Inflation of elastomeric circular membranes using network constitutive equations

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Abstract

The present paper deals with the use of network-based hyperelastic constitutive equations in the context of thin membranes inflation. The study focus on the inflation of plane circular membranes and the materials are assumed to obey Gaussian and non-Gaussian statistical chains network models. The governing equations of the inflation of axisymmetric thin rubber-like membranes are briefly recalled. The material models are implemented in a numerical tool that incorporates an efficient B-spline interpolation method and a coupled Newton-Raphson/arc-length solving algorithm. Two numerical examples are studied: the homogeneous inflation of spherical balloons and the inflation of initially plane circular membranes. In the second example, the inflation profiles and the distributions of extension ratios along the membrane are extensively analyzed during the inflation process. Both examples highlight the need of an accurate modeling of the strain-hardening phenomenon in elastomers.

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1 Introduction

In the last few years, classical phenomenological constitutive equations for rubber-like solids, such as Mooney-Rivlin or Ogden models, are progressively replaced by more physical models based on statistical considerations, in various engineering applications [1]. The determination of the material parameters of the constitutive models is often performed using classical homogeneous strain experiments (uniaxial extension or pure shear tests for example). For biaxial deformation, authors use frequently the bubble inflation technique, that consists in inflating an initially plane circular thin membrane (see [16,22] and more recently [8,20]). In this type of experiments, deformations are not homogeneous and the analysis of experimental data needs efficient numerical method to solve the inflation problem. The problem of circular membranes inflation is classically solved for phenomenological rubber-like models. Here, we present some solutions corresponding with new statistical constitutive equations.

The governing equations of the large strains inflation of axisymmetric membranes were established by Green and Adkins [11]. Then, some authors solve the corresponding problem by directly integrating the system of ordinary differential equations using shooting methods. Both hyperelastic [17,33] and viscoelastic [30,10] constitutive equations are considered. Nevertheless, most of the works are based on numerical discretized methods, such as finite element analysis [19,5,15,28]. Most of the papers focus on the numerical methods and consider simple phenomenological constitutive equations. Very recently, Hassager *et al.* study the response of polymeric fluid membranes under inflation using the advanced Doi-Edwards and Tom-Pom models [12].

The aim of the present paper is the analysis of the inflation of thin elastomeric membranes using network based rubber-like constitutive equations. In Section 2, the governing equations of the axisymmetric inflation problem are recalled and the constitutive equations are presented. Four models are studied: the classical neo-Hookean model based on Gaussian statistics and three chains models that are based on non-Gaussian statistics. Section 3 is devoted to the solving procedure. The numerical scheme and our original B-spline interpolation, detailed in [29], are briefly recalled. The numerical analysis of spherical balloons and circular plane membranes are presented in Section 4. Finally, concluding remarks are proposed in Section 5.

2 Problem formulation

2.1 Governing equations

We consider an axially symmetric hyperelastic membrane of uniform thickness h_0 in its undeformed configuration. The geometry of this membrane can be described by the cylindrical coordinates of its mid-surface. In each θ-constant plane, both undeformed and deformed configurations are reduced to one-dimensional curves, as shown in Figure 1. Consider a particle of undeformed coordinates (r_0, z_0) of the undeformed membrane. This point is displaced to a new position (r, z) in the deformed configuration. The thickness of the deformed membrane is denoted h. Every geometrical data r_0, z_0, r, z and h are functions of the arc-length coordinate in the undeformed configuration s, that varies between 0 and l_0, the length of the initial membrane. As shown by Yang and Feng [33], the principal stretch ratios in meridian, circumferential and normal directions are given by:

$$\lambda_m = \sqrt{\frac{r_{,s}^2 + z_{,s}^2}{r_{0,s}^2 + z_{0,s}^2}} \quad , \quad \lambda_c = \frac{r}{r_0} \quad , \quad \lambda_n = \frac{h}{h_0} \tag{1}$$

in which $\cdot_{,s}$ stands for the differentiation with respect to s. As the material is assumed incompressible, the thickness of the deformed body is simply related to the thickness in the natural state by:

$$h = \frac{1}{\lambda_m \lambda_c} h_0 \tag{2}$$

Under a quasi-static pressure load, the Principle of Virtual Work can be written in the following form:

$$\int_{V_0} \delta W \, dV_0 - \int_S \delta \mathbf{u} \, P \, \mathbf{n} \, dS = 0 \qquad \forall \, \delta \mathbf{u} \tag{3}$$

where V_0 and S are respectively the volume of the undeformed membrane and the surface of the deformed membrane. W is the strain energy, $P\,\mathbf{n}$ is the outward oriented blowing pressure and $\delta\mathbf{u}$ is an admissible virtual displacement vector. Noting that the components of $\mathbf{n}$ are $(z_{,s}/\sqrt{r_{,s}^2 + z_{,s}^2}\,;\, -r_{,s}/\sqrt{r_{,s}^2 + z_{,s}^2})$, the previous equation (3) simplifies:

$$\int_0^{l_0} 2\,\pi\, r_0\, h_0 \;(t_m\, \delta\lambda_m + t_c\, \delta\lambda_c)\; ds - \int_0^{l_0} 2\,\pi\, P\, r \;(\delta u\, z_{,s} - \delta v\, r_{,s})\; ds = 0 \tag{4}$$

where t_m and t_c are the principal first Piola-Kirchhoff stresses respectively in the meridian and circumferential directions, and $(\delta u, \delta v)$ stand for the components of the virtual displacement vector.

2.2 Network constitutive equations for elastomers

As mentioned above, the use of statistical network models for rubber-like models increases in finite elements tools. These constitutive equations present some major advantages: they necessitate a small number of parameters and they agree well with experiments in various modes of deformation [32].

In the statistical approach, polymers are considered as networks of long flexible chains randomly oriented and joined together by chemical cross-links. More details on this subject can be found in [26,9].

2.2.1 Gaussian statistics model

Treloar first proposed a statistical treatment of rubber elasticity. He considered that the configuration of polymer chains can be described by Gaussian statistics. This leads to the well-known neo-Hookean constitutive equation [25]. The corresponding strain energy function is:

$$W = \frac{1}{2} nkT(I_1 - 3) \tag{5}$$

where n denotes the average number of polymer chains per unit of volume, k is the Boltzmann constant, T is the absolute temperature and finally I_1 is the first invariant of the stretch tensor, said its trace. This model only depends on a unique parameter C_R defined as:

$$C_R = nkT \tag{6}$$

2.2.2 Non-Gaussian statistics models

In order to overcome the limitations of the previous model that is restricted to small strains, authors use non-Gaussian statistics theory to describe molecular polymer chains configurations.

2.2.2.1 Single chain elasticity In 1942, Kuhn and Grün use the non-Gaussian statistics theory to model the limited extension of chains [18]. Their approach is based on the random walk statistics of an ideal phantom chain.

Consider a molecular chain composed by N monomer segments of length l. Its average unstretched length is $\sqrt{N}l$ and its maximum stretched length is Nl. Moreover, the strain energy function of the chain w can be written in the following form:

$$w = NkT\left[\frac{\lambda}{\sqrt{N}}\beta + \ln\frac{\beta}{\sinh\beta}\right] \tag{7}$$

where λ is the extension of the chain and with $\beta = \mathcal{L}^{-1}\left(\lambda/\sqrt{N}\right)$. $\mathcal{L}^{-1}$ is the inverse of the Langevin function defined by $\mathcal{L}(x) = \coth(x) - 1/x$. The first Piola-Kirchhoff stress in this single chain is obtained by derivation of the strain energy with respect to the extension:

$$t = \frac{\partial w}{\partial \lambda} = kT\sqrt{N}\,\mathcal{L}^{-1}\left(\frac{\lambda}{\sqrt{N}}\right) \tag{8}$$

2.2.2.2 *p*-chains models In order to develop constitutive equations, a network of chains which strain energy functions are given by (7) should be considered. Recently, Wu and van der Giessen integrate the previous stress (8) on a unit sphere defined by its chains density [32]. Their full network model agrees well with experiments but necessitates numerical integration procedures on the sphere. This numerical integration does not lead to an efficient implementation in numerical softwares.

Other authors used the non-Gaussian statistics theory to develop simpler models which do not need numerical integration. The most employed are the 3-chains and the 8-chains models. Moreover, we mention the approached full network model, that accurately approximates the general theory of Wu and van der Giessen presented above. These constitutive equations are developed by considering privileged directions in the unit sphere. Denoting n the density of polymer chains as introduced for the neo-Hookean model, it is assumed that n/p chains per unit of volume are oriented in each of the p privileged directions in the undeformed configuration. Then, the previous numerical integration of the strain energy on the unit sphere is advantageously replaced by a sum of p single chain strain energy functions weighted by the factor n/p. Particular cases of p-chains models are presented in Figure 2.

The simplest p-chains model is defined by considering the three principal strain axes as privileged directions. It was derived by James and Guth [13] and is widely known as the *3-chains model*. Figure 2(a) presents this model. Principal stresses $(t_i^{\text{3-chains}})_{i=1,3}$ are expressed as functions of principal stretch ratios $(\lambda_i)_{i=1,3}$ by:

$$t_i^{\text{3-chains}} = -\frac{q}{\lambda_i} + \frac{1}{3} C_R \sqrt{N}\, \mathcal{L}^{-1}\left(\frac{\lambda_i}{\sqrt{N}}\right) \tag{9}$$

where q is the hydrostatic pressure introduced by the incompressibility assumption.

A more recent model based on non-Gaussian statistics is the *8-chains model* developed by Arruda and Boyce [2]. Privileged directions are defined by the half diagonals of a cube contained in the unit sphere. Figure 2(b) shows this chains distribution of the model. Its major property is its symmetry with respect to the three principal axes. Therefore, the eight chains are stretched with the same extension ratio, λ^{ch}, that corresponds to the stretching of each half diagonals of the cube:

$$\lambda^{\text{ch}} = \sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)/3} \tag{10}$$

This leads to a simple stress-stretch relationship:

$$t_i^{\text{8-chains}} = -\frac{q}{\lambda_i} + \frac{1}{3} C_R \sqrt{N} \frac{\lambda_i}{\lambda^{\text{ch}}} \, \mathcal{L}^{-1} \left(\frac{\lambda^{\text{ch}}}{\sqrt{N}} \right) \tag{11}$$

Finally, we can mention the *approached full network model* proposed by Wu and van der Giessen [31]. Authors use a linear combination of the stresses of the 3-chains and 8-chains models to approximate the full network stresses by:

$$t_i^{\text{full network}} = (1-\rho) \, t_i^{\text{3-chains}} + \rho \, t_i^{\text{8-chains}} \tag{12}$$

where ρ is a parameter that is related to the maximal principal stretch ratio:

$$\rho = \frac{0.85}{\sqrt{N}} \max(\lambda_1, \lambda_2, \lambda_3) \tag{13}$$

in which the factor 0.85 is chosen to give the best correlation with the full integration on the unit sphere.

3 Solution procedure

In this part, the resolution procedure used to integrate the problem (4) associated with one of the stress-strain relationships defined above is briefly presented. For more details, the reader can refer to [29].

3.1 Discretized equations and solution procedure

Whatever the interpolation method adopted, the membrane is discretized by considering $n+1$ nodes, $(N_i)_{i=0,n}$. Each node N_i is defined by its undeformed arc-length coordinate s_i. Its displacements in radial and axial direction are respectively u_i and v_i. The assembled nodal displacements vector, that includes all nodal displacements, is denoted $\mathbf{U}$ with:

$$U_{2i-1} = u_i \text{ and } U_{2i} = v_i \qquad \forall\, i = 0, \ldots, n \tag{14}$$

Using interpolation formulas (detailed in the next paragraph), the Principle of Virtual Work (4) becomes:

$$\delta\mathbf{U}^T \left[\mathbf{F}_{\text{int}}(\mathbf{U}) - \mathbf{F}_{\text{ext}}(\mathbf{U}, p)\right] = 0 \qquad \forall\, \delta\mathbf{U}^T \tag{15}$$

where $\mathbf{F}_{\text{int}}(\mathbf{U})$ and $\mathbf{F}_{\text{ext}}(\mathbf{U}, p)$ are the internal and external forces vectors respectively, $\delta\mathbf{U}$ is the virtual nodal displacement vector and $\cdot^T$ denotes the transposition. $\mathbf{F}_{\text{int}}$ is the discretized counterpart of the first left-hand side term in Eq. (4) and $\mathbf{F}_{\text{ext}}$ corresponds to the second term in this equation. It is obvious that the system (15) is highly non-linear both geometrically and by the constitutive equation. Thus, an iterative method must be employed to solve this system and the tangent operator must be derived. This operator is denoted $\mathbf{K}$ and represents the derivative of the out-of-balance force vector $(\mathbf{F}_{\text{int}} - \mathbf{F}_{\text{ext}})$ with respect to $\mathbf{U}$:

$$\mathbf{K} = \frac{\partial \mathbf{F}_{\text{int}}}{\partial \mathbf{U}} - \frac{\partial \mathbf{F}_{\text{ext}}}{\partial \mathbf{U}} \tag{16}$$

The detailed ready-to-program formulas of $\mathbf{F}_{\text{int}}$, $\mathbf{F}_{\text{ext}}$ and $\mathbf{K}$ are given in [23] for the classical two nodes membrane finite element and in [29] for our original B-spline interpolation.

As shown by Beatty for static inflation [4] and Verron *et al.* for dynamic inflation [27], the pressure-displacement curves corresponding to rubber-like membranes inflation exhibit limit points. Consequently a continuation method has to be adopted to compute the equilibrium path [24]. Here, the arc-length method is implemented. It consists in completing the system of equilibrium equations with an additional relation between the load (here the pressure) and the displacement increments (see [6] for various versions of this method). In the present case, the displacement control method is considered [3] and the additional equation is:

$$\|\Delta \mathbf{U}\|^2 - da^2 = 0 \tag{17}$$

where $\Delta \mathbf{U}$ is the displacement increment vector between two equilibrium points and da is the prescribed arc-length. This equation is added to the previous system (15) as a bordered equation and the assembled system is solved by the classical Newton-Raphson algorithm [21].

3.2 B-splines interpolation method

In this study, the membrane is interpolated by two cubic splines defined on the set of nodes $(N_i)_{i=0,n}$, one for each coordinate r_0 and z_0:

$$r_0(s) = \sum_{i=-1}^{n+1} \alpha_i \, B_i(s/l_0) \tag{18}$$

$$z_0(s) = \sum_{i=-1}^{n+1} \beta_i \, B_i(s/l_0) \tag{19}$$

where $(\alpha_i)_{i=-1,n+1}$ and $(\beta_i)_{i=-1,n+1}$ are the parameters of the two splines and functions B_i are B-spline functions. Denoting ξ the reduced arc-length, said

s/l_0, these functions are given by [7]:

$$B_i(\xi) = \begin{cases} 0 & \xi \le \xi_{i-2} \\ \frac{(\xi-\xi_{i-2})^3}{(\xi_{i+1}-\xi_{i-2})(\xi_i-\xi_{i-2})(\xi_{i-1}-\xi_{i-2})} & \xi_{i-2} \le \xi < \xi_{i-1} \\ \frac{(\xi-\xi_{i-2})^2(\xi_i-\xi)}{(\xi_{i+1}-\xi_{i-2})(\xi_i-\xi_{i-2})(\xi_i-\xi_{i-1})} + \frac{(\xi_{i+1}-\xi)(\xi-\xi_{i-1})(\xi-\xi_{i-2})}{(\xi_{i+1}-\xi_{i-1})(\xi_i-\xi_{i-1})(\xi_{i+1}-\xi_{i-2})} + \frac{(\xi-\xi_{i-1})^2(\xi_{i+2}-\xi)}{(\xi_{i+1}-\xi_{i-2})(\xi_i-\xi_{i-1})(\xi_{i+2}-\xi_{i-1})} & \xi_{i-1} \le \xi < \xi_i \\ \frac{(\xi_{i+1}-\xi)^2(\xi-\xi_{i-2})}{(\xi_{i+1}-\xi_{i-2})(\xi_{i+1}-\xi_{i-1})(\xi_{i+1}-\xi_i)} + \frac{(\xi_{i+2}-\xi)(\xi_{i+1}-\xi)(\xi-\xi_{i-1})}{(\xi_{i+2}-\xi_{i-1})(\xi_{i+1}-\xi_{i-1})(\xi_{i+1}-\xi_i)} + \frac{(\xi_{i+2}-\xi)^2(\xi-\xi_i)}{(\xi_{i+2}-\xi_{i-1})(\xi_{i+2}-\xi_i)(\xi_{i+1}-\xi_i)} & \xi_i \le \xi < \xi_{i+1} \\ \frac{(\xi_{i+2}-\xi)^3}{(\xi_{i+2}-\xi_{i-1})(\xi_{i+2}-\xi_i)(\xi_{i+2}-\xi_{i+1})} & \xi_{i+1} \le \xi < \xi_{i+2} \\ 0 & \xi_{i+2} \le \xi \end{cases} \tag{20}$$

where ξ_i is the reduced arc-length coordinate of the node N_i and where the following conventions are adopted:

$$\xi_i = s_0/l_0 \text{ for } i \le 0 \tag{21}$$

$$\xi_i = s_n/l_0 \text{ for } i \ge n \tag{22}$$

With these two conventions some denominators are equal to 0 and we adopt the equality: $0/0 = 0$.

In Eqs (18, 19), the $n+3$ parameters of each splines are defined in a unique manner using the undeformed coordinates of the $n+1$ nodes and by taking into account end conditions to determine the two extra parameters. Depending on the mechanical boundary conditions, different mathematical end conditions for the splines can be adopted:

1. if the extreme point does not lies in the symmetry axis, both r_0 and z_0 interpolation admit natural end conditions, i.e. the second derivative of both functions are equal to zero at this point,
2. if the extreme point lies in the symmetry axis, the z_0-spline admits natural end conditions but the r_0-spline admits Hermite end conditions, i.e. the first derivative of the function is prescribed at the point and, in our case, is set to zero.

These boundary conditions provide two additional linear relations between parameters for both splines, and interpolation formulas (18,19) reduce to:

$$r_0(s) = \sum_{i=0}^{n} \alpha_i \, B_{r_i}(s/l_0) \tag{23}$$

$$z_0(s) = \sum_{i=0}^{n} \beta_i \, B_{z_i}(s/l_0) \tag{24}$$

where $(B_{r_i})_{i=0,n}$ and $(B_{z_i})_{i=0,n}$ are the new interpolation basis, defined as linear combinations of the classical B-spline functions B_i.

Similarly to isoparametrical finite elements, the displacement field (u,v) is interpolated by two splines developed on the same basis functions:

$$u(s) = \sum_{i=0}^{n} \gamma_i \, B_{r_i}(s/l_0) \tag{25}$$

$$v(s) = \sum_{i=0}^{n} \delta_i \, B_{z_i}(s/l_0) \tag{26}$$

in which $(\gamma_i)_{i=0,n}$ and $(\delta_i)_{i=0,n}$ are the parameters of these two splines.

Some calculations are now necessary to establish the discretized problem (15,16). This derivation is detailed in [29].

4 Numerical examples

In this section, some numerical examples are presented.

The aim of this paper is the comparison of different network models. Thus, the different constitutive equations have to be fitted for the same material using the same type of experimental data. Here, experimental data corresponding to the uniaxial extension of a natural rubber as reported by James *et al.* are considered [14]. Moreover, we adopt the material parameters previously determined by Wu and van der Giessen [32]. It is to note that network parameters N and C_R must be determined for each model. But, as the small strain behaviour is governed primarily by C_R, Wu and van der Giessen first determine it, and consider an identical value of C_R for all models, said $C_R = 0.4$ MPa. Then afterwards, the average number of monomers per chain is fitted for each constitutive equation: $N = 80$ for the 3-chains model, $N = 25$ for the 8-chains model, and finally $N = 50$ for the approached full network model.

4.1 Spherical balloons

Before examining the case of circular membranes, we consider the simple homogeneous inflation of spherical balloons. This problem was widely studied for classical hyperelastic models, both in statics [4] and dynamics [27]. As the deformation remains biaxial in every material points of the sphere, the problem reduces to a simple analytical governing equation that relies the blowing pressure and the deformed radius. Here, our numerical approach is used to solve the problem and the numerical results were successfully compared with the analytical relation, that validates the present method.

Consider a thin spherical membrane of natural rubber. The initial radius is equal to 1.0 and the uniform thickness is 0.01. The semi-spherical cap is discretized using 21 nodes. It is assumed that the balloon remains spherical during the inflation, so that no bifurcating phenomenon occurs. Consequently, the extension ratios are:

$$\lambda_m = \lambda \quad , \quad \lambda_c = \lambda \quad , \quad \lambda_c = \frac{1}{\lambda^2} \tag{27}$$

where λ is the adimensional deformed radius r/r_0. The curves pressure versus λ are presented in Figure 3 for the four constitutive equations studied here. Values of extension ratios and blowing pressures that correspond to the two limit points, (λ_I, P_I) and (λ_{II}, P_{II}), are presented in Table 1. As shown elsewhere, the neo-Hookean balloon exhibits only one limit point that divides the curve into a stable path and an unstable path. For the three other balloons, the curves admit two limit points and three paths, two stable (increasing parts of the curves) and one unstable (decreasing parts of the curves). A similar behaviour was highlighted previously for the Mooney-Rivlin model [27]. Let us examine more precisely the first part of the curves that correspond to small strains, i.e. for $\lambda \leq 2.0$. First limit points occur approximately for the same values of the extension ratio, said $\lambda = 1.39$. This value can be obtained analytically in the case of the neo-Hookean model and is equal to $\sqrt[6]{7}$. This result verifies our previous statement that the small strains behaviour only depends on the material parameter C_R, that is identical for the four models. For larger strains, inflation curves differ. In the case of the neo-Hookean material, there is no strain-hardening (because of the Gaussian assumption) and the pressure slowly decreases to zero as the membrane continues to inflate. For the three other models, polymer chains reach their limit of extensibility for different values of the undeformed radius. Last points (circles) of the curves in Fig. 3

represent the last configurations in which the simulation was able to converge. The limit values of the adimensional radius, i.e. values for which the limiting stretch of chains is reached, can be simply predicted for both 3-chains and 8-chains models. In the case of the 3-chains model, the limit of extensibility is reached when one of the extension ratios λ_i tends to $\sqrt{N}$ in Eq. (9). Here, recalling the equibiaxial deformations (27), we have:

$$\lambda_1 = \lambda_m = \lambda \quad , \quad \lambda_2 = \lambda_c = \lambda \quad , \quad \lambda_3 = \lambda_n = 1/\lambda^2 \tag{28}$$

Thus the limiting extensibility is obtained for $\lambda = \sqrt{80} \approx 8.94$. Note that in our numerical simulations, we are not able to overcome $\lambda \approx 8.0$ due to convergence problems. In the case of the 8-chains model, the stretching of the chains λ^{ch} is defined by Eq. (10). Using (28), it reduces to:

$$\lambda^{\mathrm{ch}} = \sqrt{(2\lambda^2 + 1/\lambda^4)/3} \tag{29}$$

and the limit chain extension is obtained by:

$$\lambda^{\mathrm{ch}} = \sqrt{N} \tag{30}$$

with $N = 25$. The solution of this equation is $\lambda \approx 6.12$. This value can be seen as the asymptotic vertical line corresponding to the pressure curve of the 8-chains model in Fig. 3. Finally, the response of the approached full network model is considered. As the model is a linear combination of the 3-chains and the 8-chains models, the model reaches its limiting extensibility when one of them reaches its own limit. Thus, considering now that $N = 50$, the limiting extensibility is obtained for $\lambda = \sqrt{50} \approx 7.07$ for the 3-chains model. For the 8-chains model, Eqs (29) and (30) are solved with $N = 50$ and the limiting extensibility is approximately equal to 8.66. The stresses (12) tends to infinity for the smallest value, said $\lambda \approx 7.07$ which corresponds to the 3-chains stress, as shown by the corresponding curve in Fig. 3.

4.2 Circular plane membranes

Consider now an initially plane circular membrane of radius 1.0 and uniform thickness 0.01. The exterior edge of the membrane is clamped. The membrane is meshed using 21 nodes. In this problem, the deformation is not homogeneous in the membrane. At the pole (the node in which $r = 0.$), the deformation is equibiaxial:

$$\lambda_m \quad , \quad \lambda_c = \lambda_m \quad , \quad \lambda_n = \frac{1}{\lambda_m^2} \tag{31}$$

and at the clamped rim, the membrane is under pure shear conditions:

$$\lambda_m \quad , \quad \lambda_c = 1 \quad , \quad \lambda_n = \frac{1}{\lambda_m} \tag{32}$$

Moreover, it is obvious that the pole lies in the symmetry axis. With our numerical approach, there is no difficulty to satisfy the tangency condition, said $r' = 0$, because this boundary condition is directly taken into account in the B_{r_i} interpolation functions (see Section 3.2). Using classical finite elements, this condition can not be strictly imposed, because there is no degree of freedom that corresponds to r' (see [12] for example).

First, we study the curves that show the inflating pressure versus the axial coordinate of the pole (denoted z_{pole} through the rest of the paper) for the different constitutive equations. These results are shown in Figure 4. The corresponding values of limit points $(z_{\text{pole}\ I}, P_I)$ and $(z_{\text{pole}\ II}, P_{II})$ are presented in Table 2. General aspects of the curves are similar to those observed for the spherical balloons with both increasing and decreasing paths. In the present case, the two values of the pressure that correspond to limit points are not very different for each chains models. Moreover, the strain-hardening of the non-Gaussian constitutive equations is highly remarkable and the limiting chain extension is rapidly reached as it can be shown by the final vertical parts of curves presented in Fig. 4.

Let us now examine the inflation profiles shown in Figure 5. For the four models, we present profiles that correspond to z_{pole} =1.0, 2.0, 3.0 and 4.0. Moreover, the last profiles obtained before numerical divergence are also drawn for the non-Gaussian constitutive equations. It can be seen that for z_{pole} equal to 1.0 and 2.0 the profiles corresponding to the four models are similar. Nevertheless, for higher values of the axial coordinate of the pole, said $z_{\text{pole}} \geq 3.0$, the profiles corresponding to the three non-Gaussian models are similar, but they highly differ from the shape of the deformed neo-Hookean membranes. The neo-Hookean bubble is more "spherical" than the other ones. This example shows that the strain-hardening of the material has a very significant influence of the shape of inflated membranes.

This remark is confirmed by the thickness distributions presented in Figure 6. The thickness is plotted as function of the undeformed radius r_0. As observed by Hassager *et al.*, the neo-Hookean membrane exhibits a strong thinning at the pole [12]. Here, we show that the same observations can be made for non-Gaussian statistical models. As predicted by the examination of the profiles, the three non-Gaussian thickness distributions are almost similar at the pole, and are of the same order of magnitude along the membrane for a given value of z_{pole}. The last profile of each non-Gaussian membrane exhibits the same thickness distribution that can be seen as the ultimate state before the occurrence of chains rupture. Moreover, the difference between the neo-Hookean membrane and the non-Gaussian membranes is very significant. For values of

z_{pole} of 2.0, 3.0 and 4.0, the thickness of the neo-Hookean membrane at the clamped rim are identical and values at the pole tend to zero (see Fig. 6(a)). In the case of the three other models, the thickness at the clamped rim evolves significantly with respect to z_{pole} and the thinning at the pole remains smaller than in the neo-Hookean case.

In order to highlight the influence of the strain-hardening modeled by the inverse Langevin function in constitutive equations (9), (11) and (12), the distribution of the in-plane extension ratios λ_m and λ_c along the membranes are presented in Figure 7. First, we verify that the circumferential stretch ratio λ_c varies between λ_m at the pole and 1.0 at the clamped rim. Fig. 7(a) reveals that, for the neo-Hookean bubble, both distributions are similar, as previously demonstrated by Yang and Feng [33]. Moreover, for $z_{\text{pole}} = 5.0$, extensions ratios are equal to 25.0 at the pole. This result has no physical meaning and proves the importance of a good modeling of strain-hardening of elastomers at large strains. In the case of the non-Gaussian models, distributions of λ_c for the three models are comparable for the same deformation level. Distributions of λ_m are also similar until the material reaches its strain-hardening behaviour. In this case, the 8-chains models distribution (Fig. 7(c) for $z_{\text{pole}} = 4.02$) differs from the 3-chains and the approached full network distributions (Fig. 7(b) for $z_{\text{pole}} = 5.60$ and Fig. 7(d) for $z_{\text{pole}} = 3.87$). In the former case, λ_m is greater at the clamped extremity than at the pole; in the later, λ_m is approximately uniform for both 3-chains and approached full network membranes. This last observation can be explained by examining the maximum chain extensibility along the membrane for the three non-Gaussian bubbles. Recall that for the 3-chains model, the maximum chain extension at a material point is defined by:

$$\lambda^{\text{ch max}} = \max(\lambda_m, \lambda_c, \lambda_n) \tag{33}$$

and in our case, it is always equal to λ_m. For the 8-chains model, it is given by:

$$\lambda^{\text{ch max}} = \sqrt{(\lambda_m^2 + \lambda_c^2 + \lambda_n^2)/3} \tag{34}$$

and finally, for the approached full network model, it is the greater values of the 3-chains and 8-chains data:

$$\lambda^{\text{ch max}} = \max(\lambda_m, \lambda_c, \lambda_n, \sqrt{(\lambda_m^2 + \lambda_c^2 + \lambda_n^2)/3}) \tag{35}$$

and here, it is always equal to λ_m. In the three cases, as $\lambda^{\text{ch max}}$ tends to $\sqrt{N}$, the material hardens and the asymptotic vertical line in the pressure versus z_{pole} diagram is approached (Fig. 4). The distributions of $\lambda^{\text{ch max}}$ along the membranes are plotted versus the undeformed radius in Figure 8, for the three non-Gaussian network models. They are highly similar for the three models: at the beginning of the inflation the distribution is uniform, then the membrane extends more significantly in the neighbourhood of the pole than near the clamped rim, and finally for higher values of z_{pole} the chain extensibility tends to become uniform and approaches the limit value $\sqrt{N}$, while few evolution of strains takes place at the pole. This can be seen as the inflation of a circular membrane with a rigid inclusion at the pole, considering that the radius of this inclusion increases during the inflation.

5 Concluding remarks

In this paper, we have presented some results concerning the inflation of rubber-like circular membranes using physical networks hyperelastic constitutive equations. The simulations are performed using a new B-spline model that was proved to be very efficient for large strains inflation of axisymmetric membranes. The non-Gaussian models are implemented and compared with the classical neo-Hookean constitutive equation. The influence of strain-hardening on the membrane response is highlighted. It is shown that both inflation profiles and thickness distribution are highly influenced by the nature of the material model.

These results are of interest in the context of the identification of material parameters in biaxial conditions. In order to perform biaxial experiments on elastomers or heat-softened plastics, the bubble inflation technique is widely used. In most of the cases, only experimental data obtained at the pole are considered to determine parameters of constitutive equations, because the deformation is equibiaxial at this point. The present work shows that the evolution of the membrane profile during inflation and some data concerning the thickness distribution may be of great interest in the choice of the constitutive equation that must be adopted, and in the determination of the corresponding material parameters. Thus, the numerical analysis of the entire membrane is highly suitable in the identification procedure.

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Fig. 1. Problem description.

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Fig. 4. Pressure versus axial coordinate of the pole for the plane membranes.

Fig. 5. Inflation profiles of the circular plane membranes: (a) neo-Hookean, (b) 3-chains, (c) 8-chains, (d) approached full network. Numbers on the curves stand for the corresponding inflating pressure.

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Fig. 7. Extension ratios in the circular plane membranes (—) λ_m, $(\cdots)$ λ_c: (a) neo-Hookean, (b) 3-chains, (c) 8-chains, (d) approached full network. Numbers on the curves stand for the corresponding value of z_{pole}.

Fig. 8. Maximum chains extensibility in the circular plane membranes (—) distribution in the membrane, (- -) limit value $\sqrt{N}$: (a) 3-chains, (b) 8-chains, (c) approached full network. Numbers on the curves stand for the corresponding value of z_{pole}.

Tables

Model	λ_I	P_I	λ_{II}	P_{II}
Neo-hookean	1.40	4.96e-03	·	·
3-chains	1.40	5.04e-03	5.71	1.97e-03
8-chains	1.39	5.13e-03	3.92	2.88e-03
Approached full network	1.39	5.08e-03	5.10	2.26e-03

Table 1

Limit points for spherical balloons. Circles represent the analytical solutions.

Model	$z_{\text{pole } I}$	P_I	$z_{\text{pole } II}$	P_{II}
Neo-hookean	1.16	7.52e-03	·	·
3-chains	1.23	7.73e-03	3.16	6.09e-03
8-chains	1.17	8.01e-03	2.17	7.64e-03
Approached full network	1.26	7.84e-03	2.83	6.66e-03

Table 2

Limit points for circular plane membranes.