Calibration of a stock's beta using options prices

Sofiene El Aoud, Frédéric Abergel

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Abstract

We present in our work a continuous time Capital Asset Pricing Model where the volatilities of the market index and the stock are both stochastic. Using a singular perturbation technique, we provide approximations for the prices of european options on both the stock and the index. These approximations are functions of the model parameters. We show then that existing estimators of the parameter beta, proposed in the recent literature, are biased in our setting because they are all based on the assumption that the idiosyncratic volatility of the stock is constant. We provide then an unbiased estimator of the parameter beta using only implied volatility data. This estimator is a forward measure of the parameter beta in the sense that it represents the information contained in derivatives prices concerning the forward realization of this parameter, we test then its capacity of prediction of forward beta and we draw a conclusion concerning its predictive power.

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Contents

1 Introduction

The notion of a stock's β was first introduced in the theory of Capital Asset Pricing Model by Sharpe. This model extended previous works Markowitz did on portfolio construction theory (see [12]). The CAPM model was considered to be original and innovative because it introduced the concept of systematic and specific risk and facilitated then the understanding of the equity market (see [5]). The parameter β, which is a key parameter in this model, enables to separate the stock risk into two parts. The first part represents the systematic risk implied by the market risk, while the second part is the idiosyncratic risk that reflects the specific performance of the stock. The parameter β is of great use and its estimation is crucial in the construction of stock portfolios (see [16], [1], [13]). This parameter was traditionally estimated using historical data of daily returns of the stock and the market index (see [18], [9]). In this approach, the estimator of the parameter is obtained as the slope of the linear regression of stock returns on market index returns. This approach is backward-looking as it estimates the realized value of the parameter in the past using historical data. This characteristic can be considered as a weakness of the method. In fact, the value of the realized beta in the future can be remarkably different from its realization in the past, so the method lacks a predictive power.

In the recent literature, different authors have focused on the estimation of the β coefficient using options data. This methodology provides a different way to estimate the parameter β. In fact, whereas classical methods allow an historical estimation of this parameter, the method based on the use of options prices enables us to obtain a "forward looking" measure of this parameter. In fact, the obtained estimator represents the information contained in derivatives prices and then summarizes the expectation of market participants for the forward realization of this parameter.

In [4], Christoffersen, Jacobs and Vainberg provided an estimation of this parameter using the risk-neutral variance and skewness of the stock and the index. More recently, Fouque and Kollman proposed in [6] a continuous-time CAPM model in which the market index has a stochastic volatility driven by a fast mean-reverting process. Using a singular perturbation method, they managed to obtain an approximation of the beta parameter depending on the skews of implied volatilities of both the stock and the index. Fouque and Tashman introduced also in [7] a "Stressed-Beta model" in which the parameter β can take two values depending on the market regime. Using this model, Fouque et *al* provided a method to price options on index and stock. This method enables also to estimate the parameter β based on options data. In [3], Carr and Madan used the CAPM model to price options on the stock when options on the index are liquid. Their approach didn't aim to estimate the parameter beta using option prices, but to price options on the stock given the parameter beta and options prices on the market index.

This work is inspired mainly from [6] and [10]. We look here into the estimation of the coefficient β using options prices. The paper is organized as follows : in the first part, we make a brief reminder of the results obtained in the recent literature mainly in [6]. In the second part, we focus on the case where the index volatility and the stock's idiosyncratic volatility are both stochastic. This model is more likely to reproduce the stylised facts observed in the market and to capture realistic relations between the stock and the market index. We provide an estimator for the beta coefficient by means of a singular perturbation technique. In the third part, we make an empirical study in order to test the predictive power of our estimator for the forward realized β.

2 Literature review

2.1 The model of Fouque et *al*

Fouque et *al* proposed in [6] a continuous time Capital Asset Pricing Model in a stochastic volatility environment. They supposed that the volatility of the index is driven by a fast mean-reverting Ornstein-Uhlenbeck process. Under the historic probability measure P, we have :

$$\begin{aligned} \frac{dI_t}{I_t} &= \mu_I dt + f(Y_t) dW_t^{(1)}, \\ \frac{dS_t}{S_t} &= \mu_S dt + \beta \frac{dI_t}{I_t} + \sigma dW_t^{(2)}, \\ dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^{(3)}, \end{aligned}$$

where $W_t^{(3)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(4)}$ and $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(4)} \end{pmatrix}$ is a Wiener process under P

Let $\lambda_t = \begin{pmatrix} \frac{\mu_I - r}{f(Y_t)} \\ \frac{\mu_S + r(\beta-1)}{\sigma} \\ \gamma(Y_t) \end{pmatrix}$, P^* be a probability measure equivalent to P defined as:

$$\frac{dP^*}{dP}_{|\mathcal{F}_t} = exp(-\int_0^t \lambda_u' dW_u - \frac{1}{2}\int_0^t |\lambda_u|^2 du),$$

and $W^* = \begin{pmatrix} W^{*,(1)} \\ W^{*,(2)} \\ W^{*,(4)} \end{pmatrix}$ such that: $W_t^* = W_t + \int_0^t \lambda_u du$. By Girsanov's theorem, W^* is a brownian motion under P^*.

Under continuity and boundedness conditions on the function γ, P^* is a risk-neutral probability-measure under which we have:

$$\begin{aligned} \frac{dI_t}{I_t} &= rdt + f(Y_t) dW_t^{*,(1)}, \\ \frac{dS_t}{S_t} &= rdt + \beta f(Y_t) dW_t^{*,(1)} + \sigma dW_t^{*,(2)}, \\ dY_t &= (\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\chi(Y_t))dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^{*,(3)}, \end{aligned}$$

where $\chi(Y_t) = \rho\frac{\mu_I - r}{f(Y_t)} + \sqrt{1-\rho^2}\gamma(Y_t)$ and $W_t^{*,(3)} = \rho W_t^{*,(1)} + \sqrt{1-\rho^2}W_t^{*,(4)}$.

We can notice that there is an infinity of risk-neutral probability measures. The choice of the function γ determines the risk-neutral probability under which we price options.

2.2 Calibration of implied beta

Using a singular perturbation method with respect to the small parameter ϵ, Fouque et *al* managed to obtain an approximation $\tilde{P}^{I,\epsilon}(K_I, T)$ for the price of an european call on the index with strike K_I and maturity T, and an approximation $\tilde{P}^{S,\epsilon}(K_S, T)$ for the price of an european call on the stock with strike K_S and maturity T. Afterwards, by doing a Taylor expansion in $\sqrt{\epsilon}$ for the implied volatility of the stock and the index, they managed to approximate the shape of the implied volatility surfaces as follows:

$$\begin{aligned}\Sigma_I(K_I, T) &= b_I + a_I\frac{ln(\frac{F_I}{K_I})}{T},\\ \Sigma_S(K_S, T) &= b_S + a_S\frac{ln(\frac{F_S}{K_S})}{T},\end{aligned}$$

where F_I and F_S are the forward prices for maturity T of the index and stock respectively. The quantities b_I, a_I, b_S, a_S are functions of the model parameters. The parameter β can be approximated by $\hat{\beta}$ which is defined as :

$$\hat{\beta} = (\frac{a_S}{a_I})^{\frac{1}{3}}\frac{b_S}{b_I}. \tag{2.1}$$

2.3 Limits of the model

In the model described so far, Fouque et *al* made the assumption that the idiosyncratic volatility of the stock is constant. This hypothesis is too simplistic. In fact, a stock has an idiosyncratic volatility that varies significantly, especially when the market reacts to specific news of the company (earning expectation, restructuring projects,...). The idiosyncratic volatility has then its own dynamics (see [17], [2], [11], [15]).
By way of example, we give here the graph of the idiosyncratic volatility σ of XLF when projected on the SPX index, from 01/01/2008 to 31/12/2012. The parameter σ is obtained here by computing the standard deviation of errors in the linear regression of stock's returns (XLF) on index returns (SPX) with a sliding window of 1 month.

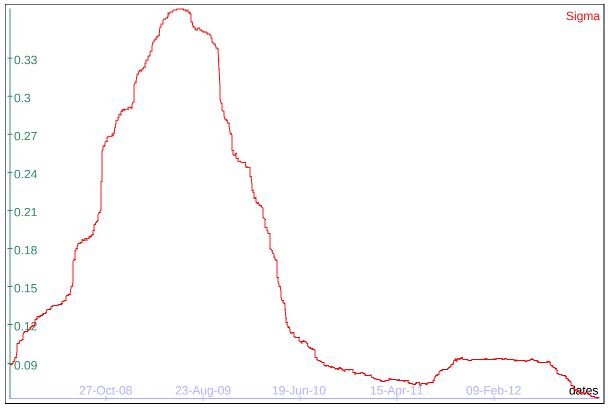


Figure 1: Evolution of the idiosyncratic volatility of XLF with respect to the SPX index

The inspection of the graph above shows that the parameter σ is not constant and then we should take this characteristic into consideration.

3 Model with fast mean-reverting idiosyncratic volatility

In this section, we propose a new model in which the idyosyncratic volatility of the stock is driven by a fast mean-reverting Ornstein-Uhlenbeck process. We explain then how we could calibrate the parameter β using options prices.

3.1 Presentation of the model

Let us assume that under the historic probability measure P, the stock and the index have the the following dynamics:

$$\begin{aligned}
\frac{dI_t}{I_t} &= \mu_I dt + f_1(Y_t) dW_t^{(1)}, \\
\frac{dS_t}{S_t} &= \mu_S dt + \beta \frac{dI_t}{I_t} + f_2(Z_t) dW_t^{(2)}, \\
dY_t &= \frac{1}{\epsilon}(m_Y - Y_t)dt + \frac{\nu_Y \sqrt{2}}{\sqrt{\epsilon}} dW_t^{(3)}, \\
dZ_t &= \frac{\alpha}{\epsilon}(m_Z - Z_t)dt + \frac{\nu_Z \sqrt{2\alpha}}{\sqrt{\epsilon}} dW_t^{(4)},
\end{aligned}$$

where $W_t^{(3)} = \rho_Y W_t^{(1)} + \sqrt{1-\rho_Y^2} W_t^{(5)}$, $W_t^{(4)} = \rho_Z W_t^{(2)} + \sqrt{1-\rho_Z^2} W_t^{(6)}$ and $W = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(5)} \\ W^{(6)} \end{pmatrix}$ is a Wiener process under P.

Let $\lambda_t = \begin{pmatrix} \frac{\mu_I - r}{f_1(Y_t)} \\ \frac{\mu_S + r(\beta-1)}{f_2(Z_t)} \\ \gamma_1(Y_t) \\ \gamma_2(Z_t) \end{pmatrix}$ and P^* a probability measure equivalent to P and defined such that:

$$\frac{dP^*}{dP} = exp(-\int_0^t \lambda'_u dW_u - \frac{1}{2}\int_0^t |\lambda_u|^2 du).$$

Let us define $W^* = \begin{pmatrix} W^{*,(1)} \\ W^{*,(2)} \\ W^{*,(5)} \\ W^{*,(6)} \end{pmatrix}$ such that $W_t^* = W_t + \int_0^t \lambda_u du$. Using Girsanov's theorem, W^* is a brownian motion under P^*.

P^* is a risk-neutral probability measure under which we have:

$$\begin{aligned} \frac{dI_t}{I_t} &= rdt + f_1(Y_t)dW_t^{*,(1)}, \\ \frac{dS_t}{S_t} &= rdt + \beta f_1(Y_t)dW_t^{*,(1)} + f_2(Z_t)dW_t^{*,(2)}, \\ dY_t &= \frac{1}{\epsilon}(m_Y - Y_t)dt - \frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}}\chi_1(Y_t)dt + \frac{\nu_Y\sqrt{2}}{\sqrt{\epsilon}}dW_t^{*,(3)}, \\ dZ_t &= \frac{\alpha}{\epsilon}(m_Z - Z_t)dt - \frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\epsilon}}\chi_2(Z_t)dt + \frac{\nu_Z\sqrt{2\alpha}}{\sqrt{\epsilon}}dW_t^{*,(4)}, \end{aligned}$$

where: $\chi_1(Y_t) = \rho_Y \frac{\mu_I - r}{f_1(Y_t)} + \sqrt{1-\rho_Y^2}\gamma_1(Y_t)$ and $\chi_2(Z_t) = \rho_Z \frac{\mu_S + r(\beta-1)}{f_2(Z_t)} + \sqrt{1-\rho_Z^2}\gamma_2(Z_t)$.

$W^{*,(3)}$ and $W^{*,(4)}$ are brownian motions under P^* such that:

$$\begin{aligned} W^{*,(3)} &= \rho_Y W^{*,(1)} + \sqrt{1-\rho_Y^2}W^{*,(5)}, \\ W^{*,(4)} &= \rho_Z W^{*,(2)} + \sqrt{1-\rho_Z^2}W^{*,(6)}. \end{aligned}$$

3.2 Pricing options on the index and the stock

3.2.1 Approximation formula for index option price

We call $P^{I,\epsilon}(K_I,T) = E^{P^*}(e^{-r(T-t)}(I_T - K_I)^+|\mathcal{F}_t)$ the price of an european call on the index with strike K_I and maturity T. We can easily see that the pricing of options on the index remains the same as in Fouque's model where the idiosyncratic volatility of the stock is constant. This is due to the fact that the diffusion equations of the processes (I) and (Y) are still the same.

By doing a singular perturbation method as in [6], Fouque et *al* obtained an approximation $\tilde{P}^{I,\epsilon}(K_I,T)$ for the price $P^{I,\epsilon}(K_I,T)$. For simplification purposes, we will use the notation

$\tilde{P}^{I,\epsilon}$ instead of $\tilde{P}^{I,\epsilon}(K_I, T)$. The proof of the approximation result is given in [6] and also in Appendix 1 for completeness. We just recall the result here.

$$\tilde{P}^{I,\epsilon} = \tilde{P}_0^{I,\epsilon} - (T-t)V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t}\left(I_t^2 \frac{\partial^2 \tilde{P}_0^{I,\epsilon}}{\partial I_t^2}\right), \tag{3.1}$$

where the quantities $\tilde{P}_0^{I,\epsilon}$ and $V_3^{I,\epsilon}$ are defined as:

$$\tilde{P}_0^{I,\epsilon} = P_{BS}^I(\bar{\sigma}_I^*) \tag{3.2}$$

$$V_2^{I,\epsilon} = -\frac{\sqrt{\epsilon}}{\sqrt{2}} \nu_Y < \phi_I' \chi_1 >_1, \tag{3.3}$$

$$V_3^{I,\epsilon} = \frac{\sqrt{\epsilon}}{\sqrt{2}} \rho_Y \nu_Y < \phi_I' f_1 >_1, \tag{3.4}$$

$$(\bar{\sigma}_I^*)^2 = < f_1^2 >_1 - 2V_2^{I,\epsilon}. \tag{3.5}$$

We should precise here that $< . >_1$ is the average with respect to the invariant distribution of the Ornstein-Uhlenbeck process (Y_1) whose dynamics are described by :

$$dY_{1,t} = (m_Y - Y_{1,t})dt + \nu_Y \sqrt{2} dW_t^{(3)}.$$

ϕ_I is defined as the solution of the following Poisson equation :

$$\mathcal{L}_0^I \phi_I(y) = f_1^2(y) - < f_1^2 >_1, \tag{3.6}$$

where $\mathcal{L}_0^I$ is the infinitesimal generator of the process (Y_1) :

$$\mathcal{L}_0^I = \frac{\partial}{\partial y}(m_Y - y) + \nu_Y^2 \frac{\partial^2}{\partial y^2}.$$

3.2.2 Approximation formula for stock option price

Let $P_t^{S,\epsilon}(K_S, T)$ be the price at time t of an european call on the stock with strike K_S and maturity T:

$$P_t^{S,\epsilon}(K_S, T) = E^{P^*}(e^{-r(T-t)}(S_T - K_S)^+ | \mathcal{F}_t).$$

To simplify the notations, we will use the notation $P_t^{S,\epsilon}$ instead of $P_t^{S,\epsilon}(K_S, T)$.
Using a singular perturbation technique on the small parameter ϵ, we can obtain an approximation $\tilde{P}^{S,\epsilon}$ for the option's price $P_t^{S,\epsilon}$. This approximation can be detailed as follows:

Proposition 3.1

$$\tilde{P}^{S,\epsilon} = \tilde{P}_0^{S,\epsilon} - (T-t)V_3^{S,\epsilon} S_t \frac{\partial}{\partial S_t}\left(S_t^2 \frac{\partial^2 \tilde{P}_0^{S,\epsilon}}{\partial S_t^2}\right), \tag{3.7}$$

where the quantities $\tilde{P}_0^{S,\epsilon}$ and $V_3^{S,\epsilon}$ are defined as:

$$\tilde{P}_0^{S,\epsilon} = P_{BS}^S(t, S_t, \bar{\sigma}_S^*), \tag{3.8}$$
$$(\bar{\sigma}_S^*)^2 = \bar{\sigma}_S^2 - 2V_2^{S,\epsilon}, \tag{3.9}$$
$$V_2^{S,\epsilon} = -\frac{\sqrt{\epsilon}}{\sqrt{2}}(\beta^2\nu_Y < \phi_I'\chi_1 > +\nu_Z\sqrt{\alpha} < \phi_{Idios}'\chi_2 >), \tag{3.10}$$
$$V_3^{S,\epsilon} = \frac{\sqrt{\epsilon}}{\sqrt{2}}(\beta^3\rho_Y\nu_Y < \phi_I' f_1 > +\rho_Z\nu_Z\sqrt{\alpha} < \phi_{Idios}' f_2 >). \tag{3.11}$$

Proof:

$$P_t^{S,\epsilon} = E^{P^*}(e^{-r(T-t)}(S_T - K_S)^+ | S_t = x, Y_t = y, Z_t = z).$$

Using Itô's Lemma, we have:

$$\mathcal{L}^S P_t^{S,\epsilon} = 0.$$

We then expand $\mathcal{L}^S$ in powers of $\sqrt{\epsilon}$, and obtain:

$$\mathcal{L}^S = \mathcal{L}_2^S + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^S + \frac{1}{\epsilon}\mathcal{L}_0^S,$$

where:

$$\begin{aligned}
\mathcal{L}_0^S &= (m_Y - y)\frac{\partial}{\partial y} + \nu_Y^2\frac{\partial^2}{\partial y^2} + \alpha(m_Z - z)\frac{\partial}{\partial z} + \alpha\nu_Z^2\frac{\partial^2}{\partial z^2}, \\
\mathcal{L}_1^S &= -\nu_Y\sqrt{2}\chi_1(y)\frac{\partial}{\partial y} + \beta S_t f_1(y)\sqrt{2}\rho_Y\nu_Y\frac{\partial^2}{\partial S\partial y} - \nu_Z\sqrt{2\alpha}\chi_2(z)\frac{\partial}{\partial z} + S_t f_2(z)\sqrt{2\alpha}\rho_Z\nu_Z\frac{\partial^2}{\partial S\partial z}, \\
\mathcal{L}_2^S &= \frac{\partial}{\partial t} + r(\frac{\partial}{\partial S}S_t - .) + \frac{1}{2}\frac{\partial^2}{\partial S_t^2}S_t^2(\beta^2 f_1(y)^2 + f_2(z)^2).
\end{aligned}$$

We can note that $\mathcal{L}_0^S$ is the infinitesimal generator of the two-dimensional Ornstein-Uhlenbeck process $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ having the following dynamics :

$$d\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}(\begin{pmatrix} m_Y \\ m_Z \end{pmatrix} - \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix})dt + \sqrt{2}\begin{pmatrix} \nu_Y & 0 \\ 0 & \sqrt{\alpha}\nu_Z \end{pmatrix} d\begin{pmatrix} W_t^{(3)} \\ W_t^{(4)} \end{pmatrix}.$$

We define here:

- $< . >_1$ denotes the averaging with respect to the invariant distribution of the process $(Y_{1,t})_t$.
- $< . >_2$ denotes the averaging with respect to the invariant distribution of the process $(Y_{2,t})_t$.
- $< . >_{1,2}$ denotes the averaging with respect to the invariant distribution of $\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix}_t$.

We formally expand $P^{S,\epsilon}$ in powers of $\sqrt{\epsilon}$:

$$P^{S,\epsilon} \;=\; \sum_{i=0}^{\infty}(\sqrt{\epsilon})^i P_i^{S,\epsilon},$$

then, we expand the term $\mathcal{L}^S P_t^{S,\epsilon}$:

$$(\mathcal{L}_2^S + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^S + \frac{1}{\epsilon}\mathcal{L}_0^S)(\sum_{i=0}^{\infty}(\sqrt{\epsilon})^i P_i^{S,\epsilon}) \;=\; 0.$$

By classifying the terms of the last equation by powers of $\sqrt{\epsilon}$, we obtain :

$$(0) \;:\; \mathcal{L}_2^S P_0^{S,\epsilon} + \mathcal{L}_1^S P_1^{S,\epsilon} + \mathcal{L}_0^S P_2^{S,\epsilon} = 0, \tag{3.12}$$
$$(-1) \;:\; \mathcal{L}_1^S P_0^{S,\epsilon} + \mathcal{L}_0^S P_1^{S,\epsilon} = 0, \tag{3.13}$$
$$(-2) \;:\; \mathcal{L}_0^S P_0^{S,\epsilon} = 0, \tag{3.14}$$
$$(1) \;:\; \mathcal{L}_2^S P_1^{S,\epsilon} + \mathcal{L}_1^S P_2^{S,\epsilon} + \mathcal{L}_0^S P_3^{S,\epsilon} = 0, \tag{3.15}$$
$$(2) \;:\; \mathcal{L}_2^S P_2^{S,\epsilon} + \mathcal{L}_1^S P_3^{S,\epsilon} + \mathcal{L}_0^S P_4^{S,\epsilon} = 0. \tag{3.16}$$

The term of order (-2) in $\sqrt{\epsilon}$ states that $\mathcal{L}_0^S P_0^{S,\epsilon} = 0$. Given that $\mathcal{L}_0^S$ contains only derivatives with respect to y and z, we can solve this equation by choosing $P_0^{S,\epsilon} = P_0^{S,\epsilon}(t, S_t)$ independent of Y_t and Z_t.

The term of order (-1) in $\sqrt{\epsilon}$ states that $\mathcal{L}_1^S P_0^{S,\epsilon} + \mathcal{L}_0^S P_1^{S,\epsilon} = 0$. $\mathcal{L}_1^S$ contains first and second order derivatives with respect to y and z, then $\mathcal{L}_1^S P_0^{S,\epsilon} = 0$. The equation becomes then $\mathcal{L}_0^S P_1^{S,\epsilon} = 0$. The equation is satisfied if $P_1^{S,\epsilon} = P_1^{S,\epsilon}(t, S_t)$ independent of Y_t and Z_t.

Consequently, $P_0^{S,\epsilon}$ and $P_1^{S,\epsilon}$ are independent of Y_t and Z_t, and we have:

$$\mathcal{L}_0^S P_0^{S,\epsilon} = \mathcal{L}_1^S P_0^{S,\epsilon} = \mathcal{L}_0^S P_1^{S,\epsilon} = \mathcal{L}_1^S P_1^{S,\epsilon} = 0.$$

Given that $\mathcal{L}_1^S P_1^{S,\epsilon} = 0$, the term of order 0 in $\sqrt{\epsilon}$ becomes:

$$\mathcal{L}_2^S P_0^{S,\epsilon} + \mathcal{L}_0^S P_2^{S,\epsilon} = 0.$$

This is a Poisson equation for $P_2^{S,\epsilon}$ with respect to $\mathcal{L}_0^S$. The solvability condition for this equation is:

$$< \mathcal{L}_2^S P_0^{S,\epsilon} >_{1,2} = < \mathcal{L}_2^S >_{1,2} P_0^{S,\epsilon} = 0.$$

We can notice that the average $< \mathcal{L}_2^S >$ of the generator $\mathcal{L}_2^S$ is equal to :

$$< \mathcal{L}_2^S >_{1,2} \;=\; \frac{\partial}{\partial t} + r(\frac{\partial}{\partial S_t} S_t - .) + \frac{1}{2}\frac{\partial^2}{\partial S_t^2} S_t^2 < \beta^2 f_1^2(y) + f_2^2(z) >_{1,2} .$$

Then, we can deduce that: $< \mathcal{L}_2^S >_{1,2} = \mathcal{L}_{BS}^S(\bar{\sigma}_S)$ where $\bar{\sigma}_S^2 = \beta^2 < f_1^2 >_{1,2} + < f_2^2 >_{1,2}$.

Consequently, $P_0^{S,\epsilon}$ is the solution of the following problem :

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma}_S)P_0^{S,\epsilon} &= 0, \\ P_0^{S,\epsilon}(T,S_T) &= h(S_T). \end{aligned}$$

We deduce that $P_0^{S,\epsilon}$ is the Black-Scholes price of the option with implied volatility equal to $\bar{\sigma}_S$, meaning that:

$$P_0^{S,\epsilon} = P_{BS}^S(t, S_t, \bar{\sigma}_S).$$

The term of order 1 in $\sqrt{\epsilon}$ is a Poisson equation for $P_3^{S,\epsilon}$ with respect to $\mathcal{L}_0^S$. The solvability condition for this equation is:

$$< \mathcal{L}_2^S >_{1,2} P_1^{S,\epsilon} \quad = \quad - < \mathcal{L}_1^S P_2^{S,\epsilon} >_{1,2} = < \mathcal{L}_1^S(\mathcal{L}_0^S)^{-1}(\mathcal{L}_2^S - < \mathcal{L}_2^S >_{1,2}) > P_0^{S,\epsilon}. \quad (3.17)$$

$P_1^{S,\epsilon}$ is the solution of the last equation with terminal condition $P_1^{S,\epsilon}(T,S_t) = 0$.

Given that f_1 is independent of z and f_2 is independent of y, we have: $< f_1^2 >_{1,2} = < f_1^2 >_1$ and $< f_2^2 >_{1,2} = < f_2^2 >_2$. We recall here that ϕ_I, the solution of (3.6), doesn't depend on z, we deduce that:

$$\mathcal{L}_0^S \phi_I(y) \quad = \quad \mathcal{L}_0^I \phi_I(y) = f_1^2(y) - < f_1^2 >_{1,2} .$$

Let ϕ_{Idios} the solution of the following equation:

$$\mathcal{L}_0^S \phi_{Idios}(z) = f_2^2(z) - < f_2^2 >_{1,2} . \quad (3.18)$$

Given that ϕ_{Idios} doesn't depend on y, we obtain that:

$$\mathcal{L}_0^S(\beta^2\phi_I(y) + \phi_{Idios}(z)) \quad = \quad \beta^2(f_1^2(y) - < f_1^2 >_{1,2}) + (f_2^2(z) - < f_2^2 >_{1,2}).$$

Then we deduce that:

$$\mathcal{L}_1^S(\mathcal{L}_0^S)^{-1}(\mathcal{L}_2^S - < \mathcal{L}_2^S >_{1,2}) \quad = \quad (\beta^2\mathcal{L}_1^S\phi_I(y) + \mathcal{L}_1^S\phi_{Idios}(z))\frac{1}{2}S_t^2\frac{\partial^2}{\partial S_t^2}.$$

By developing the right term in the previous equation, we obtain:

$$\begin{aligned} \frac{1}{2}(\beta^2 < \mathcal{L}_1^S\phi_I(y) >_{1,2} + < \mathcal{L}_1^S\phi_{Idios}(z) >_{1,2}) = {} & (\frac{\beta^3\nu_Y\rho_Y}{\sqrt{2}} < \phi_I' f_1 >_{1,2} + \frac{\rho_Z\nu_Z\sqrt{\alpha}}{\sqrt{2}} < \phi_{Idios}' f_2 >_{1,2})S_t\frac{\partial}{\partial S_t} \\ & - (\frac{\beta^2\nu_Y}{\sqrt{2}} < \phi_I'\chi_1 >_{1,2} + \frac{\nu_Z\sqrt{\alpha}}{\sqrt{2}} < \phi_{Idios}'\chi_2 >_{1,2}) \end{aligned}$$

In order to simplify the notations, we define the quantities $V_2^{S,\epsilon}$ and $V_3^{S,\epsilon}$:

$$\begin{aligned} V_3^{S,\epsilon} &= \frac{\sqrt{\epsilon}}{\sqrt{2}}(\beta^3\nu_Y\rho_Y < \phi_I' f_1 > + \rho_Z\nu_Z\sqrt{\alpha} < \phi_{Idios}' f_2 >), \\ V_2^{S,\epsilon} &= -\frac{\sqrt{\epsilon}}{\sqrt{2}}(\beta^2\nu_Y < \phi_I'\chi_1 > + \nu_Z\sqrt{\alpha} < \phi_{Idios}'\chi_2 >). \end{aligned}$$

The equation (3.17) becomes then:

$$< \mathcal{L}_2^S >_{1,2} \sqrt{\epsilon} P_1^{S,\epsilon} \quad = \quad V_2^{S,\epsilon} S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2} + V_3^{S,\epsilon} S_t \frac{\partial}{\partial S_t} (S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2}). \tag{3.19}$$

Consequently, $P_1^{S,\epsilon}$ is the solution of (3.19) with the final condition $P_1^{S,\epsilon}(T, S_T) = 0$. Given that $< \mathcal{L}_2^S >_{1,2} = \mathcal{L}_{BS}^S(\bar{\sigma}_S)$ commits with the operators $\mathcal{D}_{1,S} = S_t \frac{\partial}{\partial S_t}$ and $\mathcal{D}_{2,S} = S_t^2 \frac{\partial^2}{\partial S_t^2}$, and that $< \mathcal{L}_2^S >_{1,2} P_0^{S,\epsilon} = 0$, the solution to the last problem can be given explicitly by :

$$\sqrt{\epsilon} P_1^{S,\epsilon} = -(T-t)(V_2^{S,\epsilon} S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2} + V_3^{S,\epsilon} S_t \frac{\partial}{\partial S_t} (S_t^2 \frac{\partial^2 P_0^{S,\epsilon}}{\partial S_t^2})). \tag{3.20}$$

In order to check the validity of the solution, we can easily make the following verification:

$$\begin{aligned}
< \mathcal{L}_2^S >_{1,2} \sqrt{\epsilon} P_1^{S,\epsilon} &= (V_2^{S,\epsilon} \mathcal{D}_{2,S} P_0^{S,\epsilon} + V_3^{S,\epsilon} \mathcal{D}_{1,S} \mathcal{D}_{2,S} P_0^{S,\epsilon}) < \mathcal{L}_2^S >_{1,2} ((t-T)) \\
&- (T-t)(V_2^{S,\epsilon} \mathcal{D}_{2,S} < \mathcal{L}_2^S >_{1,2} (P_0^{S,\epsilon}) + V_3^{S,\epsilon} \mathcal{D}_{1,S} \mathcal{D}_{2,S} < \mathcal{L}_2^S >_{1,2} (P_0^{S,\epsilon})) \\
&= V_2^{S,\epsilon} \mathcal{D}_{2,S} P_0^{S,\epsilon} + V_3^{S,\epsilon} \mathcal{D}_{1,S} \mathcal{D}_{2,S} P_0^{S,\epsilon}.
\end{aligned}$$

By neglecting terms of order higher or equal to 1 in $\sqrt{\epsilon}$, we can approximate the stock option's price by $(P_0^{S,\epsilon} + \sqrt{\epsilon} P_1^{S,\epsilon})$. In order to reduce the number of the parameters in the approximation, we prove that:

$$P_0^{S,\epsilon} + \sqrt{\epsilon} P_1^{S,\epsilon} \quad = \quad \tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon} \tilde{P}_1^{S,\epsilon} + o(\epsilon) \, ,$$

where $\tilde{P}_0^{S,\epsilon}$ is defined as:

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}_S^*) \tilde{P}_0 &= 0, \\
\tilde{P}_0^{S,\epsilon}(T, S_T) &= (S_T - K_S)^+,
\end{aligned}$$

and $\mathcal{L}_{BS}(\bar{\sigma}_S^*)$ is the Black-Scholes differential operator with volatility $\bar{\sigma}_S^*$ such that:

$$(\bar{\sigma}_S^*)^2 \quad = \quad \bar{\sigma}_S^2 - 2V_2^{S,\epsilon},$$

which gives $\tilde{P}_0^{S,\epsilon} = P_{BS}(t, S_t, \bar{\sigma}_S^*)$. On the other hand, $\tilde{P}_1^{S,\epsilon}$ is defined as:

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}_S^*) \sqrt{\epsilon} \tilde{P}_1^{S,\epsilon} &= V_3^{S,\epsilon} S_t \frac{\partial}{\partial S_t} (S_t^2 \frac{\partial^2}{\partial S_t^2} \tilde{P}_0^{S,\epsilon}), \\
\tilde{P}_1^{S,\epsilon}(T, S_T) &= 0.
\end{aligned}$$

We can notice that:

$$\mathcal{L}_{BS}(\bar{\sigma}_S^*) = \mathcal{L}_{BS}(\bar{\sigma}_S) - V_2^{S,\epsilon} S_t^2 \frac{\partial^2}{\partial S_t^2}.$$

We start by studying the term $(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon})$. Using (3.21), we can then easily prove that:

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}_S)(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon}) &= -V_2^{S,\epsilon} S_t^2 \frac{\partial^2 \tilde{P}_0^{S,\epsilon}}{\partial S_t^2}, \\
(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon})(T, S_T) &= 0.
\end{aligned}$$

Since the source term is $O(\sqrt{\epsilon})$ because of $V_2^{S,\epsilon}$, then the difference term $(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon})$ is also $O(\sqrt{\epsilon})$. Next, we write:

$$|P^{S,\epsilon} - (\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon})| \leq |P^{S,\epsilon} - (P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon})| + |(P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon}) - (\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon})|$$

The first term $|P^{S,\epsilon,\delta} - (P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon})|$ is already $o(\epsilon)$. We focus then on the second term. To simplify the notations, we introduce the error term:

$$\mathcal{R} = (P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon}) - (\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}),$$

and the differential operators:

$$\begin{aligned}
\mathcal{H}_\epsilon &= V_2^{S,\epsilon}\mathcal{D}_{2,S} + V_3^{S,\epsilon}\mathcal{D}_{1,S}\mathcal{D}_{2,S}, \\
\mathcal{H}_\epsilon^* &= V_3^{S,\epsilon}\mathcal{D}_{1,S}\mathcal{D}_{2,S}.
\end{aligned}$$

We can then compute the term:

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}_S)\mathcal{R} &= \mathcal{L}_{BS}(\bar{\sigma}_S)((P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon}) - (\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon})), \\
&= \mathcal{H}_\epsilon P_0^{S,\epsilon} - (\mathcal{L}_{BS}(\bar{\sigma}_S^*) + V_2^{S,\epsilon}\mathcal{D}_{2,S})(\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}), \\
&= \mathcal{H}_\epsilon P_0^{S,\epsilon} - \mathcal{H}_\epsilon^*\tilde{P}_0^{S,\epsilon} - V_2^{S,\epsilon}\mathcal{D}_{2,S}(\tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}), \\
&= \mathcal{H}_\epsilon^*(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon}) - V_2^{S,\epsilon}\mathcal{D}_{2,S}(\tilde{P}_0^{S,\epsilon} - P_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}).
\end{aligned}$$

Knowing that:

- $(P_0^{S,\epsilon} - \tilde{P}_0^{S,\epsilon})$ is $O(\sqrt{\epsilon})$.
- $\mathcal{H}_\epsilon^*$ is $O(\sqrt{\epsilon})$
- $V_2^{S,\epsilon}\mathcal{D}_{2,S}$ is $O(\sqrt{\epsilon})$
- $\sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}$ is $O(\sqrt{\epsilon})$

and additionally $\mathcal{R}(T) = 0$, then we can obtain that:

$$\mathcal{R} = O(\epsilon)\,.$$

This concludes the derivation of the following result:

$$P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon} = \tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon} + O(\epsilon)\,.$$

So up to order 1 in $\sqrt{\epsilon}$, we can approximate the option price by $\tilde{P}^{S,\epsilon}$ which is defined as:

$$\tilde{P}^{S,\epsilon} = \tilde{P}_0^{S,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{S,\epsilon}.$$

Fouque and *al* proved that the estimation error obtained, when approximating $P^{S,\epsilon}$ by $\tilde{P}^{S,\epsilon}$, is at order 1 in ϵ. The reader can refer to Appendix 2 for a brief review of the proof.

3.3 Calibration of implied beta using options prices

We have already given an approximation for the price of an european option on the index in (3.1) and an approximation for the price of an european option on the stock in (3.7). In this section, an estimator of the parameter β using options prices is provided.

3.3.1 Approximation formula of the implied volatility smile

We derive an approximation of the smile of implied volatility for both the stock and the index in function of model parameters. As suggested in [6], we perform an expansion of the implied volatility in powers of $\sqrt{\epsilon}$. This enables us to approximate the shape of the implied volatility surfaces for both the stock and the index and to obtain the following results:

Proposition 3.2 *The implied volatility of an european call on the index with strike K_I and maturity T can be approximated, at order 1 in $\sqrt{\epsilon}$, by $\Sigma_I(K_I, T)$ which can be given as:*

$$\Sigma_I(K_I, T) = b_I + a_I \frac{ln(\frac{F_I}{K_I})}{T},$$

where $b_I = \bar{\sigma}_I^* - \frac{V_3^{I,\epsilon}}{2\bar{\sigma}_I^*}$, $a_I = \frac{V_3^{I,\epsilon}}{(\bar{\sigma}_I^*)^3}$ and $F_I = I_t e^{r(T-t)}$.
On the other hand, the implied volatility of an european call on the stock with strike K_S and maturity T can be approximated, at order 1 in $\sqrt{\epsilon}$, by $\Sigma_S(K_S, T)$ which is equal to:

$$\Sigma_S(K, T) = b_S + a_S \frac{ln(\frac{F_S}{K_S})}{T},$$

where $b_S = \bar{\sigma}_S^* - \frac{V_3^{S,\epsilon}}{2\bar{\sigma}_S^*}$, $a_S = \frac{V_3^{S,\epsilon}}{(\bar{\sigma}_S^*)^3}$ and $F_S = S_t e^{r(T-t)}$.

A derivation of this result is included here for the purpose of making this paper as self-contained as possible.

Proof:
We use here A to denote either the stock S or the index I. We have :

$$P^{A,\epsilon} \quad = \quad \tilde{P}_0^{A,\epsilon} - (T-t)V_3^{A,\epsilon} A_t \frac{\partial}{\partial A_t}(A_t^2 \frac{\partial^2 \tilde{P}_0^{A,\epsilon}}{\partial A_t^2}) + o(\epsilon)\,, \tag{3.21}$$

where $\tilde{P}_0^{A,\epsilon}$ is defined as:

$$\tilde{P}_0^{A,\epsilon} \quad = \quad P_{BS}(t, A_t, \bar{\sigma}_A^*).$$

$P^{A,\epsilon}$ could represent the price of the option on the index $P^{I,\epsilon}$ if $A = I$ or the price of the option on the stock $P^{S,\epsilon}$ in the case where $A = S$.

Let I_A be the implied volatility associated to the asset's option price $P^{A,\epsilon}$, then we have $P^{A,\epsilon} = P_{BS}(t, A_t, I_A(K_A, T))$. We can do an expansion of $I_A(K, T)$ around $\bar{\sigma}_A^*$ in powers of $\sqrt{\epsilon}$:

$$I_A(K_A, T) = \bar{\sigma}_A^* + \sqrt{\epsilon} I_1(K_A, T) + O(\epsilon)\,.$$

We have then:

$$P^{A,\epsilon} = P_{BS}(t, A_t, \bar{\sigma}_A^*) + \frac{\partial P_{BS}}{\partial \sigma}_{|\sigma=\bar{\sigma}_A^*} \sqrt{\epsilon} I_1 + o(\epsilon)\,. \tag{3.22}$$

If we combine the equations (3.21) and (3.22), we get:

$$\frac{\partial P_{BS}(t, A_t, \bar{\sigma}_A^*)}{\partial \sigma}_{|\sigma=\bar{\sigma}_A^*} \sqrt{\epsilon} I_1(K, T) = -(T-t) V_3^{A,\epsilon} A_t \frac{\partial}{\partial A_t}\left(A_t^2 \frac{\partial^2 \tilde{P}_0}{\partial A_t^2}\right). \tag{3.23}$$

By performing simple computations on the derivatives of the Black-Scholes price, we can obtain that:

$$\mathcal{D}_{2,A} \tilde{P}_0 \;\; = \;\; \frac{1}{\bar{\sigma}_A^*(T-t)} \frac{\partial P_{BS}}{\partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*).$$

We apply then the operator $\mathcal{D}_{1,A}$ to the last equation:

$$\mathcal{D}_{1,A} \mathcal{D}_{2,A} \tilde{P}_0 \;\; = \;\; \frac{A_t}{\bar{\sigma}_A^*(T-t)} \frac{\partial^2 P_{BS}}{\partial A_t \partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*).$$

Using closed-form formulas of Black-Scholes greeks, we can write that:

$$A_t \frac{\partial^2 P_{BS}}{\partial A_t \partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*) = -\frac{d_2}{\bar{\sigma}_A^* \sqrt{T-t}} \frac{\partial P_{BS}}{\partial \bar{\sigma}_A^*}(t, A_t, \bar{\sigma}_A^*).$$

The equation (3.23) can be written as:

$$\sqrt{\epsilon} I_1(K_A, T) = -\frac{V_3^{A,\epsilon}}{\bar{\sigma}_A^*} \frac{A_t \frac{\partial^2 \tilde{P}_0}{\partial A_t \partial \bar{\sigma}_A^*}}{\frac{\partial \tilde{P}_0}{\partial \bar{\sigma}_A^*}}.$$

Then, it is straightforward that:

$$\sqrt{\epsilon} I_1(K_A, T) = \frac{V_3^{A,\epsilon} d_2(K_A, T)}{(\bar{\sigma}_A^*)^2 \sqrt{T-t}},$$

where:

$$d_2(K_A, T) \;\; = \;\; \frac{\log\left(\frac{A_t e^{r(T-t)}}{K_A}\right) - \frac{(\bar{\sigma}_A^*)^2}{2}(T-t)}{\bar{\sigma}_A^* \sqrt{T-t}}.$$

We can then approximate the implied volatility by the following formula:

$$I_A(K_A,T) = \bar{\sigma}^*_A - \frac{V_3^{A,\epsilon}}{2\bar{\sigma}^*_A} + \frac{V_3^{A,\epsilon}}{(\bar{\sigma}^*_A)^3}\frac{ln(\frac{F_A(T)}{K_A})}{T-t}. \quad (3.24)$$

Then, we obtain the following smile approximation expression:

$$I_A(K_A,T) = b_A + a_A\frac{ln(\frac{F_A(T)}{K_A})}{T-t},$$

with:

$$\begin{aligned} b_A &= \bar{\sigma}^*_A - \frac{V_3^{A,\epsilon}}{2\bar{\sigma}^*_A}, \\ a_A &= \frac{V_3^{A,\epsilon}}{(\bar{\sigma}^*_A)^3}. \end{aligned}$$

3.3.2 Comparison with the model with constant idiosyncratic volatility

We use the approximations given previously in order to estimate the parameter β. Based on the definitions of $V_3^{S,\epsilon}$ and $V_3^{I,\epsilon}$, we can write that:

$$\begin{aligned} V_3^{S,\epsilon} &= \beta^3 V_3^{I,\epsilon} + \frac{\sqrt{\epsilon}}{\sqrt{2}}\rho_Z\nu_Z\sqrt{\alpha} < \phi'_{Idios}f_2 >, \\ \frac{V_3^{S,\epsilon}}{V_3^{I,\epsilon}} &= \beta^3 + \frac{\rho_Z\nu_Z\sqrt{\alpha}}{\rho_Y\nu_Y}\frac{< \phi'_{Idio}f_2 >}{< \phi'_I f_1 >}. \end{aligned}$$

The estimator $\hat{\beta}$ proposed in [6] and introduced also in (2.1) verifies:

$$\hat{\beta}^3 = \frac{V_3^{S,\epsilon}}{V_3^{I,\epsilon}} = \beta^3 + \frac{\rho_Z\nu_Z\sqrt{\alpha}}{\rho_Y\nu_Y}\frac{< \phi'_{Idio}f_2 >}{< \phi'_I f_1 >}.$$

We deduce that in the case where the idiosyncratic volatility of the stock is stochastic, $\hat{\beta}$ is a biased estimator of the parameter β.

3.3.3 Alternative method for the estimation

We recall here that:

$$\bar{\sigma}_S^2 = \beta^2\bar{\sigma}_I^2 + < f_2^2 >_{1,2} .$$

We can then write that:

$$\beta^2 = \frac{(\bar{\sigma}^*_S)^2 - < f_2^2 >_{1,2}}{(\bar{\sigma}^*_I)^2 + 2V_2^{I,\epsilon}} + \frac{2V_2^{S,\epsilon}}{(\bar{\sigma}^*_I)^2 + 2V_2^{I,\epsilon}}.$$

Given the values of $V_2^{S,\epsilon}$ and $V_2^{I,\epsilon}$, we can then write β as following:

$$\beta = \sqrt{\frac{(\bar{\sigma}_S^*)^2 - < f_2^2 >_{1,2}}{(\bar{\sigma}_I^*)^2 + 2V_2^{I,\epsilon}} + \frac{2V_2^{S,\epsilon}}{(\bar{\sigma}_I^*)^2 + 2V_2^{I,\epsilon}}}. \tag{3.25}$$

Using the smile approximation formula (3.24), we can write, for the asset A which denotes either the stock S or the index I, the following relation:

$$b_A = \bar{\sigma}_A^* - \frac{1}{2} a_A (\bar{\sigma}_A^*)^2.$$

This second order equation in $\bar{\sigma}_A^*$ has two admissible solutions x_1 and x_2 :

$$x_1 = \frac{1 - \sqrt{1 - 2a_A b_A}}{a_A},$$
$$x_2 = \frac{1 + \sqrt{1 - 2a_A b_A}}{a_A}.$$

Since $V_2^{A,\epsilon}$ and $V_3^{A,\epsilon}$ are of order 1 in $\sqrt{\epsilon}$, then a_A is of order 1 in $\sqrt{\epsilon}$ and $b_A = \bar{\sigma}_A^* + o(\sqrt{\epsilon})$. We can then deduce that the appropriate solution is x_1. Consequently, we have:

$$\bar{\sigma}_S^* = \frac{1 - \sqrt{1 - 2a_S b_S}}{a_S},$$
$$\bar{\sigma}_I^* = \frac{1 - \sqrt{1 - 2a_I b_I}}{a_I}.$$

Then, we can approximate β by $\tilde{\beta}$ which is given by:

$$\tilde{\beta} = \sqrt{\frac{(\frac{1-\sqrt{1-2a_S b_S}}{a_S})^2 - < f_2^2 >_{1,2}}{(\frac{1-\sqrt{1-2a_I b_I}}{a_I})^2 + 2V_2^{I,\epsilon}} + \frac{2V_2^{S,\epsilon}}{(\frac{1-\sqrt{1-2a_I b_I}}{a_I})^2 + 2V_2^{I,\epsilon}}}. \tag{3.26}$$

In order to compute $\tilde{\beta}$, we follow the methodology given below:

- We approximate $\bar{\sigma}_I$ and $\bar{\sigma}_S$ by the historical volatilities of the market index and the stock respectively using underlying log-returns.
- We calibrate an affine function in the composite variable called "log-moneyness-to-maturity ratio" to the implied volatility surfaces of the index and the stock across different strikes and maturities. We get then the estimated slopes a_S, a_I and the intercepts b_S, b_I.
- From the estimated slope a_S, the intercept b_S and the effective volatility $\bar{\sigma}_S$, we calculate the following quantities:
  - $\bar{\sigma}_S^* = \frac{1-\sqrt{1-2a_S b_S}}{a_S}$
  - $V_2^{S,\epsilon} = \frac{\bar{\sigma}_S^2 - (\bar{\sigma}_S^*)^2}{2}$

- $V_3^{S,\epsilon} = a_S(\bar{\sigma}_S^*)^3$

- Likewise, from the estimated slope a_I, the intercept b_I and the effective volatility $\bar{\sigma}_I$, we calculate the following quantities:
  - $\bar{\sigma}_I^* = \frac{1-\sqrt{1-2a_I b_I}}{a_I}$
  - $V_2^{I,\epsilon} = \frac{\bar{\sigma}_I^2-(\bar{\sigma}_I^*)^2}{2}$
  - $V_3^{I,\epsilon} = a_I(\bar{\sigma}_I^*)^3$

- We use the time series of the log-returns of the stock (S) and the index (I) in order to deduce the time serie of the idiosyncratic volatility $(f(Z_t))_t$. We calibrate the parameters m_Z, ν_Z and ρ_Z on the time serie of $(Z_t)_t$. For this purpose, we use the maximum likelihood estimation method suggested in [8]. We evaluate then the quantity $< f_2^2 >_{1,2}$. If we consider the case where f_2 denotes the exponential function, then we have:

$$< f_2^2 >_{1,2} \quad = \quad \int_{-\infty}^{+\infty} e^{2z} \frac{1}{\sqrt{2\pi}} \exp(-\frac{(z-m_Z)^2}{2\nu_Z^2})dz = e^{2m_Z+2\nu_Z^2}.$$

- We compute finally the value of $\tilde{\beta}$ using (3.26).

4 Empirical study

4.1 Numerical illustrations

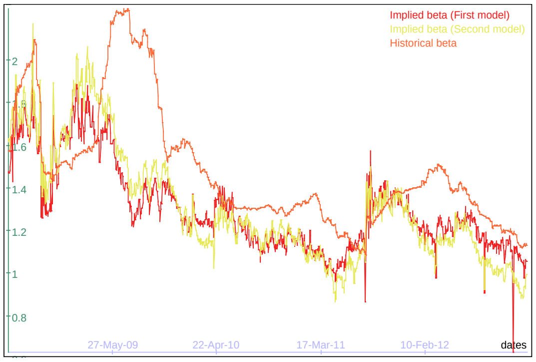
In this section, we will estimate the parameter β according to the methods (2.1) and (3.26) using real data. For this purpose, we use options prices whose maturities are equal to $T = 0.5$ and whose moneyness range from 80% to 120%.

Furthermore, we introduce a third estimator, the historical beta β_H, which is used traditionally as an estimator of the parameter β. β_H is a backward measure of the realization of the parameter β. It is obtained, at a given date t, as the slope of the linear regression of the stock daily returns on the index daily returns between $t - L$ and t. We intentionally fixed L, the length of the backward window on which we perform the linear regression, to be equal to the maturity of the options we use to compute $\tilde{\beta}$ and $\hat{\beta}$. We have then $L = T = 0.5$

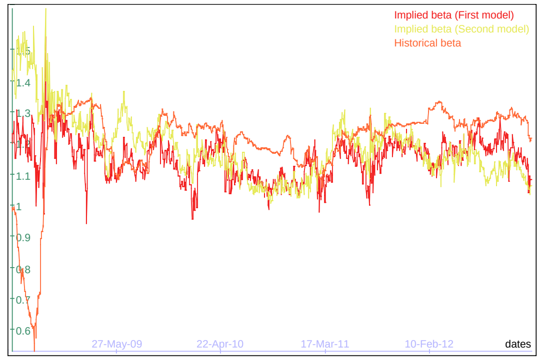
We use here spot and options data on a sample of SPX 500 stocks and ETF and the SPX 500 market index. The sample includes Financial Select Sector (XLF), Energy Select Sector (XLE), Materials Select Sector (XLB), Technology Select Sector (XLK), Industrial Select Sector (XLI), Goldman Sachs (GS), Microsoft (MSFT), General Electric (GE), Google (GOOG), IBM (IBM), JP Morgan (JPM), Cisco Systems (CSCO), Bank Of America (BAC), Intel Corporation (INTC), Chevron Corporation (CVX), Caterpillar Inc(CAT), Apple Inc (AAPL), Alcoa Inc (AA). The data used for this empirical study range between 01/01/2008 and 31/12/2012.

For every date t of the sample, we compute the estimators $\hat{\beta}(t)$ and $\tilde{\beta}(t)$, according to (2.1) and (3.26) respectively, by using options prices with maturity $T = 6M$ and moneyness M between 80% and 120%. We use also the linear regression method on a backward window with size L equal to T in order to compute $\beta_H(t)$.

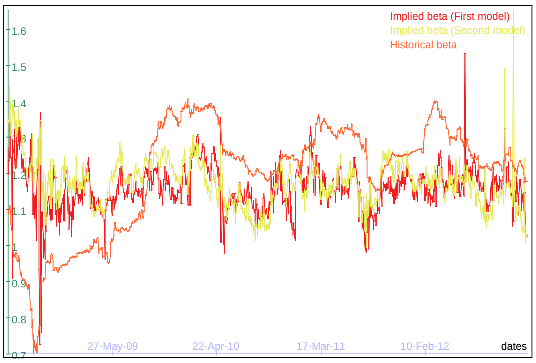
The graphs below represent the time series $(\hat{\beta}(t))_{t_0 \leq t \leq t_N}$, $(\tilde{\beta}(t))_{t_0 \leq t \leq t_N}$ and $(\beta_H(t))_{t_0 \leq t \leq t_N}$, for $t_0 = 01/01/2008$ and $t_N = 31/12/2012$.



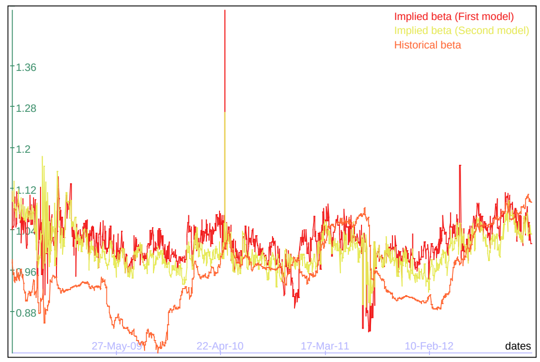
(a) XLF vs SPX



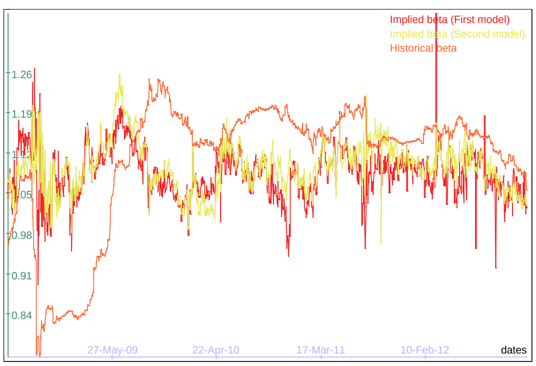
(b) XLE vs SPX



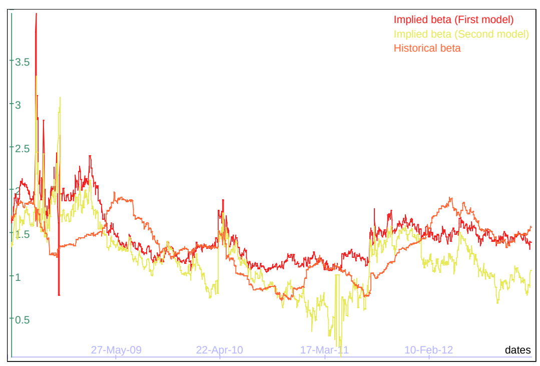
(c) XLB vs SPX



(d) XLK vs SPX

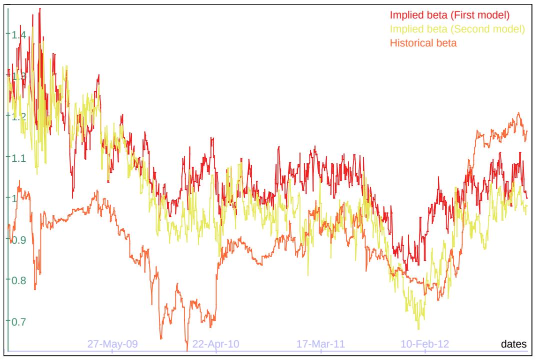


(e) XLI vs SPX

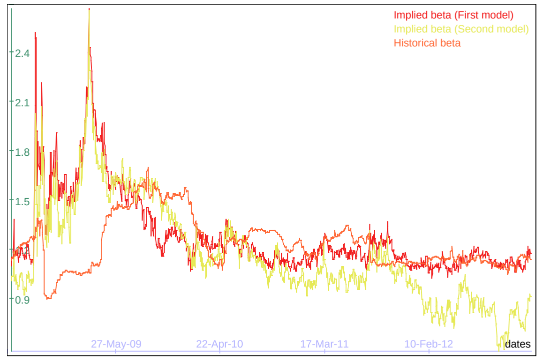


(f) GS vs SPX

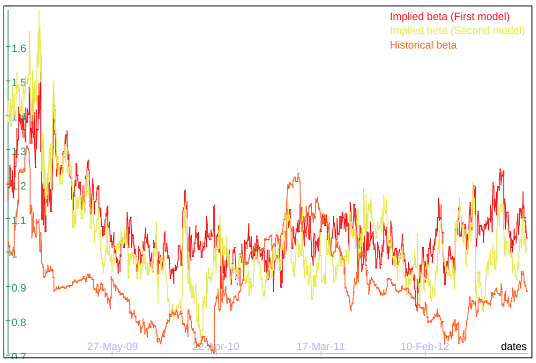
Figure 2: Comparison between $\hat{\beta}$, $\tilde{\beta}$ and β_H



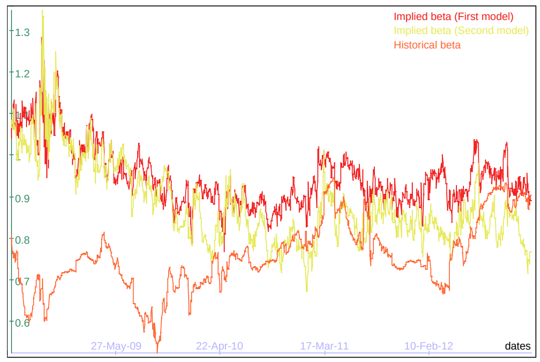
(a) MSFT vs SPX



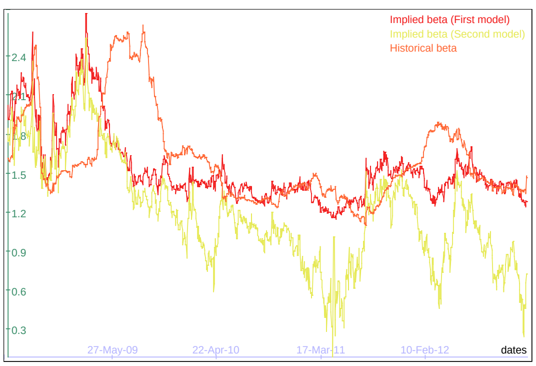
(b) GE vs SPX



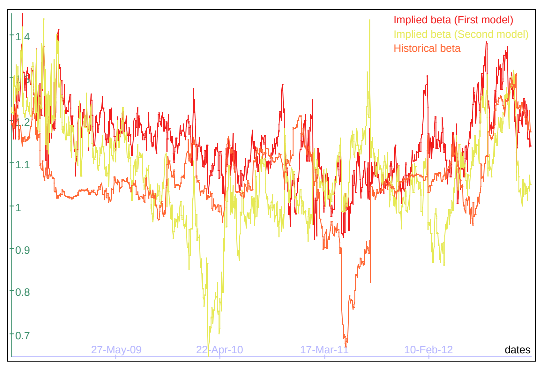
(c) GOOG vs SPX



(d) IBM vs SPX

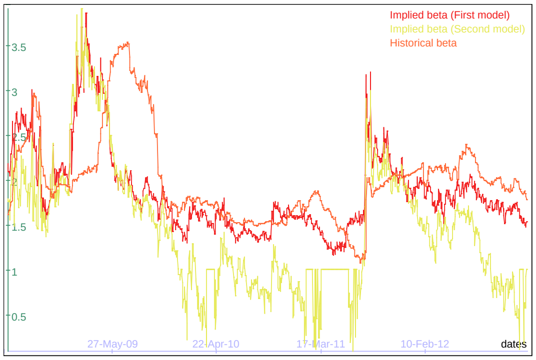


(e) JPM vs SPX

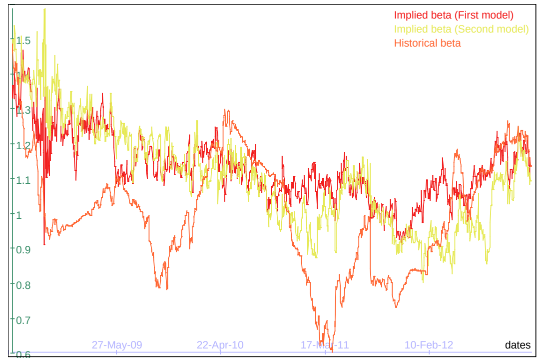


(f) CSCO vs SPX

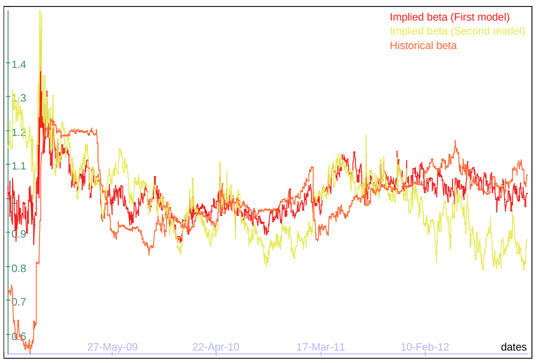
Figure 3: Comparison between $\hat{\beta}$, $\tilde{\beta}$ and β_H



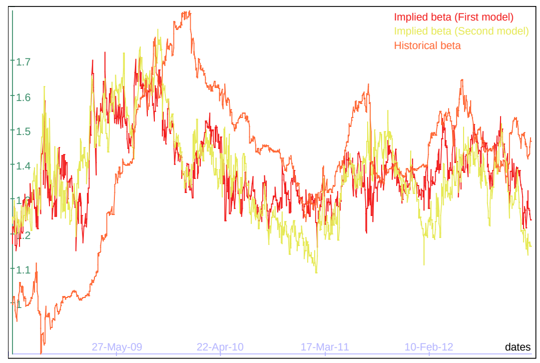
(a) BAC vs SPX



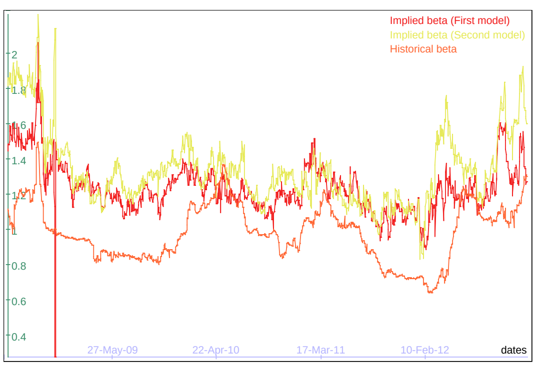
(b) INTC vs SPX



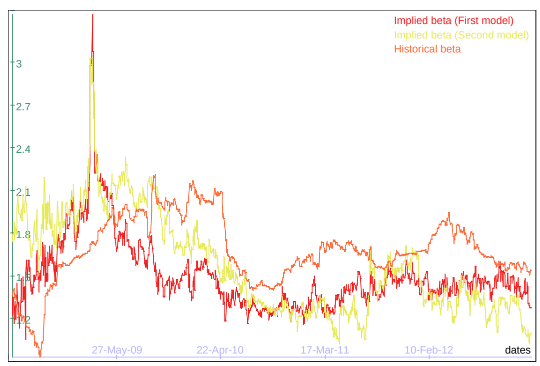
(c) CVX vs SPX



(d) CAT vs SPX



(e) AAPL vs SPX



(f) AA vs SPX

Figure 4: Comparison between $\hat{\beta}$, $\tilde{\beta}$ and β_H

We define the parameter β_F as the forward realized beta. At a given date t, $\beta_F(t)$ is defined as the slope of the linear regression of the stock log-returns on the index log-returns between t and $t+T$. This means that $\beta_F(t)$ is a measure of the effective realization of the parameter β on the forward window with length T.

We aim then to measure the prediction error relative to each of the estimators $\hat{\beta}$, $\tilde{\beta}$ and β_H. We use here the bias and the root-mean-square error as measures of the estimation error.

Stock	$E(\hat{\beta} - \beta_F)$	$E(\tilde{\beta} - \beta_F)$	$E(\beta_H - \beta_F)$
XLF.US	-0.148	-0.121	0.054
XLE.US	-0.05	0.0094	-0.0313
XLB.US	-0.0231	0.0002	-0.0345
XLK.US	0.0679	0.0542	-0.0158
XLI.US	-0.0258	-0.0183	-0.0215
AAPL.US	0.2649	0.4073	0.0005
AA.US	-0.1814	-0.0738	-0.0333
BAC.US	-0.1414	-0.3557	0.0222
CAT.US	-0.0453	-0.0574	-0.0512
CSCO.US	0.0854	0.0216	-0.0073
CVX.US	0.0252	0.0366	-0.0187
GE.US	0.0375	-0.0534	0.0016
GOOG.US	0.1693	0.1742	0.0193
GS.US	0.1306	-0.0962	0.0113
IBM.US	0.2063	0.1575	-0.0233
INTC.US	0.1462	0.1451	-0.0069
JPM.US	-0.0415	-0.347	0.0383
MSFT.US	0.1717	0.1102	-0.0234

Table 1: Bias of the estimators $\hat{\beta}$, $\tilde{\beta}$ and β_H

Stock	$\sqrt{E((\frac{\hat{\beta}-\beta_F}{\beta_F})^2)}$	$\sqrt{E((\frac{\tilde{\beta}-\beta_F}{\beta_F})^2)}$	$\sqrt{E((\frac{\beta_H-\beta_F}{\beta_F})^2)}$
XLF.US	0.143	0.138	0.195
XLE.US	0.183	0.207	0.128
XLB.US	0.196	0.209	0.116
XLK.US	0.132	0.124	0.095
XLI.US	0.128	0.112	0.108
AAPL.US	0.41	0.547	0.252
AA.US	0.187	0.188	0.181
BAC.US	0.234	0.407	0.337
CAT.US	0.118	0.123	0.149
CSCO.US	0.159	0.162	0.147
CVX.US	0.198	0.238	0.165
GE.US	0.154	0.189	0.158
GOOG.US	0.277	0.31	0.165
GS.US	0.341	0.292	0.303
IBM.US	0.369	0.348	0.134
INTC.US	0.279	0.294	0.244
JPM.US	0.158	0.292	0.265
MSFT.US	0.297	0.27	0.163

Table 2: RMSE of the estimators $\hat{\beta}$, $\tilde{\beta}$ and β_H

4.2 Conclusion

We can see through the tables above that the historical beta estimator β_H is not biased for all the tested stocks and has in average the lowest RMSE compared to the two other estimators $\hat{\beta}$ and $\tilde{\beta}$. On the other hand, the forward looking beta estimator $\tilde{\beta}$ has a positive bias for the stocks AAPL, GOOG, IBM, INTC, MSFT and a negative bias for XLF and BAC. For those same stocks, the RMSE of the estimator $\tilde{\beta}$ is consequently higher than the one of β_H. This means that for some stocks, the market is expecting a realization of the parameter β which is consistenty shifted to the effective realized one. We can say that for those stocks, the parameter β is misspecified under the risk-neutral distribution.

5 Appendix

5.1 Appendix 1: Pricing options on the index

Let $P_t^{I,\epsilon} = E(h(I_T)|I_t = x, Y_t = y)$ be the price of a european call on the index with strike K and maturity T.

$$P_t^{I,\epsilon} = P^{I,\epsilon}(t, I_t, Y_t).$$

By applying Itô's lemma, we obtain:

$$\mathcal{L}^I P_t^{I,\epsilon} = 0.$$

We can then classify the elements of the generator $\mathcal{L}^I$ by powers of $\sqrt{\epsilon}$.

$$\begin{aligned}
\mathcal{L}^I &= \mathcal{L}_2^I + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^I + \frac{1}{\epsilon}\mathcal{L}_0^I, \\
\mathcal{L}_0^I &= \frac{\partial}{\partial y}(m_Y - y) + \nu_Y^2 \frac{\partial^2}{\partial y^2}, \\
\mathcal{L}_1^I &= -\nu_Y \sqrt{2}\chi_1(y)\frac{\partial}{\partial y} + \sqrt{2}\rho_Y \nu_Y I_t f_1(y) \frac{\partial^2}{\partial I \partial y}, \\
\mathcal{L}_2^I &= \frac{\partial}{\partial t} + r(\frac{\partial}{\partial I_t} I_t - .) + \frac{1}{2}\frac{\partial^2}{\partial I_t^2} I_t^2 f_1(y)^2.
\end{aligned}$$

We can notice here that $\mathcal{L}_0^I$ is the infinitesimal generator of the Ornstein-Uhlenbeck process $(Y_{1,t})_t$ that diffuses according to the following SDE:

$$dY_{1,t} = (m_Y - Y_{1,t})dt + \nu_Y \sqrt{2} dW_t.$$

We expand then $P^{I,\epsilon}$ in powers of $\sqrt{\epsilon}$:

$$P^{I,\epsilon} = \sum_{i=0}^{\infty} (\sqrt{\epsilon})^i P_i^{I,\epsilon}.$$

We can now do the expansion of the term $\mathcal{L}^I P_t^{I,\epsilon} = 0$ and classify it by powers of $\sqrt{\epsilon}$. We give below the terms of orders $0, -1, -2, 1$ respectively in $\sqrt{\epsilon}$.

$$\begin{aligned}
(0) &: \mathcal{L}_2^I P_0^{I,\epsilon} + \mathcal{L}_1^I P_1^{I,\epsilon} + \mathcal{L}_0^I P_2^{I,\epsilon} = 0, \\
(-1) &: \mathcal{L}_1^I P_0^{I,\epsilon} + \mathcal{L}_0^I P_1^{I,\epsilon} = 0, \\
(-2) &: \mathcal{L}_0^I P_0^{I,\epsilon} = 0, \\
(1) &: \mathcal{L}_2^I P_1^{I,\epsilon} + \mathcal{L}_1^I P_2^{I,\epsilon} + \mathcal{L}_0^I P_3^{I,\epsilon} = 0,
\end{aligned}$$

We have $P_0^{I,\epsilon}$ is a solution of $\mathcal{L}_0^I P_0^{I,\epsilon} = 0$ with final condition $P_0^{I,\epsilon}(T, I_T, y_T) = h(I_T)$ (independent of y_T). Then, by solving this equation, we find:

$$P_0(t, I_t, y_t) = C_1(t, I_t) \int_0^y e^{\frac{u^2}{2\mu_Y^2} - \frac{u m_Y}{\mu^2}} du + C_2(t, I_t)$$

If C_1 is not the null function then the solution diverges when $y \to +\infty$. We know that, in the case of a call option, we have $0 \leq P(t, I_t) \leq I_t$. Then we will choose C_1 to be null, and then P_0 is independent of y.
We have then $\mathcal{L}_1^I P_0^{I,\epsilon} + \mathcal{L}_0^I P_1^{I,\epsilon} = 0$ which reduces to $\mathcal{L}_0^I P_1^{I,\epsilon} = 0$ (since P_0 doesn't depend on y). Using the same reasoning as before, we can prove that $P_1^{I,\epsilon} = P_1^{I,\epsilon}(t, I_t)$ is independent of y. Consequently, we have:

$$\begin{aligned}
\mathcal{L}_0^I P_0^{I,\epsilon} &= \mathcal{L}_1^I P_0^{I,\epsilon} = 0, \\
\mathcal{L}_0^I P_1^{I,\epsilon} &= \mathcal{L}_1^I P_1^{I,\epsilon} = 0.
\end{aligned}$$

Consequently, the terms of order (-1) and (-2) in $\sqrt{\epsilon}$ are null. We focus now on the term of order 0 in $\sqrt{\epsilon}$. We have : $\mathcal{L}_1^I P_1^{I,\epsilon} = 0$, then this term becomes:

$$\mathcal{L}_0^I P_2^{I,\epsilon} + \mathcal{L}_2^I P_0^{I,\epsilon} = 0.$$

This equation is a Poisson equation for $P_2^{I,\epsilon}$. The solvability condition for this equation is :

$$< \mathcal{L}_0^I P_2^{I,\epsilon} >_1 + < \mathcal{L}_2^I P_0^{I,\epsilon} >_1 = 0.$$

$< . >_1$ is then the average with respect to the invariant distribution $N(m_Y, \nu_Y^2)$ of the Ornstein-Uhlenbeck process $(Y_{1,t})_t$.
Since $\mathcal{L}_0^I$ is the infinitesimal generator of the process (Y_1), we can check easily that:

$$< \mathcal{L}_0^I P_2^{I,\epsilon} >_1 = 0.$$

Then, the solvability condition reduces to:

$$< \mathcal{L}_2^I >_1 P_0^{I,\epsilon} = 0. \tag{5.1}$$

We can then compute the average of the operator $\mathcal{L}_2^I$:

$$< \mathcal{L}_2^I >_1 = \frac{\partial}{\partial t} + r(\frac{\partial}{\partial I} I - .) + \frac{1}{2} \frac{\partial^2}{\partial I^2} I^2 < f_1^2 >_1 .$$

If we note $\bar{\sigma}_I^2 =< f_1^2 >_1$ then $< \mathcal{L}_2^I >_1 = \mathcal{L}_{BS}(\bar{\sigma}_I)$. We can then define $P_0^{I,\epsilon}$ as the solution to the following problem :

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}_I) P_0^{I,\epsilon} &= 0, \\
P_0^{\epsilon}(T, I_T) &= h(I_T).
\end{aligned}$$

We deduce then that $P_0^{I,\epsilon} = P_{BS}(t, I_t, \bar{\sigma}_I)$ meaning that $P_0^{I,\epsilon}$ is the Black-Scholes price of the index option with implied volatility equal to $\bar{\sigma}_I$.
We can then write $P_2^{I,\epsilon}$ as:

$$P_2^{I,\epsilon} = -(\mathcal{L}_0^I)^{-1} (\mathcal{L}_2^I - < \mathcal{L}_2^I >_1) P_0^{I,\epsilon}.$$

The term of order 1 in $\sqrt{\epsilon}$ is a poisson equation for $P_3^{I,\epsilon}$. The solvability condition for this equation is :

$$\begin{aligned}
< \mathcal{L}_2^I P_1^{I,\epsilon} >_1 &= - < \mathcal{L}_1^I P_2^{I,\epsilon} >_1, & (5.2) \\
< \mathcal{L}_2^I >_1 P_1^{I,\epsilon} &= < \mathcal{L}_1^I (\mathcal{L}_0^I)^{-1} (\mathcal{L}_2^I - < \mathcal{L}_2^I >_1) > P_0^{I,\epsilon}. & (5.3)
\end{aligned}$$

In order to solve the previous equation, we compute the difference between the operator $\mathcal{L}_2^I$ and its average:

$$\mathcal{L}_2^I - <\mathcal{L}_2^I>_1 = \frac{1}{2}(f_1^2(y) - <f_1^2>_1) I_t^2 \frac{\partial^2}{\partial I_t^2}.$$

Let ϕ_I the solution of:

$$\mathcal{L}_0 \phi_I(y) = f_1^2(y) - <f_1^2>_1 \ . \tag{5.4}$$

Using (5.4), we can write that:

$$(\mathcal{L}_0^I)^{-1}(\mathcal{L}_2 - <\mathcal{L}_2>_1) = \frac{1}{2}\phi_I(y) I_t^2 \frac{\partial^2}{\partial I_t^2}.$$

By applying the operator $\mathcal{L}_1$ to the last equation, we get:

$$\mathcal{L}_1^I(\mathcal{L}_0^I)^{-1}(\mathcal{L}_2 - <\mathcal{L}_2>_1) = (-\nu_Y\sqrt{2} <\chi_1\phi_I'>_1 + \sqrt{2}\rho_Y\nu_Y <f_1\phi_I'>_1 I_t\frac{\partial}{\partial I_t})\frac{1}{2}I_t^2\frac{\partial^2}{\partial I_t^2}.$$

In order to simplify the notation, we define $V_2^{I,\epsilon}$ and $V_3^{I,\epsilon}$ as below:

$$\begin{aligned}
V_2^{I,\epsilon} &= -\frac{\sqrt{\epsilon}}{\sqrt{2}}\nu_Y <\phi_I'\chi_1>_1, \\
V_3^{I,\epsilon} &= \frac{\sqrt{\epsilon}}{\sqrt{2}}\rho_Y\nu_Y <\phi_I' f>_1 \ .
\end{aligned}$$

Using $V_2^{I,\epsilon}$ and $V_3^{I,\epsilon}$, the equation (5.3) becomes:

$$<\mathcal{L}_2^I>_1 \sqrt{\epsilon}P_1^{I,\epsilon} = V_2^{I,\epsilon} I_t^2\frac{\partial^2 P_0}{\partial I_t^2} + V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t}(I_t^2\frac{\partial^2 P_0}{\partial I_t^2}).$$

We can then deduce that $P_1^{I,\epsilon}$ is the solution for the following problem :

$$\begin{aligned}
<\mathcal{L}_2^I>_1 \sqrt{\epsilon}P_1^{I,\epsilon} &= V_2^{I,\epsilon} I_t^2\frac{\partial^2 P_0}{\partial I_t^2} + V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t}(I_t^2\frac{\partial^2 P_0}{\partial I_t^2}), \qquad (5.5) \\
P_1^{I,\epsilon}(T, I_T) &= 0. \qquad (5.6)
\end{aligned}$$

We define the following operators:

$$\begin{aligned}
\mathcal{D}_{1,I} &= I_t\frac{\partial}{\partial I_t}, \\
\mathcal{D}_{2,I} &= I_t^2\frac{\partial^2}{\partial I_t^2}.
\end{aligned}$$

Given that $<\mathcal{L}_2^I>_1 = \mathcal{L}_{BS}(\bar{\sigma}_I)$ commits with $\mathcal{D}_{1,I}$ and $\mathcal{D}_{2,I}$, and that $<\mathcal{L}_2^I>_1 P_0 = 0$, the solution to the last problem can be given explicitly by:

$$\sqrt{\epsilon}P_1^{I,\epsilon} = -(T-t)(V_2^{I,\epsilon} I_t^2\frac{\partial^2 P_0}{\partial I_t^2} + V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t}(I_t^2\frac{\partial^2 P_0}{\partial I_t^2})).$$

If we neglect terms of order higher or equal to 2 in $\sqrt{\epsilon}$, we can approximate the option's price by $(P_0^{I,\epsilon} + \sqrt{\epsilon}P_1^{I,\epsilon})$. As it was proven by Fouque et Al in [6], we can carry out a parameter reduction method and approximate $P^{I,\epsilon}$ by the following formula:

$$P^{I,\epsilon} \sim \tilde{P}_0^{I,\epsilon} + \sqrt{\epsilon}\tilde{P}_1^{I,\epsilon},$$

such as:

$$\tilde{P}_0^{I,\epsilon} = P_{BS}(\bar{\sigma}_I^*), \tag{5.7}$$

$$(\bar{\sigma}_I^*)^2 = \bar{\sigma}_I^2 - 2V_2^{I,\epsilon}, \tag{5.8}$$

$$\sqrt{\epsilon}\tilde{P}_1^{I,\epsilon} = -(T-t)V_3^{I,\epsilon} I_t \frac{\partial}{\partial I_t}(I_t^2 \frac{\partial^2 \tilde{P}_0}{\partial I_t^2}). \tag{5.9}$$

5.2 Appendix 2: Accuracy of the approximation

We want to prove here that the approximation $(P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon})$ of the price $P(t, S_t, Y_t)$ is of order 1 in ϵ meaning that:

$$|P(t, S_t, Y_t) - (P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon})| \leq C\epsilon.$$

The proof of this property is given in [10] in the case where the payoff h is smooth. We will give here a summary of this proof in order to make this paper self contained.

Let us introduce the quantity $Z^{S,\epsilon}$ which verifies that:

$$P(t, S_t, Y_t) = P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon} + \epsilon P_2^{S,\epsilon} + \epsilon^{\frac{3}{2}} P_3^{S,\epsilon} - Z^{S,\epsilon}.$$

Given that $\mathcal{L}^S P(t, S_t, Y_t) = 0$, we have:

$$\mathcal{L}^S Z^{S,\epsilon} = \mathcal{L}_S(P_0^{S,\epsilon} + \sqrt{\epsilon}P_1^{S,\epsilon} + \epsilon P_2^{S,\epsilon} + \epsilon^{\frac{3}{2}} P_3^{S,\epsilon})$$

We recall that $\mathcal{L}^S = \mathcal{L}_2^S + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^S + \frac{1}{\epsilon}\mathcal{L}_0^S$. We can develop then $\mathcal{L}^S Z^{S,\epsilon}$ and regroup the terms by orders of $\sqrt{\epsilon}$:

$$\begin{aligned}\mathcal{L}^S Z^{S,\epsilon} = \frac{1}{\epsilon}\mathcal{L}_0^S P_0^{S,\epsilon} + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0^S P_1^{S,\epsilon} + \mathcal{L}_1^S P_0^{S,\epsilon}) + (\mathcal{L}_0^S P_2^{S,\epsilon} + \mathcal{L}_1^S P_1^{S,\epsilon} + \mathcal{L}_2^S P_0^{S,\epsilon}) \\ + \sqrt{\epsilon}(\mathcal{L}_0^S P_3^{S,\epsilon} + \mathcal{L}_1^S P_2^{S,\epsilon} + \mathcal{L}_2^S P_1^{S,\epsilon}) + \epsilon(\mathcal{L}_1^S P_3^{S,\epsilon} + \mathcal{L}_2^S P_2^{S,\epsilon}) + \epsilon^{\frac{3}{2}}\mathcal{L}_2^S P_3^{S,\epsilon}.\end{aligned}$$

Since $P_0^{S,\epsilon}, P_1^{S,\epsilon}$ and $P_2^{S,\epsilon}$ are chosen to nullify the first four terms in the previous equation. Then we obtain:

$$\mathcal{L}^S Z^{S,\epsilon} = \epsilon(\mathcal{L}_1^S P_3^{S,\epsilon} + \mathcal{L}_2^S P_2^{S,\epsilon}) + \epsilon^{\frac{3}{2}}\mathcal{L}_2^S P_3^{S,\epsilon},$$

with the final condition:

$$Z^{S,\epsilon}(T, S_T, Y_T, Z_T) = \epsilon P_2(T, S_T, Y_T, Z_T) + \epsilon^{\frac{3}{2}} P_3(T, S_T, Y_T, Z_T).$$

Using the Feynman-Kac theorem, we obtain:

$$Z^{S,\epsilon}(t,x,y,z) = \epsilon E(e^{-r(T-t)}(P_2(T,S_T,Y_T,Z_T) + \epsilon^{\frac{1}{2}} P_3(T,S_T,Y_T,Z_T))$$
$$- \int_t^T e^{-r(u-t)}((\mathcal{L}_1^S P_3^{S,\epsilon} + \mathcal{L}_2^S P_2^{S,\epsilon}) + \epsilon^{\frac{1}{2}} \mathcal{L}_2^S P_3^{S,\epsilon})(u,S_u,Y_u,Z_u)du | S_t = x, Y_t = y, Z_t = z)$$

Under the following assumptions:

- The payoff function h is smooth.
- The functions f, χ_1 and χ_2 are bounded, so that the solutions of the poisson equation are at most linearly growing in $|y|$ and $|z|$

We can deduce that $Z^{S,\epsilon}$ is at most linearly growing in $|y|$ and $|z|$ and bounded uniformly in x, and we have:

$$Z^{S,\epsilon}(t,x,y,z) \quad = \quad O(\epsilon)\,.$$

Consequently: $P(t,S_t,Y_t,Z_t) = P_0^{S,\epsilon} + \sqrt{\epsilon} P_1^{S,\epsilon} + O(\epsilon)$. In order to drop the condition on the boundedness of the payoff function h, we should have ϵ independent estimates for moments of S^ϵ. This would be the case when the processes Y and Z have enougn time to fluctuate in order to attain respectively their invariant distributions. Consequently, when the payoff function is not bounded, the approximation $P_0^{S,\epsilon} + \sqrt{\epsilon} P_1^{S,\epsilon}$ is of order 1 in ϵ when the maturity of the option (of the payoff function h) is sufficiently larger than ϵ.

The demonstration of the accuracy of the approximation for a non smooth payoff h (as in the case of a call option) is derived in [14].

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