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## MULTISCALE MODEL REDUCTION FOR THE SOLUTION OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH LOCALIZED SOURCES OF UNCERTAINTIES

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### ABSTRACT

*The presence of numerous localized sources of uncertainties in stochastic models leads to high dimensional and multiscale problems. A numerical strategy is here proposed to propagate the uncertainties through such models. It is based on a multiscale domain decomposition method that exploits the localized side of uncertainties. The separation of scales has the double benefit of improving the conditioning of the problem as well as the convergence of tensor based methods (namely Proper Generalized Decomposition methods) used within the strategy for the separated representation of high dimensional stochastic parametric solutions.*

### 1 INTRODUCTION

The propagation of uncertainties through multiscale stochastic models remains today a challenging issue. In the presence of numerous localized sources of uncertainties, dedicated approaches have to be developed in order to handle the high dimensionality and complexity of associated multiscale models. We here propose a numerical strategy based on a multiscale domain decomposition method that exploits the localized side of uncertainties and incidentally improves the conditioning of the problem by operating a separation of scales. An efficient iterative

algorithm is proposed that requires the solution of a sequence of simple global problems at a macro scale, involving a deterministic operator, and local problems at a micro scale for which we have the possibility to use fine approximation spaces. This method is an extension of a “numerical zoom” method dedicated to multimodel problems that has been proposed in [1] in the deterministic framework. Global and local problems are solved using tensor based approximation methods, here the Proper Generalized Decomposition methods [2, 3], allowing the representation of high dimensional stochastic parametric solutions. Convergence properties of these tensor based methods, which are closely related to spectral decompositions [4], benefit from the separation of scales. Different types of uncertainties are considered at the micro level. They may be associated with some variability in the operator or source terms, or even with some geometrical variability. In the latter case, specific reformulations of local problems using fictitious domain methods are introduced [5].

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## 2 PROBLEM WITH LOCALIZED VARIABILITIES

### 2.1 Model problem

We consider a diffusion problem defined on a domain  $\Omega \subset \mathbb{R}^d$ :

$$\begin{aligned} -\nabla \cdot (K \nabla u) &= f \quad \text{on } \Omega \\ K \nabla u \cdot n &= g \quad \text{on } \Gamma_N \\ u &= 0 \quad \text{on } \Gamma_D \end{aligned} \quad (1)$$

where  $K$  is a diffusion parameter,  $\Gamma_D$  and  $\Gamma_N$  are the Dirichlet and Neumann boundaries respectively. We introduce a set of random parameters  $\xi$  with values in  $\Xi$  that model the uncertainties in the problem, and  $(\Xi, \mathcal{B}, P_\xi)$  is the associated probability space, where  $P_\xi$  is the probability law of  $\xi$ .

### 2.2 Initial weak formulation of the problem

We introduce the Hilbert space  $\mathcal{V} = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_D\}$  equipped with the inner product norm  $|u|_\Omega = (\int_\Omega \nabla u \cdot \nabla u)^{1/2}$ . Let  $\mathcal{V}^\Xi = L_{P_\xi}^2(\Xi; \mathcal{V})$  be the Hilbert space equipped with the norm  $\|\cdot\|_\Omega = \mathbb{E}(|\cdot|_\Omega^2)^{1/2}$  where  $\mathbb{E}$  is the mathematical expectation:  $\mathbb{E}(v) = \int_\Xi v(y) dP_\xi(y)$ .

The classical weak formulation of problem (1) writes:

$$u \in \mathcal{V}^\Xi, \quad a_\Omega(u, \delta u) = \ell_\Omega(\delta u) \quad \forall \delta u \in \mathcal{V}^\Xi \quad (2)$$

with

$$\begin{aligned} a_\Omega(u, \delta u) &= \mathbb{E} \left( \int_\Omega K \nabla u \cdot \nabla \delta u \right) \\ \ell_\Omega(\delta u) &= \mathbb{E} \left( \int_\Omega f \delta u + \int_{\Gamma_N} g \delta u \right) \end{aligned}$$

### 2.3 Patch containing localized variabilities

We consider the case where the sources of uncertainties are located inside a part  $\Lambda \subset \Omega$ ,  $\Lambda$  is called a patch. The localized uncertainties can stem from the diffusion coefficient  $K$ , the source term  $f$  or even an uncertain part of the boundary  $\partial\Omega$  so that  $\Lambda$  may depend on  $\xi$  and is such that

$$\Omega(\xi) = (\Omega \setminus \Lambda) \cup \Lambda(\xi)$$

with  $\Omega \setminus \Lambda$  deterministic. Let  $\Gamma = \partial(\Omega \setminus \Lambda) \cap \partial\Lambda$  be the deterministic interface between  $\Omega \setminus \Lambda$  and the patch  $\Lambda$ , as represented in Fig. 1.

Therefore for  $x \in \Omega \setminus \Lambda$  we have  $K(x, \xi) = K_0(x)$  and  $f(x, \xi) = f_0(x)$ .

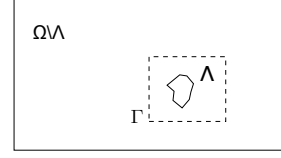


FIGURE 1. Patch  $\Lambda \subset \omega$  with localized variabilities.

### 2.4 Domain decomposition

The solution  $u \in \mathcal{V}^\Xi$  is split as follows

$$u = \begin{cases} U & \text{on } \Omega \setminus \Lambda \\ w & \text{on } \Lambda \end{cases}$$

where  $U$  and  $w$  verify the following continuity conditions on  $\Gamma$  interpreted in a weak sense:

$$w = U \quad \text{and} \quad \frac{\partial U}{\partial n} = \frac{\partial w}{\partial n} \quad \text{on } \Gamma$$

We consider that  $\Gamma \cap \Gamma_D = \emptyset$  and we split the Dirichlet boundary  $\Gamma_D$  into the Dirichlet boundary of the patch  $\Gamma_D^\Lambda = \Gamma_D \cap \partial\Lambda$  (eventual internal boundary of the patch) and the complementary part  $\Gamma_D^{\Omega \setminus \Lambda} = \Gamma_D \setminus \Gamma_D^\Lambda$ . We then introduce the following spaces

$$\begin{aligned} \mathcal{U} &= \{U \in H^1(\Omega \setminus \Lambda); U = 0 \text{ on } \Gamma_D^{\Omega \setminus \Lambda}\} \\ \mathcal{W} &= \{w \in H^1(\Lambda); w = 0 \text{ on } \Gamma_D^\Lambda\} \\ \mathcal{M} &= H^{-1/2}(\Gamma) \end{aligned}$$

and the associated spaces  $\mathcal{U}^\Xi$ ,  $\mathcal{W}^\Xi$  and  $\mathcal{M}^\Xi$ , where for a Hilbert space  $\mathcal{H}$ ,  $\mathcal{H}^\Xi$  denotes the Bochner space  $L_{P_\xi}^2(\Xi; \mathcal{H})$ . Problem (2) is then equivalent to finding  $(U, w, \lambda) \in \mathcal{U}^\Xi \times \mathcal{W}^\Xi \times \mathcal{M}^\Xi$  such that for all  $(\delta U, \delta w, \delta \lambda) \in \mathcal{U}^\Xi \times \mathcal{W}^\Xi \times \mathcal{M}^\Xi$ ,

$$\begin{cases} a_{\Omega \setminus \Lambda}(U, \delta U) + b_\Gamma(\lambda, \delta U) = \ell_{\Omega \setminus \Lambda}(\delta U) \\ a_\Lambda(w, \delta w) - b_\Gamma(\lambda, \delta w) = \ell_\Lambda(\delta w) \\ b_\Gamma(\delta \lambda, U - w) = 0 \end{cases} \quad (3)$$

where for one subdomain  $\mathcal{O} \subset \Omega$ ,

$$a_{\mathcal{O}}(v, \delta v) = \mathbb{E} \left( \int_{\mathcal{O}} K \nabla v \cdot \nabla \delta v \right), \quad \ell_{\mathcal{O}}(v) = \mathbb{E} \left( \int_{\mathcal{O}} f v \right)$$

and where

$$b_\Gamma(\lambda, v) = \mathbb{E} \left( \int_\Gamma \lambda v \right) \quad (4)$$

$\lambda$  represents a Lagrange multiplier ensuring the weak continuity of  $u$  on the interface  $\Gamma$ .

### 3 A GLOBAL-LOCAL ITERATIVE ALGORITHM WITH OVERLAPPING DOMAINS

#### 3.1 Reformulation with overlapping domains

In the case where  $\Lambda$  contains geometrical details, *i.e.* if  $\partial\Lambda \setminus \Gamma \neq \emptyset$ , we introduce a fictitious patch  $\tilde{\Lambda} \supset \Lambda$  such that  $\partial\tilde{\Lambda} = \Gamma$  (see Fig. 2).  $\tilde{\Lambda}$  is then deterministic and does not contain any internal boundaries (no geometrical details). We also define the deterministic fictitious domain  $\tilde{\Omega} = (\Omega \setminus \Lambda) \cup \tilde{\Lambda}$  which contains  $\Omega$  and is such that  $\tilde{\Omega} \setminus \tilde{\Lambda} = \Omega \setminus \Lambda$ .

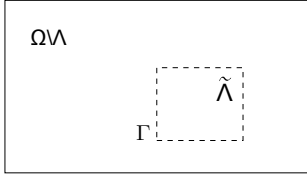


FIGURE 2. Domain with fictitious patch

We now consider a prolongation of function  $U$  from  $\Omega \setminus \Lambda$  to  $\tilde{\Omega}$ . We introduce on  $\tilde{\Omega}$  a fictitious diffusion field  $\tilde{K}$  such that

$$\tilde{K} = K \quad \text{on} \quad \Omega \setminus \Lambda \quad (5)$$

We then introduce the new function space  $\tilde{\mathcal{U}}^\Xi$  with

$$\tilde{\mathcal{U}} = \{U \in H^1(\tilde{\Omega}) ; U = 0 \text{ on } \Gamma_D^{\Omega \setminus \Lambda}\}$$

For a subset  $\mathcal{O} \subset \tilde{\Omega}$ , we define a bilinear form  $c_{\mathcal{O}} : \tilde{\mathcal{U}}^\Xi \times \tilde{\mathcal{U}}^\Xi \rightarrow \mathbb{R}$  associated with the fictitious diffusion coefficient:

$$c_{\mathcal{O}}(U, \delta U) = \mathbb{E} \left( \int_{\mathcal{O}} \tilde{K} \nabla U \cdot \nabla \delta U \right)$$

We notably have

$$a_{\Omega \setminus \Lambda}(U, \delta U) = c_{\Omega \setminus \Lambda}(U, \delta U) = c_{\tilde{\Omega}}(U, \delta U) - c_{\tilde{\Lambda}}(U, \delta U) \quad (6)$$

Problem (3) can now be reformulated as follows: find  $(U, w, \lambda) \in \tilde{\mathcal{U}}^\Xi \times \mathcal{W}^\Xi \times \mathcal{M}^\Xi$  such that for all  $(\delta U, \delta w, \delta \lambda) \in \tilde{\mathcal{U}}^\Xi \times \mathcal{W}^\Xi \times \mathcal{M}^\Xi$ ,

$$\begin{cases} c_{\tilde{\Omega}}(U, \delta U) - c_{\tilde{\Lambda}}(U, \delta U) + b_{\Gamma}(\lambda, \delta U) = \ell_{\Omega \setminus \Lambda}(\delta U) \\ a_{\Lambda}(w, \delta w) - b_{\Gamma}(\lambda, \delta w) = \ell_{\Lambda}(\delta w) \\ b_{\Gamma}(\delta \lambda, U - w) = 0 \end{cases} \quad (7)$$

#### 3.2 Global-Local algorithm

For the solution of problem (7), we introduce an iterative algorithm that defines a sequence  $\{(U_n, w_n, \lambda_n)\}_{n \geq 0}$ . Starting with  $w_0 = 0$ ,  $\lambda_0 = 0$  and  $U_0 = 0$  and for  $n \geq 1$ , we define  $(U_n, w_n, \lambda_n)$  as follows.

**Global step** We compute  $\hat{U}_n \in \tilde{\mathcal{U}}^\Xi$  such that for all  $\delta U \in \tilde{\mathcal{U}}^\Xi$ ,

$$c_{\tilde{\Omega}}(\hat{U}_n, \delta U) = c_{\tilde{\Lambda}}(U_{n-1}, \delta U) - b_{\Gamma}(\lambda_{n-1}, \delta U) + \ell_{\Omega \setminus \Lambda}(\delta U) \quad (8)$$

A relaxation step is introduced

$$U_n = \rho \hat{U}_n + (1 - \rho) U_{n-1}$$

with  $\rho > 0$ . Global step (8) is a problem defined on domain  $\tilde{\Omega}$ , with an arbitrary material parameter  $\tilde{K}$  in  $\tilde{\Lambda}$  and which does not contain any geometrical details in  $\tilde{\Lambda}$ . The term  $\lambda_{n-1}$  appears as an imposed source term on an internal surface in  $\tilde{\Omega}$ .

**Local step** We then compute  $(w_n, \lambda_n) \in \mathcal{W}^\Xi \times \mathcal{M}^\Xi$  such that for all  $(\delta w, \delta \lambda) \in \mathcal{W}^\Xi \times \mathcal{M}^\Xi$ ,

$$\begin{cases} a_{\Lambda}(w_n, \delta w) - b_{\Gamma}(\lambda_n, \delta w) = \ell_{\Lambda}(\delta w) \\ b_{\Gamma}(\delta \lambda, w_n) = b_{\Gamma}(\delta \lambda, U_n) \end{cases} \quad (9)$$

Local step (9) is a problem defined on patch  $\Lambda$ , with the true material parameter  $K$  and the true geometrical details.  $\lambda_n$  can be interpreted as a Lagrange multiplier that ensures in a weak sense  $w_n = U_n$  on  $\Gamma$ .

#### 3.3 Multiple patches with independent variabilities

We now consider that the patch  $\Lambda$  is the disjoint union of  $N_{\Lambda}$  patches  $\{\Lambda_s\}_{s=1}^{N_{\Lambda}}$  that contain uncertainties that are statistically independent between patches so that the set of parameters can be split into disjoint subsets of parameters  $\xi_s \in \Xi^s \subset \mathbb{R}^{d_s}$ ,  $s = 1, \dots, N_{\Lambda}$ , and

$$\Lambda(\xi) = \Lambda_1(\xi_1) \cup \dots \cup \Lambda_{N_{\Lambda}}(\xi_{N_{\Lambda}}),$$

The random field  $K$  and source term  $f$  also only depend on  $\xi_s$  on the subdomain  $\Lambda_s$ , *i.e.* for  $s = 1, \dots, N_{\Lambda}$ ,

$$\begin{aligned} K(x, \xi) &= K_s(x, \xi_s) \quad \text{for } x \in \Lambda_s(\xi_s) \\ f(x, \xi) &= f_s(x, \xi_s) \quad \text{for } x \in \Lambda_s(\xi_s) \end{aligned}$$

The local step is then composed of  $N_{\Lambda}$  independent problems on  $\Lambda_s$  of the same type as problem (9). These problems can be solved in parallel.

### 3.4 General features on the algorithm

It can be shown that the sequence  $u_n$  defined by  $u_n = U_n$  on  $\Omega \setminus \Lambda$  and  $u_n = w_n$  on  $\Lambda$  converges towards  $u$  for a sufficiently small relaxation parameter  $\rho > 0$ . Note that the final solution does not depend on the choice of  $\tilde{K}$ . However, this choice can influence the behavior of the iterative algorithm. Note that although problem (7) does not define the prolongation  $U$  inside the fictitious patch  $\tilde{\Lambda}$  (this problem admits an infinite number of solutions), it can be shown that the sequence  $U_n$  remains in a particular subspace of  $\tilde{\mathcal{U}}^\Xi$  in which the problem admits a unique solution.

## 4 APPROXIMATE SOLUTION OF LOCAL AND GLOBAL PROBLEMS

### 4.1 Approximation spaces

We introduce finite element approximation spaces at the deterministic level  $\tilde{\mathcal{U}}_H \subset \tilde{\mathcal{U}}$ ,  $\mathcal{W}_h \subset \mathcal{W}$  and  $\mathcal{M}_h \subset \mathcal{M}$ . Non conforming finite element meshes can be used in order to well describe localized quantities in the patches by introducing fine meshes within these patches. The Lagrange multiplier approximation space  $\mathcal{M}_h$  is simply chosen as the trace on  $\Gamma$  of the approximation space  $\mathcal{W}_h$ . For the construction of suitable approximation spaces  $\mathcal{M}_h$  in the case of non conforming meshes, see [6] and the references therein. A finite element approximation space  $\mathcal{H}$  with dimension  $N$  and basis  $\{\varphi_i\}_{i=1}^N$  being given, an element  $v \in \mathcal{H}$  writes  $v = \sum_{i=1}^N v_i \varphi_i$  and an element  $v \in \mathcal{H}^\Xi$  is such that  $v(\xi) = \sum_{i=1}^N v_i(\xi) \varphi_i$ .

We also introduce approximation spaces at the stochastic level

$$\mathcal{S}_P = \left\{ v(\xi) = \sum_{k=1}^P v_k \Psi_k(\xi) \right\} \subset \mathcal{S} := L_{P_\xi}^2(\Xi)$$

An element  $v \in \mathcal{H} \otimes \mathcal{S}_P$  can then be written  $v = \sum_{i=1}^N \sum_{k=1}^P v_{k,i} \varphi_i \otimes \Psi_k$ .

### 4.2 Global problem

The approximate global problem consists in finding  $\hat{U}_n \in \tilde{\mathcal{U}}_H \otimes \mathcal{S}_P$  such that  $\forall \delta U \in \tilde{\mathcal{U}}_H \otimes \mathcal{S}_P$ ,

$$c_{\tilde{\Omega}}(\hat{U}_n, \delta U) = c_{\tilde{\Lambda}}(U_{n-1}, \delta U) - b_\Gamma(\lambda_{n-1}, \delta U) + \ell_{\Omega \setminus \Lambda}(\delta U)$$

If the fictitious diffusion parameter  $\tilde{K}$  does not depend on the uncertain parameters, the operator is deterministic for it also does not depend on the uncertain parameters.

### 4.3 Local problems

Regarding the local problem, the operator and right-hand side are uncertain. The approximate local problem consists

in finding  $(w_n, \lambda_n) \in (\mathcal{W}_h \otimes \mathcal{S}_P) \times (\mathcal{M}_h \otimes \mathcal{S}_P)$  such that  $\forall \delta w \in \mathcal{W}_h \otimes \mathcal{S}_P$  and  $\forall \delta \lambda \in \mathcal{M}_h \otimes \mathcal{S}_P$ ,

$$\begin{cases} a_\Lambda(w_n, \delta w) - b_\Gamma(\lambda_n, \delta w) = \ell_\Lambda(\delta w) \\ b_\Gamma(\delta \lambda, w_n) = b_\Gamma(\delta \lambda, U_n) \end{cases} \quad (10)$$

### 4.4 Local problems with geometrical variabilities

The solution method presented in section 4.3 is valid when the patch contains geometrical details with a fixed internal boundary  $\Gamma_\Lambda := \partial\Lambda \setminus \Gamma$ . It suffices to introduce a conforming approximation space  $\mathcal{W}_h$  in  $\mathcal{W}$ . However, if the patch  $\Lambda$  contains uncertain geometrical details, that means an uncertain internal boundary  $\Gamma_\Lambda(\xi)$ , the previous construction of approximation spaces is not possible since the space  $\mathcal{W}^\Xi$  has no more a tensor product structure. Different strategies have been proposed in the literature for solving such problems [4, 5, 7, 8]. We here use reformulations on a fictitious domain, which is simply taken as the patch  $\tilde{\Lambda}$ . Depending on the type of boundary conditions (Dirichlet or Neumann) that are imposed on the internal boundary  $\Gamma^\Lambda = \partial\Omega \cap \partial\Lambda$ , different reformulations are introduced.

If  $\Gamma^\Lambda$  is a Neumann boundary with homogeneous data, the local problem (10) is reformulated on the fictitious domain  $\tilde{\Lambda}$  by prolongating bilinear form  $a_\Lambda$  and linear form  $\ell_\Lambda$  to functions defined on  $\tilde{\Lambda}$ . In practice, we make use of the indicator function  $I_\Lambda$  of the random domain  $\Lambda$ . For further details on the numerical solution of such problems, see [4, 5].

In the case where  $\Gamma^\Lambda$  is a Dirichlet boundary with homogeneous data we propose a reformulation based on the use of a characteristic function method [9]. Approximation  $w_n$  is prolonged on the fictitious patch  $\tilde{\Lambda}$  and is searched under the form  $w_n = U_n^\diamond + \psi z_n$ , with  $U_n^\diamond = U_n$  in a weak sense on  $\Gamma$ , and with  $z_n = 0$  on  $\Gamma$  and where  $\psi : \tilde{\Lambda} \times \Xi \rightarrow \mathbb{R}$  is a characteristic function such that

$$\begin{aligned} \text{(c1)} \quad & \psi(x, \xi) > 0 && \text{for } x \in \Lambda(\xi) \\ \text{(c2)} \quad & \psi(x, \xi) = 0 && \text{for } x \in \Gamma_\Lambda(\xi) \\ \text{(c3)} \quad & |\nabla \psi(x, \xi)| \neq 0 && \text{for } x \in \Gamma_\Lambda(\xi) \end{aligned}$$

A function  $(\psi z)$  with  $z \in \tilde{\mathcal{W}}_h \otimes \mathcal{S}_P$  then verifies the Dirichlet boundary conditions on  $\Gamma^\Lambda$ .

The prolongation of function  $w$  on the fictitious patch  $\tilde{\Lambda}$  enables the definition of  $w$  in  $\tilde{\mathcal{W}}^\Xi$ , with  $\tilde{\mathcal{W}} = H^1(\tilde{\Lambda})$ . Function space  $\tilde{\mathcal{W}}^\Xi$  can now be identified with tensor product space  $\tilde{\mathcal{W}} \otimes \mathcal{S}$  and approximation spaces can be introduced such as in section 4.1, i.e. by introducing a finite element approximation space  $\tilde{\mathcal{W}}_h \subset \tilde{\mathcal{W}}$  and the approximation space  $\mathcal{S}_P \subset \mathcal{S}$ .

## 5 SOLUTION OF LOCAL AND GLOBAL PROBLEMS USING SEPARATED REPRESENTATIONS

### 5.1 Tensor structure of stochastic function spaces

If  $\mathcal{H}$  is a deterministic space, then

$$\mathcal{H}^\Xi = \mathcal{H} \otimes \mathcal{S}$$

with  $\mathcal{S} = L_{P_\xi}^2(\Xi)$ . An element in  $\mathcal{H}^\Xi$  can then be interpreted as a two-order tensor. When the set of random variables  $\xi$  is a collection of  $d$  independent sets  $\xi_s$  of random variables with  $d = N_\Lambda$ , then probability measure  $P_\xi = P_{\xi_1} \otimes \dots \otimes P_{\xi_{N_\Lambda}}$ , and the space  $\mathcal{S}$  has the following tensor product structure:

$$\mathcal{S} = \mathcal{S}^1 \otimes \dots \otimes \mathcal{S}^{N_\Lambda}$$

with  $\mathcal{S}^s = L_{P_{\xi_s}}^2(\Xi_s)$ . This tensor product structure of stochastic function spaces will be here exploited.

We denote by  $\mathcal{R}_1(\mathcal{H}^\Xi)$  the set of rank-one elements in tensor product space  $\mathcal{H} \otimes \mathcal{S}^1 \otimes \dots \otimes \mathcal{S}^{N_\Lambda}$ , defined by

$$\mathcal{R}_1(\mathcal{H}^\Xi) = \{z = Z \otimes \phi^1 \otimes \dots \otimes \phi^{N_\Lambda} : Z \in \mathcal{H}, \phi^s \in \mathcal{S}^s\}$$

and by  $\mathcal{R}_m(\mathcal{H}^\Xi) = \{\sum_{i=1}^m z_i : z_i \in \mathcal{R}_1(\mathcal{H}^\Xi)\}$  the set of rank- $m$  elements.

### 5.2 Proper Generalized Decomposition

We now propose a construction of a separated representation of the solution of a variational problem (global problem or local problems). Convergence results can be found in [10]. Let us consider a problem which admits the following variational form

$$u \in \mathcal{H}^\Xi, \quad A(u, \delta u) = L(\delta u) \quad \forall \delta u \in \mathcal{H}^\Xi$$

with  $A$  symmetric continuous and coercive. The problem is then equivalent to

$$u = \arg \min_{v \in \mathcal{H}^\Xi} \mathcal{J}(v) \quad \text{with} \quad \mathcal{J}(v) = \frac{1}{2} A(v, v) - L(v)$$

We then define a progressive separated representation as follows: knowing  $u_{m-1} \in \mathcal{R}_{m-1}(\mathcal{H}^\Xi)$ , we define a new element  $z_m \in \mathcal{R}_1(\mathcal{H}^\Xi)$  by

$$\mathcal{J}(u_{m-1} + z_m) \approx \min_{z \in \mathcal{R}_1(\mathcal{H}^\Xi)} \mathcal{J}(u_{m-1} + z)$$

where the symbol  $\approx$  means that we compute an approximation  $z_m$  of an optimal element in  $\mathcal{R}_1(\mathcal{H}^\Xi)$ . In practice, we obtain an element  $z = Z \otimes \phi^1 \otimes \dots \otimes \phi^{N_\Lambda}$  by an alternating minimization algorithm, consisting in solving alternatively:

- One minimization problem on  $\mathcal{H}$ , which is a deterministic problem:

$$Z = \arg \min_{Z \in \mathcal{H}} \mathcal{J}(u_{m-1} + Z \otimes \phi^1 \otimes \dots \otimes \phi^{N_\Lambda})$$

- $N_\Lambda$  minimization problems on  $\mathcal{S}^s$  for  $s = 1, \dots, N_\Lambda$ :

$$\phi^s = \arg \min_{\phi^s \in \mathcal{S}^s} \mathcal{J}(u_{m-1} + Z \otimes \phi^1 \otimes \dots \otimes \phi^s \otimes \dots \otimes \phi^{N_\Lambda})$$

## 6 ILLUSTRATION

### 6.1 Description of the problem

We consider the diffusion problem (1) with  $\Gamma_N = \emptyset$ . The diffusion coefficient is locally uncertain on two patches  $\Lambda_1$  and  $\Lambda_2$  illustrated in figure 3. We take

$$K = \begin{cases} 1 & \text{on } \Omega \setminus (\Lambda_1 \cup \Lambda_2) \\ K_1 = 1 + \sum_{k=1}^5 \xi_1^k f_1^k(x) & \text{on } \Lambda_1 \\ K_2 = 1 + \sum_{k=1}^5 \xi_2^k f_2^k(x) & \text{on } \Lambda_2 \end{cases}$$

where  $\xi_1^k$  and  $\xi_2^k$ , for  $k = 1, \dots, 5$ , are independent uniform random variables on  $\Xi_1^k = (0, 1)$  and  $\Xi_2^k = (0, 1)$  respectively. We denote  $\Xi_1 = \times_{k=1}^5 \Xi_1^k \subset \mathbb{R}^5$  and  $\Xi_2 = \times_{k=1}^5 \Xi_2^k \subset \mathbb{R}^5$  the sets of elementary events associated with patches  $\Lambda_1 = \tilde{\Lambda}_1$  and  $\Lambda_2 = \tilde{\Lambda}_2$  respectively.

Domain  $\Omega(\xi)$  also presents geometrical details with homogeneous Dirichlet conditions on circular boundaries of random radii  $r_3 = 0.2 + 0.1 \xi_3$  and  $r_4 = 0.2 + 0.1 \xi_4$  in patches  $\Lambda_3$  and  $\Lambda_4$  respectively.  $\xi_3$  and  $\xi_4$  are independent uniform random variables on  $\Xi_3 = (0, 1)$  and  $\Xi_4 = (0, 1)$  respectively.

The solution is computed using the multiscale strategy with patches. We introduce fictitious patches  $\tilde{\Lambda}_3$  and  $\tilde{\Lambda}_4$  and fictitious domain  $\tilde{\Omega}$  such that

$$\begin{aligned} \tilde{\Omega} &= (0, 5) \times (0, 5) \\ \Omega(\xi) &= (\tilde{\Omega} \setminus (\tilde{\Lambda}_3 \cup \tilde{\Lambda}_4)) \cup (\Lambda_3(\xi_3) \cup \Lambda_4(\xi_4)) \end{aligned}$$

The global problem is defined on domain  $\tilde{\Omega}$  which does not contain any geometrical detail and it is associated with the fol-

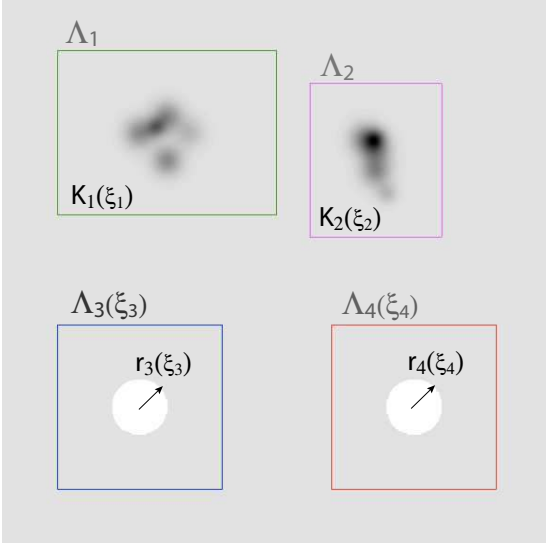


FIGURE 3. Diffusion problem

lowing deterministic diffusion coefficient:

$$\tilde{K} = \begin{cases} 1 & \text{on } \tilde{\Omega} \setminus (\Lambda_1 \cup \Lambda_2) \\ K_1 = \mathbb{E}(K_1) & \text{on } \Lambda_1 \\ K_2 = \mathbb{E}(K_2) & \text{on } \Lambda_2 \end{cases}$$

The global problem has thus a deterministic operator.

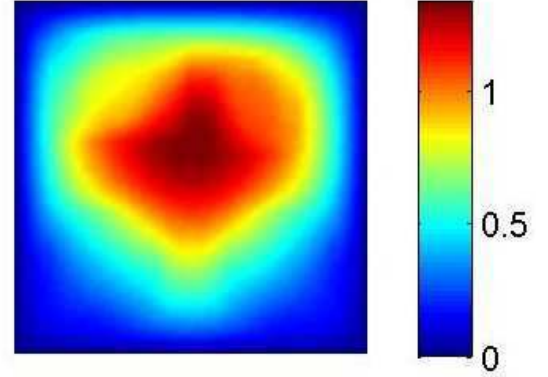
The local problems on  $\Lambda_3(\xi)$  and  $\Lambda_4(\xi)$  are solved using the characteristic function method. The characteristic function  $\psi_s(x, \xi_s)$  is chosen as the random level-set function whose iso-zero in patch  $\tilde{\Lambda}_s$  (for  $s = 3, 4$ ) defines the random boundary inside  $\tilde{\Lambda}_s$ .

We introduce a finite element approximation space  $\tilde{\mathcal{U}}_H$  in  $\tilde{\mathcal{U}}$  with dimension  $N = 3721$ . We here use conforming finite element meshes in the patches although non conforming meshes could be used. Thus, denoting by  $\tilde{\mathcal{W}}_h^s$  the approximation spaces associated with patches  $\tilde{\Lambda}_s$ , we have  $\tilde{\mathcal{W}}_h^s \subset \tilde{\mathcal{U}}_H$ . We also introduce the stochastic approximation spaces  $\mathcal{S}_{P_s}^s$  with dimensions  $P_1 = P_2 = 56$  and  $P_3 = P_4 = 60$ , using multidimensional polynomial bases for  $s = 1$  and  $2$ , and piecewise polynomial bases for  $s = 3$  and  $4$ .

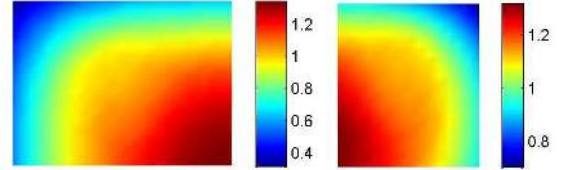
Global and local problems are solved with tensor approximation methods with a given tolerance. The global problem is solved exactly since it only involves a deterministic operator, and the resulting global iterate is compressed using classical tensor approximation methods. The PGD method is used for solving local problems. The solutions are searched in  $\mathcal{H} \otimes \mathcal{S}_{P_1}^1 \otimes \mathcal{S}_{P_2}^2 \otimes \mathcal{S}_{P_3}^3 \otimes \mathcal{S}_{P_4}^4$ , with  $\mathcal{H}$  referring to  $\tilde{\mathcal{U}}_H$  or  $\tilde{\mathcal{W}}_h^s$  depending on the problem.

Figure 4 shows a random sample of the “converged” global and local solutions obtained using a relaxation parameter  $\rho = 0.5$  and for a given tolerance of  $10^{-4}$  for local and global solvers. Figure 5 shows the reconstructed solution

$$u = \begin{cases} U & \text{on } \Omega \setminus \Lambda \\ w_s & \text{on } \Lambda_s \text{ for } s = 1, \dots, 4 \end{cases}$$

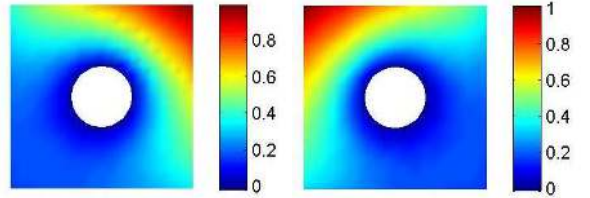


(a)  $U$  (rank 8)



(b)  $w_1$  (rank 47)

(c)  $w_2$  (rank 42)



(d)  $w_3 = w_U + \psi z_{22}$  (rank 22)

(e)  $w_4 = w_U + \psi z_{24}$  (rank 24)

FIGURE 4. Random sample of global solution  $U$  and local solutions  $w_s$  for  $s = 1, \dots, 4$ .

In order to illustrate the robustness of the method with respect to approximations, the residual error is plotted in figure 6 as a function of the iterations of the method. We see that the iterative method converges until the finite precision, in terms of the residual error, introduced in the solutions of local and global problems is reached.

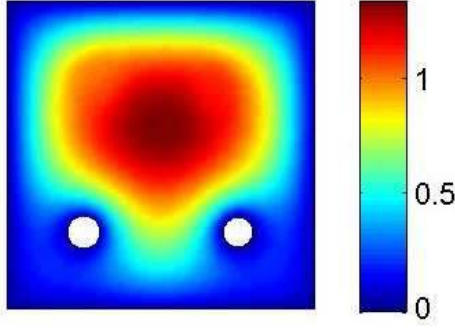


FIGURE 5. Random sample of the reconstructed solution  $u$ .

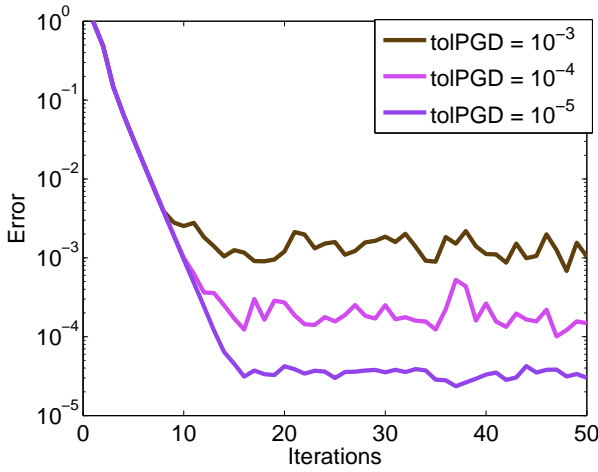


FIGURE 6. Convergence of the iterative algorithm for different tolerances of global and local solvers (PGD)

## 7 CONCLUSION

We have proposed a dedicated method to treat with localized sources of uncertainties. It is based on a domain decomposition method associated with a local/global algorithm that has nice convergence properties. It profits tensor based methods, as the PGD method, for the *a priori* construction of separated representations of the solutions of global and local problems. When dealing with parametric domains, we introduce fictitious domain methods that enable the formulation of the problem on a tensor product space. The multiscale approach enables the introduction of refined local approximation to well describe local quantities. Further studies on the convergence of the global-local algorithm with respect to the parameters introduced within the iterative method are under study for an optimal use of the approach. First results on the influence of the size of the patch notably show that when chosen too small, that is to say when the patch does not hold the main effects of the localized uncertainties on the response, the convergence rate is deteriorated.

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