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# Posterior concentration rates for empirical Bayes procedures, with applications to Dirichlet Process mixtures

## Supplementary material

Sophie Donnet\*, Vincent Rivoirard†, Judith Rousseau‡ and Catia Scricciolo§

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### Abstract

## 1 Gibbs algorithm

We detail the algorithm used to sample from the posterior distribution  $(\bar{\lambda}, A, \gamma) | N$  in the Poisson process context, in the most complete case (with a hierarchical level on  $\gamma$ ). In case where  $\gamma$  is set to a fixed value, then the corresponding part in the algorithm is removed. As a standard Gibbs algorithm, the simulation is decomposed into three steps:

$$[1] \quad \bar{\lambda} | A, \gamma, N \quad [2] \quad A | \bar{\lambda}, \gamma, N \quad [3] \quad \gamma | A, \bar{\lambda}, N.$$

where  $N$  is the observed Poisson process over  $[0, T]$ , namely a number of jumps  $N(T)$  and jump instants  $(T_1, \dots, T_{N(T)})$ . In order to avoid an artificial truncation in  $\bar{\lambda}$ , we use the slice sampler strategy proposed by Fall and Barat (2012). More precisely, we consider the stick breaking representation of  $\bar{\lambda}$ . Let  $c_i$  be the affectation variable of data  $W_i$ . The DPM model is written as:

$$W_i | c_i, \theta^* \sim h_{\theta_{c_i}^*}, \quad P(c_i = k) = w_k, \forall k \in \mathbb{N}^* \quad (w_k)_{k \in \mathbb{N}^*} \sim \text{Stick}(A), \quad (\theta_k^*)_{k \in \mathbb{N}^*} \sim_{i.i.d} G_\gamma.$$

The slice sampler strategy consists in introducing a latent variable  $u_i$  such that the joint distribution of  $(W_i, u_i)$  is  $p(W_i, u_i | \omega, \theta^*) = \sum_{k=1}^{\infty} w_k h_{\theta_k^*}(W_i) \frac{1}{\xi_k} \mathbb{1}_{[0, \xi_k]}(u_i)$  with  $\xi_k = \min(w_k, \zeta)$ , which can be reformulated as:

$$p(W_i, u_i | \omega, \theta^*) = \frac{1}{\zeta} \mathbb{1}_{[0, \zeta]}(u_i) \sum_{k=1, w_k > \zeta}^{\infty} w_k h_{\theta_k^*}(W_i) + \sum_{k=1, u_i \leq w_k \leq \zeta}^{\infty} h_{\theta_k^*}(W_i) \mathbb{1}_{[0, w_k]}(u_i) \quad (1.1)$$

$(w_k)_{k \geq 1}$  verifying  $\sum_{k \geq 1} w_k = 1$  (implying  $\lim_{k \rightarrow \infty} w_k = 0$ ), the cardinal of  $\{k, w_k > \varepsilon\}$  is finite for every  $\varepsilon > 0$ , and the sum in (1.1) is finite.

**Remark 1** Note that the number of non-null terms in (1.1) is rigorously dependent of the observation index  $i$  (denoted  $K_i^*$ ). But in the algorithm we will deal with the maximum of the  $K_i^*$ :

$$K^* = \max\{K_i^*, i = 1 \dots N(T)\}$$

The Gibbs algorithm with the Slice sampler strategy now takes into account the latent variable  $\mathbf{u} = (u_1, \dots, u_{N(T)})$  which is sampled conjointly with  $\bar{\lambda}$ , resulting into the following steps:

$$[1^*] \quad \bar{\lambda}, \mathbf{u} | A, \gamma, N \quad [2^*] \quad A | \bar{\lambda}, \mathbf{u}, \gamma, N \quad [3^*] \quad \gamma | A, \bar{\lambda}, \mathbf{u}, N.$$

We now detail steps  $[1^*]$ ,  $[2^*]$  and  $[3^*]$ .

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## 1.1 Details of the algorithm

*Initialisation* The Gibbs algorithms are initialized on  $A^{(0)} = 10$  (0 referring to the iteration number of the Gibbs algorithm). We set:  $K^{(0)} = N_T$ . for  $k = 1 \dots K$ ,  $(\theta^*)_k^{(0)} \sim G_\gamma$ . When the hierarchical approach is considered on  $\gamma$  ( $\gamma \sim \Gamma(a_\gamma, b_\gamma)$ ), we initialize  $\gamma$  on its prior mean value:  $\gamma^{(0)} = \frac{a_\gamma}{b_\gamma}$ .

[1\*] **Sampling from**  $\bar{\lambda}, \mathbf{u}|A, \gamma, N$

Step [1\*] of the Gibbs algorithm is decomposed into 5 steps which are detailed below. Let  $\mathbf{u} = (u_1, \dots, u_{N(T)})$ ,  $\mathbf{c} = (c_1, \dots, c_{N(T)})$ ,  $\boldsymbol{\theta}^* = (\theta_1, \dots, \theta_{K^*})$ ,  $\boldsymbol{\omega} = (w_1, \dots, w_{K^*}) - \mathbf{c}$ ,  $\boldsymbol{\theta}^*$  and  $\boldsymbol{\omega}$  representing  $\bar{\lambda}$  be the current object. We denote by  $K_{N(T)}$  the number of non-empty classes:

$$K_{N(T)} = \#\{k \in \{1 \dots K^*\} | \exists j \in \{1, \dots, N(T)\} \text{ such that } c_j = k\}$$

$\boldsymbol{\theta}^*$  and  $\boldsymbol{\omega}$  are ordered such that the elements indexed from  $K_{N(T)} + 1$  to  $K^*$  correspond to empty classes.  $\mathbf{u}$ ,  $\mathbf{c}$ ,  $\boldsymbol{\theta}^*$  and  $\boldsymbol{\omega}$  are iteratively sampled as follows:

[1\*.a ] First we sample  $\mathbf{u}|\boldsymbol{\omega}, \boldsymbol{\theta}^*, \mathbf{c}, N, A, \gamma$  using the following identities:

$$\begin{aligned} p(\mathbf{u}|\boldsymbol{\omega}, \boldsymbol{\theta}^*, \mathbf{c}, N, A, \gamma) &\propto p(\mathbf{u}, N|\boldsymbol{\omega}, \boldsymbol{\theta}^*, \mathbf{c}) = p(\mathbf{u}, N|\boldsymbol{\theta}^*, \mathbf{c}) = \prod_{i=1}^{N(T)} p(u_i, W_i|c_i, \theta_{c_i}^*) \\ &= \prod_{i=1}^{N(T)} h_{\theta_{c_i}^*}(W_i) \frac{1}{\xi_{c_i}} \mathbb{1}_{[0, \xi_{c_i}]}(u_i) \propto \prod_{i=1}^{N(T)} \frac{1}{\xi_{c_i}} \mathbb{1}_{[0, \xi_{c_i}]}(u_i) \end{aligned}$$

where  $\xi_{c_i} = \min(w_{c_i}, \zeta)$ . So for every  $i = 1 \dots N(T)$ ,  $u_i \sim \mathcal{U}_{[0, \min(w_{c_i}, \zeta)]}$ .

[1\*.b ] Secondly, we sample the weights of the empty classes  $(w_k)_{k \geq K_{N(T)}+1} | N, \boldsymbol{\omega}, \mathbf{c}, \boldsymbol{\theta}^*, A, \gamma$ . Without the slice sampler strategy, there is an infinite number to sample. But, thanks to the slice sampling, we only need to sample a finite number  $K^*$ . The weights of the empty classes follow the prior distribution (stick breaking). For  $k > K_{N(T)}$ ,

$$\begin{aligned} v_k &\sim \mathcal{B}(1, A) \\ w_k &= v_k r_{k-1} \\ r_k &= r_{k-1}(1 - v_k) \end{aligned}$$

As explained in Fall and Barat (2012), we know that we have to represent all the components such that their weights  $w_k > u_i$  for all the  $u_i$ . Assume that we have sampled  $w_1, \dots, w_k$ , then the weights of the following components can not exceed the rest  $1 - \sum_{j=1}^k w_j = r_k$ . So if  $r_k$  is such that  $r_k < u_i$ , for all  $i = 1 \dots n$ , i.e. if  $r_k < u^*$  with  $u^* = \min\{u_1, \dots, u_{N(T)}\}$  we are sure that there is no ‘‘interesting component’’ after that, ‘‘interesting’’ meaning that they won’t appear in joint the distribution of  $(N, \mathbf{u})$ . We can stop and get

$$K^* = \min\{k, r_k < u^*\}.$$

In the end we have sampled  $(w_{K_{N(T)}+1}, \dots, w_{K^*})$ .

[1\*.c ] Sample the parameters of the empty classes,  $(\theta_{K_{N(T)}+1}^*, \dots, \theta_{K^*}^*)$

$$\forall k = K_{N(T)} + 1, \dots, K^*, \quad \theta_k^* \sim_{i.i.d} G_\gamma$$

[1\*.d ] Sample the index  $\mathbf{c} = (c_1, \dots, c_{N(T)}) | N, \mathbf{u}, \boldsymbol{\theta}^*, \boldsymbol{\omega}$ , i.e. affect the observations to the classes  $\{1, \dots, K^*\}$ . Note that we will get new empty classes. The affectations are done using the following probabilities:

$$\begin{aligned} p(\mathbf{c}|\boldsymbol{\theta}^*, \boldsymbol{\omega}, \mathbf{u}, N) &\propto p(\mathbf{c}, \boldsymbol{\theta}^*, \boldsymbol{\omega}, \mathbf{u}, N) = p(N, \mathbf{u}, \mathbf{c}|\boldsymbol{\theta}^*, \boldsymbol{\omega}) p(\boldsymbol{\omega}) p(\boldsymbol{\theta}^*) \\ &\propto \prod_{i=1}^{N(T)} p(W_i, u_i, c_i|\boldsymbol{\theta}^*, \boldsymbol{\omega}) \end{aligned}$$

So, the  $c_i$  are independent and, for  $i = 1 \dots N(T)$ , for  $k = 1 \dots, K^*$

$$p(c_i = k | \boldsymbol{\theta}^*, \omega, \mathbf{u}_i, W_i) = w_{i,k} \propto h_{\theta_k^*}(W_i) \frac{w_k}{\min(\zeta, w_k)} \mathbb{1}_{\{k | u_i < \min(\zeta, w_k)\}}(k)$$

We obtain a new  $K_{N(T)}$ , which is the number of non-empty classes. We re-arrange the weights and the parameters by order of appearance in this affectation.

[1\*.e ] Update  $(w_1, \dots, w_{K_{N(T)}})$  and  $(\theta_1, \dots, \theta_{K_{N(T)}})$  for the non-empty classes.

$$p(\theta_k | \mathbf{u}, N, \omega, \mathbf{c}) \propto G_\gamma(\theta_k) \prod_{i=1, c_i=k}^n h_{\theta_k}(W_i), \quad \forall k = 1 \dots K_{N(T)} \quad (1.2)$$

$$w_1, \dots, w_{K_{N(T)}}, r_{K_{N(T)}} \sim \text{Dir}(n_1, \dots, n_{K_{N(T)}}, A)$$

where  $n_k = \#\{i | c_i = k\}$  and  $r_{K_{N(T)}} = 1 - \sum_{k=1}^{K_{N(T)}} w_k$ .

Note that when  $G_\gamma$  is the inverse of the translated inverse exponential distribution,  $p(\theta_k | \mathbf{u}, N, \omega, \mathbf{c})$  given in equation (1.2) is:

$$p(\theta_k | \mathbf{u}, N, \omega, \mathbf{c}) \propto \frac{1}{\left(\frac{1}{\theta_k} - \frac{1}{T}\right)^{a-1}} e^{-\frac{\gamma}{\theta_k - \frac{1}{T}}} \frac{1}{\theta_k^{n_k}} \mathbb{1}_{\left[\frac{1}{\max_{i | c_i=k} W_i}, +\infty\right]}(\theta_k) \quad (1.3)$$

Its simulation can not be performed directly and we resort to an accept-reject procedure to simulate exactly under this distribution.

**Remark 2** *The accept-reject procedure we propose is detailed and discussed here after. Note that  $(\theta_k)$  changes of dimension at each iteration of the algorithm. As a consequence, a Metropolis-Hastings procedure can not be considered easily, since it could jeopardize the theoretical and practical convergence properties of the algorithm.*

[2\*.] **Sampling from  $A | \omega, \boldsymbol{\theta}^*, \mathbf{c}, \mathbf{u}, \gamma, N$**

Let  $K_{N(T)}$  be the current number of non-empty classes. West (1992) proves that under the prior distribution  $A \sim \Gamma(a_\alpha, b_\alpha)$ , we have

$$A | x, K_{N(T)} \sim \pi_x \Gamma(a_A + K_{N(T)}, b_A - \log(x)) + (1 - \pi_x) \Gamma(a_A + K_{N(T)} - 1, b_A - \log(x)) \quad (1.4)$$

where

$$x | A, K_{N(T)} \sim \mathcal{B}(A + 1, N(T))$$

$$\frac{\pi_x}{1 - \pi_x} = \frac{a_A + K_{N(T)} - 1}{n(b_A - \log(x))}$$

Note that the generation of the new value of  $A$  relies on the current value of  $A$ .

[3\*.] **Sampling from  $\gamma | \omega, \boldsymbol{\theta}^*, \mathbf{c}, \mathbf{u}, A, N$**  If a hierarchical level is set on  $\gamma$ , we have to sample

$$\gamma | \boldsymbol{\theta}^*, \omega, \mathbf{u}, N, \mathbf{c}$$

Using the previous conditional distributions, we have:

$$p(\gamma | \boldsymbol{\theta}^*, \omega, \mathbf{u}, N, \mathbf{c}) \propto p(\mathbf{u}, N, \boldsymbol{\theta}^*, \omega, \mathbf{c} | \gamma) \pi(\gamma) = p(\mathbf{u}, N | \boldsymbol{\theta}^*, \mathbf{c}) p(\mathbf{c} | \omega) p(\boldsymbol{\theta}^* | \gamma) \pi(\gamma) = \pi(\gamma) \prod_{k=1}^{K^*} G_\gamma(\theta_k)$$

$$\propto \gamma^{a_\gamma - 1} e^{-b_\gamma \gamma} \prod_{k=1}^{K^*} \gamma^a e^{-\gamma / (\frac{1}{\theta_k} - \frac{1}{T})} = \gamma^{a_\gamma + a K^* - 1} e^{-\gamma \left( b_\gamma + \sum_{k=1}^{K^*} \frac{1}{(\frac{1}{\theta_k} - \frac{1}{T})} \right)}$$

where  $K^*$  is the total number of classes used to represent  $\bar{\lambda}$ . Finally, we get:

$$\gamma|\boldsymbol{\theta}^*, \omega, \mathbf{u}, N, \mathbf{c} \sim \Gamma\left(a_\gamma + aK^*, b_\gamma + \sum_{k=1}^{K^*} \frac{1}{\left(\frac{1}{\theta_k^*} - \frac{1}{T}\right)}\right) \quad (1.5)$$

## References

- Fall, M. D. and Barat, É. (2012). Gibbs sampling methods for Pitman-Yor mixture models. MAP5 2012-30.
- West, M. (1992). Hyperparameter estimation in Dirichlet Process Mixture models. Technical report, Duke University.