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Uniform time-decay of semigroups of contractions

Xue Ping Wang

Abstract. We discuss the uniform time-decay of semi-groups generated by dissipative Schrödinger operators in the semiclassical regime.

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High frequency analysis of propagation of waves in media with variable absorption index leads to the following dissipative Schrödinger equation:

$$\begin{cases} ih \frac{\partial}{\partial t} u^h(x, t) = P(h)u^h(x, t), \\ u^h(x, 0) = u_0^h(x), \end{cases}$$

where $P(h) = -h^2\Delta + V_1(x) - ihV_2(x)$, $x \in \mathbb{R}^n$, $h \in]0, h_0]$ is a small parameter proportional to wave length and V_j , $j = 1, 2$, are real functions with $V_2 \geq 0$ and $V_2 \neq 0$. Assume that V_j is smooth, satisfying for some $\rho > 0$

$$|\partial_x^\alpha V_j(x)| \leq C_\alpha \langle x \rangle^{-\rho - |\alpha|}, \quad j = 1, 2;$$

here $\langle x \rangle = (1 + |x|^2)^{1/2}$. Let $S_h(t) = e^{-itP(h)/h}$, $t \geq 0$, be the associated semigroup of contractions in $L^2(\mathbb{R}^n)$. Then $\|S_h(t)\| \leq 1$ for all $t \geq 0$ and $h \in]0, h_0]$. The interplay between propagation along the flow of the Hamiltonian $p_1(x, \xi) = \xi^2 + V_1(x)$ and the dissipation governs the long-time behavior of solutions. A natural question in this connection is the following

Question. Can one establish a uniform time-decay estimate for the semigroup $S_h(t)$ in the form

$$\|\langle x \rangle^{-s} S_h(t) \langle x \rangle^{-s}\| \leq w(t), \quad t > 0, \quad (1)$$

uniformly in $h \in]0, h_0]$? Here $s > 0$ and $w(t)$ is independent of h and such that $w(t) \rightarrow 0$ as $t \rightarrow \infty$.

When $n = 3$ and $\rho > 2$, one can show that for each fixed $h > 0$, one can take $w(t) = C_h \langle t \rangle^{-r}$ for any $r < s$ and $r \in [0, 3/2]$. In the regime $h \rightarrow 0$, long-time behaviors of the quantum evolution are closely related to the classical dynamics. Let $(x(t; y, \eta), \xi(t; y, \eta))$ denote the classical Hamiltonian flow of

$p_1(x, \xi)$ with initial data (y, η) . Making use of Egorov's theorem, one can deduce that a necessary condition for (1) to be true is that

$$|\langle x(t; y, \eta) \rangle^{-s} e^{-2 \int_0^t V_2(x(\tau; y, \eta)) d\tau} \langle y \rangle^{-s}| \leq w(t) \quad (2)$$

for all $(y, \eta) \in \mathbb{R}^{2n}$. Since $w(t)$ tends to zero, estimate (2) implies that each bounded classical trajectory should pass through the open set $\{x; V_2(x) > 0\}$. In addition, one derives from (2) that $w(t) \geq C \langle t \rangle^{-s\sigma}$ with $\sigma = \min\{1, \tau_0\}$, where τ_0 is the divergence rate in t of non-trapping trajectories with energy 0. Is the condition (2) sufficient for a uniform time-decay estimate of the form (1)? If yes, can one take $w(t) = C \langle t \rangle^{-\min\{\frac{n}{2}, s\sigma\}}$? The restriction on the decay rate by $\frac{n}{2}$ comes from the threshold behavior of the semigroup for fixed h .

Recall that in the selfadjoint case ($V_2 = 0$) and with a localization in energies away from \mathbb{R}_- , the result is true with $w(t) = C_s \langle t \rangle^{-s}$ for any $s > 0$. More precisely, let $U(t, h) = e^{-itP_1(h)/h}$, $P_1(h) = -h^2\Delta + V_1$, $I =]a, b[$ with $a > 0$, $\chi \in C_0^\infty(I)$. Then the estimate

$$\|\langle x \rangle^{-s} \chi(P_1(h)) U(t, h) \langle x \rangle^{-s}\| \leq C_s \langle t \rangle^{-\epsilon}, \quad t \in \mathbb{R}, \quad (3)$$

holds for some $s, \epsilon > 0$ and uniformly in $h \in]0, h_0]$ if and only if every energy E in $\text{supp } \chi$ is non-trapping. If the latter is satisfied, (3) holds with $\epsilon = s$ and for any $s > 0$; see [2].

For non-selfadjoint operators, one can no longer use compactly supported cut-off. The main difficulty to prove a result like (1) is the semiclassical analysis near the threshold zero for dissipative Schrödinger operators. A closely related problem is a global limiting absorption principle on the whole real axis from the the upper half-complex plane and a nice resolvent estimate in $h > 0$. For energies away from zero and under the condition (2), this was recently obtained by J. Royer (see [1]). The question is open near the threshold zero.

References

- [1] J. Royer, *Limiting absorption principle for the dissipative Helmholtz equation*, Comm. Part. Diff. Equ. 35(2010), no.8, 1458-1489.
- [2] X. P. Wang, *Time-decay of scattering solutions and classical trajectories*, Ann. I.H.P., Sect. A. 47(1987), no.1, 25-37.

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