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# Micromechanical modeling of internal erosion

**Stéphane Bonelli\*** — **Didier Marot\*\***

\* *Cemagref, Unité de Recherche Ouvrages Hydrauliques  
3275, route de Cézanne, CS 400061, F-13182 Aix-en-Provence Cedex 5  
stephane.bonelli@cemagref.fr*

\*\* *GeM UMR CNRS 6183, Institut de Recherche en génie civil et mécanique  
Université de Nantes, IUT de Saint-Nazaire  
58, rue Michel Ange, BP 420, F-44606 Saint-Nazaire Cedex  
didier.marot@univ-nantes.fr*

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*ABSTRACT. Suffusion is the internal erosion process by which finer soil particles are detached from the solid matrix, and transported through constrictions by seepage flow. At the macroscopic scale, this is a bulk erosion process and corresponds to a source term in the mass balance equations. This paper constitutes a step towards bridging the gap between the counter intuitive bulk erosion model and our intuition that erosion is essentially an interfacial process. We suggest that a framework within suffusion can be viewed as a clay/water interface erosion at the microscopic scale. The coefficient of surface erosion appears to be a relevant parameter for the suffusion bulk erosion law. The comparison between the results of the present modeling study and previously published experimental data supports the validity of our approach.*

*RÉSUMÉ. La suffusion est le processus d'érosion interne de détachement de particules fines de sol, et de transport de ces particules par écoulement d'eau entre les particules les plus grosses. A l'échelle macroscopique, ce phénomène est une érosion de volume, et il est représenté par un terme source dans les équations de conservation de la masse. Notre intuition est que l'érosion est un phénomène intrinsèquement interfacial. Ce travail vise à comprendre ce paradoxe apparent. Nous proposons un cadre de travail qui permet de considérer la suffusion comme une érosion d'interface argile/eau à l'échelle microscopique. Le coefficient d'érosion de surface apparaît comme un paramètre pertinent de la loi d'érosion de volume de suffusion. La comparaison entre les résultats de cette modélisation et des résultats expérimentaux publiés confirme la pertinence de notre approche.*

*KEYWORDS: internal erosion, suffusion, interfacial erosion, homogenization.*

*MOTS-CLÉS: érosion interne, suffusion, érosion interfaciale, homogénéisation.*

## 1. Introduction

Erosion is the removal of material caused by the eroding power of the flow and is essentially an interfacial process. Suffusion (or suffosion) is an internal erosion process by which finer soil particles are detached from the solid matrix and transported through pore constrictions by seepage flow. The term suffusion (L. suffossio, from suffodere, to dig under) was introduced by Pavlov in 1898. The internal instability that results from suffusion gives rise to a wide range of hazards including piping and sinkholes (Fell and Fry, 2007; Gutiérrez *et al.*, 2008).

Two scales must be considered: a micro-scale defined at the level of the pore constrictions, and a macro-scale corresponding to the Representative Elementary Volume (REV). The REV is typically a laboratory sample, or a spatial integration point in the Finite Element Method (*e.g.* a Gauss point).

The process of bulk erosion may be considered at the macro-scale as a transition from solid-like to fluid-like behaviour. This transition is smooth, and usually described by using a three-phase model (solid, fluid and fluidized solid). These three phases interact while being constrained by the balance equations. For the sake of clarity, the equations are written here in one-dimensional evolution and dilute suspension flow as follows (Papamichos *et al.*, 2001; Papamichos and Vardoulakis, 2005; Papamichos, 2010; Wan and Wang, 2004):

$$\frac{\partial \phi}{\partial t} = \frac{r}{\rho_{Clay}} \quad (\text{solid mass balance equation}) \quad [1]$$

$$\frac{\partial q}{\partial X} = 0 \quad (\text{pore-fluid mass balance equation}) \quad [2]$$

$$-\frac{\partial p}{\partial X} = F \quad (\text{pore-fluid momentum balance equation}) \quad [3]$$

$$F = \frac{\eta_w}{\kappa} q \quad (\text{Darcy constitutive law}) \quad [4]$$

In these equations,  $\phi$  is the porosity,  $\rho_{Clay}$  is the eroded material density,  $q$  is the seepage velocity,  $p$  is the water pressure,  $F$  represents the mechanical interaction between the pore-fluid and the solid matrix,  $\eta_w$  is the water viscosity and  $\kappa$  is the geometric permeability.

The term  $r$  is introduced into the mass balance Equation [1] to describe the detachment of particles via an erosion law. The description of suffusion at the macroscopic scale therefore corresponds to a bulk erosion process. The erosion constitutive law is introduced as usual by phenomenological analysis, at the macroscale.

Few attempts have been made to relate macroscopic and microscopic quantities in a multi-scale description of the suffusion process. This paper is intended as a further step in this direction, on the basis of the results published by Bouddour *et al.* (1996). What follows is not a program aimed at reformulating the theory of suffusion as a whole based on homogenization, rather attention focuses explicitly on the goal of bridging the gap between the counter intuitive bulk erosion model and our intuition that erosion is essentially an interfacial process.

## 2. Description of the representative elementary volume

Let us consider a representative elementary volume  $\Omega$  (REV) of a soil partitioned into three disjoint regions  $\Omega = \Omega_{Pore} \cup \Omega_{Clay} \cup \Omega_{Sand}^s$  (Figure 1):

i) a connected meso-pore domain  $\Omega_{Pore}$  of boundary  $\Gamma_{Pore} = \partial\Omega_{Pore}$ , with volume fraction  $\phi$  and specific surface  $S_{Pore}$

$$\phi = \frac{|\Omega_{Pore}|}{|\Omega|}, \quad S_{Pore} = \frac{|\Gamma_{Pore}|}{|\Omega_{Pore}|} \quad [5]$$

ii) a clay matrix domain  $\Omega_{Clay}$  of volume fraction of boundary  $\Gamma_{Clay} = \partial\Omega_{Clay}$ , with volume fraction  $\phi_V$  and specific surface  $S_{Clay}$

$$\phi_V = \frac{|\Omega_{Clay}|}{|\Omega|}, \quad S_{Clay} = \frac{|\Gamma_{Clay}|}{|\Omega_{Clay}|} \quad [6]$$

iii) a sand grain domain  $\Omega_{Sand}^s$  of volume fraction of boundary  $\Gamma_{Sand} = \partial\Omega_{Sand}$ , with porosity  $\phi_{Sand}$  and specific surface  $S_{Sand}$

$$1 - \phi_{Sand} = \frac{\Omega_{Sand}^s}{|\Omega|}, \quad S_{Sand} = \frac{|\Gamma_{Sand}|}{|\Omega_{Sand}^s|} \quad [7]$$

The meso-pore domain contains an incompressible two-phase flow of water and clay particles, assumed to be a dilute suspension (density  $\rho_w$ , viscosity  $\eta_w$ ). The clay matrix, which is considered as a continuum, consists of two phases: water, and clays particles  $\Omega_{Clay}^s$  (volume fraction  $1 - \phi_{Clay} = \Omega_{Clay}^s / \Omega_{Clay}$ ).

Here, the superscript  $s$  in  $\Omega_{Sand}^s$  and  $\Omega_{Clay}^s$  denotes the solid constituent, and not the solid phase. For the clay domain, the solid phase  $\Omega_{Clay}$  is the sum of the solid constituent  $\Omega_{Clay}^s$  and the micro-pores, saturated by water. All solid constituents are of density  $\rho_s$ . The sand grains are homogeneous and impervious. The clay matrix

density is  $\rho_{Clay} = (1 - \phi_{Clay})\rho_s + \phi_{Clay}\rho_w$ . From Figure 1, we infer the important connection:

$$\varphi_V = \phi_{sand} - \phi \quad [8]$$

The porosity  $\phi + \varphi_V\phi_{Clay}$  can be divided into macro-porosity  $\phi$  corresponding to the meso-pore domain, and micro-porosity  $\phi_{Clay}$  in the clay matrix. This description accounts for two observation scales:

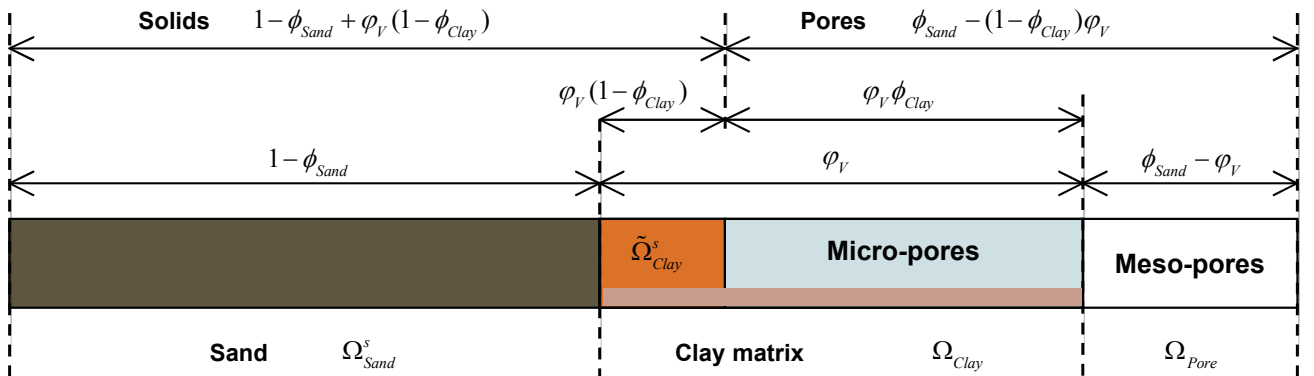
(i) local (microscopic) scale, which is associated with the characteristic size of heterogeneities (clay aggregates, sand grains, meso-pores);

(ii) the macroscopic scale, which corresponds to entire soil layers and is most important for practical purposes.

The scale corresponding to clay particles and micro-pores in the clay matrix is not considered. This type of heterogeneous soil is similar to the classical double-porosity model for modelling flow in fractured porous media, developed in the early 1960s. However, while the meso-pore domain makes up only a small percentage of the total pore volume, it transmits a major portion of the flow through the REV, and the fluid exchange between the clay matrix and the meso-pore domain can be neglected.

We assume that the soil is a clayey sand (Revil and Cathles, 1999; Revil *et al.*, 2002), and that the size of the meso-pores is much greater than the size of the micro-pores. Therefore,  $\varphi_V < \phi_{Sand}$  and the sand matrix is connected while the clay matrix may not be.

We assume that erosion occurs only at the interface  $\Gamma_{Clay} \cap \Gamma_{Pore}$  of the clay matrix and the meso-pore domain (sand grains are not erodible) (Figure 2). The erosion process is therefore external to the clay matrix, and does not affect its porosity  $\phi_{Clay}$ , nor its density  $\rho_{Clay}$ .



**Figure 1.** Relative proportions of each constituent for a sand-clay mixture in the clayey sand domain

As illustrated in Figure 3, suffusion may occur when  $0 < \phi_V < \phi_{sand}$ , whereas no suffusion occurs when  $\phi_V = 0$  (no clay) or  $\phi_V = \phi_{Sand}$  (the pore constrictions are too small).

### 3. Microscopic equations of Stokes flow with interfacial erosion

The mechanical state is such that effective stresses and matrix deformations are negligible. Physico-chemical effects (like dissolution or deposition) are not considered. The porosities  $\phi_{Sand}$  and  $\phi_{Clay}$  are thus constant. The Reynolds number is small and inertia is neglected.

We take  $a(\mathbf{x})$  to denote any quantity  $a$  varying at the micro-scale. The Stokes equations for the pore flow within  $\Omega_{Pore}$  are:

$$\underbrace{\nabla \cdot \mathbf{u}} = 0 \quad \text{in } \Omega_{Pore} \quad [9]$$

Fluid mass balance equation

$$\underbrace{\nabla \cdot \mathbf{T}} = 0 \quad \text{in } \Omega_{Pore} \quad [10]$$

Fluid momentum balance equation

$$\underbrace{\mathbf{T} = -p\mathbf{I} + 2\eta_w \mathbf{D}(\mathbf{u}), \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + {}^T \nabla \mathbf{u})}_{\text{Fluid constitutive law}} \quad \text{in } \Omega_{Pore} \quad [11]$$

The jump equations on the sand/water interface are

$$\underbrace{\mathbf{u} \cdot \mathbf{n}} = 0 \quad \text{on } \Gamma_{Sand} \cap \Gamma_{Pore} \quad [12]$$

Total mass jump equation

$$\underbrace{[[\mathbf{T} \cdot \mathbf{n}]]} = 0 \quad \text{on } \Gamma_{Sand} \cap \Gamma_{Pore} \quad [13]$$

Momentum jump equation

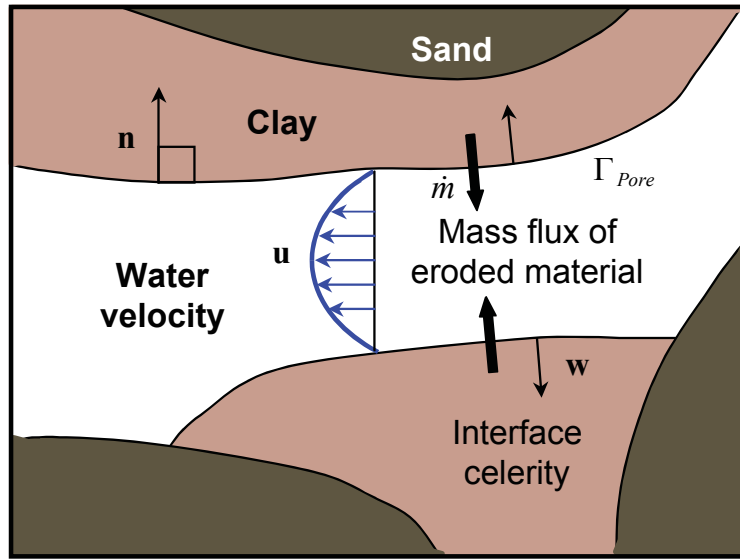
The jump equations on the clay/water interface are (Bonelli and Brivois, 2006; Brivois *et al.*, 2007)

$$\underbrace{[[\rho(\mathbf{w} - \mathbf{u}) \cdot \mathbf{n}]]} = 0 \quad \text{on } \Gamma_{Clay} \cap \Gamma_{Pore} \quad [14]$$

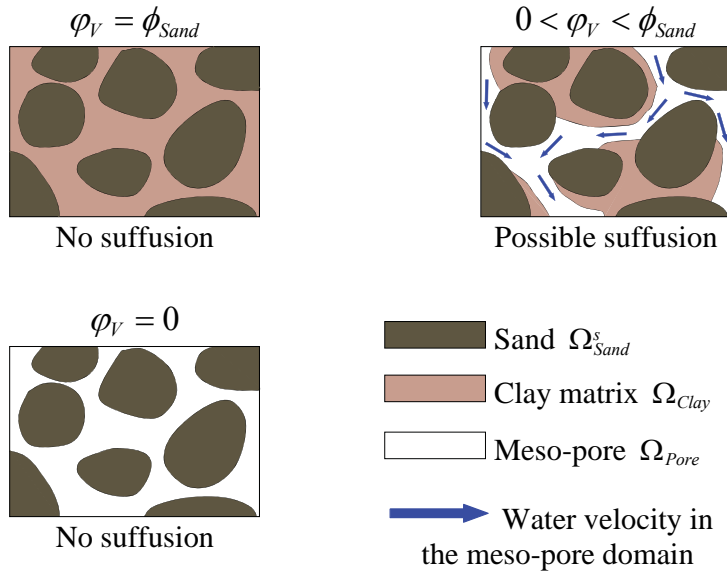
Total mass jump equation

$$\underbrace{[[\rho \mathbf{u}(\mathbf{w} - \mathbf{u}) \cdot \mathbf{n}]]} = -[[\mathbf{T} \cdot \mathbf{n}]] \quad \text{on } \Gamma_{Clay} \cap \Gamma_{Pore} \quad [15]$$

Momentum jump equation



**Figure 2.** Schematic of the suffusion process viewed as the surface erosion of the clay matrix



**Figure 3.** Occurrence of suffusion in a clayey sand

In these equations,  $\mathbf{u}$  is the mass-weighted average velocity,  $\mathbf{T}$  the Cauchy stress tensor,  $p$  the pressure,  $\mathbf{n}$  the normal unit vector of  $\Gamma_{Pore}$ ,  $\mathbf{w}$  the velocity of  $\Gamma_{Pore}$ , and  $[[a]]$  the jump of any physical variable  $a$  across  $\Gamma_{Pore}$ . The total flux of eroded material (both particles and water) crossing  $\Gamma_{Pore}$  is  $\dot{m} = \rho_{Clay} \mathbf{w} \cdot \mathbf{n}$ .

Shear induced interfacial erosion is usually described with threshold laws such as (Ariathurai, 1986; Knapen, 2007; Partheniades, 1965; Zhu, 2001):

$$\dot{m} = k_{er} H[\tau - \tau_c](\tau - \tau_c) \text{ on } \Gamma_{Clay} \cap \Gamma_{Pore} \quad [16]$$

where  $\tau_c$  (Pa) is the threshold stress,  $k_{er}$  (s/m) is the coefficient of surface erosion,  $H[\bullet]$  is the Heaviside step function,  $\tau = \|\boldsymbol{\tau}\|$  is the tangential shear stress at the interface, with  $\boldsymbol{\tau} = [\mathbf{I} - \mathbf{n} \otimes \mathbf{n}] \cdot \mathbf{T} \cdot \mathbf{n}$ .

An additional (and usual) constitutive assumption on the interface is that all tangential velocities are assumed to be continuous across  $\Gamma_{Pore}$ , implying the continuity of  $\boldsymbol{\tau}$  across  $\Gamma_{Pore}$ :

$$[[\mathbf{u}_T]] = 0, \quad \mathbf{u}_T = [\mathbf{I} - \mathbf{n} \otimes \mathbf{n}] \cdot \mathbf{u} \text{ on } \Gamma_{Pore} \quad [17]$$

#### 4. Inferring a suffusion law by means of spatial averaging

The complete set of Equations [9]-[17] has been previously used to study different systems by means of periodic homogenization (Bouddour *et al.*, 1996). Under the dilute flow assumption, the system obtained is formally similar to Equations [1]-[4]. Bouddour *et al.* (1996) established that the global volume flux of eroded material  $r$  Equation [1] is related to the local surface flux  $\dot{m}$  Equation [16] of eroded material as follows:

$$\frac{1}{|\Omega|} \int_{\Gamma_{Pore}} \dot{m} d\Gamma = r \quad [18]$$

The surface of eroded material is

$$\Gamma_{er} = \left\{ \mathbf{x} \in \Gamma_{Pore} \cap \Gamma_{Clay} / \tau(\mathbf{x}) > \tau_c(\mathbf{x}) \right\} \quad [19]$$

Equation [18] involves the spatial average of the stress  $\tau - \tau_c$  on  $\Gamma_{er}$ . As  $\Gamma_{er}$  depends on  $\tau$ , this spatial average is not straightforward. The threshold stress  $\tau_c$  is not considered, as the average of a local threshold law involving local stresses is known not to be a threshold law involving the average of the local stresses. We propose a first order approximation leading to:

$$\int_{\Gamma_{Pore}} H[\tau - \tau_c](\tau - \tau_c) d\Gamma = \frac{|\Gamma_{er}|}{|\Gamma_{Pore}|} \int_{\Gamma_{Pore}} \tau d\Gamma \quad [20]$$

In order to model the average shear stress on  $\Gamma_{Pore}$ , let  $|\Delta P / L| \mathbf{e}$  the macroscopic pressure gradient denoted by  $\partial p / \partial X$  in Equation [3], with magnitude  $|\Delta P / L|$  and orientation  $\mathbf{e}$  ( $\|\mathbf{e}\| = 1$ ). This is the leading term of the microscopic Stokes flow, as the microscopic balance equation is  $\nabla \cdot \mathbf{T} = |\Delta P / L| \mathbf{e}$ . As  $p$  in Equation [11] now denotes only the pressure fluctuation, this gives the well-known result:



$$\frac{1}{|\Omega|} \int_{\Gamma_{Pore}} 2\eta_w \mathbf{D}(\mathbf{u}) \cdot \mathbf{n} d\Gamma = \phi \left| \frac{\Delta P}{L} \right| \mathbf{e} \quad [21]$$

The local (viscous) shear stress is defined as follows (Bouddour *et al.*, 1996):

$$2\eta_w \mathbf{D}(\mathbf{u}) \cdot \mathbf{n} = \xi \left| \frac{\Delta P}{L} \right|, \quad \xi_i = \left( \frac{\partial K_{ip}}{\partial x_q} + \frac{\partial K_{qp}}{\partial x_i} \right) e_p n_q \quad [22]$$

where  $\mathbf{K}$  is the microscopic geometric permeability tensor. The volume average of  $\mathbf{K}$  gives the macroscopic permeability tensor. Equation [21] shows that the spatial average of  $\xi$  is equal to  $\phi \mathbf{e}$ , which gives:

$$\left\| \frac{1}{|\Omega|} \int_{\Gamma_{Pore}} \xi d\Gamma \right\| = \phi \quad [23]$$

The spatial average of  $\tau$  is more complicated:

$$\frac{1}{|\Omega|} \int_{\Gamma_{Pore}} \tau d\Gamma = \zeta \phi \left| \frac{\Delta P}{L} \right| \quad [24]$$

$$\zeta = \frac{\int_{\Gamma_{Pore}} \|\xi_T\| d\Gamma}{\left\| \int_{\Gamma_{Pore}} \xi d\Gamma \right\|} = \frac{1}{|\Omega_{Pore}|} \int_{\Gamma_{Pore}} \|\xi_T\| d\Gamma \quad [25]$$

$$\xi_T = [\mathbf{I} - \mathbf{n} \otimes \mathbf{n}] \cdot \xi \quad \text{on } \Gamma_{Pore} \quad [26]$$

Thorough analysis performed by means of numerical simulations (Golay *et al.*, 2010; Golay *et al.*, 2011), which is not detailed here for conciseness, shows that the normal component of the viscous stress vector on  $\Gamma$  is usually several orders of magnitude lower than the tangential component ( $2\eta_w |\mathbf{n} \cdot \mathbf{D}(\mathbf{u}) \cdot \mathbf{n}| \ll \tau$ ). At the first order we can consider that  $\|\xi_T\| \approx \|\xi\|$ , and that  $\zeta \geq 1$ .

The orientation of the microscopic velocity is given by  $\mathbf{u} \propto \mathbf{K} \cdot \mathbf{e}$  as  $\mathbf{u} = -\eta_w^{-1} \mathbf{K} \cdot \mathbf{e} |\Delta P / L|$  (Bouddour *et al.*, 1996). If this microscopic velocity has mostly the same direction as the macroscopic pressure gradient ( $\mathbf{u} \propto \mathbf{e}$ ), then the use of Equation [22] shows that  $\zeta \approx 1$  (considering that  $\|\xi_T\| \approx \|\xi\|$ ). This situation can be viewed as an isotropic behaviour with weak inhomogeneities and low tortuosity.

In reality, the porous domain of a soil comprises a mixture of pores of different shapes and orientations. This porous domain is modified by the microscopic erosion process. Suffusion is likely to induce anisotropy, but permeability anisotropy is also likely to affect the suffusion process. The quantity  $\zeta$  accounts for the relationship

between the local permeability gradient and the global pressure gradient. This parameter certainly plays a role in describing the strong coupling between the evolution of the microstructure due to erosion, and the macroscopic suffusion law.

We take  $i_w$  to denote the hydraulic gradient:

$$i_w = \frac{1}{\gamma_w} \left| \frac{\Delta P}{L} \right| \quad [27]$$

where  $\gamma_w = \rho_w g$  is the water specific weight. Assuming a constant coefficient of erosion  $k_{er}$ , Equation [18] is re-written as follows:

$$r = \gamma_w k_{er} \phi \zeta \frac{|\Gamma_{er}|}{|\Gamma_{Pore}|} i_w \quad [28]$$

The macroscopic volume erosion appears here to be driven by the global pressure gradient, and not by the seepage velocity, as earlier suggested by Einstein (1937) and Sakthivadivel *et al.* (1966). In addition, the suffusion process and the evolution of the permeability anisotropy are coupled. Modelling this coupling phenomenon is, however, beyond the scope of this paper. For the sake of simplicity, we assume that  $\zeta$  is a constant.

A model for  $|\Gamma_{er}| / |\Gamma_{Pore}|$  is now introduced.

## 5. Modeling the evolution of internal surfaces

The dimensionless quantity  $|\Gamma_{er}| / |\Gamma_{Pore}|$  can be split as follows:

$$\frac{|\Gamma_{er}|}{|\Gamma_{Pore}|} = \frac{|\Gamma_{Clay} \cap \Gamma_{Pore}|}{|\Gamma_{Pore}|} \frac{|\Gamma_{er}|}{|\Gamma_{Clay} \cap \Gamma_{Pore}|} \quad [29]$$

From the identity

$$|\Gamma_{Clay}| = |\Gamma_{Clay} \cap \Gamma_{Pore}| + |\Gamma_{Clay} \cap \Gamma_{Sand}| \quad [30]$$

we obtain

$$\frac{|\Gamma_{Clay} \cap \Gamma_{Pore}|}{|\Gamma_{Pore}|} = \frac{|\Gamma_{Clay}|}{|\Gamma_{Pore}|} - \alpha \frac{|\Gamma_{Sand}|}{|\Gamma_{Pore}|} \quad [31]$$

where

$$\alpha = \frac{|\Gamma_{Clay} \cap \Gamma_{Sand}|}{|\Gamma_{Sand}|} \quad [32]$$

Inserting the identity

$$|\Gamma_{Pore}| = |\Gamma_{Clay}| + |\Gamma_{Sand}| - 2|\Gamma_{Clay} \leftrightarrow \Gamma_{Sand}| \quad [33]$$

into Equation [32] yields

$$\alpha = \frac{1}{2} \left( 1 + \frac{|\Gamma_{Clay}|}{|\Gamma_{Sand}|} - \frac{|\Gamma_{Pore}|}{|\Gamma_{Sand}|} \right) \quad [34]$$

Now inserting this result into Equation [31] gives

$$\frac{|\Gamma_{Clay} \cap \Gamma_{Pore}|}{|\Gamma_{Pore}|} = \frac{|\Gamma_{Clay}| + |\Gamma_{Pore}| - |\Gamma_{Sand}|}{2|\Gamma_{Pore}|} \quad [35]$$

From Equations [5], [6], [7] and [35] we finally obtain

$$\frac{|\Gamma_{Clay} \cap \Gamma_{Pore}|}{|\Gamma_{Pore}|} = \frac{S_{Clay} \phi_V + S_{Pore} \phi - S_{Sand} (1 - \phi_{Sand})}{2S_{Pore} \phi} \quad [36]$$

We now assume that a domain surface area scales with the corresponding domain volume raised to the power of  $2/3$ . This is a simple surface/volume scaling: for example, the surface of a sphere scales with its volume with this exponent. The expressions  $S_{Clay} \phi_V$  and  $S_{Pore} \phi$  can therefore be modelled as follows:

$$\frac{S_{Clay} \phi_V}{S_{Clay}^0 \phi_V^0} = \frac{\Gamma_{Clay}}{\Gamma_{Clay}^0} = \left( \frac{\Omega_{Clay}}{\Omega_{Clay}^0} \right)^{2/3} = \left( \frac{\phi_V}{\phi_V^0} \right)^{2/3} \quad [37]$$

$$\frac{S_{Pore} \phi}{S_{Pore}^0 \phi^0} = \frac{\Gamma_{Pore}}{\Gamma_{Pore}^0} = \left( \frac{\Omega_{Pore}}{\Omega_{Pore}^0} \right)^{2/3} = \left( \frac{\phi}{\phi^0} \right)^{2/3} = \left( \frac{\phi_{Sand} - \phi_V}{\phi_{Sand} - \phi_V^0} \right)^{2/3} \quad [38]$$

Here, the superscript 0 in any quantity  $a^0$  denotes the initial value of  $a$ .

From Equations [5], [6], [7] and [33] we obtain:

$$S_{Pore} \phi = S_{Clay} \phi_V + S_{Sand} (1 - \phi_{Sand}) (1 - 2\alpha) \quad [39]$$

The quantity  $\alpha$  of Equation [32] must be equal to zero if  $\phi_V = 0$  ( $\Gamma_{Clay} = \emptyset$ ). By using Equations [38] and [39], we infer the identity:

$$S_{Pore}^0 (\phi^0)^{1/3} (\phi_{Sand})^{2/3} = S_{Sand} (1 - \phi_{Sand}) \quad [40]$$

By using Equation [40], Equations [37] and [38] can now be written as follows:

$$S_{Clay} \phi_V = S_{Sand} (1 - \phi_{Sand}) \left( \frac{\phi_V}{\phi_{Sand}} \right)^{2/3} \quad [41]$$

$$S_{Pore} \phi = S_{Sand} (1 - \phi_{Sand}) \left( 1 - \frac{\phi_V}{\phi_{Sand}} \right)^{2/3} \quad [42]$$

Inserting the above results into Equation [36] finally yields:

$$\frac{|\Gamma_{Clay} \cap \Gamma_{Pore}|}{|\Gamma_{Pore}|} = \frac{\left( \frac{\phi_V}{\phi_{Sand}} \right)^{2/3} + \left( 1 - \frac{\phi_V}{\phi_{Sand}} \right)^{2/3} - 1}{2 \left( 1 - \frac{\phi_V}{\phi_{Sand}} \right)^{2/3}} \quad [43]$$

For the sake of simplicity, we retain a first order approximation:

$$\frac{|\Gamma_{Clay} \cap \Gamma_{Pore}|}{|\Gamma_{Pore}|} = \frac{1}{2} \left( \frac{\phi_V}{\phi_{Sand}} \right)^{2/3} \quad [44]$$

A model for  $|\Gamma_{er}| / |\Gamma_{Clay} \cap \Gamma_{Pore}|$  is now introduced. In the erosion process of the enlargement of a pipe (Bonelli *et al.*, 2006; Bonelli and Brivois, 2008), the limitation of the amount of eroded material is a structural piece of information (it may be the earth-dam height for example). For suffusion, the situation is entirely different. This time the amount of eroded material (the clay) is limited, and depends on the hydraulic gradient, as established by experimental results (Bendahmane *et al.*, 2006; Bendahmane *et al.*, 2008; Sterpi, 2003). Parameter  $|\Gamma_{er}|$  appears to be the relevant quantity accounting for this material information. Consequently, we assume that the surface fraction of eroded material  $|\Gamma_{er}|$  scales with the clay volume fraction as follows:

$$\frac{|\Gamma_{er}|}{|\Gamma_{Clay} \cap \Gamma_{Pore}|} = H[\phi_V - \phi_V^{stop}] \left[ 1 - \left( \frac{\phi_V^{stop}}{\phi_V} \right)^{2/3} \right] \quad [45]$$

The final clay volume fraction  $\phi_V^{stop}$  is assumed to depend on the hydraulic gradient:

$$\phi_V^{stop} = \phi_V^0 \left( \frac{i_c^0}{i_c^0 + i_w} \right)^{3/2} \quad [46]$$

The parameter  $i_c^0$  is a phenomenological parameter which is not explicitly connected to the local threshold  $\tau_c$ . It is a threshold hydraulic gradient to stop

erosion, as erosion stops if  $\varphi_V = \varphi_V^{stop}$ . It is also a threshold hydraulic gradient for the onset of erosion, as erosion occurs if

$$i_w > i_c^0 \left[ \left( \frac{\varphi_V^0}{\varphi_V} \right)^{2/3} - 1 \right] \quad [47]$$

## 5. Parametric analysis

We consider a homogeneous sample, and we assume dilute flow and no deposition or clogging. As a result, the pore-fluid viscosity and the geometric permeability are homogenous. The pressure gradient is therefore homogeneous. The system Equations [1] and [2] (balance equations) and [28]-[46] (erosion constitutive law) leads to the following differential equation:

$$\begin{cases} \frac{d}{d\tilde{t}} \tilde{\varphi}_V = - \frac{\tilde{i}_w (\tilde{\phi}_{sand} - \tilde{\varphi}_V)}{\tilde{\phi}_{sand}^{2/3}} \left[ \tilde{\varphi}_V^{2/3} - \frac{1}{1 + \tilde{i}_w} \right] \\ \tilde{\varphi}_V(0) = 1 \end{cases} \quad [48]$$

In Equation [48],  $\tilde{\varphi}_V$  is the dimensionless clay volume fraction,  $\tilde{\phi}_{sand}$  the dimensionless sand porosity, and  $\tilde{i}_w$  the dimensionless hydraulic gradient defined as follows:

$$\tilde{\varphi}_V = \frac{\varphi_V}{\varphi_V^0}, \quad \tilde{\phi}_{sand} = \frac{\phi_{sand}}{\varphi_V^0}, \quad \tilde{i}_w = \frac{i_w}{i_c^0} \quad [49]$$

The dimensionless time  $\tilde{t}$  is defined by means of a characteristic time of erosion  $t_{er}$ :

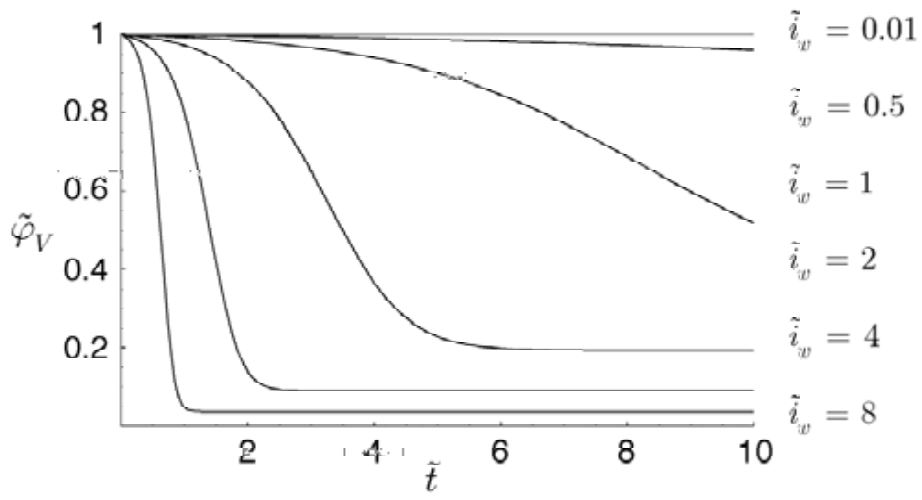
$$\tilde{t} = \frac{t}{t_{er}}, \quad t_{er} = \frac{2\rho_{Clay}}{\zeta\rho_w g i_c^0 k_{er}} \quad [50]$$

This characteristic time is the main result of this work: it is established here that the global pressure gradient for suffusion plays the same role as the pressure drop  $\Delta p / L$  in a pipe of length  $L$ , where a similar characteristic time has been defined (Bonelli *et al.*, 2006; Bonelli and Brivois, 2008).

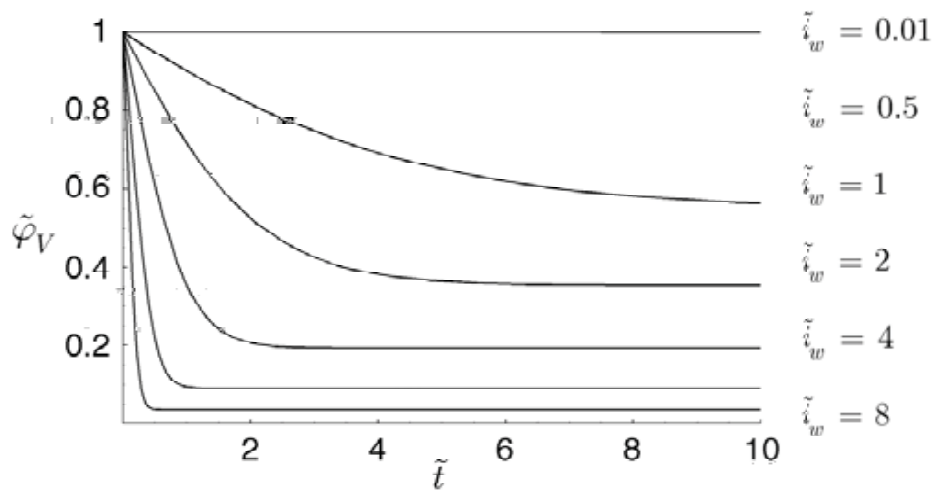
The effects of varying the dimensionless hydraulic gradient  $\tilde{i}_w$  are shown in Figure 4 for  $\tilde{\phi}_{sand} = 1/0.99$ , and Figure 5 for  $\tilde{\phi}_{sand} = 2$ . Figure 6 and Figure 7 show the effects of varying the initial dimensionless amount of clay  $\tilde{\phi}_{sand}$  for  $\tilde{i}_w = 1$ , and  $\tilde{i}_w = 10$  respectively.

For a given clayey sand ( $i_c^0, k_{er}$ ), the model captures two important aspects:

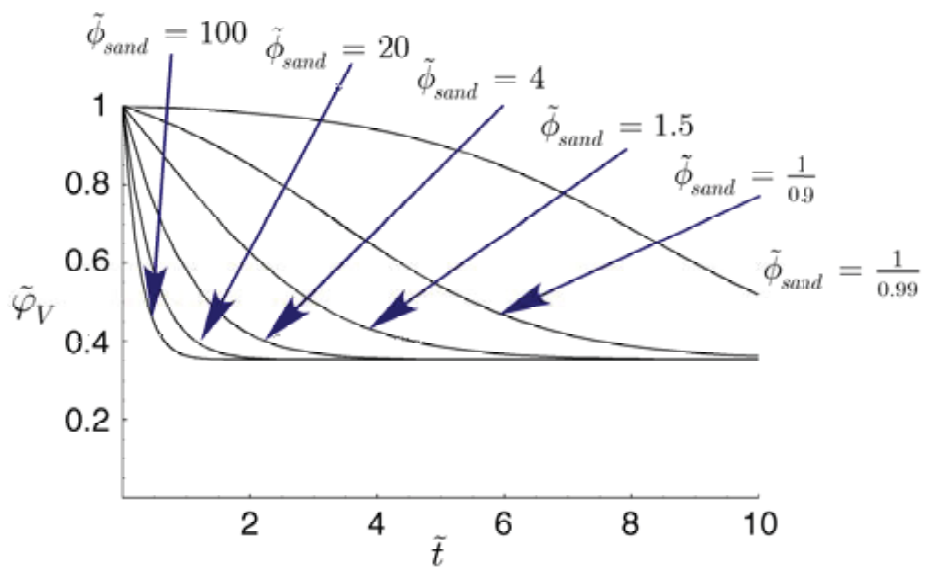
- the greater the initial clay volume fraction  $\varphi_V^0$ , the slower the erosion;
- the greater the hydraulic mechanical state  $i_w$ , the faster the erosion.



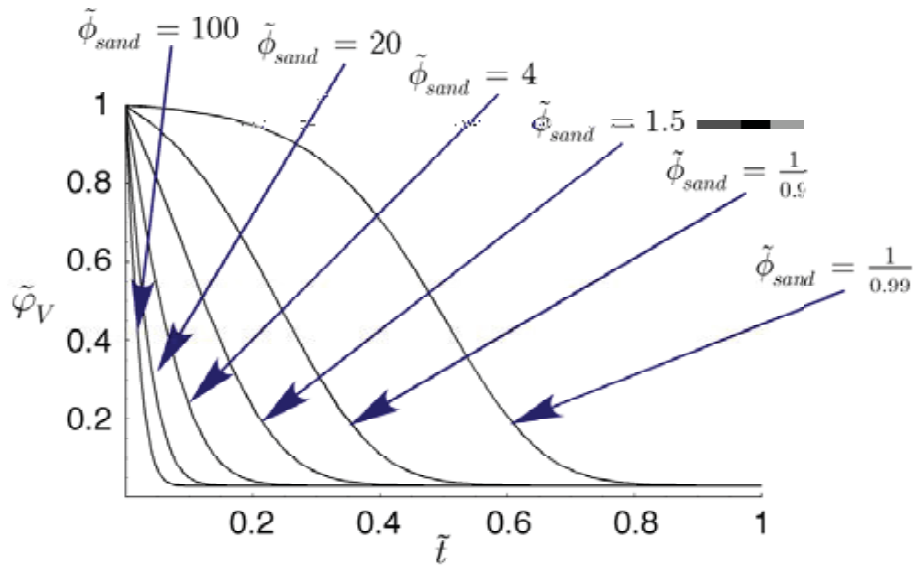
**Figure 3.** Effects of varying  $\tilde{i}_w$  on  $\tilde{\varphi}_V$  for  $\tilde{\phi}_{sand} = 1 / 0.99$



**Figure 4.** Effects of varying  $\tilde{i}_w$  on  $\tilde{\varphi}_V$  for  $\tilde{\phi}_{sand} = 2$



**Figure 5.** Effects of varying  $\tilde{\phi}_{sand}$  on  $\tilde{\varphi}_V$  for  $\tilde{i}_w = 1$



**Figure 6.** Effects of varying  $\tilde{\phi}_{sand}$  on  $\tilde{\varphi}_V$  for  $\tilde{i}_w = 10$

## 6. Comparison with experimental results

The suffusion law is now compared with previously published data, giving the mass fraction of eroded material as a function of time for several hydraulic gradients (Sterpi, 2003).

The mass fraction of eroded material is a function of the clay volume fraction as follows:

$$\mu(t) = \frac{(1 - \phi_{clay}) [\varphi_V^0 - \varphi_V(t)]}{(1 - \phi_{clay}) \varphi_V^0 + (1 - \phi_{sand})} \quad [51]$$

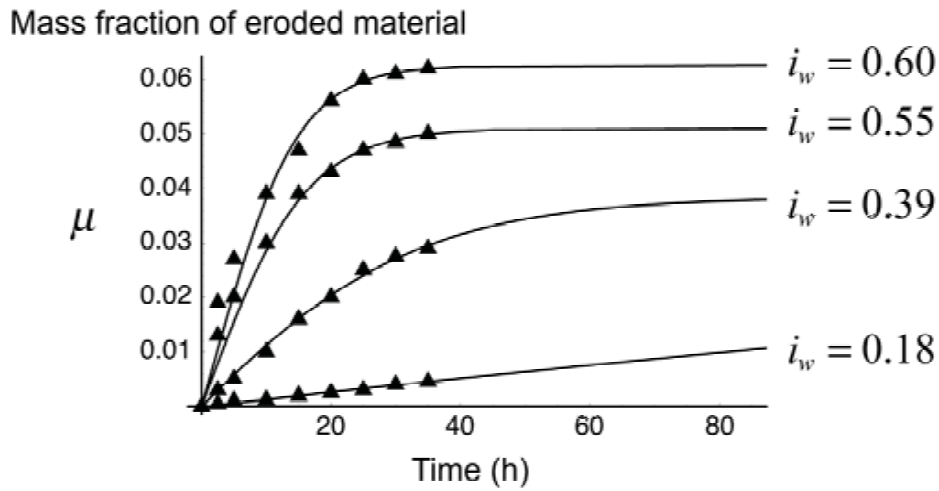
Table 1 summarizes the parameters of the suffusion equation. Parameters  $\phi_{sand}$  and  $\varphi_V^0$  were inferred from Sterpi (2003), while a characteristic value of  $\phi_{clay}$  was chosen as this value was not available.

Equation [46] suggests that it is possible to estimate the parameter  $i_c^0$  from the total amount of eroded material in a given test, if at least equilibrium is attained, which was not the case in the available data (Sterpi, 2003). The results of the identification are mean values and standard deviations for  $i_c^0$  and  $k_{er}$ .

It is notable that although most of the research on suffusion has been directed towards the threshold stress, the rate of erosion is at least equally as important. This rate of erosion appears to be primarily driven by  $t_{er}$ . Equation [50], which is essentially affected by the coefficient of surface erosion  $k_{er}$ . For piping erosion,  $k_{er}$  was found to range from  $10^{-6}$  s/m to  $10^{-3}$  s/m for fine-grained soils (Bonelli *et al.*, 2006; Bonelli and Brivois, 2008).

**Table 1.** Numerical values of the parameters

$\phi_{sand}$	Sand porosity	0.49	
$\phi_{clay}$	Clay porosity	0.60	
$\phi_V^0$	Initial clay volume fraction	38%	
$\rho_w$	Water density	1 000	kg.m <sup>-3</sup>
$\rho_{Clay}$	Clay saturated density	1 688	kg.m <sup>-3</sup>
$\zeta$	Anisotropy and tortuosity	2	
$g$	Gravitational constant	9.81	m.s <sup>-2</sup>
$k_{er}$	Coefficient of erosion	$4.33 \times 10^{-5} \pm 0.30 \times 10^{-5}$	s.m <sup>-1</sup>
$i_c^0$	Threshold hydraulic gradient	$2.95 \pm 0.64$	



**Figure 6.** Suffusion tests with a constant pressure drop, mass fraction of eroded clay is shown as a function of time, test results (symbols, (Sterpi, 2003)) versus model results (continuous lines)

The coefficient of erosion is about  $10^{-5}$  s/m. This is a low value, which corresponds to a slow erodable soil for surface erosion (Bonelli *et al.*, 2006; Bonelli and Brivois, 2008). This finding is consistent with previous experimental results: internal erosion rates have been found to be smaller than surface erosion rates (Reddi *et al.*, 2000).

Figure 6 gives the increase in the mass fraction of eroded material as a function of time. The data are well-described by our model over the full range of hydraulic gradients explored.



## 7. Conclusion

Erosion, which is the removal of material caused by the eroding power of the flow, is essentially an interfacial process. The objective of this work was to understand the bulk erosion process, which is counter intuitive at the pore scale, as an interfacial erosion process.

By means of homogenization reasoning, the macroscopic volume flux of eroded material is related to the microscopic surface flux of eroded material, at the clay water interface. This surface erosion is considered as shear stress driven. The macroscopic volume erosion is therefore driven by the global pressure gradient. In addition, the coefficient of clay surface erosion appears to be a relevant parameter to describe the kinetics of the suffusion process.

The total clay/water interface and the eroded clay/water interface are related to the clay volume fraction. The amount of erodable clay is related to the hydraulic gradient. Comparisons with published experimental results show that this model gives good results over the full range of hydraulic gradients explored.

This paper was not intended to propose a new suffusion law, or to reformulate the theory of suffusion as a whole based on homogenization. Other important phenomena should be considered in a more comprehensive model of internal erosion: permeability anisotropy induced by microstructure erosion, particle transport and filtration, deposition and clogging, two-phase seepage flow, concentrated flow, and dissolution and physico-chemical effects.

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