



HAL
open science

Improving BEM Solvers: The Proper Generalized Decomposition Boundary Element Method for Solving Parabolic Problems

Gaël Bonithon, Pierre Joyot, Francisco Chinesta, Pierre Villon

► **To cite this version:**

Gaël Bonithon, Pierre Joyot, Francisco Chinesta, Pierre Villon. Improving BEM Solvers: The Proper Generalized Decomposition Boundary Element Method for Solving Parabolic Problems. International Conference on Boundary Element Method, 2009, Athènes, Greece. hal-01008300

HAL Id: hal-01008300

<https://hal.science/hal-01008300>

Submitted on 8 Jun 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Improving BEM solvers: The Proper Generalized Decomposition Boundary Element Method for solving parabolic problems

G. Bonithon^{1,4}, P. Joyot¹, F. Chinesta² and P. Villon³

¹ ESTIA-Recherche, technopole izarbel, 64210 Bidart, France, pjoyot@estia.fr

² EADS Corporate Foundation International Chair, GEM CNRS-ECN, 1 rue de la Noë BP 92101, 44321 Nantes cedex 3, France, Francisco.Chinesta@ec-nantes.fr

³ UTC-Roberval UMR 6253, 60200 Compiègne, France, pierre.villon@utc.fr

⁴ EPSILON Ingénierie, 10 rue Jean Bart, BP 97431, France, 31674 Labège Cedex

Keywords: Boundary element method, Separated representations, Proper Generalized Decomposition

Abstract. In this work we propose a new approach for solving the heat equation within the Boundary Elements method framework. This technique lies in the use of a separated representation of the unknown field that allows decoupling the space problem (that results steady state) from the temporal one (one dimensional that only involves the time coordinate).

Introduction

The Boundary Elements Method (BEM) allows efficient solution of partial differential equations whose kernel functions are known. The heat equation is one of these candidates when the thermal parameters are assumed constant (linear model). When the model involves large physical domains and time simulation intervals the amount of information that must be stored increases significantly. We propose here an alternative strategy able to change the nature of the problem. Thus, the temperature field involved by the so called heat equation is approximated using a separated representation involving products of space and time functions. This kind of approximation is not new, in fact proper orthogonal decomposition [1] allows such one decomposition, but in this case this decomposition must be applied a posteriori, i.e. on the transient solution of the considered model.

The technique that we propose in this paper allows to transform the transient model in a sequence of space problems (all of them steady state) and time problems (that only involve the time coordinate). This iteration procedure lead to a proper space-time generalized decomposition of the model solution. The efficiency of such one approach was proven in [2, 3, 4]. In our knowledge, this technique has never been coupled with a BEM for solving the resulting steady problem defined in the physical domain.

A Proper Generalized Decomposition Boundary Element Method

Let $\Omega \subset \mathbb{R}^n$ be a domain of boundary Γ and let $\tau \subset \mathbb{R}^+$ a time interval in which the model is defined. The strong form of the heat equation with homogeneous initial and boundary conditions writes: *Find the temperature field (assuming known the heat source $b(\mathbf{x}, t)$) such that:*

$$\partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = b(\mathbf{x}, t) \quad \mathbf{x} \in \Omega, t \in \tau \quad (1)$$

$$u(\mathbf{x}, t) = 0 \quad \mathbf{x} \in \Gamma, t \in \tau \quad (2)$$

$$u(\mathbf{x}, 0) = 0 \quad (3)$$

In the proposed technique the transient equation is not solved by applying the standard BEM technique. On the contrary, we are using the fact that $u(\mathbf{x}, t)$ can be expressed into the separated representation form:

$$u(\mathbf{x}, t) \approx \sum_{k=1}^n R_k(\mathbf{x}) S_k(t)$$

We are looking for the functional couples $(R_k(\mathbf{x}), S_k(t))$. We must recall that the efficiency of the method depends on a number of functional couples n small enough. However, the solution of many models can be written from a reduced number of functional couples.

First, we should give the equations that must verify the different functional couples. For the sake of simplicity we introduce the notation:

$$\mathcal{R}_{i,j}(\mathbf{x}) = (S_i(t), \partial_t S_j(t))_{\tau} R_j(\mathbf{x}) - (S_i(t), S_j(t))_{\tau} \Delta R_j(\mathbf{x})$$

$$\mathcal{S}_{i,j}(t) = (R_i(\mathbf{x}), R_j(\mathbf{x}))_{\Omega} \partial_t S_j(t) - (R_i(\mathbf{x}), \Delta R_j(\mathbf{x}))_{\Omega} S_j(t)$$

where $(u, v)_{\tau} = \int_{\tau} u(t)v(t)dt$ and $(u, v)_{\Omega} = \int_{\Omega} u(\mathbf{x})v(\mathbf{x})d\mathbf{x}$.

We can prove that

Proposition 1 (Galerkin method for separated representations). *The couples $(R_k(\mathbf{x}), S_k(t))$ that verify the strong form of the heat equation are also solution of the non-linear systems:*

$$\begin{cases} \mathcal{R}_{1,1}(\mathbf{x}) + \dots + \mathcal{R}_{1,n}(\mathbf{x}) = (S_1(t), b(\mathbf{x}, t))_{\tau} \\ \vdots \\ \mathcal{R}_{n,1}(\mathbf{x}) + \dots + \mathcal{R}_{n,n}(\mathbf{x}) = (S_n(t), b(\mathbf{x}, t))_{\tau} \\ \mathcal{S}_{1,1}(t) + \dots + \mathcal{S}_{1,n}(t) = (R_1(\mathbf{x}), b(\mathbf{x}, t))_{\Omega} \\ \vdots \\ \mathcal{S}_{n,1}(t) + \dots + \mathcal{S}_{n,n}(t) = (R_n(\mathbf{x}), b(\mathbf{x}, t))_{\Omega} \end{cases} \quad (4)$$

Proof. Let H_{τ} be a Hilbert space with the scalar product

$$(u(t), v(t))_{\tau} = \int_{\tau} u(t)v(t) dt$$

We assume that the functions $S_k(t)$ define a basis B_{τ} of H_{τ} . By applying the Galerkin method to the heat equation, we are looking for the functions $R_k(\mathbf{x})$ such that

$$\left(\sum_{k=1}^n R_k(\mathbf{x}) \partial_t S_k(t) - \Delta R_k(\mathbf{x}) S_k(t), S_i(t) \right)_{\tau} = 0 \quad \forall i = 1 \dots n$$

that can be rewritten as:

$$\sum_{k=1}^n \mathcal{R}_{i,k}(\mathbf{x}) = (S_i(t), b(\mathbf{x}, t))_{\tau} \quad \forall i = 1 \dots n \quad (5)$$

Now, we consider another Hilbert space H_Ω with the associated scalar product

$$(u(\mathbf{x}), v(\mathbf{x}))_\Omega = \int_\Omega u(\mathbf{x})v(\mathbf{x}) d\mathbf{x}$$

We consider that functions $R_k(\mathbf{x})$ define the basis B_Ω of H_Ω . Thus, we look for the functions $S_k(t)$ such that

$$\left(\sum_{k=1}^n R_k(\mathbf{x}) \partial_t S_k(t) - \Delta R_k(\mathbf{x}) S_k(t), R_i(\mathbf{x}) \right)_\Omega = 0 \quad \forall i = 1 \dots n$$

that can be rewritten as:

$$\sum_{k=1}^n \mathcal{S}_{i,k}(t) = (R_i(\mathbf{x}), b(\mathbf{x}, t))_\Omega \quad \forall i = 1 \dots n \quad (6)$$

Equations eq (5) and eq (6) correspond to the system eq (4). □

In the context of the proposed strategy, the number of functional couples $R_k(\mathbf{x})S_k(t)$ is "a priori" unknown. We propose to build-up this basis iteratively, until reaching convergence.

Thus, at the first iteration we consider the basis B_Ω (resp. B_τ) composed of a single element $R_1(\mathbf{x})$, (resp. $S_1(t)$). Thus, we look for $R_1(\mathbf{x}), S_1(t)$ such that:

$$\begin{aligned} \mathcal{R}_{1,1}(\mathbf{x}) &= (S_1(t), b(\mathbf{x}, t))_\tau \\ \mathcal{S}_{1,1}(t) &= (R_1(\mathbf{x}), b(\mathbf{x}, t))_\Omega \end{aligned}$$

At iteration p , the basis B_Ω (resp B_τ) is composed by the functions previously computed $R_1(\mathbf{x})$ to $R_{p-1}(\mathbf{x})$, (resp. $S_1(t)$ to $S_{p-1}(t)$) and by $R_p(\mathbf{x})$ and $S_p(t)$ verifying

$$\begin{aligned} \mathcal{R}_{p,p}(\mathbf{x}) &= (S_p(t), b(\mathbf{x}, t))_\tau - \sum_{i=1}^{p-1} \mathcal{R}_{p,i}(\mathbf{x}) \\ \mathcal{S}_{p,p}(t) &= (R_p(\mathbf{x}), b(\mathbf{x}, t))_\Omega - \sum_{i=1}^{p-1} \mathcal{S}_{p,i}(t) \end{aligned} \quad (7)$$

Proposition 2. *First, we introduce the notation $b_i(\mathbf{x}, t) = R_i(\mathbf{x}) \partial_t S_i(t) - \Delta R_i(\mathbf{x}) S_i(t)$. If at iteration n , we have $b(\mathbf{x}, t) = \sum_{k=1}^n b_k(\mathbf{x}, t)$, then $u(\mathbf{x}, t) = \sum_{k=1}^n R_k(\mathbf{x}) S_k(t)$ is solution of eq (1). We can affirm that the method converges. In that case the couples $R_k(\mathbf{x})S_k(t)$ verify the system eq (4), as well as the conditions:*

$$\begin{aligned} \sum_{i=p+1}^n (S_p(t), b_i(\mathbf{x}, t))_\tau &= 0 \\ \sum_{i=p+1}^n (R_p(\mathbf{x}), b_i(\mathbf{x}, t))_\Omega &= 0 \end{aligned} \quad 1 \leq p \leq n-1$$

that can be written as:

$$\begin{aligned} \sum_{i=p+1}^n \mathcal{R}_{p,i}(\mathbf{x}) &= 0 \\ \sum_{i=p+1}^n \mathcal{S}_{p,i}(t) &= 0 \end{aligned} \quad 1 \leq p \leq n-1$$

Proof. Because $b(\mathbf{x}, t) = \sum_{k=1}^n b_k(\mathbf{x}, t)$, we have

$$(S_p(t), b(\mathbf{x}, t))_\tau = \sum_{i=1}^n (S_p(t), b_i(\mathbf{x}, t))_\tau = \sum_{i=1}^n \mathcal{R}_{p,i}(\mathbf{x})$$

At iteration p , $R_p(\mathbf{x})$ is computed from eq (7) that also writes

$$\mathcal{R}_{p,p}(\mathbf{x}) = \sum_{i=1}^n \mathcal{R}_{p,i}(\mathbf{x}) - \sum_{i=1}^{p-1} \mathcal{R}_{p,i}(\mathbf{x})$$

This implies $\sum_{i=p+1}^n \mathcal{R}_{p,i}(\mathbf{x}) = 0$. □

Remark 1. The proposed iteration scheme stops when the relation

$$\left\| b(\mathbf{x}, t) - \sum_{k=1}^n R_k(\mathbf{x}) \partial_t S_k(t) - \Delta R_i(\mathbf{x}) S_i(t) \right\|_{\Omega \times \tau} \leq \varepsilon$$

holds, where ε is a small enough threshold value.

In order to enforce the homogeneous boundary and initial conditions eq (2), one should enforce $R_k(\mathbf{x}) = 0 \quad \mathbf{x} \in \Gamma$ and $S_k(t) = 0 \quad t \in \tau$.

In order to solve the resulting non-linear problem eq (7) at iteration p , we use a fixed point strategy.

We denote by $R_p^{(q)}(\mathbf{x})$ and $S_p^{(q)}(t)$ the values of $R_p(\mathbf{x})$ and $S_p(t)$ at iteration q . We consider that $R_p^{(q)}(\mathbf{x})$ and $S_p^{(q)}(t)$ are known, and we determine $R_p^{(q+1)}(\mathbf{x})$ from

$$\begin{aligned} & \left(S_p^{(q)}(t), \partial_t S_p^{(q)}(t) \right)_\tau R_p^{(q+1)}(\mathbf{x}) - \left(S_p^{(q)}(t), S_p^{(q)}(t) \right)_\tau \Delta R_p^{(q+1)}(\mathbf{x}) = \\ & \left(S_p^{(q)}(t), b(\mathbf{x}, t) \right)_\tau - \sum_{i=1}^{p-1} \mathcal{R}_{p,i}^{(q)}(\mathbf{x}) \end{aligned} \quad (8)$$

In fact is that equation related to the boundary conditions on $R_p(\mathbf{x})$ that we are solving by applying the BEM. However, as the kernel is different depending on the sign of $\left(S_p^{(q)}(t), \partial_t S_p^{(q)}(t) \right)_\tau$, we rewrite that equation as:

$$\begin{aligned} & - \left(S_p^{(q)}(t), S_p^{(q)}(t) \right)_\tau \Delta R_p^{(q+1)}(\mathbf{x}) = \\ & \left(S_p^{(q)}(t), \partial_t S_p^{(q)}(t) \right)_\tau R_p^{(q)}(\mathbf{x}) + \left(S_p^{(q)}(t), b(\mathbf{x}, t) \right)_\tau - \sum_{i=1}^{p-1} \mathcal{R}_{p,i}^{(q)}(\mathbf{x}) \end{aligned} \quad (9)$$

where the first term of the righthand member of eq (8) is computed at iteration q instead of $q+1$. Thus, $R_p^{(q+1)}$ will be the solution of a Poisson equation.

We are now calculating $S_p^{(q+1)}(t)$ as a function of $S_p^{(q)}(t)$ and $R_p^{(q+1)}(\mathbf{x})$ from the equation

$$\begin{aligned} & \left(R_p^{(q+1)}(\mathbf{x}), R_p^{(q+1)}(\mathbf{x}) \right)_\Omega \partial_t S_p^{(q+1)}(t) - \left(R_p^{(q+1)}(\mathbf{x}), \Delta R_p^{(q+1)}(\mathbf{x}) \right)_\Omega S_p^{(q+1)}(t) = \\ & \left(R_p^{(q+1)}(\mathbf{x}), b(\mathbf{x}, t) \right)_\Omega - \sum_{i=1}^{p-1} \mathcal{S}_{p,i}^{(q)}(t) \end{aligned} \quad (10)$$

This equation is a simple ODE whose solution does not represent any difficulty.

The method needs the computation of integrals $\left(R_i(\mathbf{x}), R_j(\mathbf{x}) \right)_\Omega$, $\left(R_i(\mathbf{x}), \Delta R_j(\mathbf{x}) \right)_\Omega$, and for that we define an approximation in Ω such that $R_i(\mathbf{x}) = \mathbf{N}^T(\mathbf{x})\mathbf{s}$, $\mathbf{N}^T(x)$ is the matrix containing the shape functions and \mathbf{s} the vector containing the nodal values. We can choose the finite element shape functions, or the ones associated with the MLS or the RBF. Vector \mathbf{s} is obtained using the BEM representation formula.

We proceed in the same way for the integral $(S_i(t), S_i(t))_\tau$, $(S_i(t), \partial_t S_i(t))_\tau$.

Numerical example

In this section we validate the just proposed approach. We consider the square domain $\Omega = [0, 1] \times [0, 1]$ and the time interval $\tau = [0, 0.2]$. The source term is constant $b(\mathbf{x}, t) = 1$ and both boundary and initial conditions homogeneous.

The exact solution of the problem writes: (see [5])

$$u^{as}(\mathbf{x}, t) = \int_\tau \int_0^1 \int_0^1 G(x, y, \bar{x}, \bar{y}, \bar{t}) d\bar{x} d\bar{y} d\bar{t}$$

where

$$G(x, y, \bar{x}, \bar{y}, \bar{t}) = 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin(n\pi x) \sin(m\pi y) \sin(n\pi \bar{x}) \sin(m\pi \bar{y}) \exp(-\pi^2 (n^2 + m^2) \bar{t})$$

The Poisson's equation eq (9) is solved by the BEM. the domain boundary Γ is discretized by using $n_\Gamma \times n_\Gamma$ segments Γ_i , the field $u(\mathbf{x}, t)$ being constant on each one.

To compute de integrals $(R_i(\mathbf{x}), R_j(\mathbf{x}))_\Omega$, $(R_i(\mathbf{x}), \Delta R_j(\mathbf{x}))_\Omega$ the field $u(\mathbf{x}, t)$ is reconstructed within Ω on a mesh compatible with the segments defined on the domain boundary Γ_i . Each element of the mesh defined a domain Ω_i (see figure 1). The field is approximated using quadratic MLS based shape functions.

The integrals are computed by using a single integration point located at the middle point of each segment Ω_i .

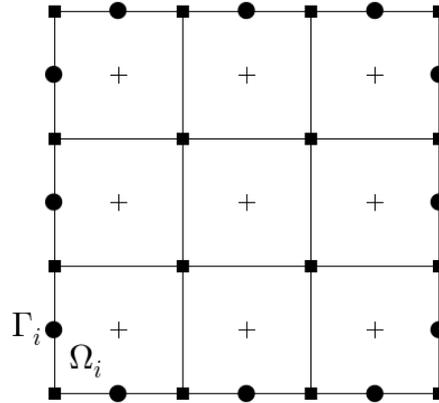


Fig. 1: Colocation nodes (●), MLS nodes (■) and integration points (+).

The time interval is discretized using linear MLS based shape functions from the n_τ nodes uniformly distributed.

We are checking the convergence for different space (i.e. n_Γ) and time (i.e. n_τ) discretizations.

Table 1 and figure 2 show the evolution of the L^2 error in time and space as a function of the number of functional couples $R_k(\mathbf{x}) S_k(t)$ involved in the approximation of $u(\mathbf{x}, t)$ and the mesh densities. We define this error as:

n_Γ	n_τ	e_1^2	e_2^2	e_3^2	e_4^2	e_5^2	e_6^2	e_7^2
5	2^5	$3.0e-2$	$1.6e-2$	$1.6e-2$	$1.5e-2$	$1.5e-2$	$1.5e-2$	$1.5e-2$
10	2^5	$2.2e-2$	$5.2e-3$	$4.7e-3$	$2.1e-3$	$2.1e-3$	$2.1e-3$	$2.1e-3$
16	2^5	$2.1e-2$	$4.3e-3$	$1.2e-3$	$1.3e-4$	$1.1e-3$	$7.7e-4$	$7.7e-4$
20	2^5	$2.1e-2$	$4.2e-3$	$1.0e-3$	$1.0e-3$	$7.0e-4$	$7.0e-4$	$7.0e-4$
32	2^8	$2.1e-2$	$4.2e-3$	$8.1e-4$	$9.6e-4$	$4.2e-4$	$4.6e-4$	$3.4e-4$

Table 1: L^2 error in time and space for different functional couples $R_k(\mathbf{x})S_k(t)$ involved in the approximation of $u(\mathbf{x}, t)$ and for different mesh densities.

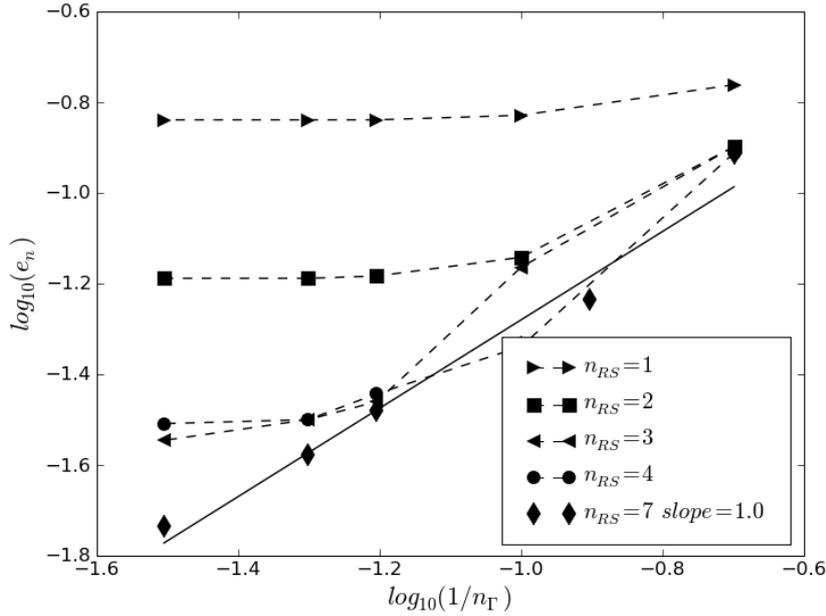


Fig. 2: Evolution of the error e_n as a function of the space discretization and the number of functional couples, n_{RS} , involved in the approximation.

$$e_n = \left\| \sum_{k=1}^n R_k(\mathbf{x})S_k(t) - u^{as}(\mathbf{x}, t) \right\|_{L^2_{\Omega \times \tau}}$$

we can notice that for a number of couples n the error e_n decreases reaching a plateau when n_Γ increases. For reducing this error we must increase n_Γ and recompute more couples $R_k(\mathbf{x})S_k(t)$. In the example here addressed, 6 couples must be considered for reaching a linear convergence of $5 \leq n_\Gamma \leq 32$.

Figures 4 depict $R_1(\mathbf{x})$, $R_2(\mathbf{x})$, $R_3(\mathbf{x})$, $R_4(\mathbf{x})$ and figures 5 show $S_1(t)$, $S_2(t)$, $S_3(t)$, $S_4(t)$ for $n_\Gamma = 5$ and $n_\tau = 32$. Figure 3 depicts the field $u(\mathbf{x}, t)$ the error $u(\mathbf{x}, t) - u^{as}(\mathbf{x}, t)$.

Conclusion

The proposed approach transform an incremental BEM procedure into a decoupled one that needs the solution of some steady problems defined in space (Poisson equation in the case here addressed), and some ordinary differential equations that only involve the time coordinate. Significant reduction of CPU time is expected due to the non-incremental nature of the proposed technique, as well as

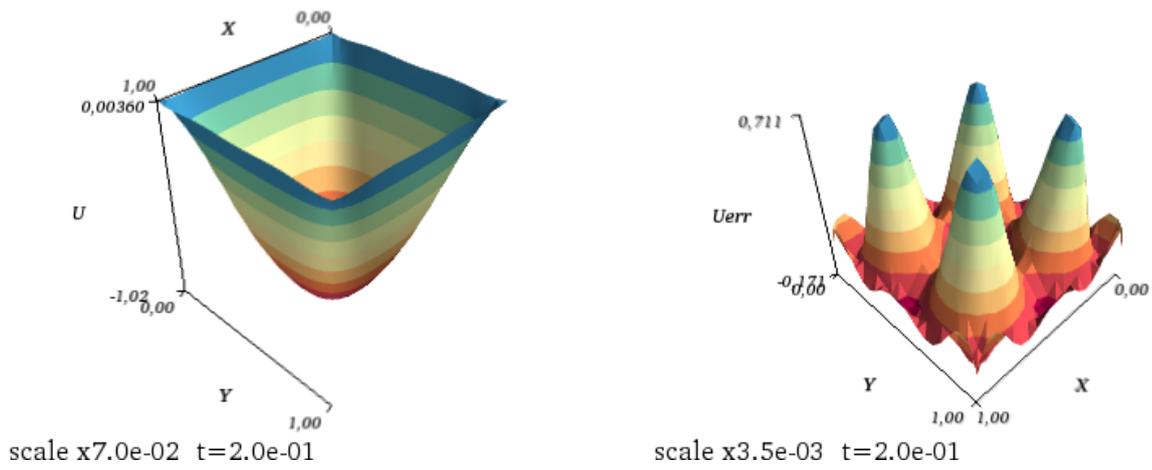


Fig. 3: $u(\mathbf{x},t)$ et $u(\mathbf{x},t) - u^{as}(\mathbf{x},t)$ for $t = 0.2s$ and $n_\Gamma = 5$

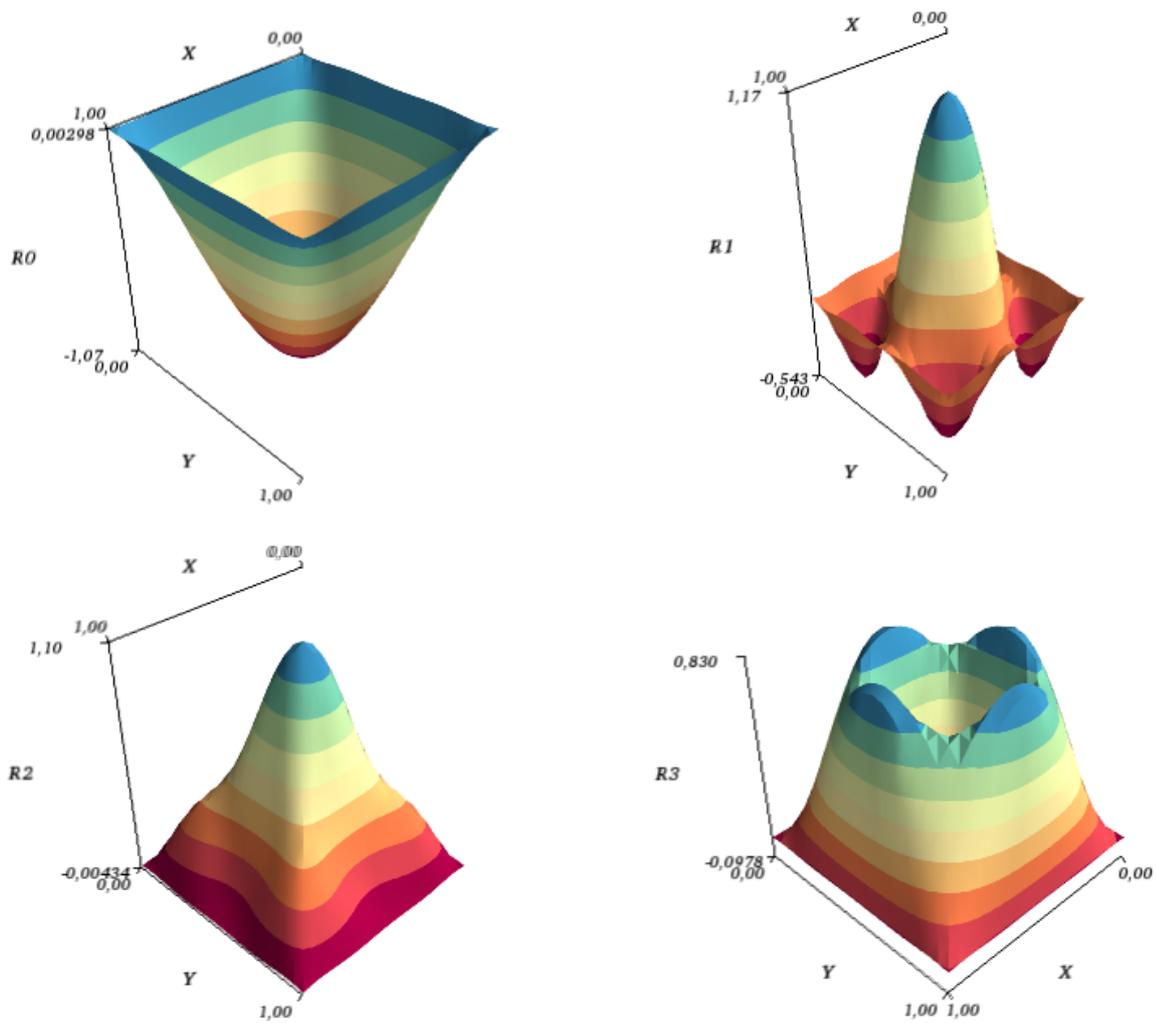


Fig. 4: $R_1(\mathbf{x}), R_2(\mathbf{x}), R_3(\mathbf{x}), R_4(\mathbf{x})$ for $n_\Gamma = 5$

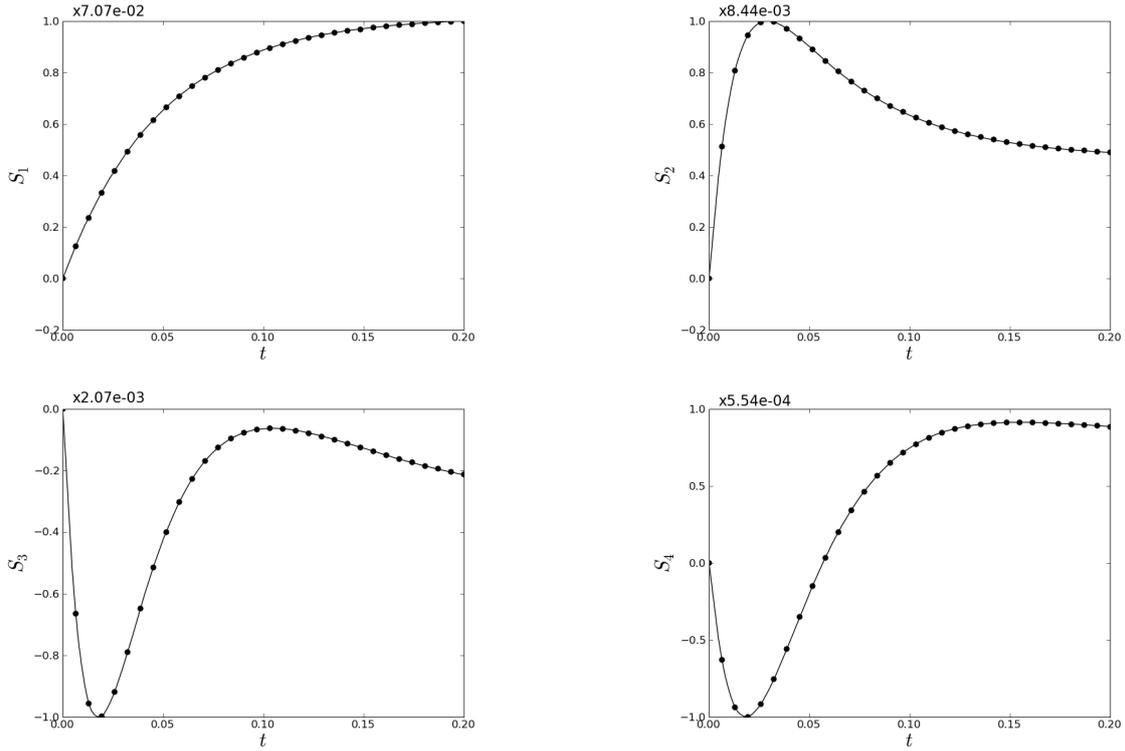


Fig. 5: $S_1(t)$, $S_2(t)$, $S_3(t)$, $S_4(t)$ for $n_\Gamma = 5$

a significant reduction of the amount of information to be stored. Moreover, this technique seems specially well adapted for the treatment of non-linear transient BEM models.

References

- [1] F. Chinesta, A. Ammar, F. Lemarchand, P. Beauchene, and F. Boust. *Alleviating mesh constraints: Model reduction, parallel time integration and high resolution homogenization*. *Computer Methods in Applied Mechanics and Engineering*, 197(5):400 – 413, 2008.
- [2] A. Ammar, F. Chinesta, and P. Joyot. *The nanometric and micrometric scales of the structure and mechanics of materials revisited: An introduction to the challenges of fully deterministic numerical descriptions*. *International Journal for Multiscale Computational Engineering*, 6(3):191–213, 2008.
- [3] A. Ammar, B. Mokdad, F. Chinesta, and R. Keunings. *A new family of solvers for some classes of multidimensional partial differential equations encountered in kinetic theory modeling of complex fluids*. *Journal of Non-Newtonian Fluid Mechanics*, 139(3):153 – 176, 2006.
- [4] A. Ammar, B. Mokdad, F. Chinesta, and R. Keunings. *A new family of solvers for some classes of multidimensional partial differential equations encountered in kinetic theory modelling of complex fluids: Part ii: Transient simulation using space-time separated representations*. *Journal of Non-Newtonian Fluid Mechanics*, 144(2-3):98 – 121, 2007.
- [5] H. S. Carslaw and J. C. Jaeger. *Conduction of Heat in Solids*. *Oxford University Press*, USA, second edition, April 1986.