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WEIGHTED RESTRICTION TYPE ESTIMATES FOR GRUSHIN OPERATORS AND APPLICATION TO SPECTRAL MULTIPLIERS AND BOCHNER-RIESZ SUMMABILITY

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Abstract. We prove weighted restriction type estimates for Grushin operators. These estimates are then used to prove sharp spectral multiplier theorems as well as Bochner-Riesz summability results with sharp exponent.

1. Introduction

We consider Grushin operators on $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}=\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''}$ defined by

$$L:=-\sum_{j=1}^{d_1}\partial^2_{x'_j}-\big(\sum_{j=1}^{d_1}|x'_j|^2\big)\sum_{k=1}^{d_2}\partial^2_{x''_k}. \tag{1.1}$$

Such operators, defined by the quadratic form technique, are self-adjoint in $L^2(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$. Let $E_L(\lambda)$ be the spectral resolution of the operator L for $\lambda\geq 0$. By the spectral theorem for every bounded Borel function $F:\mathbb{R}\to\mathbb{C}$, one can define

$$F(L)=\int_0^\infty F(\lambda)\,dE_L(\lambda). \tag{1.2}$$

The operator $F(L)$ is bounded on $L^2(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$. This paper is devoted to spectral multiplier results for L, that is, we investigate minimal sufficient condition on F under which the operator $F(L)$ extends to a bounded operator on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ for some range of p. In this context, the minimal condition on F we have in mind is the same as in the Fourier multiplier theorem, i.e., boundedness of $F(-\Delta)$ on $L^p(\mathbb{R}^d)$ where Δ is the Euclidean Laplacian. We also study the closely related question of critical exponent δ for which the Bochner-Riesz means $(1-tL)^\delta_+$ are bounded on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ uniformly in $t\in[0,\infty)$.

Spectral multipliers and Bochner-Riesz summability for Grushin operators have been studied recently by other authors. In [9], it is proved that for $\delta>\frac12(d_1+d_2)-\frac12$, the Bochner-Riesz means $(1-tL)^\delta_+$ are bounded on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ uniformly in $t\in[0,\infty)$ for all $1\leq p\leq\infty$. A previous result was proved in [10] with the condition $\delta>\frac12\max(d_1+d_2,2d_2)-\frac12$. Our aim is to get similar results for smaller values of δ, i.e. when $0<\delta<\frac12(d_1+d_2)-\frac12$. In this case, we cannot hope for $(1-tL)^\delta_+$ to be bounded on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ for all $p\in[1,\infty]$. Our aim is to prove that $(1-tL)^\delta_+$ are bounded on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ uniformly in t for p in some symmetric interval $[p_\delta,p'_\delta]$ around 2. The value p_δ depends of course on δ. Such questions have been studied for the Euclidean Laplacian in which case the optimality of δ is known but the optimality of p is

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a celebrate open problem, known as the Bochner-Riesz problem. See [15], p. 420 and [16] for more details and recent progress on this problem.

Starting from the result quoted above from [9] and [10], one can use complex interpolation between L^2 boundedness for any $\delta > 0$ and L^1 boundedness for a fixed $\delta > (d_1+d_2)/2 - 1/2$ to obtain that for $\delta > (d_1+d_2-1)|1/p-1/2|$, $(1-tL)^\delta_+$ are bounded on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ uniformly in t. Note however that this strategy does not give the optimal exponent. For example, when $L = -\Delta$ on $\mathbb{R}^n$, $(1+t\Delta)^\delta_+$ are bounded on $L^p(\mathbb{R}^n)$ uniformly when $\delta > \max\{n|1/p-1/2|-1/2, 0\}$ for $1 \le p \le (2n+2)/(n+3)$, which is better than the interpolation approach which leads to $\delta > (n-1)|1/p-1/2|$. The sharpened result for the Laplacian, i.e., $\delta > \max\{n|1/p-1/2|-1/2, 0\}$ for $1 \le p \le (2n+2)/(n+3)$, is obtained by the restriction theorem for the Fourier transform on the unit sphere. In an abstract setting, versions of the restriction estimate are introduced in [1] and we are tempted to follow [1] in order to prove boundedness of Bochner-Riesz means for L. There is however an obstacle. The restriction type estimate introduced in [1] leads to spectral multipliers using "the" homogeneous dimension $Q = d_1 + 2d_2$ rather than the topological one $d_1 + d_2$. The exponent we will get for the Bochner-Riesz means is then $\max\{Q|1/p-1/2|-1/2, 0\}$. The problem of getting sharp spectral multipliers using the topological dimension rather than the homogeneous one appeared already in the case of the Heisenberg group. See [5] and [11].

Our strategy to deal with this problem is to use a weighted version of restriction estimates for the operator L. More precisely, let F be a bounded Borel function with support $\operatorname{supp} F$ contained in $[R/4, R]$ for some $R > 0$. Then for $1 \le p \le \min\{2d_1/(d_1+2), (2d_2+2)/(d_2+3)\}$ and $0 \le \gamma < d_2(1/p-1/2)$, we prove that

$$\||x'|^\gamma F(\sqrt{L})f\|_{L^2(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})} \le CR^{(2d_2+d_1)(1/p-1/2)-\gamma}\|\delta_R F\|_{L^2(\mathbb{R})}\|f\|_{L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})}.$$

Using this weighted restriction type estimate, we prove sharp spectral multiplier results and optimal Bochner-Riesz summability stated in Theorems 1.1 and 1.2 below. We set

$$D := \max\{d_1+d_2, 2d_2\}$$

and denote as usual W_2^s the L^2 Sobolev space of order s with $\|F\|_{W_2^s} := \|(I-d_x^2)^{s/2}F\|_2$. Throughout, η is an auxiliary and non trivial C^∞ function with compact support contained in $(0,\infty)$.

Theorem 1.1. *Let $1 \le p \le \min\{2d_1/(d_1+2), (2d_2+2)/(d_2+3)\}$. Suppose that the bounded Borel function $F : \mathbb{R} \to \mathbb{C}$ satisfies*

$$\sup_{t>0}\|\eta\, F(t\cdot)\|_{W_2^s} < \infty$$

for some $s > \max\{D|1/p-1/2|, 1/2\}$. Then the spectral multiplier operator $F(L)$ is bounded on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$. In addition

$$\|F(L)\|_{L^p\to L^p} \le C_p \sup_{t>0}\|\eta\, F(t\cdot)\|_{W_2^s}.$$

For Bochner-Riesz means we prove the following result.

Theorem 1.2. *Let $1 \le p \le \min\{2d_1/(d_1+2), (2d_2+2)/(d_2+3)\}$. Suppose that $\delta > \max\{D|1/p-1/2|-1/2, 0\}$. Then the Bochner-Riesz means $(1-tL)^\delta_+$ are bounded on $L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ uniformly in $t \in [0,\infty)$.*

Theorems 1.1 and 1.2 are optimal when $d_1 \ge d_2$. In this case D coincides with the topological dimension d_1+d_2 of $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$. By the elliptic property of L in the region where $x' \ne 0$, one can use the transplantation argument described in [6] to deduce the sharpness of the above theorems

from the fact that the exponent $D|1/p - 1/2| - 1/2$ is sharp for the classical Bochner-Riesz summability on $\mathbb{R}^D$. See also [10] and [9].

Conjecture. We believe that the previous theorems are true with $D = d_1 + d_2$ instead of $D = \max(d_1 + d_2, 2d_2)$. As we mentioned above, if $p = 1$, the spectral multiplier theorem in [9] is valid for $s > \frac{1}{2}(d_1 + d_2)$. This means that the conjecture is true when $p = 1$.

Throughout, the symbols "c" and "C" will denote (possibly different) positive constants that are independent of the essential variables. The notation $A \sim B$ means that the quantities A and B satisfy $cA \leq B \leq CA$ for some positive constants c and C.

2. Riemannian distance and the heat kernel estimates

Heat kernel bounds for Grushin type operators have been proved in [12]. Here we state some basic results concerning the Riemannian distance associated with the Grushin operator L and recall the Gaussian bound for the corresponding heat kernel.

Recall that the Riemannian (quasi-)distance corresponding to the operator L can be defined by

$$\rho(x, y) = \sup_{\psi \in \mathbb{D}} (\psi(x) - \psi(y))$$

for all $x = (x', x''), y = (y', y'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ where

$$\mathbb{D} = \left\{ \psi \in W^{1,\infty}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) : \Big(\sum_{j=1}^{d_1} |\partial_{x'_j} \psi|^2 + \Big(\sum_{j=1}^{d_1} |x'_j|^2 \Big) \sum_{k=1}^{d_2} |\partial_{x''_k} \psi|^2 \Big) \leq 1 \right\} .$$

For this distance ρ and the Lebesgue measure the finite speed propagation property for the corresponding wave equation as well as Gaussian estimates for the heat kernel of L are satisfied. See [12, Proposition 4.1] for more detailed discussion and references.

Theorem 2.1. *Let ρ be Riemannian distance associated with the Grushin operator L. Then for $x = (x', x'')$, $y = (y', y'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$,*

$$\rho(x, y) \sim |x' - y'| + \begin{cases} \frac{|x'' - y''|}{|x'| + |y'|} & \text{if } |x'' - y''|^{1/2} \leq |x'| + |y'|, \\ |x'' - y''|^{1/2} & \text{if } |x'' - y''|^{1/2} \geq |x'| + |y'|. \end{cases} \tag{2.1}$$

Moreover the volume of the ball $B(x, r) := \{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : \rho(x, y) < r\}$ satisfies the following estimates

$$|B(x, r)| \sim r^{d_1 + d_2} \max\{r, |x'|\}^{d_2}, \tag{2.2}$$

and in particular, for all $\lambda \geq 0$,

$$|B(x, \lambda r)| \leq C(1 + \lambda)^Q |B(x, r)| \tag{2.3}$$

where $Q = d_1 + 2d_2$ is "the" homogenous dimension of the considered metric space. Next, there exist constants $b, C > 0$ such that, for all $t > 0$, the integral kernel p_t of the operator $\exp(-tL)$ satisfies the following Gaussian bound

$$|p_t(x, y)| \leq C|B(y, t^{1/2})|^{-1} e^{-b\rho(x,y)^2/t} \tag{2.4}$$

for all $x, y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

Proof. For the proof, we refer the reader to [12, Proposition 5.1 and Corollary 6.6]. □

3. Weighted restriction estimates

In this section, we discuss the spectral decomposition of L and then state and prove the weighted restriction estimate.

Let $\mathcal{F} : L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \to L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ be the partial Fourier transform in the variable x'', that is

$$\mathcal{F}\phi(x', \xi) = \widehat{\phi}(x', \xi) = (2\pi)^{-d_2/2} \int_{\mathbb{R}^{d_2}} \phi(x', x'')\, e^{-i\xi\cdot x''}\, dx''.$$

Then

$$\mathcal{F}L\phi(x', \xi) = L_\xi\, \mathcal{F}\phi(x', \xi), \tag{3.1}$$

where L_ξ is the Schrödinger operator defined by

$$L_\xi = -\Delta_{d_1} + |x'|^2|\xi|^2$$

acting on $L^2(\mathbb{R}^{d_1})$ where $\xi \in \mathbb{R}^{d_2}$. We have the following proposition.

Proposition 3.1. *For any integrable function F with compact support in $\mathbb{R}$, we have*

$$F(L)f(x', x'') = \mathcal{F}^{-1}(F(L_\xi)\widehat{f}(x', \xi))(x'').$$

Proof. This equality is essentially proved in [10, Proposition 5]. Alternatively, we can follow the approach used in the proof of Proposition 3.2 in [2] for a direct proof. □

Next we turn to the spectral decomposition of the operator L_ξ on $\mathbb{R}^{d_1}$. Let $L_1 = -\Delta_{d_1} + |x'|^2$ be the harmonic oscillator on $\mathbb{R}^{d_1}$, ν be a multi-index and $\Phi_\nu(x') = h_{\nu_1}(x'_1) \cdots h_{\nu_{d_1}}(x'_{d_1})$, where h_{ν_j} is the Hermite function of order ν_j. Recall that $2|\nu| + d_1$ and Φ_ν are the eigenvalues and eigenfunctions of the operator L_1. Thus $(2|\nu| + d_1)|\xi|$ and $\Phi^\xi_\nu(x') = |\xi|^{d_1/4}\Phi_\nu(\sqrt{|\xi|}x')$ are the eigenvalues and eigenfunctions of the operator L_ξ; see [10]. Then we have

$$L_\xi f = \sum_{k=0}^{\infty} (2k + d_1)|\xi| \sum_{|\nu|=k} \langle f, \Phi^\xi_\nu\rangle \Phi^\xi_\nu$$

and

$$F(L_\xi)f = \sum_{k=0}^{\infty} F((2k + d_1)|\xi|) \sum_{|\nu|=k} \langle f, \Phi^\xi_\nu\rangle \Phi^\xi_\nu.$$

We have the following restriction type estimate for L_ξ.

Proposition 3.2. *Suppose $d_1 \geq 2$. For $1 \leq p \leq 2d_1/(d_1 + 2)$,*

$$\| \sum_{|\nu|=k} \langle f, \Phi^\xi_\nu\rangle \Phi^\xi_\nu \|_{L^2(\mathbb{R}^{d_1})} \leq C|\xi|^{\frac{d_1}{2}(\frac{1}{p}-\frac{1}{2})}(2k + d_1)^{\frac{d_1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^{d_1})}. \tag{3.2}$$

Proof. From [7, Corollary 3.2] we have for $1 \leq p \leq 2d_1/(d_1 + 2)$,

$$\| \sum_{|\nu|=k} \langle f, \Phi_\nu\rangle \Phi_\nu \|_{L^2} \leq C(2k + d_1)^{\frac{d_1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f\|_{L^p}.$$

Then changing variables from the above inequality implies

$$\begin{aligned} \| \sum_{|\nu|=k} \langle f, \Phi^\xi_\nu\rangle \Phi^\xi_\nu \|_{L^2} &= \| \sum_{|\nu|=k} \left\langle f, |\xi|^{d_1/4}\Phi_\nu(\sqrt{|\xi|}\cdot)\right\rangle |\xi|^{d_1/4}\Phi_\nu(\sqrt{|\xi|}x')\|_{L^2} \\ &= \| \sum_{|\nu|=k} \left\langle f(\frac{\cdot}{\sqrt{|\xi|}}), \Phi_\nu(\cdot)\right\rangle \Phi_\nu(\sqrt{|\xi|}x')\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&= \| \sum_{|\nu|=k} \left\langle f(\frac{\cdot}{\sqrt{|\xi|}}), \Phi_\nu(\cdot) \right\rangle \Phi_\nu(x') \|_{L^2} |\xi|^{-d_1/4} \\
&\leq C|\xi|^{-d_1/4} (2k+d_1)^{\frac{d_1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f(\frac{\cdot}{\sqrt{|\xi|}})\|_{L^p} \\
&\leq C|\xi|^{\frac{d_1}{2}(\frac{1}{p}-\frac{1}{2})} (2k+d_1)^{\frac{d_1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^{d_1})}.
\end{aligned}
$$

□

In order to prove the weighted restriction estimate for L, we need the following proposition, which is essentially the same as [10, Proposition 4].

Proposition 3.3. *Let $\gamma \in [0,\infty)$ and $f \in L^2(\mathbb{R}^{d_1})$. Then*

$$\||x'|^\gamma f\|_{L^2(\mathbb{R}^{d_1})} \leq C_\gamma \||\xi|^{-\gamma} L_\xi^{\gamma/2} f\|_{L^2(\mathbb{R}^{d_1})}.$$

Here C_γ is non-decreasing in γ.

Proof. Let H be the harmonic oscillator $-d^2/du^2 + u^2$ on $\mathbb{R}$. It is obvious that

$$\||u|f\|_{L^2(\mathbb{R})} \leq \|H^{1/2} f\|_{L^2(\mathbb{R})}.$$

In addition since the first eigenvalue of H is bigger than 1,

$$
\begin{aligned}
\|\frac{d^2}{du^2} f\|_2^2 + \|u^2 f\|_2^2 &\leq \|(-\frac{d^2}{du^2} + u^2) f\|_2^2 - 2Re\langle -\frac{d^2}{du^2} f, u^2 f\rangle \\
&\leq \|Hf\|_2^2 - 2Re\langle \frac{d}{du} f, 2uf\rangle - 2\|u\frac{d}{du} f\|_2^2 \\
&\leq \|Hf\|_2^2 - 2Re\langle \frac{d}{du} f, 2uf\rangle \\
&\leq \|Hf\|_2^2 + 4\|\frac{d}{du} f\|_2 \|uf\|_2 \\
&\leq \|Hf\|_2^2 + 4\|H^{1/2} f\|_2 \|H^{1/2} f\|_2 \\
&\leq 5\|Hf\|_2^2.
\end{aligned}
$$

This implies that

$$\|u^2 f\|_{L^2(\mathbb{R})} \leq \sqrt{5} \|Hf\|_{L^2(\mathbb{R})}.$$

By iteration, we can prove that for $k \in \mathbb{N}$,

$$\|u^k f\|_{L^2(\mathbb{R})} \leq C_k \|H^{k/2} f\|_{L^2(\mathbb{R})}.$$

For details, we refer to Proposition 3.2 and 3.3 in [4]. Now by a similar approach as in in the proof of Proposition 2.2 in [2], we can prove Proposition 3.3. □

We state our weighted restriction estimate for the Grushin operator L.

Theorem 3.4. *Let F be a Borel function with* $\operatorname{supp} F \subset [R/4, R]$ *for some $R > 0$. Then for $1 \leq p \leq \min\{2d_1/(d_1+2), (2d_2+2)/(d_2+3)\}$ and $0 \leq \gamma < d_2(1/p - 1/2)$,*

$$\||x'|^\gamma F(\sqrt{L}) f\|_{L^2(\mathbb{R}^{d_1}_{x'} \times \mathbb{R}^{d_2}_{x''})} \leq C R^{(2d_2+d_1)(\frac{1}{p}-\frac{1}{2})-\gamma} \|\delta_R F\|_{L^2(\mathbb{R})} \|f\|_{L^p(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})}. \tag{3.3}$$

Moreover, when $|y'| > 4r$,

$$\||x'|^\gamma F(\sqrt{L}) P_{B(y,r)} f\|_{L^2(\mathbb{R}^{d_1}_{x'} \times \mathbb{R}^{d_2}_{x''})} \leq C R^{(d_2+d_1)(\frac{1}{p}-\frac{1}{2})} |y'|^{\gamma - d_2(\frac{1}{p}-\frac{1}{2})} \|\delta_R F\|_{L^2} \|f\|_{L^p(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})}, \tag{3.4}$$

where $y=(y',y'')\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ and $P_{B(y,r)}$ is the projection on the ball $B(y,r)$ of $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ for distance ρ.

Proof. When $p=1$, this theorem is proved in [10, Proposition 10]. So in what follows we may assume that $d_1>2$. Let $G(x)=F(\sqrt{x})$. Then $\operatorname{supp} G\subset[R^2/16,R^2]$. By a density argument, it is enough to prove the estimates (3.3) and (3.4) for functions $f\in L^2(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})\cap L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ such that $f(x',x'')=g(x')h(x'')$ where $g\in L^2(\mathbb{R}^{d_1})\cap L^p(\mathbb{R}^{d_1})$ and $h\in L^2(\mathbb{R}^{d_2})\cap L^p(\mathbb{R}^{d_2})$.

By Proposition 3.1 and Plancherel equality,

$$\begin{aligned}\||x'|^\gamma F(\sqrt{L})f\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})} &= \|\mathcal{F}^{-1}(|x'|^\gamma G(L_\xi)g(x')\widehat{h}(\xi))(x'')\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})}\\ &= \||x'|^\gamma G(L_\xi)g(x')\widehat{h}(\xi)\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{\xi})}. \qquad (3.5)\end{aligned}$$

Then by Proposition 3.3,

$$\begin{aligned}&\||x'|^\gamma G(L_\xi)g(x')\widehat{h}(\xi)\|^2_{L^2(\mathbb{R}^{d_1}_{x'})}\\ &\le \||\xi|^{-\gamma}L_\xi^{\gamma/2}G(L_\xi)g(x')\widehat{h}(\xi)\|^2_{L^2(\mathbb{R}^{d_1}_{x'})}\\ &= \||\xi|^{-\gamma}\sum_{k=0}^\infty((2k+d_1)|\xi|)^{\gamma/2}G((2k+d_1)|\xi|)\sum_{|\nu|=k}\langle g,\Phi^\xi_\nu\rangle\Phi^\xi_\nu(x')\widehat{h}(\xi)\|^2_{L^2(\mathbb{R}^{d_1}_{x'})},\end{aligned}$$

and by the orthonormal property for eigenfunctions of different eigenvalues, we have

$$\begin{aligned}&\||x'|^\gamma G(L_\xi)g(x')\widehat{h}(\xi)\|^2_{L^2(\mathbb{R}^{d_1}_{x'})}\\ &\le \sum_{k=0}^\infty\||\xi|^{-\gamma}((2k+d_1)|\xi|)^{\gamma/2}G((2k+d_1)|\xi|)\sum_{|\nu|=k}\langle g,\Phi^\xi_\nu\rangle\Phi^\xi_\nu(x')\widehat{h}(\xi)\|^2_{L^2(\mathbb{R}^{d_1}_{x'})}.\end{aligned}$$

This together with equality (3.5) implies

$$\begin{aligned}&\||x'|^\gamma F(\sqrt{L})f\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})}\\ &\le \sum_{k=0}^\infty\||\xi|^{-\gamma}((2k+d_1)|\xi|)^{\gamma/2}G((2k+d_1)|\xi|)\sum_{|\nu|=k}\langle g,\Phi^\xi_\nu\rangle\Phi^\xi_\nu(x')\widehat{h}(\xi)\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{\xi})}. \qquad (3.6)\end{aligned}$$

Let $\widetilde{G}_{k,x'}(|\xi|)$ be the function on $\mathbb{R}$ defined by

$$\widetilde{G}_{k,x'}(|\xi|)=|\xi|^{-\gamma}((2k+d_1)|\xi|)^{\gamma/2}G((2k+d_1)|\xi|)\sum_{|\nu|=k}\langle g,\Phi^\xi_\nu\rangle\Phi^\xi_\nu(x').$$

By estimate (3.6) and Plancherel equality,

$$\begin{aligned}\||x'|^\gamma F(\sqrt{L})f\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})} &\le \sum_{k=0}^\infty\|\mathcal{F}^{-1}\left(\widetilde{G}_{k,x'}(|\xi|)\widehat{h}(\xi)\right)(x'')\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})}\\ &= \sum_{k=0}^\infty\|\widetilde{G}_{k,x'}(\sqrt{-\Delta_2})h(x'')\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})}. \qquad (3.7)\end{aligned}$$

Note that $\operatorname{supp} G\subset[R^2/16,R^2]$. Thus $\operatorname{supp}\widetilde{G}_{k,x'}\subset[0,R^2/(2k+d_1)]$. Set $m=2k+d_1$ and $a=R^2/(2k+d_1)$. By restriction type estimates for $-\Delta_2$ (see e.g. [1]),

$$\|\widetilde{G}_{k,x'}(\sqrt{-\Delta_2})h(x'')\|^2_{L^2(\mathbb{R}^{d_2}_{x''})}\le Ca^{2d_2(\frac{1}{p}-\frac{1}{2})}\|\delta_a\widetilde{G}_{k,x'}\|^2_{L^2(\mathbb{R})}\|h\|^2_{L^p(\mathbb{R}^{d_2})}.$$

Thus by (3.7),

$$\||x'|^\gamma F(\sqrt{L})f\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})} \le C\sum_{m=1}^{\infty} a^{2d_2(\frac{1}{p}-\frac{1}{2})}\|\delta_a\widetilde{G}_{k,x'}\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}_{|\xi|})}\|h\|^2_{L^p(\mathbb{R}^{d_2})}. \tag{3.8}$$

By Proposition 3.2,

$$\begin{aligned}\|\delta_a\widetilde{G}_{k,x'}\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}_{|\xi|})} &\le \int_{\mathbb{R}_{|\xi|}} |a\xi|^{-2\gamma}(am|\xi|)^\gamma|G(am|\xi|)|^2\|\sum_{|\nu|=k}\langle g,\Phi^{a\xi}_\nu\rangle\Phi^{a\xi}_\nu\|^2_{L^2(\mathbb{R}^{d_1}_{x'})}\,d|\xi|\\ &\le C\int_{\mathbb{R}_{|\xi|}} |a\xi|^{-2\gamma}(am|\xi|)^\gamma|G(am|\xi|)|^2|a\xi|^{d_1(\frac{1}{p}-\frac{1}{2})}m^{d_1(\frac{1}{p}-\frac{1}{2})-1}\|g\|^2_{L^p(\mathbb{R}^{d_1})}\,d|\xi|\\ &\le Cm^{2\gamma-1}\int_{\mathbb{R}} t^{-2\gamma}t^\gamma|G(t)|^2t^{d_1(\frac{1}{p}-\frac{1}{2})}\frac{dt}{ma}\|g\|^2_{L^p(\mathbb{R}^{d_1})}\\ &\le Cm^{2\gamma-1}R^{2d_1(\frac{1}{p}-\frac{1}{2})-2\gamma}\|\delta_{R^2}G\|^2_{L^2(\mathbb{R})}\|g\|^2_{L^p(\mathbb{R}^{d_1})}. \end{aligned}\tag{3.9}$$

Combing estimates (3.8) and (3.9) and noting that $\gamma < d_2(1/p-1/2)$ yields

$$\begin{aligned}&\||x'|^\gamma F(\sqrt{L})f\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})}\\ &\le C\sum_{m=1}^{\infty} R^{4d_2(\frac{1}{p}-\frac{1}{2})+2d_1(\frac{1}{p}-\frac{1}{2})-2\gamma}m^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})-1}\|\delta_{R^2}G\|^2_{L^2(\mathbb{R})}\|g\|^2_{L^p(\mathbb{R}^{d_1})}\|h\|^2_{L^p(\mathbb{R}^{d_2})}\\ &\le CR^{4d_2(\frac{1}{p}-\frac{1}{2})+2d_1(\frac{1}{p}-\frac{1}{2})-2\gamma}\|\delta_R F\|^2_{L^2(\mathbb{R})}\|f\|^2_{L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})}.\end{aligned}$$

This proves the estimate (3.3).

Next we prove (3.4). Similarly to the above derivation, we have

$$\begin{aligned}&\||x'|^\gamma F(\sqrt{L})P_{B(y,r)}f\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})}\\ &\le C\sum_{m=1}^{\infty} a^{2d_2(\frac{1}{p}-\frac{1}{2})}\|\delta_a\widetilde{G}_{k,x'}\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}_{|\xi|})}\|h\|^2_{L^p(\mathbb{R}^{d_2})}\\ &\le C\sum_{m=1}^{\infty} a^{2d_2(\frac{1}{p}-\frac{1}{2})}\int_{\mathbb{R}_{|\xi|}} |a\xi|^{-2\gamma}(am|\xi|)^\gamma|G(am|\xi|)|^2\|\sum_{|\nu|=k}\langle g,\Phi^{a\xi}_\nu\rangle\Phi^{a\xi}_\nu\|^2_{L^2(\mathbb{R}^{d_1}_{x'})}\,d|\xi|\|h\|^2_{L^p(\mathbb{R}^{d_2})}\\ &\le C\sum_{m=1}^{\infty} a^{2d_2(\frac{1}{p}-\frac{1}{2})}\int_{\mathbb{R}} m^{2\gamma}t^{-\gamma}|G(t)|^2\|\sum_{|\nu|=k}\langle g,\Phi^{t/m}_\nu\rangle\Phi^{t/m}_\nu\|^2_{L^2(\mathbb{R}^{d_1}_{x'})}\,dt/(am)\|h\|^2_{L^p(\mathbb{R}^{d_2})}\\ &\le C\int_{\mathbb{R}} R^{4d_2(\frac{1}{p}-\frac{1}{2})-2}t^{-\gamma}|G(t)|^2\sum_{m=1}^{\infty} m^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})}\|\sum_{|\nu|=k}\langle g,\Phi^{t/m}_\nu\rangle\Phi^{t/m}_\nu\|^2_{L^2(\mathbb{R}^{d_1}_{x'})}\,dt\|h\|^2_{L^p(\mathbb{R}^{d_2})},\end{aligned}$$

where the function g has compact support such that $\operatorname{supp} g \subset B(y',r)$ which is the standard ball defining by Euclidean distance in $\mathbb{R}^{d_1}$. Note that $\operatorname{supp} G \subset [R^2/16, R^2]$. Thus $R^2 \sim t$ in the last integral and then

$$\begin{aligned}&\||x'|^\gamma F(\sqrt{L})P_{B(y,r)}f\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})}\\ &\le C\int_{\mathbb{R}} R^{2d_2(\frac{1}{p}-\frac{1}{2})-2}t^{d_2(\frac{1}{p}-\frac{1}{2})-\gamma}|G(t)|^2\sum_{m=1}^{\infty} m^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})}\|\sum_{|\nu|=k}\langle g,\Phi^{t/m}_\nu\rangle\Phi^{t/m}_\nu\|^2_{L^2(\mathbb{R}^{d_1}_{x'})}\,dt\|h\|^2_{L^p(\mathbb{R}^{d_2})}.\end{aligned}\tag{3.10}$$

Next we claim that for $0 \le \gamma < d_2(\frac{1}{p}-\frac{1}{2})$,

$$\sum_{m=1}^{\infty} |y'|^{2d_2(\frac{1}{p}-\frac{1}{2})-2\gamma} t^{d_2(\frac{1}{p}-\frac{1}{2})-d_1(\frac{1}{p}-\frac{1}{2})-\gamma} m^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})} \| \sum_{|\nu|=k} \langle g, \Phi_\nu^{t/m}\rangle \Phi_\nu^{t/m} \|^2_{L^2(\mathbb{R}^{d_1}_{x'})} \le C\|g\|^2_{L^p}, \tag{3.11}$$

where C is independent of t and y'.

In order to prove (3.11) we split the sum into two parts: $m \le \sqrt{t}|y'|/4$ and $m > \sqrt{t}|y'|/4$.

If $m > \sqrt{t}|y'|/4$, by Proposition 3.2,

$$\begin{aligned}
&\sum_{m>\sqrt{t}|y'|/4}^{\infty} |y'|^{2d_2(\frac{1}{p}-\frac{1}{2})-2\gamma} t^{d_2(\frac{1}{p}-\frac{1}{2})-d_1(\frac{1}{p}-\frac{1}{2})-\gamma} m^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})} \| \sum_{|\nu|=k} \langle g, \Phi_\nu^{t/m}\rangle \Phi_\nu^{t/m} \|^2_{L^2(\mathbb{R}^{d_1}_{x'})} \\
&\le C \sum_{m>\sqrt{t}|y'|/4}^{\infty} |y'|^{2d_2(\frac{1}{p}-\frac{1}{2})-2\gamma} t^{d_2(\frac{1}{p}-\frac{1}{2})-d_1(\frac{1}{p}-\frac{1}{2})-\gamma} m^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})} (t/m)^{d_1(\frac{1}{p}-\frac{1}{2})} m^{d_1(\frac{1}{p}-\frac{1}{2})-1} \|g\|^2_{L^p} \\
&\le C \sum_{m>\sqrt{t}|y'|/4}^{\infty} (\sqrt{t}|y'|)^{2d_2(\frac{1}{p}-\frac{1}{2})-2\gamma} m^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})-1} \|g\|^2_{L^p} \\
&\le C\|g\|^2_{L^p}.
\end{aligned}$$

If $m \le \sqrt{t}|y'|/4$ and $x' \in B(y', r)$, then $|x'| \ge |y'|/2$ and $m \le \sqrt{t}|x'|/2$. Moreover, $|m^{-1/2}\sqrt{t}x'|^2 \ge 4m$. By [10, Lemma 8], we know that $\sum_{|\nu|=k} |\Phi_\nu(x')|^2 \le C\exp(-c|x'|^2)$ when $|x'|^2 \ge 2(2k+d_1)$. Hence

$$\begin{aligned}
\sum_{|\nu|=k} |\Phi_\nu^{t/m}(x')|^2 &= |t/m|^{d_1/2} \sum_{|\nu|=k} |\Phi_\nu(m^{-1/2}\sqrt{t}x')|^2 \\
&\le C|t/m|^{d_1/2} e^{-ct|x'|^2/m} \\
&\le C|t/m|^{d_1/2} e^{-ct|y'|^2/m}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\| \sum_{|\nu|=k} \langle g, \Phi_\nu^{t/m}\rangle \Phi_\nu^{t/m} \|^2_{L^2(\mathbb{R}^{d_1}_{x'})} &\le \|g\|^2_{L^p} \sum_{|\nu|=k} \|\Phi_\nu^{t/m}\|^2_{L^{p'}(B(y',r))} \\
&\le \|g\|^2_{L^p} \| (\sum_{|\nu|=k} |\Phi_\nu^{t/m}|^2)^{1/2} \|^2_{L^{p'}(B(y',r))} \\
&\le C\|g\|^2_{L^p} |t/m|^{d_1/2} e^{-ct|y'|^2/m} r^{2d_1(1-\frac{1}{p})}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{m\le\sqrt{t}|y'|/4}^{\infty} |y'|^{2d_2(\frac{1}{p}-\frac{1}{2})-2\gamma} t^{d_2(\frac{1}{p}-\frac{1}{2})-d_1(\frac{1}{p}-\frac{1}{2})-\gamma} m^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})} \| \sum_{|\nu|=k} \langle g, \Phi_\nu^{t/m}\rangle \Phi_\nu^{t/m} \|^2_{L^2(\mathbb{R}^{d_1}_{x'})} \\
&\le \sum_{m\le\sqrt{t}|y'|/4}^{\infty} (\sqrt{t}|y'|)^{2d_1(1-\frac{1}{p})+2d_2(\frac{1}{p}-\frac{1}{2})-2\gamma} m^{-2d_2(\frac{1}{p}-\frac{1}{2})-d_1/2} e^{-ct|y'|^2/m} \|g\|^2_{L^p} \\
&\le \sum_{m=1}^{\infty} \sup_{u>m} u^{d_1(1-\frac{1}{p})+d_2(\frac{1}{p}-\frac{1}{2})-\gamma} m^{d_1(1-\frac{1}{p})-d_1/2} e^{-cu} \|g\|^2_{L^p} \\
&\le C\|g\|^2_{L^p},
\end{aligned}$$

which complete the proof of the claim (3.11).

Combining (3.11) and (3.10), we obtain

$$\begin{aligned}
&\||x'|^\gamma F(\sqrt{L})P_{B(y,r)}f\|^2_{L^2(\mathbb{R}^{d_1}_{x'}\times\mathbb{R}^{d_2}_{x''})}\\
&\le C\int_{\mathbb{R}} R^{2d_2(\frac{1}{p}-\frac{1}{2})-2}|y'|^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})}t^{d_1(\frac{1}{p}-\frac{1}{2})}|G(t)|^2\,dt\|g\|^2_{L^p}\|h\|^2_{L^p(\mathbb{R}^{d_2})}\\
&\le CR^{2(d_2+d_1)(\frac{1}{p}-\frac{1}{2})}|y'|^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})}\int_{\mathbb{R}}|G(t)|^2\,dt/R^2\|g\|^2_{L^p}\|h\|^2_{L^p(\mathbb{R}^{d_2})}\\
&\le CR^{2(d_2+d_1)(\frac{1}{p}-\frac{1}{2})}|y'|^{2\gamma-2d_2(\frac{1}{p}-\frac{1}{2})}\|\delta_R F\|^2_{L^2}\|f\|^2_{L^p(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})}.
\end{aligned}$$

This completes the proof of estimate (3.4) and so the proof of Theorem 3.4. □

Remark 3.5. Under the assumptions of Theorem 3.4, when $\gamma=0$, the estimate (3.3) holds for all $1\le p\le(2d_2+2)/(d_2+3)$, which means that the condition $p<2d_1/(d_1+2)$ is not necessary in this case. Actually, in our proof, if $\gamma=0$, we do not need the sharp order $\frac{d_1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}$ for $2k+d_1$ in the estimate (3.2). We only need that for all $1\le p\le 2$

$$\|\sum_{|\nu|=k}\langle f,\Phi^\xi_\nu\rangle\Phi^\xi_\nu\|_{L^2(\mathbb{R}^{d_1})}\le C|\xi|^{\frac{d_1}{2}(\frac{1}{p}-\frac{1}{2})}(2k+d_1)^{(\frac{d_1}{2}-1)(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^p(\mathbb{R}^{d_1})},$$

which can be achieved by interpolation between $p=1$ and the fact that

$$\|\sum_{|\nu|=k}\langle f,\Phi^\xi_\nu\rangle\Phi^\xi_\nu\|_{L^2}\le C\|f\|_{L^2}.$$

See also [8].

4. Spectral multipliers for compactly supported functions

As mentioned in Section 2, the heat kernel of the operator L satisfies a Gaussian upper bound given in terms of the distance ρ. In addition, L satisfies the Davies-Gaffney estimate and the finite speed propagation property, see [12]. On the other hand, we proved restriction type estimates for the operator L in Section 3. Therefore we may follow ideas in [1], Sections 3 and 4, to prove spectral multiplier results as well as Bochner-Riesz summability results.

Define the multiplication operator w_γ on $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ by

$$w_\gamma f(x',x'')=|x'|^\gamma f(x',x'').$$

Lemma 4.1. *Let $F:[0,\infty)\to\mathbb{C}$ be a bounded Borel function. We denote by $K_{F(L)}$ the Schwartz kernel of $F(L)$. Assume that*

$$\operatorname{supp}K_{F(L)}\subset\mathcal{D}_r=\{(x,y)\in(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})\times(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}):\rho(x,y)\le r\}$$

for some $r>0$. Then for $1\le p\le 2$, there exists a constant $C=C_p$ such that for $\gamma\in[0,d_1(1/p-1/2))$

$$\begin{aligned}
\|F(L)\|_{p\to p}\ \le\ & C\sup_{|y'|\le 4r}\{r^{(2d_2+d_1)(\frac{1}{p}-\frac{1}{2})-\gamma}\|w_\gamma F(L)P_{B(y,r)}\|_{p\to 2}\}\\
&+C\sup_{|y'|>4r}\{r^{(d_2+d_1)(\frac{1}{p}-\frac{1}{2})}|y'|^{d_2(\frac{1}{p}-\frac{1}{2})-\gamma}\|w_\gamma F(L)P_{B(y,r)}\|_{p\to 2}\}.
\end{aligned}$$

Proof. First we choose a sequence $(x_n) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ such that $\rho(x_i, x_j) > r/10$ for $i \neq j$ and $\sup_{x\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \inf_i \rho(x, x_i) \leq r/10$. Such sequence exists because $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is separable under the new distance ρ. Second we let $B_i = B(x_i, r)$ and define $\widetilde{B_i}$ by the formula

$$\widetilde{B_i} = \bar{B}\left(x_i, \frac{r}{10}\right) \setminus \bigcup_{j<i} \bar{B}\left(x_j, \frac{r}{10}\right),$$

where $\bar{B}(x, r) = \{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \colon \rho(x, y) \leq r\}$. Third we put $\chi_i = \chi_{\widetilde{B}_i}$, where $\chi_{\widetilde{B}_i}$ is the characteristic function of the set $\widetilde{B}_i$. Note that for $i \neq j$, $B(x_i, \frac{r}{20}) \cap B(x_j, \frac{r}{20}) = \emptyset$. Hence

$$K := \sup_i \#\{j : \ \rho(x_i, x_j) \leq 2r\} \leq \sup_x \frac{|B(x, (2 + \frac{1}{20})r)|}{|B(x, \frac{r}{20})|} < C41^{d_1+2d_2} < \infty.$$

It is not difficult to see that

$$\mathcal{D}_r \subset \bigcup_{\{i,j\colon \rho(x_i,x_j)<2r\}} \widetilde{B}_i \times \widetilde{B}_j \subset \mathcal{D}_{4r}.$$

Therefore,

$$F(L)f = \sum_{i,j\colon \rho(x_i,x_j)<2r} P_{\widetilde{B}_i} F(L) P_{\widetilde{B}_j} f.$$

Hence by Hölder's inequality

$$\begin{aligned}
\|F(L)f\|_p^p &= \| \sum_{i,j\colon \rho(x_i,x_j)<2r} P_{\widetilde{B}_i} F(L) P_{\widetilde{B}_j} f\|_p^p \\
&= \sum_i \| \sum_{j\colon \rho(x_i,x_j)<2r} P_{\widetilde{B}_i} F(L) P_{\widetilde{B}_j} f\|_p^p \\
&\leq CK^{p-1} \sum_i \sum_{j\colon \rho(x_i,x_j)<2r} \|P_{\widetilde{B}_i} F(L) P_{\widetilde{B}_j} f\|_p^p \\
&\leq CK^{p-1} \sum_i \sum_{j\colon \rho(x_i,x_j)<2r} \||x'|^{-\gamma}\|_{L^q(\widetilde{B}_i)}^p \||x'|^\gamma P_{\widetilde{B}_i} T P_{\widetilde{B}_j} f\|_2^p \qquad (4.1)
\end{aligned}$$

where $1/q = 1/p - 1/2$.

Now we estimate $\||x'|^{-\gamma}\|_{L^q(\widetilde{B}_i)}^p$.

Suppose $|x'_j| > 4r$. Since $|x'_i - x'_j| < \rho(x_i, x_j) < 2r$, we have $|x'_i| > 2r$ and $2|x'_j| > |x'_i| > |x'_j|/2$. Thus, for $x \in \widetilde{B}_i$, that is, $\rho(x, x_i) \leq r/10$, we have $|x' - x'_i| \leq r/10$. This implies $|x'| > r$ and $|x'| > |x'_i|/2 > |x'_j|/4$. Then by (2.2)

$$\||x'|^{-\gamma}\|_{L^q(\widetilde{B}_i)}^p \leq C|x'_j|^{-p\gamma} \mu(\widetilde{B}_i)^{p/q} \leq C|x'_j|^{-p\gamma} r^{p(d_1+d_2)(\frac{1}{p}-\frac{1}{2})} |x'_j|^{pd_2(\frac{1}{p}-\frac{1}{2})}. \qquad (4.2)$$

If $|x'_j| \leq 4r$, $|x'_i| \leq |x'_i - x'_j| + |x'_j| \leq 6r$. So $x \in \widetilde{B}_i$ implies $|x'| \leq 7r$ and $|x'' - x''_i| \leq 13r^2$. Then

$$\||x'|^{-\gamma}\|_{L^q(\widetilde{B}_i)}^q \leq \int_{|x''-x''_i|\leq 13r^2} \int_{|x'|\leq 7r} |x'|^{-q\gamma} dx' dx'' \leq Cr^{2d_2+d_1-q\gamma}. \qquad (4.3)$$

Substituting estimates (4.2) and (4.3) in (4.1) finishes the proof of Lemma 4.1.

□

Now we can state and prove the following multiplier theorem for compactly supported functions. Recall that $D = \max(d_1 + d_2, 2d_2)$.

Theorem 4.2. *Suppose that a bounded Borel function $F : \mathbb{R} \to \mathbb{C}$ with compact support in $[1/4, 1]$ satisfies*

$$\|F\|_{W_2^s} < \infty$$

for some $s > D|1/p - 1/2|$. Then the operator $F(tL)$ is bounded on $L^p(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. In addition

$$\sup_{t>0} \|F(tL)\|_{L^p \to L^p} \le C_p \|F\|_{W_2^s}.$$

Proof. Let $\eta \in C_c^\infty(\mathbb{R})$ be even and such that $\operatorname{supp} \eta \subseteq \{\xi : 1/4 \le |\xi| \le 1\}$ and

$$\sum_{\ell \in \mathbb{Z}} \eta(2^{-\ell}\lambda) = 1 \qquad \forall \lambda > 0.$$

Then we set $\eta_0(\lambda) = 1 - \sum_{\ell>0} \eta(2^{-\ell}\lambda)$,

$$F^{(0)}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \eta_0(t) \hat{F}(t) \cos(t\lambda)\, dt \tag{4.4}$$

and

$$F^{(\ell)}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \eta(2^{-\ell}t) \hat{F}(t) \cos(t\lambda)\, dt. \tag{4.5}$$

Note that in virtue of the Fourier inversion formula

$$F(\lambda) = \sum_{\ell \ge 0} F^{(\ell)}(\lambda)$$

and by [1, Lemma 2.1],

$$\operatorname{supp} K_{F^{(\ell)}(t\sqrt{L})} \subset \mathcal{D}_{2^\ell t}.$$

Now by Lemma 4.1

$$\begin{aligned} \big\|F(t\sqrt{L})\big\|_{p\to p} &\le \sum_{\ell\ge 0} \big\|F^{(\ell)}(t\sqrt{L})\big\|_{p\to p} \\ &\le C \sum_{\ell\ge 0} \sup_{|y'|\le 42^\ell t} \{(2^\ell t)^{(2d_2+d_1)(\frac{1}{p}-\frac{1}{2})-\gamma} \|w_\gamma F^{(\ell)}(t\sqrt{L}) P_{B(y,2^\ell t)}\|_{p\to 2}\} \\ &\quad + C \sum_{\ell\ge 0} \sup_{|y'|> 42^\ell t} \{(2^\ell t)^{(d_2+d_1)(\frac{1}{p}-\frac{1}{2})} |y'|^{d_2(\frac{1}{p}-\frac{1}{2})-\gamma} \|w_\gamma F^{(\ell)}(t\sqrt{L}) P_{B(y,2^\ell t)}\|_{p\to 2}\}. \end{aligned} \tag{4.6}$$

Since $F^{(\ell)}$ is not compactly supported we choose a function $\psi \in C_c^\infty(1/16, 4)$ such that $\psi(\lambda) = 1$ for $\lambda \in (1/8, 2)$ and note that

$$\begin{aligned} &\big\|w_\gamma F^{(\ell)}(t\sqrt{L}) P_{B(y,2^\ell t)}\big\|_{p\to 2} \\ &\le \big\|w_\gamma \big(\psi F^{(\ell)}\big)(t\sqrt{L}) P_{B(y,2^\ell t)}\big\|_{p\to 2} + \big\|w_\gamma \big((1-\psi) F^{(\ell)}\big)(t\sqrt{L}) P_{B(y,2^\ell t)}\big\|_{p\to 2}. \end{aligned} \tag{4.7}$$

To estimate the norm $\|w_\gamma \big(\psi F^{(\ell)}\big)(t\sqrt{L}) P_{B(y,2^\ell t)}\|_{p\to 2}$, we use the weighted restriction estimates (3.3) and the fact that $\psi \in C_c(1/16, 4)$ to obtain

$$\big\|w_\gamma \big(\psi F^{(\ell)}\big)(t\sqrt{L}) P_{B(y,2^\ell t)}\big\|_{p\to 2} \le C t^{-(2d_2+d_1)(1/p-1/2)+\gamma} \big\|\delta_{t^{-1}} \big(\psi F^{(\ell)}\big)(t\cdot)\big\|_{L^2}$$

and for $|y'| \ge 2^{\ell+2} t$

$$\big\|w_\gamma \big(\psi F^{(\ell)}\big)(t\sqrt{L}) P_{B(y,2^\ell t)}\big\|_{p\to 2} \le C t^{-(d_2+d_1)(1/p-1/2)} |y'|^{\gamma - d_2(1/p-1/2)} \big\|\delta_{t^{-1}} \big(\psi F^{(\ell)}\big)(t\cdot)\big\|_{L^2}$$

for all $t > 0$.

If $|y'| > 2^{\ell+2}t$, it follows from $s > (d_1+d_2)(1/p-1/2)$ that

$$\begin{aligned}
&\sum_{\ell\geq 0}\sup_{|y'|>2^{\ell+2}t}\{(2^\ell t)^{(d_2+d_1)(\frac{1}{p}-\frac{1}{2})}|y'|^{d_2(\frac{1}{p}-\frac{1}{2})-\gamma}\|w_\gamma(\psi F^{(\ell)})(t\sqrt{L})P_{B(y,2^\ell t)}\|_{p\to 2}\}\\
&\leq C\sum_{\ell\geq 0}2^{\ell(d_1+d_2)(\frac{1}{p}-\frac{1}{2})}\big\|\delta_{t^{-1}}\big(\psi F^{(\ell)}\big)(t\cdot)\big\|_{L^2}\\
&\leq C\sum_{\ell\geq 0}2^{\ell(d_1+d_2)(\frac{1}{p}-\frac{1}{2})}\|F^{(\ell)}\|_{L^2} \qquad (4.8)\\
&\leq C\|F\|_{W_2^s}.
\end{aligned}$$

For $|y'| \leq 2^{\ell+2}t$, we take $\gamma < \min\{d_1,d_2\}(1/p-1/2)$ such that $\min\{d_1,d_2\}(1/p-1/2)-\gamma$ is small enough and $s - D(1/p-1/2) > \min\{d_1,d_2\}(1/p-1/2)-\gamma$. Then for $s > D(1/p-1/2)$

$$\begin{aligned}
&\sum_{\ell\geq 0}\sup_{|y'|\leq 2^{\ell+2}t}\{(2^\ell t)^{(2d_2+d_1)(\frac{1}{p}-\frac{1}{2})-\gamma}\|w_\gamma(\psi F^{(\ell)})(t\sqrt{L})P_{B(y,2^\ell t)}\|_{p\to 2}\}\\
&\leq C\sum_{\ell\geq 0}2^{\ell\left(D(\frac{1}{p}-\frac{1}{2})+\min\{d_1,d_2\}(1/p-1/2)-\gamma\right)}\big\|\delta_{t^{-1}}\big(\psi F^{(\ell)}\big)(t\cdot)\big\|_{L^2}\\
&\leq C\sum_{\ell\geq 0}2^{\ell\left(D(\frac{1}{p}-\frac{1}{2})+\min\{d_1,d_2\}(1/p-1/2)-\gamma\right)}\|F^{(\ell)}\|_{L^2} \qquad (4.9)\\
&\leq C\|F\|_{W_2^s}.
\end{aligned}$$

Next we show bounds for $\big\|w_\gamma\big((1-\psi)F^{(\ell)}\big)(t\sqrt{L})P_{B(y,2^\ell t)}\big\|_{p\to 2}$. Since the function $1-\psi$ is supported outside the interval $(1/8,2)$, we can choose a function $\phi \in C_c^\infty(2,8)$ such that

$$1 = \psi(\lambda) + \sum_{k\geq 0}\phi(2^{-k}\lambda) + \sum_{k\leq -6}\phi(2^{-k}\lambda) = \psi(\lambda) + \sum_{k\geq 0}\phi_k(\lambda) + \sum_{k\leq -6}\phi_k(\lambda) \qquad \forall \lambda > 0.$$

Hence

$$\big((1-\psi)F^{(\ell)}\big)(\lambda) = (\sum_{k\geq 0} + \sum_{k\leq -6})\big(\phi_k F^{(\ell)}\big)(\lambda) \qquad \forall \lambda > 0.$$

Note that by the Gaussian upper bound for the heat kernel of L, we have $E_{\sqrt{L}}\{0\} = 0$. So it follows from Theorem 3.4 that

$$\begin{aligned}
&\big\|w_\gamma\big((1-\psi)F^{(\ell)}\big)(t\sqrt{L})P_{B(y,2^\ell t)}\big\|_{p\to 2}\\
&\leq (\sum_{k\geq 0} + \sum_{k\leq -6})\big\|w_\gamma\big(\phi_k F^{(\ell)}\big)(t\sqrt{L})P_{B(y,2^\ell t)}\big\|_{p\to 2}\\
&\leq C(\sum_{k\geq 0} + \sum_{k\leq -6})(2^k t^{-1})^{(2d_2+d_1)(1/p-1/2)-\gamma}\big\|\delta_{2^{k+3}t^{-1}}\big(\phi_k F^{(\ell)}\big)(t\cdot)\big\|_\infty.
\end{aligned}$$

Note that suppF $\subset [1/4,1]$, supp $\phi \subset [2,8]$ and $\check{\eta}$ is in the Schwartz class so

$$\big\|\phi_k F^{(\ell)}\big\|_\infty = 2^\ell\big\|\phi_k(F * \delta_{2^\ell}\check{\eta})\big\|_\infty \leq C2^{-M(\ell+\max\{0,k\})}\|F\|_{L^2}$$

and similarly, $\big\|\phi_k F^{(0)}\big\|_\infty \leq C2^{-M\max\{0,k\}}\|F\|_{L^2}$. Therefore

$$\big\|w_\gamma\big((1-\psi)F^{(\ell)}\big)(t\sqrt{L})P_{B(y,2^\ell t)}\big\|_{p\to 2} \;\leq\; C2^{-M\ell}t^{-(2d_2+d_1)(1/p-1/2)+\gamma}\|F\|_{L^2}$$

and when $|y'| \geq 2^{\ell+2}t$

$$\big\|w_\gamma\big((1-\psi)F^{(\ell)}\big)(t\sqrt{L})P_{B(y,2^\ell t)}\big\|_{p\to 2} \;\leq\; C2^{-M\ell}t^{-(d_2+d_1)(1/p-1/2)}|y'|^{\gamma-d_2(1/p-1/2)}\|F\|_{L^2}$$

Then by a similar calculation as in (4.8) and (4.9),

$$(4.10)\sum_{\ell\geq 0}\sup_{|y'|>2^{\ell+2}t}\{(2^\ell t)^{(d_2+d_1)(\frac{1}{p}-\frac{1}{2})}|y'|^{d_2(\frac{1}{p}-\frac{1}{2})-\gamma}\|w_\gamma((1-\psi)F^{(\ell)})(t\sqrt{L})P_{B(y,2^\ell t)}\|_{p\to 2}\}\leq C\|F\|_{L^2}.$$

and

$$(4.11)\qquad \sum_{\ell\geq 0}\sup_{|y'|\leq 2^{\ell+2}t}\{(2^\ell t)^{(2d_2+d_1)(\frac{1}{p}-\frac{1}{2})-\gamma}\|w_\gamma((1-\psi)F^{(\ell)})(t\sqrt{L})P_{B(y,2^\ell t)}\|_{p\to 2}\}\leq C\|F\|_{L^2}.$$

Now we combine (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11) to complete the proof of Theorem 4.2. □

5. Proofs of Theorems 1.1 and 1.2

In this section, we prove our main results, i.e., Theorems 1.1 and 1.2. The two results follow from Theorem 4.2. To do this, we need a theorem from [14] which states that singular multiplier results follow from the corresponding one for compactly supported functions. We recall this explicitly.

Let (X,ρ,μ) be a metric measure space satisfying doubling condition, that is, for all $r>0$ and $\lambda>1$,

$$\mu(B(x,\lambda r))\leq C\lambda^Q\mu(B(x,r)),$$

where C and Q are positive constants. Let A be a non-negative self-adjoint operator which satisfies the following two off-diagonal estimates: for some $m\geq 2$, some $p_0\in[1,2]$ and for all $t>0$ and all $x,y\in X$

$$(DG_m)\qquad \|P_{B(x,t^{1/m})}e^{-tA}P_{B(y,t^{1/m})}\|_{2\to 2}\leq C\exp\left(-c\Big(\frac{\rho(x,y)}{t^{1/m}}\Big)^{\frac{m}{m-1}}\right)$$

and

$$(G_{p_0,2,m})\qquad \|P_{B(x,t^{1/m})}e^{-tA}P_{B(y,t^{1/m})}\|_{p_0\to 2}\leq C\mu(B(x,t^{1/m}))^{-(\frac{1}{p_0}-\frac{1}{2})}\exp\left(-c\Big(\frac{\rho(x,y)}{t^{1/m}}\Big)^{\frac{m}{m-1}}\right).$$

Let again η be a non trivial C^∞ function with compact support in $(0,\infty)$. We have

Theorem 5.1. *Let A be a non-negative self-adjoint operator on $L^2(X)$ satisfying off-diagonal estimates (DG_m) and $(G_{p_0,2,m})$ for some $1\leq p_0<2$. Assume that for any bounded Borel function H such that* $\operatorname{supp} H\subset[1/4,4]$*, the following condition holds:*

$$\sup_{t>0}\|H(t\sqrt[m]{A})\|_{p\to p}\leq C\|H\|_{W^\alpha_q}$$

for some $p\in(p_0,2)$, $\alpha>1/q$, and $1\leq q\leq\infty$. Then for any bounded Borel function F such that

$$\sup_{t>0}\|\eta F(t\cdot)\|_{W^\alpha_q}<\infty,$$

the operator $F(A)$ is bounded on $L^r(X)$ for all $p<r<p'$. In addition,

$$\|F(A)\|_{r\to r}\leq C\sup_{t>0}\|\eta F(t\cdot)\|_{W^\alpha_q}.$$

This theorem is taken from [14, Theorem 3.3]. It is stated there with the additional assumption that $\alpha>Q(\frac{1}{p}-\frac{1}{2})$ where Q is "the" homogeneous dimension. An inspection of the proof shows that this condition is not needed and the theorem is valid for $\alpha>\frac{1}{q}$ without appealing to any dimension.

Proofs of Theorems 1.1 and 1.2. Note that from Theorem 2.1, the Grushin operator satisfies Gaussian upper bound and so it satisfies off-diagonal estimates (DG_m) and $(G_{p_0,2,m})$ for $m = 1$ and $p_0 = 1$. Then Theorem 1.1 follows from Theorem 4.2 and 5.1.

To prove Theorem 1.2, we decompose the Bochner-Riesz means

$$(1 - tL)^\delta_+ = \phi(L)(1 - tL)^\delta_+ + (1 - \phi(L))(1 - tL)^\delta_+,$$

where ϕ is a smooth cutoff function on $\mathbb{R}$ with $\operatorname{supp}\phi \subset [-1/2, 1/2]$ and $\phi = 1$ on interval $[-1/4, 1/4]$. Then when $\delta > \max\{D|1/p - 1/2| - 1/2, 0\}$, $(1 - \phi(L))(1 - tL)^\delta_+$ is uniformly bounded on L^p by Theorem 4.2. For $\phi(L)(1 - tL)^\delta_+$, because the function $\phi(\lambda)(1 - t\lambda)^\delta_+$ is smooth for all $\delta > 0$, so the L^p-boundedness follows from the Gaussian bound of heat kernel of the operator L and the spectral multiplier result in [3, Theorem 3.1] or [1, Theorem 3.1].

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