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Posterior concentration rates for empirical Bayes procedures, with applications to Dirichlet Process mixtures

Supplementary material

Sophie Donnet*, Vincent Rivoirard†, Judith Rousseau‡ and Catia Scricciolo§

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Abstract

1 Gibbs algorithm

We detail the algorithm used to sample from the posterior distribution $(\bar{\lambda}, A, \gamma) | N$ in the Poisson process context, in the most complete case (with a hierarchical level on γ). In case where γ is set to a fixed value, then the corresponding part in the algorithm is removed. As a standard Gibbs algorithm, the simulation is decomposed into three steps:

$$[1] \quad \bar{\lambda} | A, \gamma, N \quad [2] \quad A | \bar{\lambda}, \gamma, N \quad [3] \quad \gamma | A, \bar{\lambda}, N.$$

where N is the observed Poisson process over $[0, T]$, namely a number of jumps $N(T)$ and jump instants $(T_1, \dots, T_{N(T)})$. In order to avoid an artificial truncation in $\bar{\lambda}$, we use the slice sampler strategy proposed by Fall and Barat (2012). More precisely, we consider the stick breaking representation of $\bar{\lambda}$. Let c_i be the affectation variable of data W_i . The DPM model is written as:

$$W_i | c_i, \boldsymbol{\theta}^* \sim h_{\theta_{c_i}^*}, \quad P(c_i = k) = w_k, \forall k \in \mathbb{N}^* \quad (w_k)_{k \in \mathbb{N}^*} \sim \text{Stick}(A), \quad (\theta_k^*)_{k \in \mathbb{N}^*} \sim_{i.i.d} G_\gamma.$$

The slice sampler strategy consists in introducing a latent variable u_i such that the joint distribution of (W_i, u_i) is $p(W_i, u_i | \omega, \boldsymbol{\theta}^*) = \sum_{k=1}^{\infty} w_k h_{\theta_k^*}(W_i) \frac{1}{\xi_k} \mathbb{1}_{[0, \xi_k]}(u_i)$ with $\xi_k = \min(w_k, \zeta)$, which can be reformulated as:

$$p(W_i, u_i | \omega, \boldsymbol{\theta}^*) = \frac{1}{\zeta} \mathbb{1}_{[0, \zeta]}(u_i) \sum_{k=1, w_k > \zeta}^{\infty} w_k h_{\theta_k^*}(W_i) + \sum_{k=1, u_i \leq w_k \leq \zeta}^{\infty} h_{\theta_k^*}(W_i) \mathbb{1}_{[0, w_k]}(u_i) \quad (1.1)$$

$(w_k)_{k \geq 1}$ verifying $\sum_{k \geq 1} w_k = 1$ (implying $\lim_{k \rightarrow \infty} w_k = 0$), the cardinal of $\{k, w_k > \varepsilon\}$ is finite for every $\varepsilon > 0$, and the sum in (1.1) is finite.

Remark 1 *Note that the number of non-null terms in (1.1) is rigorously dependent of the observation index i (denoted K_i^*). But in the algorithm we will deal with the maximum of the K_i^* :*

$$K^* = \max\{K_i^*, i = 1 \dots N(T)\}$$

The Gibbs algorithm with the Slice sampler strategy now takes into account the latent variable $\mathbf{u} = (u_1, \dots, u_{N(T)})$ which is sampled conjointly with $\bar{\lambda}$, resulting into the following steps:

$$[1^*] \quad \bar{\lambda}, \mathbf{u} | A, \gamma, N \quad [2^*] \quad A | \bar{\lambda}, \mathbf{u}, \gamma, N \quad [3^*] \quad \gamma | A, \bar{\lambda}, \mathbf{u}, N.$$

We now detail steps $[1^*]$, $[2^*]$ and $[3^*]$.

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1.1 Details of the algorithm

Initialisation The Gibbs algorithms are initialized on $A^{(0)} = 10$ (0 referring to the iteration number of the Gibbs algorithm). We set: $K^{(0)} = N_T$. for $k = 1 \dots K$, $(\theta^*)_k^{(0)} \sim G_\gamma$. When the hierarchical approach is considered on γ ($\gamma \sim \Gamma(a_\gamma, b_\gamma)$), we initialize γ on its prior mean value: $\gamma^{(0)} = \frac{a_\gamma}{b_\gamma}$.

[1*] Sampling from $\bar{\lambda}, \mathbf{u}|A, \gamma, N$

Step [1*] of the Gibbs algorithm is decomposed into 5 steps which are detailed below. Let $\mathbf{u} = (u_1, \dots, u_{N(T)})$, $\mathbf{c} = (c_1, \dots, c_{N(T)})$, $\boldsymbol{\theta}^* = (\theta_1, \dots, \theta_{K^*})$, $\boldsymbol{\omega} = (w_1, \dots, w_{K^*}) - \mathbf{c}$, $\boldsymbol{\theta}^*$ and $\boldsymbol{\omega}$ representing $\bar{\lambda}$ be the current object. We denote by $K_{N(T)}$ the number of non-empty classes:

$$K_{N(T)} = \#\{k \in \{1 \dots K^*\} | \exists j \in \{1, \dots, N(T)\} \text{ such that } c_j = k\}$$

$\boldsymbol{\theta}^*$ and $\boldsymbol{\omega}$ are ordered such that the elements indexed from $K_{N(T)} + 1$ to K^* correspond to empty classes. \mathbf{u} , \mathbf{c} , $\boldsymbol{\theta}^*$ and $\boldsymbol{\omega}$ are iteratively sampled as follows:

[1*.a] First we sample $\mathbf{u}|\boldsymbol{\omega}, \boldsymbol{\theta}^*, \mathbf{c}, N, A, \gamma$ using the following identities:

$$\begin{aligned} p(\mathbf{u}|\boldsymbol{\omega}, \boldsymbol{\theta}^*, \mathbf{c}, N, A, \gamma) &\propto p(\mathbf{u}, N|\boldsymbol{\omega}, \boldsymbol{\theta}^*, \mathbf{c}) = p(\mathbf{u}, N|\boldsymbol{\theta}^*, \mathbf{c}) = \prod_{i=1}^{N(T)} p(u_i, W_i|c_i, \theta_{c_i}^*) \\ &= \prod_{i=1}^{N(T)} h_{\theta_{c_i}^*}(W_i) \frac{1}{\xi_{c_i}} \mathbb{1}_{[0, \xi_{c_i}]}(u_i) \propto \prod_{i=1}^{N(T)} \frac{1}{\xi_{c_i}} \mathbb{1}_{[0, \xi_{c_i}]}(u_i) \end{aligned}$$

where $\xi_{c_i} = \min(w_{c_i}, \zeta)$. So for every $i = 1 \dots N(T)$, $u_i \sim \mathcal{U}_{[0, \min(w_{c_i}, \zeta)]}$.

[1*.b] Secondly, we sample the weights of the empty classes $(w_k)_{k \geq K_{N(T)}+1} | N, \boldsymbol{\omega}, \mathbf{c}, \boldsymbol{\theta}^*, A, \gamma$. Without the slice sampler strategy, there is an infinite number to sample. But, thanks to the slice sampling, we only need to sample a finite number K^* . The weights of the empty classes follow the prior distribution (stick breaking). For $k > K_{N(T)}$,

$$\begin{aligned} v_k &\sim \mathcal{B}(1, A) \\ w_k &= v_k r_{k-1} \\ r_k &= r_{k-1}(1 - v_k) \end{aligned}$$

As explained in Fall and Barat (2012), we know that we have to represent all the components such that their weights $w_k > u_i$ for all the u_i . Assume that we have sampled w_1, \dots, w_k , then the weights of the following components can not exceed the rest $1 - \sum_{j=1}^k w_j = r_k$. So if r_k is such that $r_k < u_i$, for all $i = 1 \dots n$, i.e. if $r_k < u^*$ with $u^* = \min\{u_1, \dots, u_{N(T)}\}$ we are sure that there is no ‘‘interesting component’’ after that, ‘‘interesting’’ meaning that they won’t appear in joint the distribution of (N, \mathbf{u}) . We can stop and get

$$K^* = \min\{k, r_k < u^*\}.$$

In the end we have sampled $(w_{K_{N(T)}+1}, \dots, w_{K^*})$.

[1*.c] Sample the parameters of the empty classes, $(\theta_{K_{N(T)}+1}^*, \dots, \theta_{K^*}^*)$

$$\forall k = K_{N(T)} + 1, \dots, K^*, \quad \theta_k^* \sim_{i.i.d} G_\gamma$$

[1*.d] Sample the index $\mathbf{c} = (c_1, \dots, c_{N(T)}) | N, \mathbf{u}, \boldsymbol{\theta}^*, \boldsymbol{\omega}$, i.e. affect the observations to the classes $\{1, \dots, K^*\}$. Note that we will get new empty classes. The affectations are done using the following probabilities:

$$\begin{aligned} p(\mathbf{c}|\boldsymbol{\theta}^*, \boldsymbol{\omega}, \mathbf{u}, N) &\propto p(\mathbf{c}, \boldsymbol{\theta}^*, \boldsymbol{\omega}, \mathbf{u}, N) = p(N, \mathbf{u}, \mathbf{c}|\boldsymbol{\theta}^*, \boldsymbol{\omega}) p(\boldsymbol{\omega}) p(\boldsymbol{\theta}^*) \\ &\propto \prod_{i=1}^{N(T)} p(W_i, u_i, c_i|\boldsymbol{\theta}^*, \boldsymbol{\omega}) \end{aligned}$$

So, the c_i are independent and, for $i = 1 \dots N(T)$, for $k = 1 \dots, K^*$

$$p(c_i = k | \boldsymbol{\theta}^*, \omega, \mathbf{u}_i, W_i) = w_{i,k} \propto h_{\theta_k^*}(W_i) \frac{w_k}{\min(\zeta, w_k)} \mathbb{1}_{\{k | u_i < \min(\zeta, w_k)\}}(k)$$

We obtain a new $K_{N(T)}$, which is the number of non-empty classes. We re-arrange the weights and the parameters by order of appearance in this affectation.

[1*.e] Update $(w_1, \dots, w_{K_{N(T)}})$ and $(\theta_1, \dots, \theta_{K_{N(T)}})$ for the non-empty classes.

$$p(\theta_k | \mathbf{u}, N, \omega, \mathbf{c}) \propto G_\gamma(\theta_k) \prod_{i=1, c_i=k}^n h_{\theta_k}(W_i), \quad \forall k = 1 \dots K_{N(T)} \quad (1.2)$$

$$w_1, \dots, w_{K_{N(T)}}, r_{K_{N(T)}} \sim \text{Dir}(n_1, \dots, n_{K_{N(T)}}, A)$$

where $n_k = \#\{i | c_i = k\}$ and $r_{K_{N(T)}} = 1 - \sum_{k=1}^{K_{N(T)}} w_k$.

Note that when G_γ is the inverse of the translated inverse exponential distribution, $p(\theta_k | \mathbf{u}, N, \omega, \mathbf{c})$ given in equation (1.2) is:

$$p(\theta_k | \mathbf{u}, N, \omega, \mathbf{c}) \propto \frac{1}{\left(\frac{1}{\theta_k} - \frac{1}{T}\right)^{a-1}} e^{-\frac{\gamma}{\theta_k - T}} \frac{1}{\theta_k^{n_k}} \mathbb{1}_{\left[\frac{1}{\max_{i|c_i=k} W_i}, +\infty\right]}(\theta_k) \quad (1.3)$$

Its simulation can not be performed directly and we resort to an accept-reject procedure to simulate exactly under this distribution.

Remark 2 *The accept-reject procedure we propose is detailed and discussed here after. Note that (θ_k) changes of dimension at each iteration of the algorithm. As a consequence, a Metropolis-Hastings procedure can not be considered easily, since it could jeopardize the theoretical and practical convergence properties of the algorithm.*

[2*.] **Sampling from $A | \omega, \boldsymbol{\theta}^*, \mathbf{c}, \mathbf{u}, \gamma, N$**

Let $K_{N(T)}$ be the current number of non-empty classes. West (1992) proves that under the prior distribution $A \sim \Gamma(a_\alpha, b_\alpha)$, we have

$$A | x, K_{N(T)} \sim \pi_x \Gamma(a_A + K_{N(T)}, b_A - \log(x)) + (1 - \pi_x) \Gamma(a_A + K_{N(T)} - 1, b_A - \log(x)) \quad (1.4)$$

where

$$x | A, K_{N(T)} \sim \mathcal{B}(A + 1, N(T))$$

$$\frac{\pi_x}{1 - \pi_x} = \frac{a_A + K_{N(T)} - 1}{n(b_A - \log(x))}$$

Note that the generation of the new value of A relies on the current value of A .

[3*.] **Sampling from $\gamma | \omega, \boldsymbol{\theta}^*, \mathbf{c}, \mathbf{u}, A, N$** If a hierarchical level is set on γ , we have to sample

$$\gamma | \boldsymbol{\theta}^*, \omega, \mathbf{u}, N, \mathbf{c}$$

Using the previous conditional distributions, we have:

$$p(\gamma | \boldsymbol{\theta}^*, \omega, \mathbf{u}, N, \mathbf{c}) \propto p(\mathbf{u}, N, \boldsymbol{\theta}^*, \omega, \mathbf{c} | \gamma) \pi(\gamma) = p(\mathbf{u}, N | \boldsymbol{\theta}^*, \mathbf{c}) p(\mathbf{c} | \omega) p(\boldsymbol{\theta}^* | \gamma) \pi(\gamma) = \pi(\gamma) \prod_{k=1}^{K^*} G_\gamma(\theta_k)$$

$$\propto \gamma^{a_\gamma - 1} e^{-b_\gamma \gamma} \prod_{k=1}^{K^*} \gamma^a e^{-\gamma / (\frac{1}{\theta_k^*} - \frac{1}{T})} = \gamma^{a_\gamma + a K^* - 1} e^{-\gamma \left(b_\gamma + \sum_{k=1}^{K^*} \frac{1}{(\frac{1}{\theta_k^*} - \frac{1}{T})} \right)}$$

where K^* is the total number of classes used to represent $\bar{\lambda}$. Finally, we get:

$$\gamma|\boldsymbol{\theta}^*, \omega, \mathbf{u}, N, \mathbf{c} \sim \Gamma\left(a_\gamma + aK^*, b_\gamma + \sum_{k=1}^{K^*} \frac{1}{\left(\frac{1}{\theta_k^*} - \frac{1}{T}\right)}\right) \quad (1.5)$$

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