

Inequality level set for contact

A new coupled X-FEM/ Level-Set strategy
to solve contact problems

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European Research Council

Established by the European Commission



- 1 Context and tool box
- 2 Theoretical development
- 3 Numerical example
- 4 Conclusion and upcoming





→ Summary

- 1 Context and tool box
- 2 Theoretical development
- 3 Numerical example
- 4 Conclusion and upcoming





Contact is an active research domain!

Several studies and methods :

- ▶ theoretical*,
- ▶ practical†.

Still studied, why?

- ▶ multiple contact zones,
- ▶ represent discontinuities.

*.

†.



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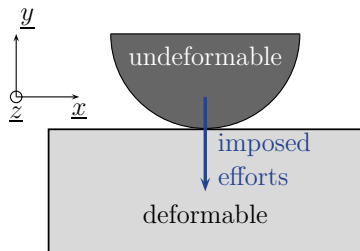
Several studies and methods :

- ▶ theoretical^{*},
- ▶ practical[†].

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To **contact** or not to **contact**, that is the question ?

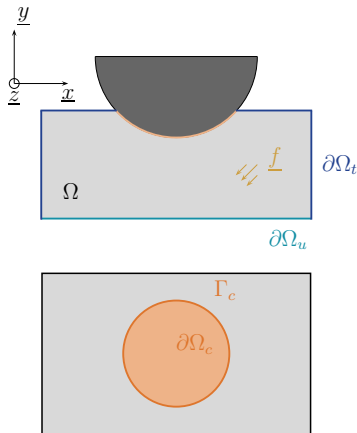


*.

†.

First let's agree on some notations :

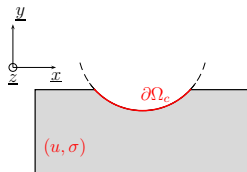
- ▶ Ω domain,
- ▶ $\partial\Omega$ its frontier,
- ▶ $\partial\Omega_c$ the part in contact,
- ▶ Γ_c its boundary,
- ▶ $\partial\Omega_t$ imposed effort t_d ,
- ▶ $\partial\Omega_u$ imposed displacement u_d ,
- ▶ f volumic forces



Our original problem :

- ▶ what is the **equilibrium state** ?
- ▶ what is the **contact zone** ?

Hard to solve !



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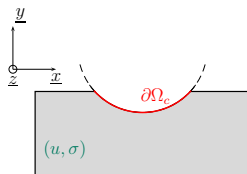
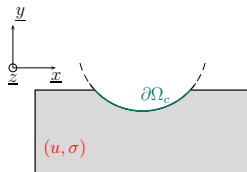
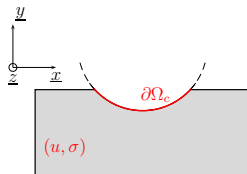
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- ▶ solve **elastic problem** guessing $\partial\Omega_c$,
- ▶ find $\partial\Omega_c$ which verifies the contact conditions.

⇒ iterative process



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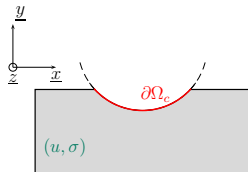
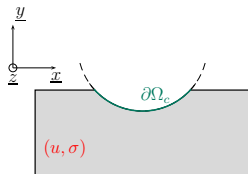
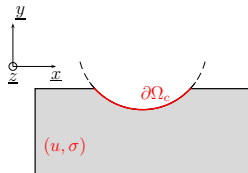
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Question : how to make $\partial\Omega_c$ evolve ?

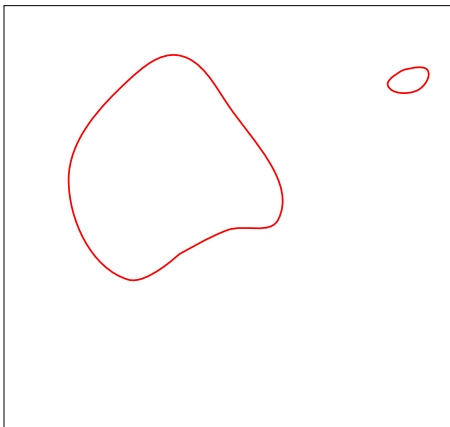




Question : what does our method need to do in order to make $\partial\Omega_c$ evolve ?

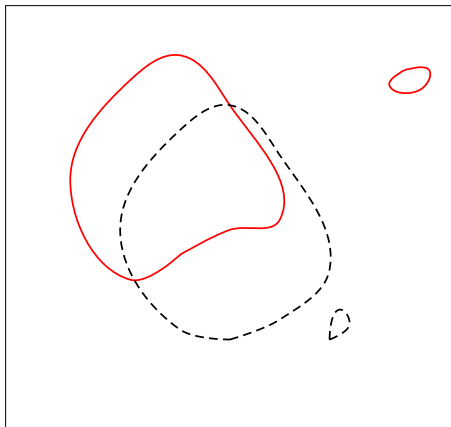


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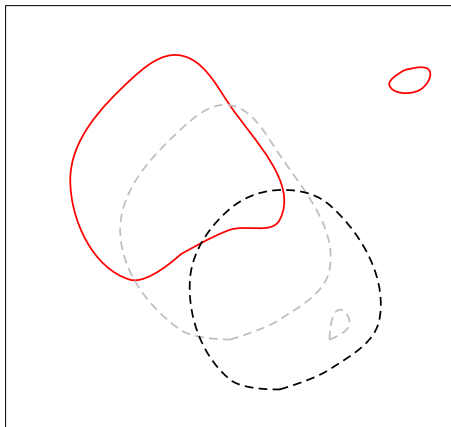




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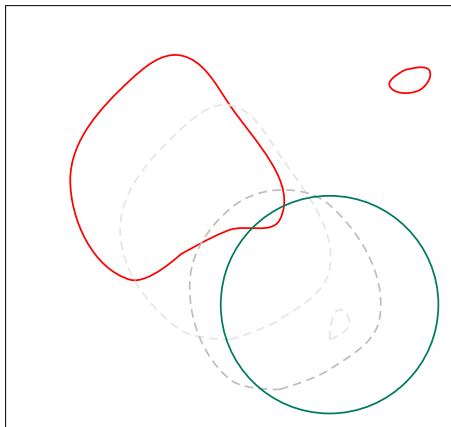


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Criteria :

- ▶ good representation of Γ_c and the phenomena on it,
- ▶ having a criterion for the “good position” of the contact zone,
- ▶ fast convergence,
- ▶ adaptability.



Question : what does our method need to do in order to make $\partial\Omega_c$ evolve ? *

Criteria :

- ▶ good representation of Γ_c and the phenomena on it,
level-set/XFEM
- ▶ having a criterion for the “good position” of the contact zone,
configurational mechanic
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- ▶ adaptability.
level-set

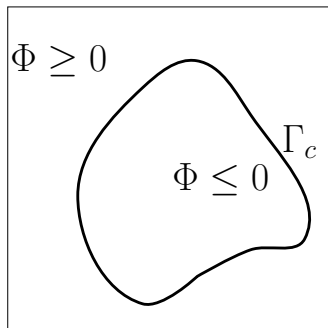


level-set*

signed function :

$$\phi(x) = \pm \min_{x_c \in \Gamma_c} \|x_c - x\| \quad (1)$$

thus the iso-zero level-set
represents $\Gamma_c = \{x / \phi(x) = 0\}$



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XFEM†

exploits partition of unity‡ :

→ take into account

discontinuities on Γ_c → differentiate contact or non
contact domain

*.

†.

‡.



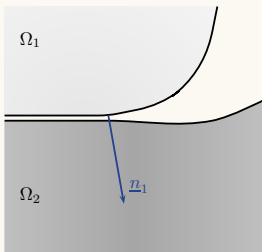
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Hypothesis

- ▶ elastic linear material
- ▶ small displacement
- ▶ frictionless contact



Strong formulation for 2 bodies

- ▶ $\text{div } \underline{\underline{\sigma}} + \underline{f} = \underline{0}$ in Ω
- ▶ \underline{u} KA and $\underline{\underline{\sigma}}$ SA with $\underline{\underline{\sigma}} = \underline{\underline{C}} \underline{\underline{\varepsilon}}$

Karush-Kuhn-Tucker(KKT) conditions :

- ▶ $(\underline{x}_2 - \underline{x}_1) \cdot \underline{n}_1 \geq 0$ on Γ_c
- ▶ $F_{n_1} = -F_{n_2} \leq 0$ on Γ_c
- ▶ $(\underline{x}_2 - \underline{x}_1) \cdot \underline{n}_1 F_{n_1} = 0$ on Γ_c



Variationnal formulation *

Find \underline{u} KA + CC

$$a(\underline{u}, \underline{v}) - l(\underline{v}) \geq 0 \quad \forall \underline{v} \in \text{KA}_0 \quad (2)$$

with :

$$\blacktriangleright a(\underline{u}, \underline{v}) = \int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{v}) d\Omega$$

$$\blacktriangleright l(\underline{v}) = \int_{\Omega} f(\underline{v}) d\Omega + \int_{\Gamma_t} t_d \underline{v} d\Gamma$$

→ variational **inequality**



Variational formulation *

If $\partial\Omega_c$ is given we can relax the inequality with Lagrange multipliers (2) becomes :

Find \underline{u} KA and $\lambda \in \Lambda$ such as :

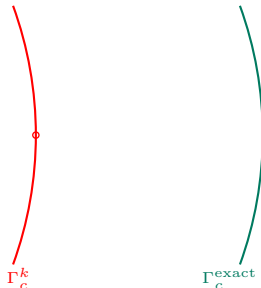
$$\begin{cases} a(\underline{u}, \underline{v}) + b(\lambda, \underline{v}) - l(\underline{v}) = 0 & \forall \underline{v} \text{ KA}_0 \\ b(q, \underline{u}) = l^b(q) & \forall q \in \Lambda \end{cases}$$

Thus we will have a solution which fulfills the **static and kinematic conditions** but not the **contact conditions**.



➔ Evolution of the interface

To fulfill the contact conditions \longrightarrow find the right $\partial\Omega_c$.

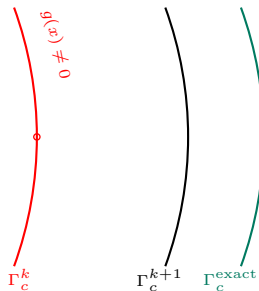




➔ Evolution of the interface

Let's define a positioning criteria g such as :

$$g(\underline{x}) = 0 \text{ on } \Gamma_c \text{ if } \Gamma_c = \Gamma_c^{\text{ex}}$$





➔ Evolution of the interface

12/28

Let's define a positioning criteria g such as :

$$g(\underline{x}) = 0 \text{ on } \Gamma_c \text{ if } \Gamma_c = \Gamma_c^{\text{ex}}$$

Starting from $g(\underline{x}) \neq 0$ we want to find $\underline{\theta}$ s.t. :

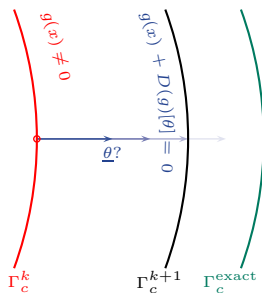
$$g(\underline{x}) + D(g)[\underline{\theta}] = 0 \text{ on } \Gamma_c$$

with the directional derivative of F :

$$DF(a)[b] = \lim_{\varepsilon \rightarrow 0} \frac{F(a + \varepsilon b) - F(a)}{\varepsilon}$$

We can reformulate the problem :

$$\int_{\Gamma_c} g q \, d\Gamma + \int_{\Gamma_c} D(g)[\underline{\theta}] q \, d\Gamma = 0 \quad \forall q \in L^2(\Gamma_c) \quad (2)$$





→ Evolution of the interface

The problem is now to find the right θ .





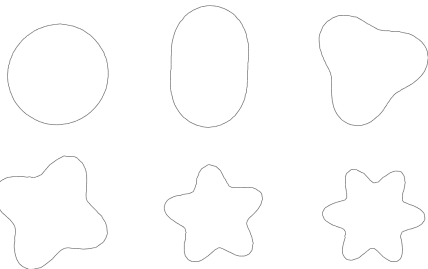
→ Evolution of the interface

The problem is now to find the right $\underline{\theta}$.

In order to do so we need to discretize it .

$$\underline{\theta} \approx \sum_i \theta_i \underline{\Theta}_i \quad (3)$$

with $\{\underline{\Theta}_i\}$, a modal basis





➔ Evolution of the interface

The problem is now to find the **right $\underline{\theta}$** .

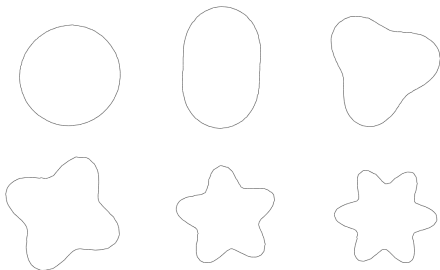
In order to do so we need to **discretize** it .

$$\underline{\theta} \approx \sum_i \theta_i \underline{\Theta}_i \quad (3)$$

with $\{\underline{\Theta}_i\}$, a modal basis

Then equation (2) can be discretized :

$$\int_{\Gamma_c} g q d\Gamma + \int_{\Gamma_c} D(g) \left[\sum_i \theta_i \underline{\Theta}_i \right] q d\Gamma = 0 \quad \forall q \in L^2(\Gamma_c) \approx [S] \{\theta\} = \{P\} \quad (4)$$





➔ Evolution of the interface

Then we need to express $[S]$.

Thus we need to be able to compute $D(g)[\Theta_i]$,

as $g = g(u, \lambda)$ we will actually compute for all i the sensibility of (u, λ)

$$D(\underline{u})[\Theta_i] = \underline{\dot{u}}^i \quad \text{and} \quad D(\underline{\lambda})[\Theta_i] = \underline{\dot{\lambda}}^i$$

To obtain $(\underline{\dot{u}}^i, \underline{\dot{\lambda}}^i)$ we are going to take the **directionnal derivative of the variational formulation** :

$$D \left(\begin{cases} a(\underline{u}, \underline{v}) + b(\underline{\lambda}, \underline{v}) - l(\underline{v}) = 0 & \forall \underline{v} \in \text{KA}_0 \\ b(q, \underline{u}) = l^b(q) & \forall q \in \Lambda \end{cases} \right) [\Theta_i] \quad (5)$$

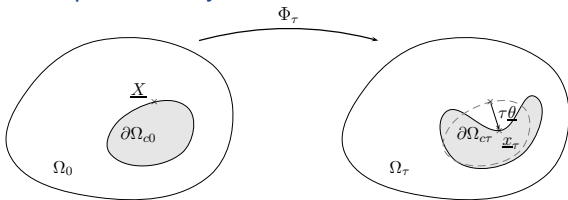


➔ Evolution of the interface

In order to easily express the directional derivative in (2) we are going to assimilate the **change of shape** to a **body deformation** *.

$$\begin{aligned} \Omega_0 &\xrightarrow{\Phi_\tau} \Omega_\tau \\ \underline{X} &\rightarrow \underline{x}_\tau = \underline{X} + \tau \underline{\theta} \end{aligned} \quad (6)$$

with $\underline{\theta} = \theta \underline{n}_{\Gamma_c}$.





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=

$$\begin{cases} a(\underline{\dot{u}}^i, \underline{v}) + b(\underline{\dot{\lambda}}^i, \underline{v}) - t(\underline{v}, \underline{\Theta}_i) = 0 & \forall \underline{v} \in \dot{\text{KA}}_0 \\ b(q, \underline{\dot{u}}^i) = t^b(q, \underline{\Theta}_i) & \forall q \in \dot{\Lambda} \end{cases} \quad (7)$$

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For all i :

find $\underline{\dot{u}}^i \in \text{KA}$ and $\underline{\dot{\lambda}}^i \in \Lambda$ such as :

$$\begin{cases} a(\underline{\dot{u}}^i, \underline{v}) + b(\underline{\dot{\lambda}}^i, \underline{v}) - t(\underline{v}, \underline{\Theta}_i) = 0 & \forall \underline{v} \in \text{KA}_0 \\ b(q, \underline{\dot{u}}^i) = t^b(q, \underline{\Theta}_i) & \forall q \in \Lambda \end{cases} \quad (7)$$

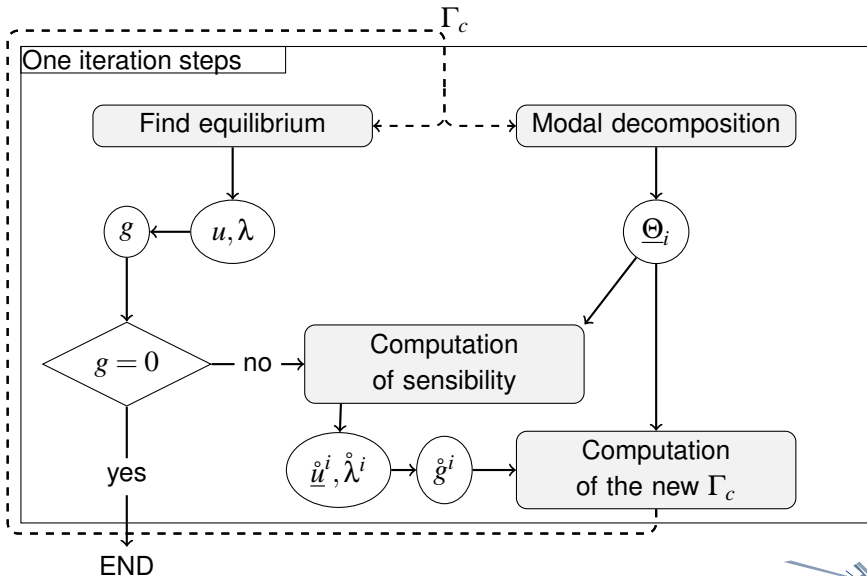
*



$$\begin{aligned}
 t(\underline{v}, \underline{\Theta}_i) &= \frac{1}{4} \int_{\Omega} (\nabla \underline{u} \nabla \underline{\Theta}_i + \nabla \underline{\Theta}_i^T \nabla \underline{u}^T) \underline{\underline{\underline{C}}} (\nabla \underline{v} + \nabla \underline{v}^T) \, d\Omega \\
 &+ \frac{1}{4} \int_{\Omega} (\nabla \underline{u} + \nabla \underline{u}^T) \underline{\underline{\underline{C}}} (\nabla \underline{v} \nabla \underline{\Theta}_i + \nabla \underline{\Theta}_i^T \nabla \underline{v}^T) \, d\Omega \\
 &- \frac{1}{4} \int_{\Omega} (\nabla \underline{u} + \nabla \underline{u}^T) \underline{\underline{\underline{C}}} (\nabla \underline{v} + \nabla \underline{v}^T) \operatorname{div} \underline{\Theta}_i \, d\Omega \\
 &- \int_{\Omega_c} \lambda \underline{v} \operatorname{div} \underline{\Theta}_i \, d\Omega + \int_{\Omega} \underline{\underline{f}} \underline{v} + \underline{f} \underline{v} \operatorname{div} \underline{\Theta}_i \, d\Omega + \int_{\Gamma_c} \underline{\underline{t}}_d \underline{v} + \underline{t}_d \underline{v} \operatorname{div} \underline{\Theta}_i \, d\Gamma
 \end{aligned}$$



Sum up





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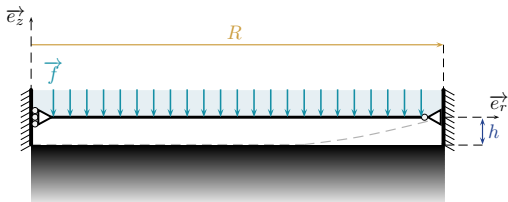
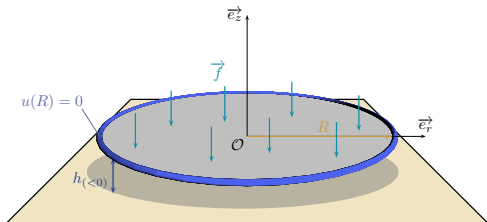
⊕ Membrane with homogeneous loading

$$\begin{cases} T\Delta u + f = 0 & \text{in } \Omega \\ u \leq h & \text{in } \Omega \end{cases} \quad (8)$$

- ▶ T the tension
- ▶ f the surface forces

Taking into account the axisymmetry :

$$\begin{cases} T \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + f = 0 & \text{on } \Omega \\ u \leq h & \text{on } \Omega \end{cases} \quad (9)$$



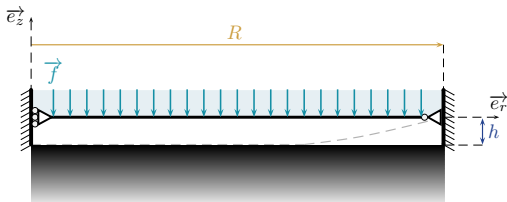
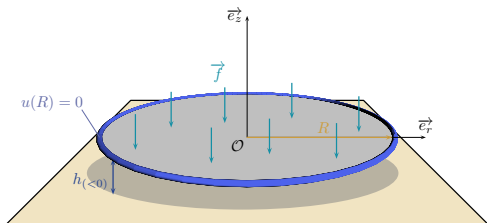
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Why did we pick this problem ?

- ▶ analytical solution (r_c limit of the contact zone)

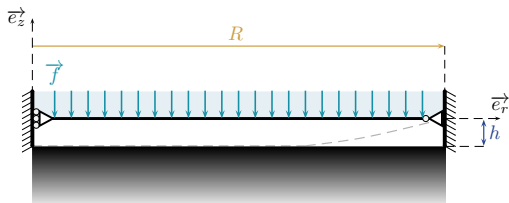
$$u = -\frac{fr^2}{4T} + A \ln\left(\frac{r}{R}\right) + B \quad (10)$$

$$\begin{cases} A = \frac{f(R^2 - r_c^2) - 4hT}{4T \ln(R/r_c)} \\ B = \frac{fR^2}{4T} \\ r_c = R \exp\left(\frac{1}{2} \left(1 + W_{-1}\left(\frac{4hT-f}{ef}\right)\right)\right) \end{cases} \quad (11)$$

- ▶ easy



⊙ Axisymmetric case



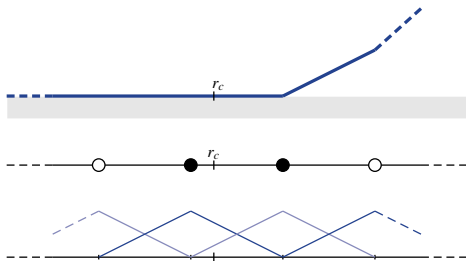


➔ Axisymmetric case

Unknowns of the problem are

u

the displacement.



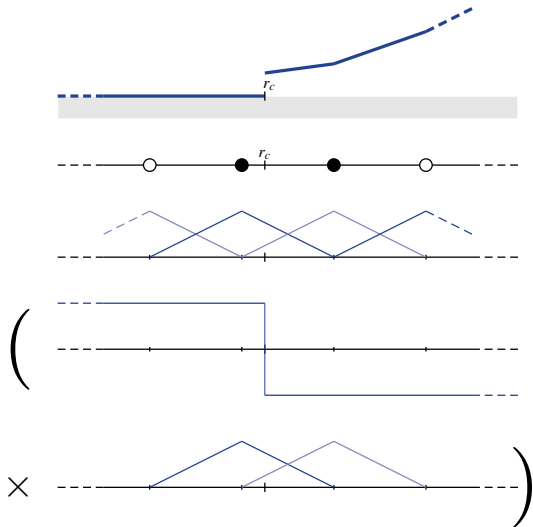


Axisymmetric case

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Axisymmetric case

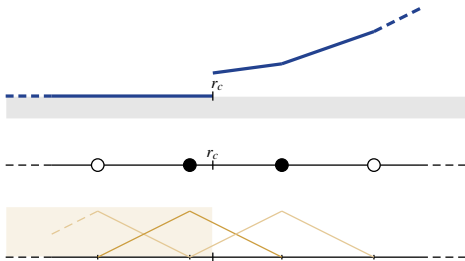
Unknowns of the problem are

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the displacement.

λ

the Lagrange multiplier field
on the left hand side.





⊙ Axisymmetric case

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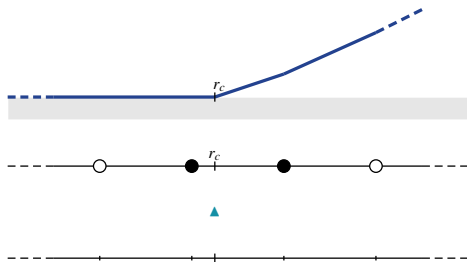
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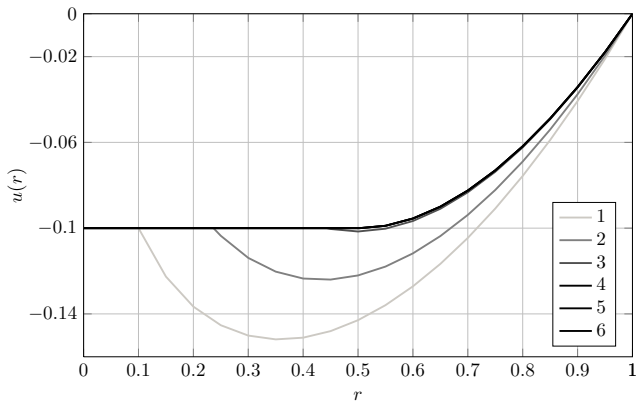
$$\lambda_c = g$$

the Lagrange multiplier at r_c ,
(positioning criteria).





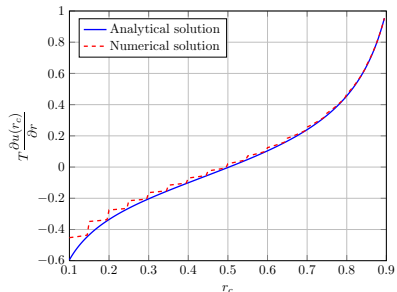
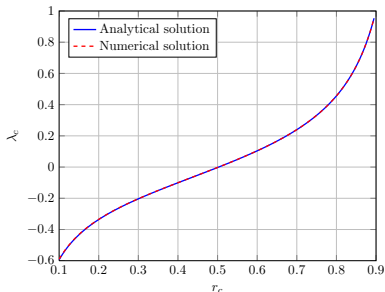
Results



Evolution of the solution with iterations



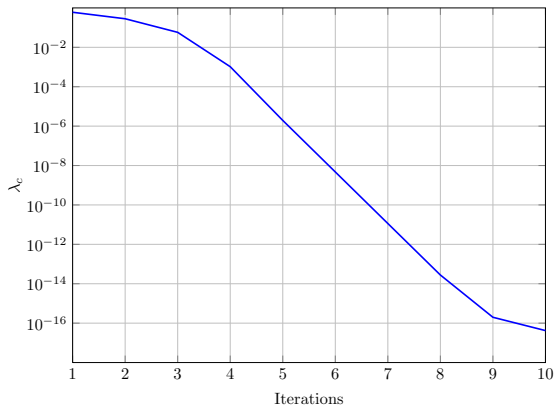
Results



Comparison of some possible positioning criterion showing that λ_c is the **right pick**



→ Results



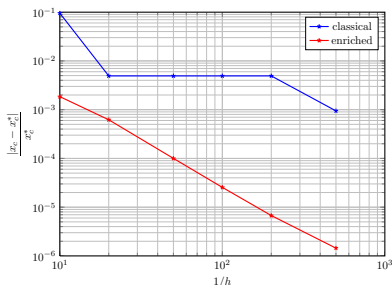
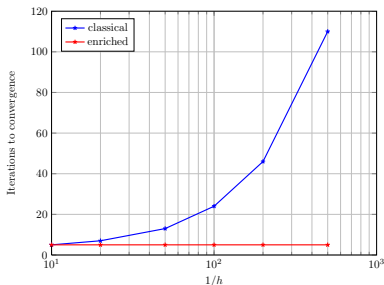
Evolution of the error on λ_c with iterations → quadratic





Comparison with a classical method

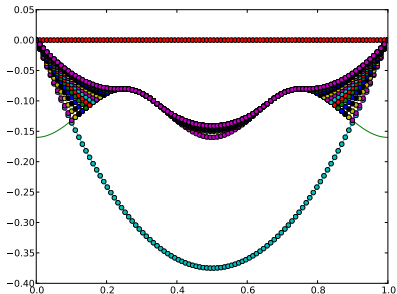
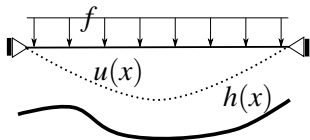
Classical method : if $\lambda > 0$ remove contact, if $u < h$ add contact



Comparison of the number of iterations (left) and the convergence (right) of our method and a rough one.

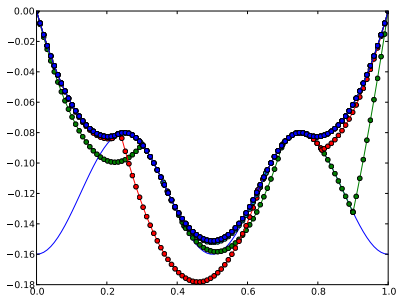
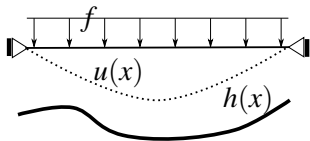


➔ Another example : a rope under self-weight



Result and iterations with a classic active set method.

➔ Another example : a rope under self-weight



Result and iterations with the Inequality Level Set method. (starting from a very wrong guess)



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→ Conclusion and upcoming

- ▶ ILS for contact is working on simple problems
- ▶ promising
 - good representation of the contact zone
 - good convergence





→ Conclusion and upcoming

- ▶ ILS for contact is working on simple problems
- ▶ promising
 - good representation of the contact zone
 - good convergence
- ▶ apply it to 2D membrane
- ▶ punch problem
- ▶ finite deformation

