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# APPLICATION OF THE LIKELIHOOD METHOD TO THE ANALYSIS OF WAVES IN ELASTIC AND VISCOELASTIC RODS

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In this paper, we are interested in separating waves in elastic and viscoelastic rods propagating in opposite direction.  $N$  strain and  $P$  velocity measurements are taken into account. This application of the likelihood method gives a solution in the frequency domain. Using the inverse Fourier transform, one can recover the strain, stress, displacement and velocity at any section of the rod. In experimental conditions, the results are stable against noise when  $N+P>2$  and  $NP \neq 0$ .

## 1 Introduction

In the classical configuration, the loading time in the SHB (Split Hopkinson Bar) system is limited by the length of the bars together with the maximum measured strain in the specimen, because of the need to separate opposite waves propagating in the bar. Hence, for many materials, it is of no interest to carry out tests with the SHB apparatus at medium strain-rates. As mechanical testing machines are limited at much lower strain rates because of sensor oscillations, alternative solutions have been already investigated, in particular the wave separation technique. They are based on a two strain measurement and they take account of wave dispersion<sup>1, 7-8</sup> or not<sup>3,4</sup>. Bussac and al.<sup>2</sup> showed that the noise is amplified on the reconstructed signals when using only two measurements. In this paper, a new separation method using  $N$  strain and  $P$  velocity measurements is presented. It is based on the Maximum of Likelihood principle<sup>2,5,6</sup>.

## 2 Theory

Let us consider an  $L$ -long elastic or viscoelastic bar. In the case of single mode propagating longitudinal waves, the Fourier transform of stress, strain, displacement and velocity are expressed as follows :

$$\begin{aligned}\tilde{\varepsilon}(x, \omega) &= A(\omega)e^{-i\xi(\omega)x} + B(\omega)e^{i\xi(\omega)x}, \\ \tilde{\sigma}(x, \omega) &= E * (\omega) (A(\omega)e^{-i\xi(\omega)x} + B(\omega)e^{i\xi(\omega)x}),\end{aligned}$$

$$\begin{aligned}\tilde{v}(x, \omega) &= \frac{\omega(-A(\omega)e^{-i\xi(\omega)x} + B(\omega)e^{i\xi(\omega)x})}{\xi(\omega)}, \\ \tilde{u}(x, \omega) &= \frac{i(A(\omega)e^{-i\xi(\omega)x} - B(\omega)e^{i\xi(\omega)x})}{\xi(\omega)},\end{aligned}$$

where  $A(\omega)$  and  $B(\omega)$  are the Fourier components of the ascendant and descendant waves at origin, respectively.  $E^*(\omega)$  is the complex Young's modulus and  $\xi(\omega) = k(\omega) + i\alpha(\omega)$  is the complex wave number. The two parameters  $E^*(\omega)$  and  $\xi(\omega)$  are only related to the bar properties (geometry and material). In the following, it is assumed that they are known.

From strain and/or speed measurements, we want to recover  $A(\omega)$  and  $B(\omega)$  so that strain, stress, displacement and velocity can be calculated at any point of the bar, in particular at both ends.

We perform  $N$  strain and  $P$  velocity measurements on the bar. The corresponding record is modelled as the superposition of the exact measurement and a Gaussian white noise:

$$\begin{aligned}\hat{\varepsilon}_J(t) &= \varepsilon(x_J, t) + W_J(t), \quad J = 1, \dots, N, \\ \hat{v}_K(t) &= v(x_K, t) + W_J(t), \quad J = N + 1, \dots, N + P.\end{aligned}$$

The  $N + P$  white noises are supposed to be two-by-two independent. The amplitudes of the noise concerning strain and velocity are denoted  $1/a_\varepsilon$  and  $1/a_v$ , respectively.

In order to estimate the two functions  $A(\omega)$  and  $B(\omega)$ , the Maximum Likelihood Method is used<sup>2</sup>.

We denote  $X_{JK}$ ,  $J = 1, \dots, N + P$  and  $K = 1, \dots, M$ , the random variable corresponding to the noise stored on the measurement made at the station  $J$  at the time  $t = K/f_{ech}$ , where  $M$  is the maximum measured points and  $f_{ech}$  is the sampling frequency.

The likelihood function is given by :

$$V(\hat{\varepsilon}_J(t), \hat{v}_J(t), A(\omega), B(\omega)) = P\left(\bigcap_{\substack{J=1, \dots, N+P \\ K=1, \dots, M}} (X_{JK} = W_J(K/f_{ech}))\right)$$

Since all the noises are white and two-by-two independent, the likelihood function is then expressed as follows :

$$V(\hat{\varepsilon}_J(t), \hat{v}_J(t), A(\omega), B(\omega)) = \prod_{\substack{J=1, \dots, N+P \\ K=1, \dots, M}} P(X_{JK} = W_J(K/f_{ech}))$$

Noises are also Gaussiens, hence :

$$V(\hat{\varepsilon}_j(t), \hat{v}_j(t), A(\omega), B(\omega)) = \prod_{\substack{j=1, \dots, N+P \\ k=1, \dots, M}} \left( \frac{1}{v_j \sqrt{2\pi}} e^{-\frac{(W_j(t) - \mu_j)^2}{2v_j^2}} \right),$$

where  $v_j = 1/a_j$  is the standard deviation and  $\mu_j$  is the mean of the noise  $W_j$ .

This method consists in writing that what is measured corresponds to the most probable event (a particular application is the least-square method). This leads to maximize the function  $V(\hat{\varepsilon}_j(t), \hat{v}_j(t), A(\omega), B(\omega))$  which is equivalent to minimizing the following function:

$$F = \int_{-\infty}^{+\infty} \left\{ \sum_{j=1}^N (a_j)^2 (\hat{\varepsilon}_j(t) - \varepsilon(x_j, t))^2 + \sum_{j=N+1}^{N+P} (a_j)^2 (\hat{v}_j(t) - v(x_j, t))^2 \right\} dt.$$

According to Parseval's theorem:

$$F = \int_{-\infty}^{+\infty} \sum_{j=1}^N (a_j)^2 \left| \tilde{\varepsilon}_j(\omega) - A(\omega) e^{-i\xi(\omega)x_j} - B(\omega) e^{i\xi(\omega)x_j} \right|^2 d\omega \\ + \int_{-\infty}^{+\infty} \sum_{j=1}^N (a_j)^2 \left| \tilde{v}_j(\omega) + \frac{\omega}{\xi(\omega)} (A(\omega) e^{-i\xi(\omega)x_j} - B(\omega) e^{i\xi(\omega)x_j}) \right|^2 d\omega.$$

The function  $F$  is minimized when:

$$A(\omega) = \frac{h_2(\omega)E_1(\omega) - g(\omega)E_2(\omega)}{h_1(\omega)h_2(\omega) - g(\omega)g(\omega)}, \\ B(\omega) = \frac{h_1(\omega)E_2(\omega) - \overline{g(\omega)}E_1(\omega)}{h_1(\omega)h_2(\omega) - g(\omega)g(\omega)},$$

where:

$$h_1(\omega) = \sum_{j=1}^N (a_j)^2 e^{-i(\xi(\omega) - \overline{\xi(\omega)})x_j} + \left| \frac{\omega}{\xi(\omega)} \right|^2 \sum_{k=1}^P (a_k)^2 e^{-i(\xi(\omega) - \overline{\xi(\omega)})y_k}, \\ h_2(\omega) = \sum_{j=1}^N (a_j)^2 e^{i(\xi(\omega) - \overline{\xi(\omega)})x_j} + \left| \frac{\omega}{\xi(\omega)} \right|^2 \sum_{k=1}^P (a_k)^2 e^{i(\xi(\omega) - \overline{\xi(\omega)})y_k}, \\ g(\omega) = \sum_{j=1}^N (a_j)^2 e^{i(\xi(\omega) + \overline{\xi(\omega)})x_j} - \left| \frac{\omega}{\xi(\omega)} \right|^2 \sum_{k=1}^P (a_k)^2 e^{i(\xi(\omega) + \overline{\xi(\omega)})y_k}, \\ E_1(\omega) = \sum_{j=1}^N (a_k)^2 e^{i\overline{\xi(\omega)}x_j} \tilde{\varepsilon}_j(\omega) - \frac{\omega}{\xi(\omega)} \sum_{k=1}^P (a_k)^2 e^{i\overline{\xi(\omega)}y_k} \tilde{v}_k(\omega), \\ E_2(\omega) = \sum_{j=1}^N (a_k)^2 e^{-i\overline{\xi(\omega)}x_j} \tilde{\varepsilon}_j(\omega) + \frac{\omega}{\xi(\omega)} \sum_{k=1}^P (a_k)^2 e^{-i\overline{\xi(\omega)}y_k} \tilde{v}_k(\omega).$$

$A(\omega)$  and  $B(\omega)$  are then calculated.

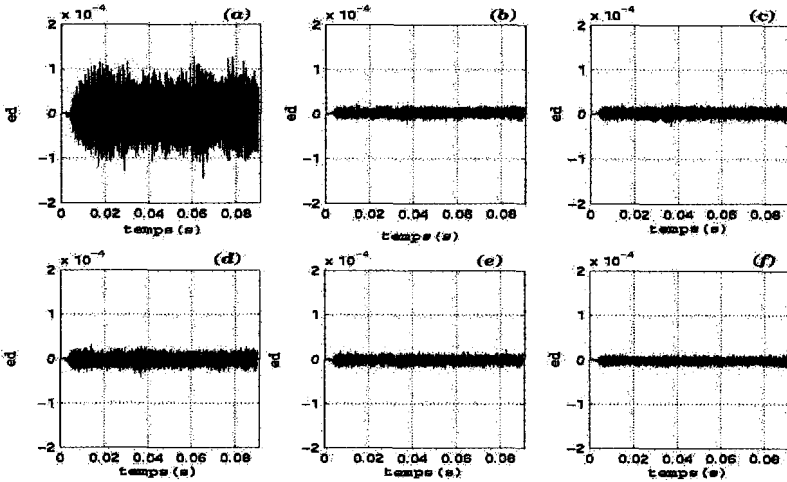
Assuming that mechanical values are constant in a bar cross-section, formulas 1 provide forces and displacements at any point, and in particular at bar ends.

### 3 Numerical simulation

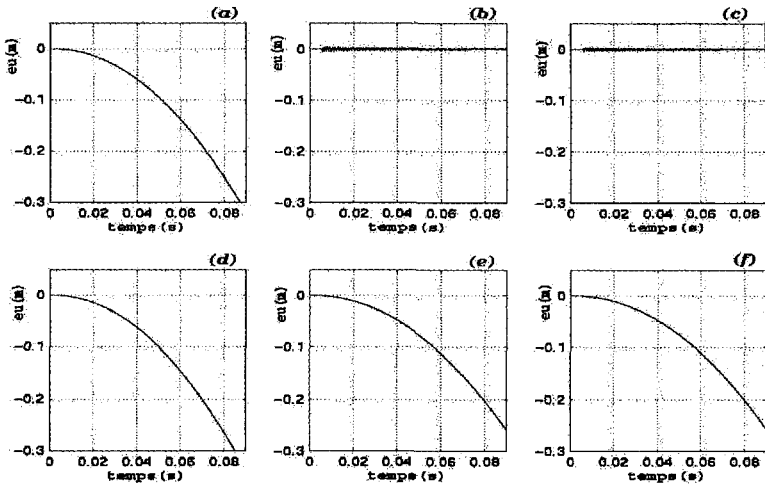
In this section we propose to validate the method with a numerical test. The bar simulated is elastic. It is 3m long and 40 mm in diameter. The Young's modulus is  $E_0 = 70\text{ GPa}$ , the Poisson's ratio is  $\nu = 0.34$ , and volumic mass is  $\rho = 2800\text{ kg/m}^3$ . A 1.2m long striker, having the same characteristic as the bar, is launched at one end of the bar ( $x = 0$ ) at a speed  $V_i = 12\text{ m/s}$ . Five strain measurements at sections  $x_1 = 0.5\text{ m}$ ,  $x_2 = 1.02\text{ m}$ ,  $x_3 = 1.4\text{ m}$ ,  $x_4 = 1.78\text{ m}$ ,  $x_5 = 2.2\text{ m}$  and two velocity measurements at sections  $x_6 = 0.8\text{ m}$ ,  $x_7 = 1.4\text{ m}$  are simulated.

Gaussian noise with amplitude 2 % of the maximum strain or the maximum velocity is added to each measurement. We suppose also that the mean of each noise on strain measurements is not zero and equals 5 ‰ of the maximum strain.

We compare results provided by the method developed in section 2. to exact simulated signals. We denote  $med$  the maximum relative error on strain and  $meu$  the maximum relative error of reconstructed displacements. Figs. 1 and 2 compare the error on strain and displacement for different values of  $N$  et  $P$ . The results show that it is sufficient to use three strain and one velocity measurement to have good accuracy on reconstruct strain and displacement. For experimental application we choose this solution (Tab. 1).



**Figure 1.** Error on strain (a)  $N=2$  and  $P=0$  - (b)  $N=3$  and  $P=1$  - (c)  $N=3$  and  $P=2$  - (d)  $N=3$  and  $P=0$  - (e)  $N=4$  and  $P=0$  - (f)  $N=5$  and  $P=0$



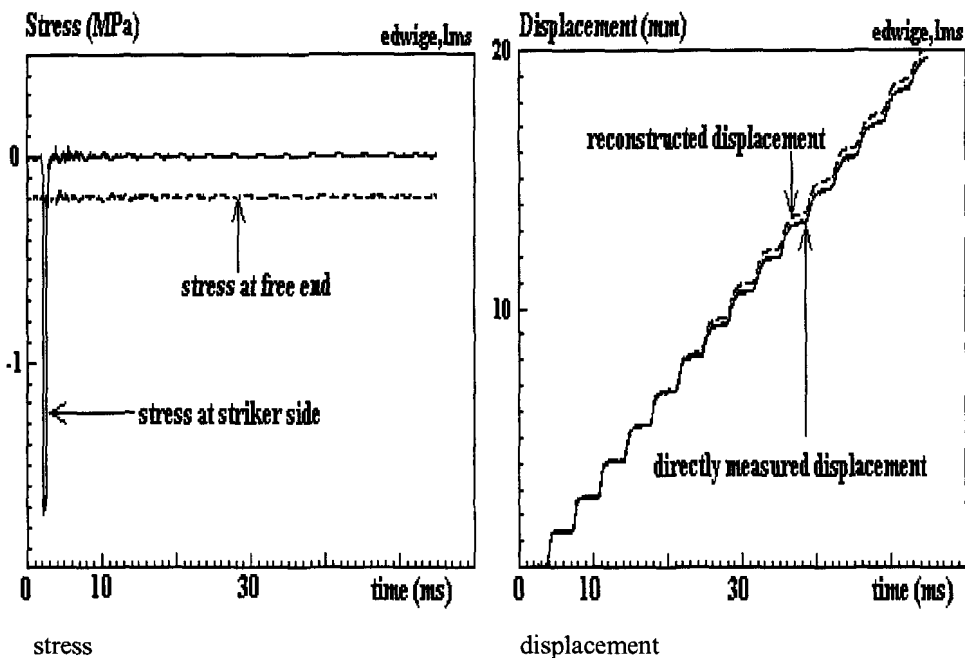
**Figure 2.** Error on displacement (a)  $N=2$  and  $P=0$  - (b)  $N=3$  and  $P=1$  - (c)  $N=3$  and  $P=2$  - (d)  $N=3$  and  $P=0$  - (e)  $N=4$  and  $P=0$  - (f)  $N=5$  and  $P=0$

$N$	2	3	4	5	3	3
$P$	0	0	0	0	1	2
<b>med (%)</b>	11,90	3,50	1,80	1,50	2,00	2,10
<b>meu (%)</b>	71,00	74,00	57,00	56,00	1,10	0,98

**Table 1.** Maximum error on strain and displacement

#### 4 Application to a Nylon bar

The validity of the method is checked using a nylon bar. A nylon striker is launched at the left end of the bar at a speed of  $3.03 \text{ m/s}$ . The right end is free. Three strain and one-velocity measurements are recorded on the bar. We use the method developed in section 2. to reconstruct the stress at two ends of the bar and the displacement at the free end.



**Figure 3.** Reconstructed stresses at the ends of the bar and displacement at the free end (The stress at the free end has been shifted down by 0.2 MPa to make the figure more readable).

The stress at the left end was almost zero as expected. At the right end, the stress became almost zero after the first incident wave. Compared to the amplitude of the impact stress, the error on the reconstructed stress was less than 3.5% (Fig 3). The reconstructed displacement was similar to the directly measured one. The relative error was less than 2.5% (Fig 3).

## 5 Conclusion

A multi-point method (multi-strain and/or multi-velocity measurements) is presented for reconstructing one-dimensional waves in bars. This method is exact when used with the single-mode dispersive propagation model commonly applied to Hopkinson bars. It yields consistent results (the inaccuracy due to imprecise measurements does not increase with time). It is illustrated here by applying it successfully to the analysis of a real test on a Nylon bar. It provides a significant increase in the observation time available when using measuring techniques based on the use of bars such as SHPB set-ups. The method would make it possible to obtain precise measurements at

medium strain rates in a test range in between that of mechanical testing machines and that of Hopkinson bars.

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