisation de contrôles CND lors de l'auscultation d'un champ stationnaire de propriétés aléatoires

# Optimization of Non Destructive testing when assessing stationary stochastic processes

**Trung Viet Tran*, Franck Schoefs*, Emilio Bastida-Arteaga*, Geraldine Villain** Xavier Derobert****

** LUNAM Université, Université de Nantes,*
*Institute for Research in Civil and Mechanical Engineering, CNRS UMR 6183/FR 3473, Nantes, France*
*Franck.Schoefs@univ-nantes.fr*
*** LUNAM Université, IFSTTAR, MACS, Nantes, France*

ABSTRACT. *The localization of weak properties or bad behavior of a structure is still a challenge for Non Destructive Testing (NDT) tools development and Structural Health Monitoring (SHM) design. In case of random loading or material properties, this challenge is arduous because of the limited number of measures and the quasi-infinite potential positions of local failures. Deterministic algorithms and specific multi-sensors systems are developed to this aim. In that case, generally, the precision of the positioning of a defect goes with a lack of sizing. However, the stochastic field is rarely a pure white noise and has a stochastic structure and probabilistic properties. These properties should be used to provide rational aid tools for optimizing the number of sensor. The paper shows that the stationary property is sufficient to find the minimum quantity of sensors and their position. A measure of the quality of information is suggested and the illustration is performed on a one-dimensional Gaussian stochastic field.*

KEYWORDS: *NDT, stochastic field, reliability, random porperties.*

## 1. Introduction

Structural health monitoring is well recognized since at least two decades to provide valuable information for:

- structural model updating;
- material property updating;
- monitoring of degradation and maintenance optimization;
- loading analysis and modelling;
- survey of critical quantities in the structure.

The scope of the paper takes place in this last family where a decision should be made from a set of measure of a material property (yield strength, elasticity modulus) or a mechanical quantity of interest (strain, stress) Z. In a lot of cases, the material properties are random and two questions must be addressed: where is the defect and what is the probability of this event. Then the decision lies on a risk analysis that combines this measure of probability with the subsequent potential consequences, directly or after a structural reliability computation. In this last case, the question of random variable updating has been widely addressed during the two last decades. Random variable updating is very useful when data from inspections or monitoring are collected for condition assessment and reliability updating (Straub and Faber, 2003; Schoefs et al, 2012). Basically, the Bayes theorem and its derivative tools (Bayesian Networks) offer the theoretical context to deal with this issue. The so-called Risk Based Inspection (RBI) generalizes these approaches in the case of non-perfect inspections by linking inspection and decisions Faber, 2002; Sorensen and Faber, 2002; Schoefs et al, 2011). RBI methods are powerful once (i) there is no stochastic field involved into the problem, or (ii) the location of the most critical defect, from a reliability point of view, is known. In circumstance (i), if the field of Z is a white noise, the number of measures should tend to infinity. In practical cases, the building of large structures (soil, concrete, composite) generates a stochastic field for Z and its probabilistic properties should be used. Recently authors have combined structural reliability computation and stochastic field description in view to carry out a complete reliability analysis Stewart and Al-Harthy, 2008; Tran et al, 2012). The input of these works is the complete description of the stochastic field.

No work combines actually the two challenges: description of the stochastic field with a limited number of data and the structural reliability. This paper aims to suggest a methodology for the first one. The stochastic field could take several forms more or less complicated. The most simple is the stationary stochastic field that can be used, for instance, to model chloride distribution or other concrete properties (Bazant and Novak, 2000a; 2000b; Bazant and Xi, 1991). Other models suggest piecewise stochastic fields (Schoefs et al, 2009a, 2009b). The complexity comes from hazards B during building itself and from the spatial distribution of external factors E -e.g. environmental conditions. Within this context, the main objective of this work is to find the optimal geo-position of sensors and their number to satisfy a given level of quality when a stationary field can describe the stochastic field and the noise of measurement can be neglected.

## 2. Probabilistic modeling of measurements in case of spatial variability

### 2.1. Scope of the paper: stakes and limits

The interest of SHM checked out by this paper is to get a direct (so called non-model based) decision from the measurement and to detect:

- even the worst case;
- or the distribution of the worst cases (probabilistic distribution tails).

The quantity of interest $Z(B(\mathbf{x},\ \theta),E(\mathbf{x},\ \theta))$ is supposed to be dependent of the hazard during building represented by the stochastic field $B(\mathbf{x},\ \theta)$ and external factors $E(\mathbf{x},\ \theta)$ where $\mathbf{x}$ denotes the vector of position and q is the hazard: $\theta \in \Omega$ where $\Omega$ is the probabilistic space supposed to generates all the event that influence even $B(\mathbf{x},\ \theta)$ or $E(\mathbf{x},\ \theta)$. For simplicity, it is written $Z(\mathbf{x},\ \theta)$ in the following. We note $\hat{z}(\mathbf{x},\theta_i)$ one realization after measurement. When the quantity of interest is spatially dependent and no additional information is available for characterizing the potential position of a weak region, the question is to select at which position the measurement should be done. When Non Destructing Testing tools are carried out during service life, adjustments of the protocol (position, setting of the device…) can be suggested progressively with time.

When we have to design an embedded network of sensors (scope of the present paper) the device should be the solution of an optimization problem where the quality of the data encourages increasing the number of sensors when the cost reduction tends to limit their quantity.

In this paper we consider that a solution can be obtained only if the structure of spatial variability is known. He we assume that the stochastic field is stationary (probability density function is the same whatever the location) with a known fluctuation parameter (correlation parameter) but an unknown probability density function. So we assume:

- the measured quantity can be modeled with a stationary stochastic field: as a consequence, $\mu_z$ and $\sigma_z$ are constant whatever $\mathbf{x}$ and parameter $\mathbf{x}$ will be used when necessary only.
- the stochastic field is assumed to be Gaussian i.e. the considered property is normally distributed whatever the position;
- the measure is perfect in the sense that statistics moments computed from the set of $\hat{z}(\theta_i)$ , $i \in [1,...,N]$ tend to the probabilistic moments of $Z(\theta)$ when $N \rightarrow \infty$;
- a quality of the measurement can be expressed even on the form [1], [2] or [3]:

*Confidence interval*, when the distribution of values around the mean value is focussed on:

$$\frac{P_{S,\hat{z}}}{P_{th}} \geq p_a, P_{I,\hat{z}} = P\left(\hat{z} \in \left[\mu_Z - \varepsilon\mu_Z ; \mu_Z + \varepsilon\mu_Z\right]\right) \quad [1]$$

where $P_{S,\hat{z}}$ is the probability computed from the sensor measurements, $P_{th}$ the theoretical probability (implicitly non null), and pa denotes the minimum acceptable probability to get a measurement $\hat{z}$ inside a given range governed by the exact value of the expectation µz and an error around this mean value computed by the percentage ε. Note that $\hat{z}$ can be replaced by its statistics like the mean value or standard deviation when statistical error (i.e. samples of small size) is investigated (Schoefs et al, 2011a; Tran et al, 2011).

*Distribution of extreme values, left side*, when extreme low values (strength) is analysed,

$$\frac{P_{S,\hat{z}}}{P_{th}} \leq p_a, P_{I,\hat{z}} = P\left(\hat{z} \leq \mu_Z - \varepsilon\mu_Z\right) \quad [2]$$

*Distribution of extreme values*, right side, when extreme high values (stress) is studied,

$$\frac{P_{S,\hat{z}}}{P_{th}} \geq p_a, P_{I,\hat{z}} = P\left(\hat{z} \leq \mu_Z + \varepsilon\mu_Z\right) \quad [3]$$

Equations [1], [2] and [3] are compatible with a lot of numerical post-treatment algorithms for detection assessment.

Note that, at this stage no mechanical model is s-used (model free or non model based approach) and we consider a one-dimension (1D) field for illustration. Of course the methodology can be expanded to any stationary stochastic fields: 2Dor 3D.

Let us assume that we get a set of independent realizations –i.e. structural components- $\hat{z}(\theta_i)$ of $Z(\theta)$. From a huge number of data (N≈1000), $P_{S,\hat{z}}$ can be estimated directly from the frequency of measurements only if it is not too small. When only a limited number of data is available (N<100), we have to assess the empirical distribution from a numerical sampling knowing the empirical distribution: it can be reached by using Monte Carlo Markov Chain. The number of components being limited the stake will be to get independent realization $N_s$ on a given component and to consider a set of components $N_t$.

Here we compute $P_{S,\hat{z}}$ by considering the probability density function with parameters $\mu_{\hat{Z}}$ and $\sigma_{\hat{Z}}$ computed from the empirical distribution of measurements:

$$\mu_{\hat{Z}}(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^{N} \hat{z}(\mathbf{x}, \theta_i) ; \sigma_{\hat{Z}}(\mathbf{x}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{z}(\mathbf{x}, \theta_i) - \mu_{\hat{Z}}(\mathbf{x})\right)} \quad [4]$$

This variable is denoted $\hat{Z}$ . In the following, the challenge will be to consider a model of spatial variability and to assess a set of independent realizations of $Z(\theta)$

## 2.2. *Spatial variability*

Risk Based Inspection analysis or reliability methods applied to real structures generally assume:

- either there is no spatial variability involved in the problem: random variables allow us to describe the hazard involved;
- or the location of the most critical defect from reliability point of view is known and the distribution of defects in its neighboring doesn't affect the reliability.

It is well known that the reality is more complex and that we should account for stochastic fields too. Then the stochastic field could take several forms more or less complicated:

- (i) the most simple is the stationary stochastic field that is able to model the chloride distribution or other properties in the concrete for instance (Bazant and Novak, 2000a; 2000b; Bazant and Xi, 1991);
- (ii) more sophisticated is the piecewise stationary process that can integrate the variability of the con-creating by steps or the corrosion of structures in contiguous but different environments;
- (iii) finally, fully non stationary fields are certainly the most acceptable for a fine representation of properties.

However, except for natural soils, materials used for construction (airplanes, bridges …) are produced following a quality process and control. We can consider that some variation are fair, for instance the spatial change of the mean value. This paper focuses on the first model (i) only.

In this paper, we used a Karhunen–Loève expansion to represent the spatial variability with this assumption of stationary (Schoefs et al, 2011c):

$$Z(\mathbf{x},\theta) = \mu_Z + \sigma_Z . \sum_{i=1}^{n} \sqrt{\lambda_i} . \xi_i(\theta) . f_i(\mathbf{x}) \qquad [5]$$

where, $n$ is the number of terms in the expansion, $\xi_i$ is set of centered reduced Gaussian random variable (standard normal variables), $\lambda_I$ and $f_i$ are respectively the eigenvalues and eigenfunctions of the covariance function: $\rho(\Delta\mathbf{x})$ . The major interest of this representation is that $\lambda_i$ and $f_i$ have analytical expressions for specific forms of the correlation function.

For instance, let us consider a one-dimensional (in space) stochastic field ($\mathbf{x}$=$x$)withan exponential form of correlation function as follows:

$$\rho(\Delta x) = \exp\left(-\frac{\Delta x}{b}\right);\ with\ \ b > 0 \qquad [6]$$

Then, it is shown that eigenvalues and eigenfunctions $\boldsymbol{\lambda}_i$ and $f_i$ have analytical expressions (see Tran et al. 2012b):

## 2.3. *Assessment of the autocorrelation function from measurements*

We assume that the stationary stochastic field can be characterized by an autocorrelation function (ACF). Table1 presents the most usual ACF considered for spatial variability of structures with their parameter, called scale of fluctuation $\delta$. A complete overview of the auto-correlation functions and their application is available in Kenshel (2009). Let us focus on the assessment of this function from experimental data (sensors or NDT tests).

Two major procedures have been reported in the literature for the estimation of $\delta$ for a spatially variable property from a digitized record of data. In the first procedure, reported by Li (2004), the Maximum Likelihood Estimate method (MLE) is used in which different values for the model parameter of the proposed ACF model is assumed and the value that maximizes the corresponding MLE is taken as the model parameter. In the second procedure, proposed by Vanmarcke (1983), a proposed ACF model (Tran et al. 2012b) can be adjusted to provide the best fit to the actual sample correlation coefficients $\rho(\Delta x)$ thereby providing estimates of the corresponding model parameter (i.e. $b$ in [6]).

In this paper, we select an exponential ACF [6].

and we use the likelihood estimate for the estimation of $b$.

$$L = \prod_{i=1}^{k}\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\nu_i^{\ 2}}{2}\right)\right) = \left(\frac{1}{\sqrt{2\pi}}\right)^k \exp\left(-\frac{\sum_{i=1}^{k}\nu_i^{\ 2}}{2}\right) \qquad [7]$$

where $\nu_i$ is the *ith* component of the vector of independent standard values obtained from equation:

$$\nu = C^{-1}\left(\frac{z-\mu_Z}{\sigma_Z}\right) \qquad [8]$$

where $\mathbf{z}$ is the vector of realizations of the random variable Z and C a lower triangular matrix such that $CC^T = \boldsymbol{\rho}$ and $\boldsymbol{\rho}$ the autocorrelation matrix. Beside, maximize $L$ is equivalent to minimize $L_1$:

$$L_1 = \sum_{i=1}^{k} \nu_i^{\ 2} \qquad [9]$$

## 3. Optimization of geo-position of sensors

### 3.1. *Optimization problem description*

In this paper, we consider a one-dimensional mechanical problem with a set of sensors to be equally distributed with distance $\delta_l$ on $N_t$ structures of length $L$(beams of a bridge, cables, wing of an airplane, …). The optimization problem is written as a minimization problem of the number of sensors for a structure with a finite length $L$. In fact the question is to find the minimum number of sensors $N_s$ in each component knowing the number of components $N_t$ under one of the constraints [1] to [3].

$$\overline{N}_S = \min\left(N_s \middle| (1) or (2) or (3)\right) \qquad [10]$$

In a complete risk analysis it can be written:

$$N = N_S * N_t = \underset{Ns,Nt}{\operatorname{argmin}}(E(C)) \qquad [11]$$

where E(C) denotes the expectation of the cost (Schoefs, 2009).

### 3.2. Study case

In this part, we consider the optimization of position and number of inspection for assessing the volume water content W (%) in a reinforced concrete beam by capacitive method (CAPA). Its main characteristics are 16 meters length, 1 meter height and 0.4 meter width. A grid has been drawn on each lateral surface and 2 lines of measurement have been selected: distance from the top line and the first line of measurements is 7 cm and distance between two lines is 20 cm. Distance between each measurement on one line is 20 cm.

### 3.3 Data analysis

With CAPA method, we get a value of the frequency F. The difference ΔF between the frequency in air and on concrete can be related to the permittivity of concrete from a calibration function given by IFSTTAR (Schoefs et al. 2012): it depends mainly on the size of the electrode. He we used a "great electrode" without Eccostock and calibration function is given in [12].

$$\Delta F = -22.7937 Eps + 26.9711 \quad [12]$$

where $\Delta F = F_c - F_{air}$, with $F_c$ and $F_{air}$ values of frequencies in RC beam and in the air on site. And, volume water content $W$ can be deduce from permittivity by a second calibration function [13] (see Schoefs et al. 2012).

$$Eps = 0.64W + 4.61 \quad [13]$$

Figure 1 presents the distribution of W after post-treatments for records along a line (left) and the spatial distribution of theses data (right). These data correspond to the mean value of 30 repetitive tests at each position. The scatter is low in comparison to the inherent scatter of the measurement: Schoefs et al. (2012) have recorded a standard deviation up to 2.5 for repetitivity tests.

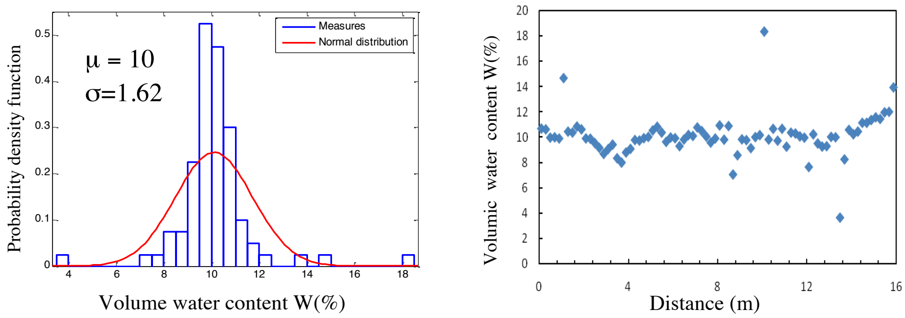


**Figure 1**: *Spatial variability of measure W and its distribution at line B*

### 3.4 Results of optimization

For the structure considered in this paper, very close values of $L_l$ [9] are obtained for the two lines: 1.63 and 1.61m respectively for the B and the A line.

For this paper, we focus to optimization positioning and number of captor on the RC beam when considering a stochastic stationary Gaussian field of volume water content $W$ with its parameter $\mu_W = 10$ and $\sigma_W = 1.62$**.**

- length of beam: L = 16 m.
- parameter of autocorrelation function: b = 1.62 m.

And the distance between two captors varied from 0.1 to 5m.

First, let us consider the case with repetitive tests (i.e. without noise of measurement), and constrain [1] were $\varepsilon$=1(%). For a normally distributed random variable $P_{th}=0.463$ and $P_{S,\hat{z}} \geq 0.42$ . Figure 2 presents the variations of $P_{S,hc}$ with the number of components (beams) $\boldsymbol{N_t}$ and distance between inspections $\boldsymbol{\delta_I}$; $\boldsymbol{\delta_I}$ varying from 0.16 ($N_s$=100) to 4 ($N_s$ =4). This figure shows that the probability increases according to the number of component $N_t$ and decreases according to distance of $\boldsymbol{\delta_I}$. Based on this result, for a given accepted value $P_{S,\hat{z}}$ , we obtain a curve that links the acceptable couples ($\delta_I$; $N_t$) (see red line in Figure 1 for $P_{S,\hat{z}}$=0.42). Note that if $N_t$ is too small there could be no solution (for instance for $P_{S,\hat{z}}$=0.42).

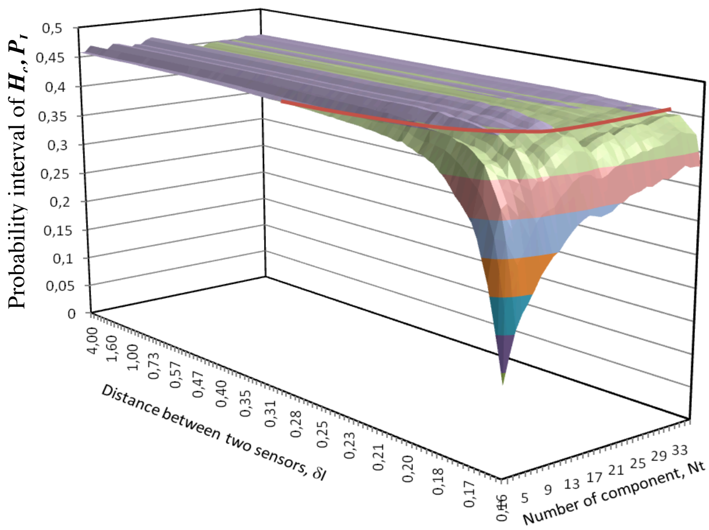


**Figure 2***: Influence of* $N_t$ *and* $\delta_I$ *in probability interval* $P_{S,\hat{z}}$ *of W*

Figure 3 presents this curve for PI=42% and its fitting with a power function (red line). Based on this result, given Nt, we obtain the distance between two captors δI following [14]:

$$\delta_I = 0.921(N_t)^{0.4327} \quad [14]$$

and the position and minimum number of sensors are deduced. For example, for Nt=10,δI=2.2m and Ns=16/2.2 = 7.27 measure on each component, and the total number of sensor is: N=10*7=70 captors.

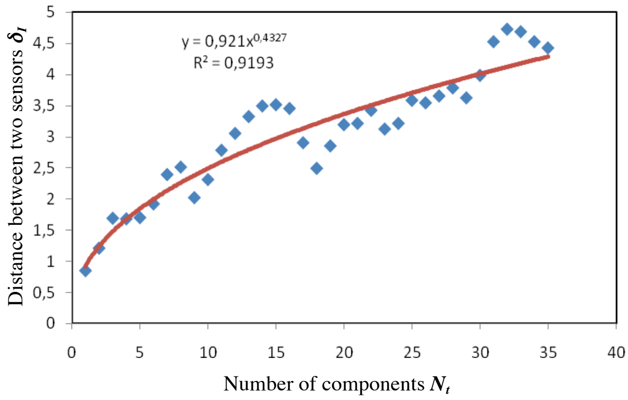


**Figure 3:** *The curve of* $N_t$ *and* $\delta_I$ *satisfying* $P_I$ *=42%*

Let us considered now the absence of repetitive tests: noise of measurement exists (see Rouhan et al. 2003 for definition). Based on preliminary results (Schoefs et al. 2012) we focus study on two value of the standard deviation of a centered normally distributed withe noise: 0.5 and 2.5. The later figure presents this result for the case $\varepsilon=3\sigma_W$ = 4.68 and for $P_I$=90% where we show the predominant role of the noise especially when it is higher than the standard deviation of the signal itself..

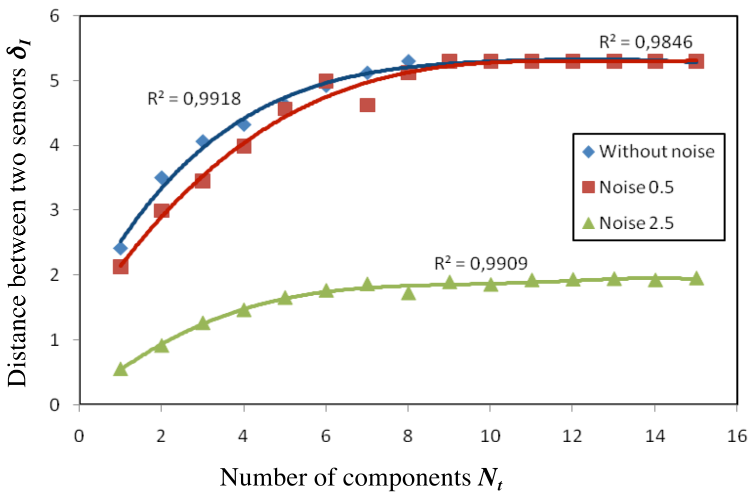


**Figure 4:** *The curve of* $N_t$ *and* $\delta_I$ *considering the noise measurement satisfying* $P_I$ *=90%*

**4. Conclusions.**

This papers aim to focus on the first step of a work dealing with uncertainty in measure assessment and intrinsic spatial variability. Only the case of stationary Gaussian fields of properties is investigated herein. It is shown how to model a stationary stochastic field and to follow up this property to get a good representation of a random variable from a limited number of NDT measurements. We developed an illustration based on capacitive measurements with and without error of measurements. This result can be exploited in further reliability updating once the first measurements are available.

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