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# Large time behaviour of mild solutions of Hamilton-Jacobi-Bellman equations in infinite dimension by a probabilistic approach

Ying Hu\*      Pierre-Yves Madec†      Adrien Richou‡

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## Abstract

We study the large time behaviour of mild solutions of HJB equations in infinite dimension by a purely probabilistic approach. For that purpose, we show that the solution of a BSDE in finite horizon  $T$  taken at initial time behaves like a linear term in  $T$  shifted with the solution of the associated EBSDE taken at initial time. Moreover we give an explicit speed of convergence, which seems to appear very rarely in literature.

## 1 Introduction

We are concerned with the large time behaviour of solutions of the Cauchy problem in an infinite dimensional real Hilbert space  $H$  :

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x) + f(x, \nabla u(t,x)G), & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u(0,x) = g(x), & \forall x \in H, \end{cases} \quad (1.1)$$

where  $u : \mathbb{R}_+ \times H \rightarrow \mathbb{R}$  is the unknown function and  $\mathcal{L}$  is the formal generator of the Kolmogorov semigroup  $\mathcal{P}_t$  of an  $H$ -valued random process solution of the following Ornstein-Uhlenbeck stochastic differential equation :

$$\begin{cases} dX_t = (AX_t + F(X_t^x))dt + GdW_t, & t \in \mathbb{R}_+, \\ X_0 = x, & x \in H, \end{cases}$$

where  $W$  is a Wiener process with values in another real Hilbert space  $\Xi$ , assumed to be separable. We recall that (formally),  $\forall h : H \rightarrow \mathbb{R}$ ,

$$(\mathcal{L}h)(x) = \frac{1}{2} \text{Tr}(GG^* \nabla^2 h(x)) + \langle Ax + F(x), \nabla h(x) \rangle .$$

Our method is purely probabilistic, which can be described as follows.

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\*IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France (ying.hu@univ-rennes1.fr)

†IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France (pierre-yves.madec@univ-rennes1.fr)

‡Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France. CNRS, IMB, UMR 5251, F-33400 Talence, France. INRIA, Equipe ALEA, F-33400 Talence, France.(adrien.richou@math.u-bordeaux1.fr)

First, let  $(v, \lambda)$  be the solution of the ergodic PDE:

$$\mathcal{L}v + f(x, \nabla v(x)G) = 0, \quad \forall x \in H.$$

Then we have the following probabilistic representation. Let  $(Y^{T,x}, Z^{T,x})$  be solution of the BSDE:

$$\begin{cases} dY_s^{T,x} = -f(X_s^x, Z_s^{T,x})ds + Z_s^{T,x}dW_s \\ Y_T^{T,x} = g(X_T^x), \end{cases}$$

and  $(Y, Z, \lambda)$  be solution of the EBSDE:

$$dY_s = -(f(X_s^x, Z_s^x) - \lambda)ds + Z_s^x dW_s.$$

Then

$$\begin{cases} Y_s^{T,x} = u(T-s, X_s^x), \\ Y_s^x = v(X_s^x). \end{cases}$$

Finally, due to Girsanov transformations and the use of an important coupling estimate result, we deduce that there is a constant  $L \in \mathbb{R}$  such that for all  $x \in H$ ,

$$Y_0^{T,x} - \lambda T - Y_0^x \xrightarrow{T \rightarrow +\infty} L,$$

i.e.

$$u(T, x) - \lambda T - v(x) \xrightarrow{T \rightarrow +\infty} L.$$

Our method is not only purely probabilistic, but also gives a speed of convergence :

$$|u(T, x) - \lambda T - v(x) - L| \leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T}.$$

The constant  $\mu$  appearing above is the polynomial growth power of  $g(\cdot)$  and  $f(\cdot, 0)$  and  $\hat{\eta}$  is linked to the dissipative constant of  $A$ .

Large time behaviour of solutions has been studied for various types of HJB equations of second order; see, e.g., [1], [7], [9] and [11]. In [1], a result in finite dimension is stated under periodic assumptions for  $f$  and a periodic and Lipschitz assumption for  $g$ . Furthermore, they assume that  $f(x, z)$  is of linear growth in  $z$  and bounded in  $x$ . In [7], some results are stated in finite dimensionnal framework, under locally Hölder conditions for the coefficients. More precisely, they assume that  $f(x, z) = H_1(z) - H_2(x)$  with  $H_1$  Lipschitz and under locally Hölder conditions for  $H_2$  and  $g$ . They also treat the case of  $H_1$  locally Lipschitz but consequently need to assume that  $H_2$  and  $g$  are Lipschitz. Furthermore, they only treat the Laplacian case, namely they assume that  $G = I_d$ . No result on speed of convergence is given in this paper. In [9], the authors deal with the problem in finite dimension. They also only treat the Laplacian case and assume that  $f(x, z)$  is convex and of quadratic growth in  $z$  and of polynomial growth in  $x$ . No result on speed of convergence is given in this paper. As far as we know, the explicit speed convergence only appears in [11], but in finite dimension and under periodic assumptions for  $f(\cdot, z)$  and  $g(\cdot)$ . Furthermore, they assume that linear growth property in  $z$  for  $f$  with derivatives in  $z$  of first and second order uniformly bounded hold.

Roughly speaking, we will assume that  $A$  is a dissipative operator,  $G : H \rightarrow H$  is invertible and bounded operator,  $g : H \rightarrow \mathbb{R}$  continuous with polynomial growth and  $f : H \times \Xi^*$  continuous, with polynomial growth in the first variable and Lipschitz in the second variable.

The paper is organised as follows: In section 2, we introduce some notations. In section 3, we recall some results about existence and uniqueness results for solutions of an Ornstein-Uhlenbeck SDE, a general BSDE and an EBSDE that will be useful for what follow in the paper. In section 4, we study the behaviour of solutions of the solution of the BSDE taken at initial time when the horizon  $T$  of the BSDE increases. In section 5, we apply our result to an optimal control problem.

## 2 Notations

We introduce some notations; let  $E, F$  be real separable Hilbert spaces. The norm and the scalar product will be denoted by  $|\cdot|$ ,  $\langle \cdot, \cdot \rangle$ , with subscripts if needed.  $L(E, F)$  is the space of linear bounded operators  $E \rightarrow F$ , with the operator norm, which is denoted by  $|\cdot|_{L(E, F)}$ . The domain of a linear (unbounded) operator  $A$  is denoted by  $D(A)$ .  $L_2(E, F)$  denotes the space of Hilbert-Schmidt operators from  $E$  to  $F$ , endowed with the Hilbert-Schmidt norm, which is denoted by  $|\cdot|_{L_2(E, F)}$ .

Given  $\phi \in B_b(E)$ , the space of bounded and measurable functions  $\phi : E \rightarrow \mathbb{R}$ , we denote by  $\|\phi\|_0 = \sup_{x \in E} |\phi(x)|$ .

We say that a function  $F : E \rightarrow F$  belongs to the class  $\mathcal{G}^1(E, F)$  if it is continuous, has a Gâteaux differential  $\nabla F(x) \in L(E, F)$  at any point  $x \in E$ , and for every  $k \in E$ , the mapping  $x \mapsto \nabla F(x)k$  is continuous from  $E$  to  $F$  (i.e.  $x \mapsto \nabla F(x)$  ) is continuous from  $E$  to  $L(E, F)$  if the latter space is endowed with the strong operator topology). In connection with stochastic equations, the space  $\mathcal{G}^1$  has been introduced in [6], to which we refer the reader for further properties.

Given a real and separable Hilbert space  $K$  and a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a filtration  $\mathcal{F}_t$ , we consider the following classes of stochastic processes.

1.  $L_{\mathcal{F}}^p(\Omega, \mathcal{C}([0, T]; K))$ ,  $p \in [1, \infty)$ ,  $T > 0$ , is the space of predictable processes  $Y$  with continuous paths on  $[0, T]$  such that

$$|Y|_{L_{\mathcal{F}}^p(\Omega, \mathcal{C}([0, T]; K))} = \mathbb{E} \sup_{t \in [0, T]} |Y_t|_K^p < \infty.$$

2.  $L_{\mathcal{F}}^p(\Omega, L^2([0, T]; K))$ ,  $p \in [1, \infty)$ ,  $T > 0$ , is the space of predictable processes  $Y$  on  $[0, T]$  such that

$$|Y|_{L_{\mathcal{F}}^p(\Omega, L^2([0, T]; K))} = \mathbb{E} \left( \int_0^T |Y_t|_K^2 dt \right)^{p/2} < \infty.$$

3.  $L_{\mathcal{F}, \text{loc}}^2(\Omega, L^2([0, \infty); K))$  is the space of predictable processes  $Y$  on  $[0, \infty)$  which belong to the space  $L_{\mathcal{F}}^2(\Omega, L^2([0, T]; K))$  for every  $T > 0$ .

4.  $L_{\mathcal{F}}^{p, \mu}(\Omega, L^2([0, T]; K))$  is the space of predictable processes  $Y$  on  $[0, T]$  with values in  $K$  such that

$$|Y|_{L_{\mathcal{F}}^{p, \mu}(\Omega, L^2([0, T]; K))} = \mathbb{E} \left( \int_0^T e^{\alpha t} |Y_t|_K^2 dt \right)^{p/2} < \infty.$$

In the following, we consider a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a cylindrical Wiener process denoted by  $(W_t)_{t \geq 0}$  with values in  $\Xi$ , a real and separable Hilbert space.  $\mathcal{F}_t$  denotes the natural filtration generated by  $(W_s)_{s \leq t}$  and augmented with  $\mathbb{P}$ -null sets of  $\mathcal{F}$ .  $H$  denotes a real and separable Hilbert space in which the SDE will take values.

### 3 Preliminaries

We will need some result about the solution of the SDE when a perturbation term  $F$  is in the drift.

#### 3.1 The perturbed forward SDE

Let us consider the following mild stochastic differential equation for an unknown process  $(X_t)_{t \geq 0}$  with values in  $H$  :

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}F(s, X_s)ds + \int_0^t e^{(t-s)A}GdW_s, \quad \forall t \geq 0, \quad \mathbb{P} - a.s. \quad (3.1)$$

Let us introduce the following hypothesis::

**Hypothesis 3.1.** 1.  $A$  is an unbounded operator  $A : D(A) \subset H \rightarrow H$ , with  $D(A)$  dense in  $H$ . We assume that  $A$  is dissipative and generates a stable  $C_0$ -semigroup  $\{e^{tA}\}_{t \geq 0}$ . By this we mean that there exist a constant  $\eta > 0$  and  $M > 0$  such that

$$\langle Ax, x \rangle \leq -\eta|x|^2, \quad \forall x \in D(A); \quad |e^{tA}|_{L(H,H)} \leq Me^{-\eta t}, \quad \forall t \geq 0.$$

2. For all  $s > 0$ ,  $e^{sA}$  is a Hilbert-Schmidt operator. Moreover  $|e^{sA}|_{L_2(H,H)} \leq Ms^{-\gamma}$  and  $\gamma \in [0, 1/2)$ .
3.  $F : \mathbb{R}_+ \times H \rightarrow H$  is bounded and measurable.
4.  $G$  is an operator in  $L(\Xi, H)$ .
5.  $G$  is invertible. We denote by  $G^{-1}$  its bounded inverse (Banach's Theorem).

**Remark 3.1.** Note that under the previous set of hypotheses, we immediately get that :

$$\begin{aligned} |e^{sA}G|_{L_2(\Xi,H)}^2 &\leq |G|_{L(\Xi,H)}^2 |e^{sA}|_{L_2(H,H)}^2 \\ &\leq |G|_{L(\Xi,H)}^2 |e^{s/2A}|_{L(H,H)}^2 |e^{s/2A}|_{L_2(H,H)}^2 \\ &\leq |G|_{L(\Xi,H)}^2 e^{-\eta s} \left(\frac{s}{2}\right)^{-2\gamma}, \end{aligned}$$

which shows that for every  $s > 0$  and  $x \in H$ ,  $e^{sA}G \in L_2(\Xi, H)$ .

**Lemma 3.2.** Assume that Hypothesis 3.1 (only points (1.)-(4.)) hold and that  $F$  is bounded and Lipschitz in  $x$ . Then for every  $p \in [2, \infty)$ , there exists a unique process  $X^x \in L^p_{\mathcal{F}}(\Omega, \mathcal{C}([0, T]; H))$  solution of (3.1). Moreover,

$$\sup_{0 \leq t < +\infty} \mathbb{E}|X_t^x|^p \leq C(1 + |x|)^p, \quad (3.2)$$

for some constant  $C$  depending only on  $p, \gamma, M$  and  $\sup_{t \geq 0} \sup_{x \in H} |F(t, x)|$ .

If  $F$  is only bounded and measurable, then the solution to equation 3.1 still exists but in the martingale sense. By this we mean, see [2], that there exists a new  $\mathcal{F}$ -Wiener process  $(\widehat{W}^x)_{t \geq 0}$  with respect to a new probability measure  $\widehat{\mathbb{P}}$  (absolutely continuous with respect to  $\mathbb{P}$ ), and an  $\mathcal{F}$ -adapted process  $\widehat{X}^x$  with continuous trajectories for which (3.1) holds with  $W$  replaced by  $\widehat{W}$ . Moreover (3.2) still holds (with respect to new probability). Finally such a martingale solution is unique in law.

*Proof.* For the first part of the Lemma, see [6]. The end of the Lemma is a simple consequence of the Girsanov Theorem.  $\square$

We define formally the Kolmogorov semigroup associated to Eq. (3.1) as follow :  $\forall \phi : H \rightarrow \mathbb{R}$

$$\mathcal{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x).$$

**Lemma 3.3** (Basic Coupling Estimates). *Assume that Hypothesis 3.1 and that  $F$  is bounded and Lipschitz. Then there exist  $\hat{c} > 0$  and  $\hat{\eta} > 0$  such that for all  $\phi : H \rightarrow \mathbb{R}$  measurable with polynomial growth (i.e.  $\exists C, \mu > 0$  such that  $\forall x \in H, |\phi(x)| \leq C(1 + |x|^\mu)$ ),  $\forall x, y \in H$ ,*

$$|\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leq \hat{c}(1 + |x|^{1+\mu} + |y|^{1+\mu})e^{-\hat{\eta}t}. \quad (3.3)$$

We stress the fact that  $\hat{c}$  and  $\hat{\eta}$  depend on  $F$  only through  $\sup_{t \geq 0} \sup_{x \in H} |F(t, x)|$ .

*Proof.* In the same manner as in the proof of Theorem 2.4 in [3], we obtain, for every  $x, y \in H$ ,

$$\mathbb{P}(X_t^x \neq X_t^y) \leq \hat{c}(1 + |x|^2 + |y|^2)e^{-\hat{\eta}t}.$$

Now write, for every  $x, y \in H$  and  $\phi : H \rightarrow \mathbb{R}$  measurable and such that  $\forall x \in H, |\phi(x)| \leq C(1 + |x|^\mu)$ ,

$$\begin{aligned} |\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| &\leq \sqrt{\mathbb{E}(|\phi(X_t^x) - \phi(X_t^y)|^2)} \sqrt{\mathbb{P}(X_t^x \neq X_t^y)} \\ &\leq C(1 + |x|^\mu + |y|^\mu)(1 + |x| + |y|)e^{-(\hat{\eta}/2)t} \\ &\leq C(1 + |x|^{1+\mu} + |y|^{1+\mu})e^{-\hat{\eta}t}. \end{aligned}$$

$\square$

**Corollary 3.4.** *Relation (3.3) can be extended to the case in which  $F$  is only bounded measurable and for all  $t \geq 0$ , there exists a uniformly bounded sequence of Lipschitz functions in  $x$  ( $F_n(t, \cdot)_{n \geq 1}$ ) (i.e.  $\forall t \geq 0, \forall n \in \mathbb{N}, F_n(t, \cdot)$  is Lipschitz and  $\sup_n \sup_t \sup_x |F_n(t, x)| < +\infty$ ) such that*

$$\lim_n F_n(t, x) = F(t, x), \quad \forall t \geq 0, \forall x \in H.$$

Clearly in this case in the definition of  $\mathcal{P}_t[\phi]$  the mean value is taken with respect to the new probability  $\hat{\mathbb{P}}$ .

*Proof.* See Corollary 2.5 in [3].  $\square$

We will need to apply the lemma above to some functions with particular form.

**Lemma 3.5.** *Let  $\zeta, \zeta' : \mathbb{R}_+ \times H \rightarrow \Xi^*$  such that for all  $s \geq 0$ ,  $\zeta(s, \cdot)$  and  $\zeta'(s, \cdot)$  are weakly\* continuous with polynomial growth. We define*

$$\Upsilon(x) = \begin{cases} \frac{\psi(x, \zeta(s, x)) - \psi(x, \zeta'(s, x))}{|\zeta(s, x) - \zeta'(s, x)|^2} (\zeta(s, x) - \zeta'(s, x)), & \text{if } \zeta(s, x) \neq \zeta'(s, x), \\ 0, & \text{if } \zeta(s, x) = \zeta'(s, x). \end{cases}$$

*There exists a uniformly bounded sequence of Lipschitz functions ( $\Upsilon_n(s, \cdot)_{n \geq 1}$ ) (i.e.  $\forall n, \Upsilon_n(s, \cdot)$  is Lipschitz and  $\sup_s \sup_n \sup_x |\Upsilon_n(s, x)| < \infty$ ) such that*

$$\lim_n \Upsilon_n(s, x) = \Upsilon(s, x), \quad \forall s \geq 0, \forall x \in H.$$

*Proof.* See the proof of Lemma 3.5 in [3].  $\square$

### 3.2 The BSDE

Let us fix  $T > 0$  and let us consider the following BSDE in finite horizon for an unknown process  $(Y_s^{T,t,x}, Z_s^{T,t,x})_{s \in [t,T]}$  with values in  $\mathbb{R} \times \Xi^*$  :

$$Y_s^{T,t,x} = g(X_T^x) + \int_t^T f(X_s^{t,x}, Z_s^{T,t,x}) ds - \int_t^T Z_s^{T,t,x} dW_s, \quad \forall s \in [t, T], \quad (3.4)$$

where  $(X_t^{t,x})_{t \geq 0}$  is the mild solution of (3.1) starting from  $x$  at time  $t > 0$ . If  $t = 0$ , we use the following standard notations  $Y_s^{T,x} := Y_s^{T,0,x}$  and  $Z_s^{T,x} := Z_s^{T,0,x}$ .

We assume the following :

**Hypothesis 3.2.** There exist  $l > 0$ ,  $\mu \geq 0$  such that the function  $f : H \times \Xi^* \rightarrow \mathbb{R}$  and  $g : H \rightarrow \mathbb{R}$  satisfy :

1.  $F : H \rightarrow H$  is a Lipschitz, bounded and belongs to the class  $\mathcal{G}^1$ ,
2.  $g(\cdot)$  is continuous and have polynomial growth : for all  $x \in H$ ,  $|g(x)| \leq C(1 + |x|^\mu)$ ,
3.  $\forall x \in H, \forall z, z' \in \Xi^*, |f(x, z) - f(x, z')| \leq l|z - z'|$ ,
4.  $f(\cdot, z)$  is continuous and  $\forall x \in H, |f(x, 0)| \leq C(1 + |x|^\mu)$ .

**Lemma 3.6.** Assume that Hypotheses (3.1) and (3.2) hold true then there exists a unique solution  $(Y^{T,t,x}, Z^{T,t,x}) \in L^p_{\mathcal{P}}(\Omega, \mathcal{C}([0, T]; \mathbb{R})) \times L^p_{\mathcal{P}}(\Omega, L^2([0, T]; \Xi^*))$  for all  $p \geq 2$  to the BSDE (3.4).

*Proof.* See [6], Proposition 4.3. □

We immediately recall the link between solution of such BSDEs and PDEs which will justify our probabilistic approach. For that purpose we start by recalling the concept of mild solution. We consider the HJB equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) + f(x, \nabla u(t,x)G) = 0, & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u(T,x) = g(x), & \forall x \in H, \end{cases} \quad (3.5)$$

where  $\mathcal{L}u(t,x) = \frac{1}{2} \text{Tr}(GG^* \nabla^2 u(t,x)) + \langle Ax + F(x), \nabla u(t,x) \rangle$ . We can define the semigroup  $(\mathcal{P}_t)_{t \geq 0}$  corresponding to  $X$  by the formula  $\mathcal{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x)$  for all measurable functions  $\phi : H \rightarrow \mathbb{R}$  having polynomial growth, and we notice that  $\mathcal{L}$  is the formal generator of  $\mathcal{P}_t$ . We give the definition of a mild solution of equation (3.5):

**Definition 3.7.** We say that a continuous function  $u : [0, T] \times H \rightarrow \mathbb{R}$  is a mild solution of the HJB equation (3.5) if the following conditions hold:

1.  $u \in \mathcal{G}^{0,1}([0, T] \times H, \mathbb{R})$
2. for all  $x \in H, h \in H, t \in [0, T]$  we have

$$|u(t,x)| \leq C(1 + |x|^C), \quad |\nabla u(t,x)h| \leq C|h|k(t)(1 + |x|^C),$$

for some constant  $C > 0$ , and some real function  $k$  satisfying  $\int_0^T k(t) dt < +\infty$ ;

the following equality holds:

$$u(t,x) = \mathcal{P}_{T-t}[g](x) + \int_t^T \mathcal{P}_{s-t}[f(\cdot, \nabla u(t, \cdot)G)](x) ds, \quad \forall t \in [0, T], \quad \forall x \in H.$$

**Lemma 3.8.** *Assume that Hypotheses (3.1) and (3.2) hold true then there exists a unique mild solution of  $u$  of the HJB equation (3.5) given by the formula*

$$u_T(t, x) = Y_t^{T, t, x}.$$

*Proof.* See Theorem 4.2 in [5]. □

**Remark 3.9.** By the following change of time:  $\tilde{u}_T(t, x) := u_T(T - t, x)$ , we remark that  $\tilde{u}_T(t, x)$  is the unique mild solution of (1.1). Now, remark that  $\tilde{u}_T(T, x) = u_T(0, x) = Y_0^{T, 0, x} = Y_0^{T, x}$ , therefore knowing the large time behaviour of  $Y_0^{T, x}$  is knowing the large time behaviour of the solution of equation (1.1).

### 3.3 The EBSDE

Let us consider the following ergodic BSDE for an unknown process  $(Y_t, Z_t, \lambda)_{t \geq 0}$  with values in  $\mathbb{R} \times \Xi^* \times \mathbb{R}$  :

$$Y_t^x = Y_T^x + \int_t^T (f(X_s^x, Z_s^x) - \lambda) ds - \int_t^T Z_s^x dW_s, \quad \forall T > 0, \forall t \in [0, T]. \quad (3.6)$$

**Hypothesis 3.3.** There exist  $l > 0, \mu \geq 0$  such that the functions  $F : H \rightarrow H$  and  $f : H \times \Xi^* \rightarrow \mathbb{R}$  satisfy :

1.  $F : H \rightarrow H$  is a Lipschitz bounded function and belongs to the class  $\mathcal{G}^1$ ,
2.  $\forall x \in H, \forall z, z' \in \Xi^*, |f(x, z) - f(x, z')| \leq l|z - z'|$ ,
3.  $f(\cdot, z)$  is continuous and  $\forall x \in H, |f(x, 0)| \leq C(1 + |x|^\mu)$ .

**Lemma 3.10** (Existence). *Assume that Hypotheses (3.1) and (3.3) hold true then there exists a solution  $(Y^x, Z^x, \lambda) \in L^2_{\mathcal{F}}(\Omega, \mathcal{C}([0, \infty[; \mathbb{R})) \times L^2_{\mathcal{F}}(\Omega, L^2([0, \infty[; \Xi^*)) \times \mathbb{R}$ , for all  $T > 0$  to the EBSDE (3.6). Moreover there exists  $v : H \rightarrow \mathbb{R}$  of class  $\mathcal{G}^1$  such that, for all  $x, x' \in H$ , for all  $t \geq 0$  :*

$$\begin{aligned} Y_t^x &= v(X_t^x) \quad \text{and} \quad Z_t^x = \nabla v(X_t^x)G, \\ v(0) &= 0, \\ |v(x) - v(x')| &\leq C(1 + |x|^{1+\mu} + |x'|^{1+\mu}), \\ |\nabla v(x)| &\leq C(1 + |x|^{1+\mu}). \end{aligned}$$

*Proof.* This can be proved in the same way as in [3]. We give the full proof in Appendix for reader convenience. □

**Lemma 3.11** (Uniqueness). *The solution  $(Y^x, Z^x, \lambda)$  of previous Lemma is unique in the class of solutions  $(Y, Z, \lambda)$  such that  $Y = v(X^x)$ ,  $|v(x)| \leq C(1 + |x|^p)$  for some  $p \geq 0$ ,  $v(0) = 0$ ,  $Z \in L^2_{\mathcal{F}, \text{loc}}(\Omega, L^2([0, \infty); \Xi^*))$ , and  $Z = \zeta(X^x)$  where  $\zeta : H \rightarrow \Xi^*$  is continuous for the weak\* topology.*

*Proof.* We give a simpler proof than in [3]. Indeed, let us consider two solutions  $(Y^1 = v^1(X^x), Z^1 = \zeta^1(X^x))$  and  $(Y^2 = v^2(X^x), Z^2 = \zeta^2(X^x), \lambda^2)$ . From Theorem 3.10 in [3], we get  $\lambda^1 = \lambda^2$ .



Then, we have

$$\begin{aligned}
v^1(x) - v^2(x) &= v^1(X_T^x) - v^2(X_T^x) + \int_0^T (f(X_s^x, Z_s^1) - f(X_s^x, Z_s^2)) ds - \int_t^T (Z_s^1 - Z_s^2) dW_s \\
&= v^1(X_T^x) - v^2(X_T^x) + \int_0^T (Z_s^1 - Z_s^2) \frac{(f(X_s^x, Z_s^1) - f(X_s^x, Z_s^2))(Z_s^1 - Z_s^2)^*}{|Z_s^1 - Z_s^2|^2} ds \\
&\quad - \int_0^T (Z_s^1 - Z_s^2) dW_s
\end{aligned}$$

Then we define

$$\beta(x) = \begin{cases} \frac{(f(x, \zeta^1(x)) - f(x, \zeta^2(x)))(\zeta^1(x) - \zeta^2(x))^*}{|\zeta^1(x) - \zeta^2(x)|^2}, & \text{if } \zeta^1(x) - \zeta^2(x) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

As  $\beta(X_s^x)$  is measurable, and bounded, one can apply Girsanov's theorem to deduce the existence of a new probability  $\mathbb{Q}$  under which  $\widetilde{W}_t = W_t - \int_0^t \beta_s ds$  is a Wiener process. Then

$$\begin{aligned}
v^1(x) - v^2(x) &= \mathbb{E}^{\mathbb{Q}}[v^1(X_T^x) - v^2(X_T^x)] \\
&= \mathbb{E}[v^1(U_T^x) - v^2(U_T^x)]
\end{aligned}$$

where  $U^x$  is the mild solution of

$$\begin{cases} dU_t^x = AU_t^x dt + F(U_t^x) dt + G\beta(U_t^x) dt + Gd\widetilde{W}_t, & t \geq 0, \\ U_0^x = x. \end{cases}$$

Now, remark that  $\beta$  satisfies hypotheses of Lemma 3.5, therefore,

$$\begin{aligned}
(v^1 - v^2)(x) - ((v^1 - v^2)(0)) &= \mathbb{E}[(v^1 - v^2)(U_T^x) - (v^1 - v^2)(U_T^0)] \\
&\leq C(1 + |x|^{p+1})e^{-\hat{\eta}T}.
\end{aligned}$$

where the last inequality is a consequence of Lemma 3.4. Then, letting  $T \rightarrow +\infty$  and noting that  $(v^1 - v^2)(0) = 0$  lead us to

$$v^1(x) = v^2(x), \quad \forall x \in H.$$

An Itô's formula applied to  $|Y_t^1 - Y_t^2|^2$  is enough to show that for all  $T > 0$

$$\mathbb{E} \int_0^T |Z_s^1 - Z_s^2|^2 ds = 0,$$

which concludes the proof of uniqueness.  $\square$

Similarly to the case of BSDE, we recall the link between solutions of such EBSDEs and ergodic HJB equations. We consider the following ergodic HJB equation for an unknown pair  $(v(\cdot), \lambda)$ ,

$$\mathcal{L}v(x) + f(x, \nabla v(x)G) - \lambda = 0, \quad \forall x \in H. \quad (3.7)$$

Since we are dealing with an elliptic equation it is natural to consider  $(v, \lambda)$  as mild solution of equation (3.7) if and only if, for arbitrary time  $T > 0$ ,  $v(x)$  coincides with the mild solution  $u(t, x)$  of the corresponding parabolic equation having  $v$  as a terminal condition:

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} + \mathcal{L}u(t, x) + f(x, \nabla u(t, x)G) - \lambda = 0, & \forall t \in [0, T], \quad \forall x \in H, \\ u(T, x) = v(x), & \forall x \in H. \end{cases}$$

Thus we are led to the following definition:

**Definition 3.12.** A pair  $(v, \lambda)$ ,  $(v : H \rightarrow \mathbb{R}$  and  $\lambda \in \mathbb{R})$  is a mild solution of the HJB equation (3.7) if the following are satisfied:

1.  $v \in \mathcal{G}^1(H, \mathbb{R})$ ;
2. there exists  $C > 0$  such that  $|\nabla v(x)| \leq C(1 + |x|^C)$  for every  $x \in H$ ;
3. for all  $0 \leq t \leq T$  and  $x \in H$ ,

$$v(x) = \mathcal{P}_{T-t}[v](x) + \int_t^T (\mathcal{P}_{s-t}[f(\cdot, \nabla v(\cdot)G)](x) - \lambda) ds$$

We recall the following result

**Lemma 3.13.** Assume that Hypotheses (3.1) and (3.3) hold true. Then equation (3.7) admits a unique mild solution which is the pair  $(v, \lambda)$  defined in Lemma 3.10.

*Proof.* See Theorem 4.1 in [3]. □

## 4 Large time behaviour of solutions

### 4.1 First behaviour

**Theorem 4.1.** Assume that Hypotheses 3.1, 3.2 and 3.3 hold true. Then,  $\forall T \geq 0$  :

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leq \frac{C(1 + |x|^{1+\mu})}{T}. \quad (4.1)$$

In particular,

$$\frac{Y_0^{T,x}}{T} \xrightarrow{T \rightarrow +\infty} \lambda,$$

uniformly in any bounded set of  $H$ .

*Proof.* For all  $x \in H$ ,  $T > 0$ :

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leq \left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| + \left| \frac{Y_0^x}{T} \right|.$$

We have :

$$\begin{aligned} Y_0^{T,x} - Y_0^x - \lambda T &= \xi^T - v(X_T^x) + \int_0^T f(X_s^x, Z_s^{T,x}) - f(X_s^x, Z_s^x) ds - \int_0^T Z_s^{T,x} - Z_s^x dW_s \\ &= \xi^T - v(X_T^x) + \int_0^T (Z_s^{T,x} - Z_s) \beta_s^T ds - \int_0^T Z_s^{T,x} - Z_s dW_s, \end{aligned}$$

where

$$\beta_s^T = \begin{cases} \frac{(f(X_s^x, Z_s^{T,x}) - f(X_s^x, Z_s^x))(Z_s^{T,x} - Z_s^x)^*}{|Z_s^{T,x} - Z_s^x|^2}, & \text{if } Z_s^{T,x} - Z_s^x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

The process  $\beta_s^T$  is progressively measurable and bounded by  $l$  therefore we can apply Girsanov Theorem to obtain that there exist a probability measure  $\mathbb{Q}^T$  under which  $\widetilde{W}_t^T = -\int_0^t \beta_s^T ds + W_t$  is a Wiener process. We recall that if we define  $M_T = \exp\left(\int_0^T \beta_s^T dW_s - \frac{1}{2} \int_0^T |\beta_s^T|^2 ds\right)$ , the following formula holds :  $d\mathbb{Q}^T = M_T d\mathbb{P}$ .

Taking the expectation with respect to  $\mathbb{Q}^T$  we get

$$Y_0^{T,x} - Y_0^x - \lambda T = \mathbb{E}^{\mathbb{Q}^T} (g(X_T^x) - v(X_T^x)). \quad (4.2)$$

Thus

$$\left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| \leq C \frac{1 + \mathbb{E}^{\mathbb{Q}^T} [C(1 + |X_T^x|^\mu)]}{T} + C \frac{1 + \mathbb{E}^{\mathbb{Q}^T} (|X_T^x|^{1+\mu})}{T}.$$

The process  $(X_t^x)_{t \geq 0}$  is the mild solution of

$$\begin{cases} dX_t^x = AX_t^x dt + F(X_t^x) dt + G\beta_t^T \mathbf{1}_{t < T} dt + G d\widetilde{W}_t^T, & t \geq 0, \\ X_0^x = x. \end{cases}$$

Thus, by Lemma (3.2), there exists a constant  $C$  which does not depend on time such that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^T} (|X_T^x|^\mu) &\leq C(1 + |x|^\mu) \\ \mathbb{E}^{\mathbb{Q}^T} (|X_T^x|^{1+\mu}) &\leq C(1 + |x|^{1+\mu}) \end{aligned}$$

therefore

$$\left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| \leq \frac{C(1 + |x|^{1+\mu})}{T}. \quad (4.3)$$

Finally note that

$$\left| \frac{Y_0^x}{T} \right| \leq \frac{C(1 + |x|^{1+\mu})}{T},$$

which gives the result.  $\square$

**Remark 4.2.** If  $G$  is possibly degenerate, Theorem 4.1 remains true under additional assumptions that  $f$  is Lipschitz in  $x$  and that  $A + F$  is dissipative. In this case, we have existence of solution to the EBSDE and  $\lambda$  is unique from [4].

**Remark 4.3.** From a deterministic point of view, the result is as follow:

$$\left| \frac{u(T, x)}{T} - \lambda \right| \leq \frac{C(1 + |x|^{1+\mu})}{T},$$

where  $u(T, x)$  is the mild solution of (1.1).

## 4.2 Second behaviour

We replace Hypothesis 3.2 by the following. Note that it is almost the same but we write it fully for reader convenience.

**Hypothesis 4.1.** There exist  $l > 0$ ,  $\mu \geq 0$  such that the function  $f : H \times \Xi^* \rightarrow \mathbb{R}$  and  $g : H \rightarrow \mathbb{R}$  satisfy :

1.  $F \equiv 0$ ,
2.  $g(\cdot)$  is continuous and have polynomial growth : there exists  $\mu > 0$  such that  $|g(x)| \leq C(1 + |x|^\mu)$ , for all  $x \in H$ ,
3.  $\forall x \in H, \forall z, z' \in \Xi^*, |f(x, z) - f(x, z')| \leq l|z - z'|$ ,
4.  $f(\cdot, z)$  is continuous and of polynomial growth, i.e.  $\forall x \in H, |f(x, 0)| \leq C(1 + |x|^\mu)$ .

**Remark 4.4.** Note that setting  $F \equiv 0$  is not restrictive. Indeed let us recall that the purpose of this paper is to study the large time behaviour of the mild solution of

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \mathcal{L}u(t, x) + f(x, \nabla u(t, x)G), & \forall (t, x) \in \mathbb{R}_+ \times H, \\ u(0, x) = g(x), & \forall x \in H. \end{cases}$$

Now remark that

$$\langle Ax + F(x), \nabla u(t, x) \rangle + f(x, \nabla u(t, x)G) = \langle Ax, \nabla u(t, x) \rangle + \tilde{f}(x, \nabla u(t, x)G),$$

where  $\tilde{f}(x, z) = f(x, z) + \langle F(x), zG^{-1} \rangle$  is Lipschitz in continuous on  $H \times \Xi^*$  and of polynomial growth in  $x$  and Lipschitz in  $z$ . Therefore, under our assumptions, we can always consider that  $F \equiv 0$  by replacing  $f$  by  $\tilde{f}$  if necessary.

**Theorem 4.5.** Assume that Hypotheses 3.1, 3.3 and 4.1 hold true. Then there exists  $L \in \mathbb{R}$  such that,

$$\forall x \in H, \quad Y_0^{T, x} - \lambda T - Y_0^x \xrightarrow{T \rightarrow +\infty} L.$$

Furthermore the following speed of convergence holds

$$|Y_0^{T, x} - \lambda T - v(x) - L| \leq C(1 + |x|^{1+\mu})e^{-\hat{\eta}T}.$$

*Proof.* Now we define

$$\begin{aligned} u_T(t, x) &:= Y_t^{T, t, x} \\ w_T(t, x) &:= u_T(t, x) - \lambda(T - t) - v(x). \end{aligned}$$

We recall that  $Y_s^{T, t, x} = u_T(s, X_s^{t, x})$  and that  $Y_s^x = v(X_s^x)$ , where  $v$  is defined in the Lemma 3.6. Remark that taking  $t = 0$  implies  $Y_s^{T, 0, x} = Y_s^{T, x}$ . We recall that for all  $T, S > 0$ ,  $u_T$  is the unique mild solution of

$$\begin{cases} \frac{\partial u_T(t, x)}{\partial t} + \mathcal{L}u_T(t, x) + f(x, \nabla u_T(t, x)G) = 0, & \forall (t, x) \in \mathbb{R}_+ \times H, \\ u_T(T, x) = g(x), & \forall x \in H, \end{cases}$$

and that  $u_{T+S}$  is the unique mild solution of

$$\begin{cases} \frac{\partial u_{T+S}(t, x)}{\partial t} + \mathcal{L}u_{T+S}(t, x) + f(x, \nabla u_{T+S}(t, x)G) = 0, & \forall (t, x) \in \mathbb{R}_+ \times H, \\ u_{T+S}(T + S, x) = g(x), & \forall x \in H, \end{cases}$$

This implies, for all  $x \in H$ ,

$$u_T(0, x) = u_{T+S}(S, x).$$

and then,

$$w_T(0, x) = w_{T+S}(S, x). \quad (4.4)$$

We are going to need some estimates on  $w_T$  given in the following Lemma.

**Lemma 4.6.** *Under the hypothesis of the Theorem 4.5, there exists  $C > 0$  such that  $\forall x, y \in H, \forall T > 0$ ,*

$$\begin{aligned} |w_T(0, x)| &\leq C(1 + |x|^{1+\mu}), \\ |\nabla_x w_T(0, x)| &\leq C(T')(1 + |x|^{1+\mu}) \frac{1 + \sqrt{T'}}{\sqrt{T'}}, \quad \forall 0 < T' \leq T \\ |w_T(0, x) - w_T(0, y)| &\leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}. \end{aligned}$$

*Proof of Lemma 4.6 .* The first inequality of the Lemma is a direct application of estimate (4.3). Indeed,  $\forall x \in H, \forall T > 0$ ,

$$\begin{aligned} |w_T(0, x)| &= |u_T(0, x) - \lambda T - v(x)| \\ &= |Y_0^{T,x} - Y_0^x - \lambda T| \\ &\leq C(1 + |x|^{1+\mu}), \end{aligned} \quad (4.5)$$

which gives the first inequality of Lemma.

Now, let us establish the gradient estimate. First remark that the inequality (4.5) still holds for  $w_T(t, x)$ , i.e.  $\forall x \in H, \forall T > t$ ,

$$|w_T(t, x)| \leq C(1 + |x|^{1+\mu}). \quad (4.6)$$

The process  $(w_T(s, X_s^{t,x}))_{t \leq s \leq T}$  satisfies the following equation, for all  $t \leq s \leq T$ ,

$$\begin{aligned} w_T(s, X_s^{t,x}) &= w_T(T, X_T^{t,x}) + \int_s^T (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_s^{t,x}, Z_s^{t,x}))ds \\ &\quad - \int_s^T (Z_r^{T,t,x} - Z_r^{t,x})dW_r. \end{aligned}$$

Now remark that for all  $t < T$  and  $t \leq s \leq T' \leq T$  the following equation hold by uniqueness of solutions,

$$\begin{aligned} w_T(s, X_s^{t,x}) &= w_T(T', X_{T'}^{t,x}) + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x}))dr \\ &\quad - \int_s^{T'} (Z_r^{T,t,x} - Z_r^{t,x})dW_r \\ &= w_{T-T'}(0, X_{T'}^{t,x}) + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x} - Z_r^x + Z_r^x) - f(X_r^{t,x}, Z_r^{t,x}))dr \\ &\quad - \int_t^{T'} (Z_r^{T,t,x} - Z_r^x)dW_r. \end{aligned}$$

where we have used the equality (4.4) for the second line.

And we recall that (see [5] Theorem 4.2 and [3] Theorem 3.8),  $\forall x, h \in H, \forall s \in [t, T]$

$$\begin{aligned} Z_s^{T,t,x} &= \nabla_x u_T(s, X_s^{t,x})G, \\ Z_s^x &= \nabla_x v(X_s^{t,x})G. \end{aligned}$$

Then :

$$Z_r^{T,t,x} - Z_r^{t,x} = \nabla_x w_T(r, X_r^{t,x})G.$$

Thus applying the Bismut-Elworthy formula (see [5], Theorem 4.2),  $\forall x, h \in H, \forall t < T$  :

$$\begin{aligned} \nabla_x w_T(t, x)h &= \mathbb{E} \int_t^T [f(X_s^{t,x}, \nabla_x w_T(r, X_r^{t,x})G - Z_s^{t,x}) - f(X_s^{t,x}, Z_s^{t,x})] U^h(s, t, x) ds \\ &\quad + \mathbb{E} [[w_{T-T'}](0, X_{T'}^{t,x})] U^h(T', t, x), \end{aligned}$$

where,  $\forall 0 \leq s \leq T, \forall x \in H$ ,

$$U^h(s, t, x) = \frac{1}{s-t} \int_t^s \langle G^{-1} \nabla_x X_u^{t,x} h, d\widetilde{W}_u \rangle.$$

Let us recall that

$$\nabla_x X_s^{t,x} h = e^{(s-t)A} h,$$

then,

$$\begin{aligned} \mathbb{E} |U^h(s, t, x)|^2 &= \frac{1}{|s-t|^2} \int_t^s |G^{-1} \nabla_x X(u, t, x) h|^2 du \\ &\leq \frac{C|h|^2}{s-t}, \end{aligned}$$

where  $C$  is independant on  $t, s$  and  $x$ .

Then,  $\forall x, h \in H, \forall t < T$ , using inequality (4.6)

$$|\nabla_x w_T(t, x)h| \leq C \int_t^{T'} \frac{\sqrt{\mathbb{E}(|\nabla_x w_T(s, X_s^{t,x})|^2)} |h|}{\sqrt{s-t}} ds + C \frac{(1 + |x|^{1+\mu}) |h|}{\sqrt{T'-t}}.$$

We define

$$\varphi(t) = \sup_x \frac{|\nabla_x w_T(t, x)|}{(1 + |x|^{1+\mu})},$$

and we remark that  $\varphi(t)$  is well defined for all  $t < T$ . Indeed  $\nabla_x w_T(t, x) = \nabla u_T(t, x) - \nabla v(x)$  and  $\nabla u_T(t, x) \leq C_T (T-t)^{-1/2} (1 + |x|^\mu)$  (see Theorem 4.2 in [6]) and  $\nabla v(x) \leq C(1 + |x|^{1+\mu})$  (see Theorem 3.8 in [3]). Then

$$|\nabla_x w_T(t, x)h| \leq C \int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}} \sqrt{\mathbb{E}((1 + |X_s^{t,x}|^{1+\mu})^2)} |h| ds + C \frac{(1 + |x|^{1+\mu}) |h|}{\sqrt{T'-t}}$$

which leads to

$$\frac{|\nabla_x w_T(t, x)h|}{(1 + |x|^{1+\mu}) |h|} \leq C \int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}} ds + \frac{C}{\sqrt{T'-t}}.$$

Taking the supremum over  $h$  and  $x$ , we have

$$\varphi(t) \leq C \int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}} ds + \frac{C}{\sqrt{T'-t}}.$$

Now remark that, we can rewrite the above inequality as follow

$$\varphi(T' - t) \leq C \int_0^t \frac{\varphi(T' - s)}{\sqrt{t-s}} ds + \frac{C}{\sqrt{t}}.$$

Then by Lemma 7.1.1 in [8] :

$$\varphi(T' - t) \leq \frac{C}{\sqrt{t}} + C\theta \int_0^t E'(\theta(t-s)) \frac{1}{\sqrt{s}} ds$$

where

$$\theta = (C\Gamma(1/2))^2, \quad E(z) = \sum_0^{\infty} \frac{z^{n/2}}{\Gamma(n/2+1)}.$$

Therefore, taking  $t = T'$  leads us to

$$\varphi(0) \leq \frac{C}{\sqrt{T'}} + C\theta \int_0^{T'} E'(\theta(T'-s)) \frac{1}{\sqrt{s}} ds$$

which implies

$$|\nabla_x w_T(0, x)| \leq C(T')(1 + |x|^{1+\mu}) \frac{1 + \sqrt{T'}}{\sqrt{T'}}.$$

For the third inequality of Lemma, we have by equation (4.2),  $\forall x \in H, \forall T > 0$  :

$$\begin{aligned} w_T(0, x) &= \mathbb{E}^{\mathbb{Q}^T} (g(X_T^x) - v(X_T^x)) \\ &= \mathbb{E}(g(U_T^x) - v(U_T^x)), \end{aligned}$$

where  $U^x$  is the mild solution of the following equation defined  $\forall t \in \mathbb{R}$  :

$$dU_t^x = [AU_t^x + G\beta^T(t, U_t^x)\mathbb{1}_{t < T}]dt + GdW_t, \quad U_0^x = x,$$

and where  $\beta^T(t, x) =$

$$\begin{cases} \frac{(f(x, \nabla u_T(t, x)G) - f(x, \nabla v(x)G))(\nabla u_T(t, x)G - \nabla v(x)G)^*}{|(\nabla u_T(t, x) - \nabla v(x))G|^2} \mathbb{1}_{t < T}, & \text{if } \nabla u_T(t, x) - \nabla v(x) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,  $\forall x \in H \forall T > 0$  :

$$|w_T(0, x) - w_T(0, y)| = |\mathbb{E}(g(U_T^x) + v(U_T^x)) - \mathbb{E}(g(U_T^y) + v(U_T^y))|$$

Then, as  $\beta^T$  is uniformly bounded in  $t$  and  $x$ , by Lemmas 3.5 and 3.4, one can apply the Corollary 3.4, we obtain, since  $(g(\cdot) + v(\cdot))$  have polynomial growth of order  $1 + \mu$  :

$$|w_T(0, x) - w_T(0, y)| \leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}, \quad (4.7)$$

which conclude the proof of the Lemma.  $\square$

Now, let us come back to the proof of the Theorem. The first estimate of Lemma 4.6 allow us to construct, by a diagonal procedure, a sequence  $(T_i)_i \nearrow +\infty$  such that for a function  $w$  defined on a countable dense subset of  $H$  (denoted by  $D$ ) :  $(w : D \rightarrow \mathbb{R})$  the following hold

$$\forall x \in D, \lim_{i \rightarrow +\infty} w_{T_i}(0, x) = w(x).$$

Then, it is possible to extend this function to the whole  $H$  thanks to the second inequality of Lemma 4.6.

Now, let us show that  $w : H \rightarrow \mathbb{R}$  is constant. We have, by the third inequality of Lemma 4.6, for all  $x, y \in H$  and  $T > 0$ ,

$$|w_T(0, x) - w_T(0, y)| \leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}.$$

Applying the previous inequality with  $T = T_i$  and taking the limit in  $i$  show us that  $x \mapsto w(x)$  is a constant function, namely there exists  $L_1 \in \mathbb{R}$  (independent of  $x$ ) such that :  $\forall x \in H$

$$\lim_i w_{T_i}(0, x) = L_1.$$

Now remark that  $\{w_T(0, \cdot); T > 1\}$  is a relatively compact subspace of the space of continuous functions  $H \rightarrow \mathbb{R}$  for the uniform distance thanks to the two first inequalities of Lemma 4.6. Therefore, if we show that  $\{w_T(0, \cdot); T > 1\}$  admits only one accumulation point, it will implies that

$$\lim_{T \rightarrow +\infty} w_T(0, x) = L_1.$$

Now we claim that the accumulation point is unique. Let us assume that there exists another subsequence  $(T'_i)_{i \in \mathbb{N}}$  such that  $\lim_i w_{T'_i}(0, x) = L_2$  (note that it must be a constant by the third inequality of Lemma 4.6).

Let us write,  $\forall x \in H, \forall T, S > 0$  :

$$\begin{aligned} w_{T+S}(0, x) &= Y_0^{T+S, x} - \lambda(T+S) - Y_0^x \\ &= Y_S^{T+S, x} - \lambda T - Y_S^x + \int_0^S (f(X_r^x, Z_r^{T+S, x}) - f(X_r^x, Z_r^x)) dr \\ &\quad - \int_0^S (Z_r^{T+S, x} - Z_r^x) dW_r \\ &= Y_S^{T+S, x} - \lambda T - Y_S^x + \int_0^S (Z_r^{T+S, x} - Z_r^x) d\widetilde{W}_r, \end{aligned}$$

$$\widetilde{W}_t^{T+S} = - \int_0^t \beta^{T+S}(s, X_s^x) ds + W_t.$$

where  $\beta^{T+S}(t, x) =$

$$\begin{cases} \frac{(f(x, \nabla u_{T+S}(t, x)G) - f(x, \nabla v(x)G))((\nabla u_{T+S}(t, x) - \nabla v(x))G)^*}{|(\nabla u_{T+S}(t, x) - \nabla v(x))G|^2} \mathbf{1}_{t < T+S}, & \text{if } \nabla u_{T+S}(t, x) - \nabla v(x) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$



Taking the expectation with respect to the probability  $\mathbb{Q}^{T+S}$  under which  $\widetilde{W}^{T+S}$  is a Brownian motion we get (using equality (4.4) for the third line):

$$\begin{aligned}
w_{T+S}(0, x) &= \mathbb{E}^{\mathbb{Q}^{T+S}}(Y_S^{T+S, x} - \lambda T - Y_S^x) \\
&= \mathbb{E}^{\mathbb{Q}^{T+S}}(w_{T+S}(S, X_S^x)) \\
&= \mathbb{E}^{\mathbb{Q}^{T+S}}(w_T(0, X_S^x)) \\
&= \mathbb{E}(w_T(0, U_S^x))
\end{aligned} \tag{4.8}$$

where  $U^x$  is the mild solution of the following equation defined  $\forall t \in \mathbb{R}_+$  :

$$dU_t^x = [AU_t^x + \beta^{T+S}(t, U_t^x)]dt + GdW_t, \quad U_0^x = x.$$

This implies, substituting  $T$  by  $T'_i$  and  $S$  by  $T_i - T'_i$ , (up to a subsequence for  $(T_i)_{i \in \mathbb{N}}$  such that  $T_i > T'_i$ )

$$w_{T_i}(0, x) = \mathbb{E}(w_{T'_i}(0, U_{T_i - T'_i}^x)).$$

We have:

$$\lim_i w_{T_i}(0, x) = L_1.$$

Furthermore,

$$\begin{aligned}
|w_{T_i}(0, U_{T_i - T'_i}^x) - L_2| &\leq |w_{T'_i}(0, U_{T_i - T'_i}^x) - w_{T'_i}(0, x)| + |w_{T'_i}(0, x) - L_2| \\
&\leq C(1 + |x|^{2+\mu} + |U_{T_i - T'_i}^x|^{2+\mu})e^{-\hat{\eta}T'_i} + |w_{T'_i}(0, x) - L_2|,
\end{aligned}$$

thanks to the third inequality of Lemma 4.6. Now remark that,  $\forall \varepsilon > 0$ ,

$$\begin{aligned}
\mathbb{P}(|U_{T_i - T'_i}^x|^{2+\mu} e^{-\hat{\eta}T'_i} > \varepsilon) &\leq \mathbb{E}[|U_{T_i - T'_i}^x|^{2+\mu} e^{-\hat{\eta}T'_i}] / \varepsilon \\
&\leq C(1 + |x|^{2+\mu}) \frac{e^{-\hat{\eta}T'_i}}{\varepsilon},
\end{aligned}$$

which shows that

$$w_{T'_i}(0, U_{T_i - T'_i}) \xrightarrow{i \rightarrow +\infty} L_2, \quad \text{in probability.}$$

Furthermore,  $w_{T'_i}(0, U_{T_i - T'_i})$  is uniformly integrable, indeed

$$\begin{aligned}
\mathbb{E}(|w_{T'_i}(0, U_{T_i - T'_i})|^2) &\leq C\mathbb{E}(1 + |U_{T_i - T'_i}|^{4+2\mu}) \\
&\leq C
\end{aligned}$$

which is independent on  $i$ . Then,

$$\lim_i \mathbb{E}(w_{T'_i}(0, U_{T_i - T'_i})) = L_2.$$

Therefore  $L_1 = L_2$ , which as mentioned before, implies that

$$\lim_{T \rightarrow +\infty} w_T(0, x) = L_1.$$

Now we prove that this convergence holds with an explicit speed of convergence. Let us write,  $\forall x \in H, \forall T > 0$ ,

$$\begin{aligned} |w_T(0, x) - L| &= \lim_{V \rightarrow +\infty} |w_T(0, x) - w_V(0, x)| \\ &= \lim_{V \rightarrow +\infty} |w_T(0, x) - \mathbb{E}(w_T(0, U_{V-T}^x))| \end{aligned}$$

thanks to equality (4.8), where  $U^x$  is the mild solution of the following equation defined  $\forall t \in \mathbb{R}_+$  :

$$dU_t^x = [AU_t^x + \beta^V(t, U_t^x)]dt + GdW_t, \quad U_0^x = x.$$

Now, thanks to the third estimate in Lemma 4.6, one have,

$$\begin{aligned} |w_T(0, x) - L| &\leq \lim_{V \rightarrow +\infty} C\mathbb{E} (1 + |x|^{2+\mu} + |U_{V-T}^x|^{2+\mu}) e^{-\hat{\eta}T} \\ &\leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T}. \end{aligned}$$

□

**Remark 4.7.** From a deterministic point of view, the result is as follow:

$$|u(T, x) - \lambda T - v(x) - L| \leq C(1 + |x|^{2+\mu}),$$

where  $u(T, x)$  is the mild solution of (1.1) and  $v(x)$  is the mild solution of (3.7).

## 5 Application to Ergodic control problem

In this section, we show how we can apply our results to an ergodic control problem. In this section we will still assume that Hypotheses 3.1, 3.3 and 4.1 hold true. We denote by  $X^x$  the mild solution of 3.1. Assume that Hypotheses 3.1, 3.3 and 4.1 hold. Let  $U$  be a separable metric space. We define a control  $u$  as an  $(\mathcal{F}_t)$ -predictable  $U$ -valued process. We will assume the following

**Hypothesis 5.1.** The functions  $R : U \rightarrow \Xi$ ,  $L : H \times U \rightarrow \mathbb{R}$  and  $g_0 : H \rightarrow \mathbb{R}$  are measurable and satisfy, for some constant  $c > 0$  such that,

1.  $|R(a)| \leq c$ , for all  $a \in U$ ,
2.  $L(\cdot, a)$  is continuous in  $x$  uniformly in  $a \in U$ . Furthermore  $|L(x, a)| \leq C(1 + |x|^\mu)$  for all  $x \in H$  and all  $a \in U$ ,
3.  $g_0(\cdot)$  is continuous and  $|g_0(x)| \leq C(1 + |x|^\mu)$  for all  $x \in H$ .

Let us denote by  $X_t^{x,a}$  the solution (unique in law) of the following SDE

$$\begin{cases} dX_t^{x,a} = (AX_t^{x,a} + F(X_t^{x,a})dt + R(a_t)dt + GdW_t, & \forall t \geq 0, \\ X_0^{x,a} = x. \end{cases}$$

We consider two costs. The first one is the cost in finite horizon :

$$J^T(x, a) := E \int_0^T L(X_s^x, a_s)ds + \mathbb{E}g_0(X_T^x).$$

The second one is called the ergodic cost and is the timed average finite horizon cost:

$$J(x, a) := \limsup_{T \rightarrow +\infty} \frac{1}{T} E \int_0^T L(X_s^x, a_s) ds.$$

We want to show how our results can be applied to such an optimisation problem to get an asymptotic expansion of the finite horizon cost involving the ergodic cost.

To apply our results, we first define the Hamiltonian in the usual way

$$f_0(x, z) = \inf_{a \in U} \{L(x, a) + zG^{-1}R(a)\} \quad (5.1)$$

and we remark that, if for all  $x, z$ , the infimum is attained in 5.1 then by the Filippov Theorem, see [10], there exists a measurable function  $\gamma : H \times \Xi^* \rightarrow U$  such that

$$f_0(x, z) = L(x, \gamma(x, z)) + zG^{-1}R(\gamma(x, z)).$$

**Lemma 5.1.** *Under the above assumptions, the Hamiltonian  $f_0$  satisfies Hypotheses on  $f$  in 3.2, 3.3 or 4.1.*

*Proof.* We first prove that  $f_0$  is continuous in  $x$ . Clearly,  $f_0$  is upper semicontinuous in  $x$ . To prove that  $f_0$  is also lower semicontinuous, let  $x_n \rightarrow x$  and take  $a_n$  such that  $L(x_n, a_n) + zG^{-1}R(a_n) \leq f_0(x, a_n) + 1/n$ , then:

$$L(x, a_n) + zG^{-1}R(a_n) \leq f_0(x_n, z) + 1/n + [L(x, a_n) - L(x_n, a_n)]$$

and, since  $L$  is continuous in  $x$  uniformly in  $a \in U$ ,

$$f_0(x, z) \leq \liminf_n f_0(x_n, z).$$

Consequently the following hold (by upper semicontinuity for the equality)

$$f_0(x, z) \leq \liminf_n f_0(x_n, z) \leq \limsup_n f_0(x_n, z) = f_0(x, z),$$

which prove the continuity of  $f_0$  in  $x$ .

It is also clear that  $|f_0(x, 0)| \leq |L(x, a)| \leq C(1 + |x|^\mu)$ .

Moreover, for all  $z_1, z_2 \in \Xi^*$ ,

$$\begin{aligned} f_0(x, z_1) &\leq L(x, a) + z_2 G^{-1}R(a) + z_1 G^{-1}R(a) - z_2 G^{-1}R(a) \\ &\leq L(x, a) + z_2 G^{-1}R(a) + c|z_1 - z_2| \end{aligned}$$

taking the infimum over  $a \in U$ , we obtain

$$f_0(x, z_1) - f_0(x, z_2) \leq c|z_1 - z_2|,$$

and exchanging  $z_1$  with  $z_2$  we get that

$$|f_0(x, z_1) - f_0(x, z_2)| \leq c|z_1 - z_2|.$$

□

We recall the following results about finite horizon cost:

**Lemma 5.2.** *Assume that Hypotheses 3.1 and 5.1 hold true, then for arbitrary control  $a$ ,*

$$J^T(x, a) \geq u(T, x),$$

where  $u(t, x)$  is the mild solution of

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \mathcal{L}u(t, x) + f_0(x, \nabla u(t, x)G), & \forall (t, x) \in \mathbb{R}_+ \times H, \\ u(0, x) = g_0(x), & \forall x \in H, \end{cases}$$

Furthermore, if for all  $x, z$  the infimum is attained in 5.1 then we have the equality:

$$J^T(x, \bar{a}^T) = u(T, x),$$

where  $\bar{a}_t^T = \gamma(X_t^{x, \bar{a}^T}, \nabla u(t, X_t^{x, \bar{a}^T})G)$ .

*Proof.* See Theorem 5.3 in [5]. □

Similarly, for the ergodic cost

**Lemma 5.3.** *Assume that Hypotheses 3.1 and 5.1 hold true, then for arbitrary control  $a$ ,*

$$J(x, a) \geq \lambda,$$

where  $(v, \lambda)$  is the mild solution of

$$\mathcal{L}v(x) + f_0(x, \nabla v(x)G) - \lambda = 0, \quad \forall x \in H.$$

Furthermore, if for all  $x, z$  the infimum is attained in 5.1 then we have the equality:

$$J(x, \bar{a}) = \lambda,$$

where  $\bar{a}_t = \gamma(X_t^{x, \bar{a}_t}, \nabla v(X_t^{x, \bar{a}_t})G)$ .

Finally, we apply our result in the following Theorem:

**Theorem 5.4.** *Assume that Hypotheses 3.1 and 5.1 hold true then, for any control  $a$ ,*

$$\lim_{T \rightarrow +\infty} \frac{J^T(x, \bar{a}^T)}{\lambda T + v(x) + L} \geq 1.$$

Furthermore, if the infimum is attained in 5.1 then

$$J^T(x, \bar{a}^T) \underset{T \rightarrow +\infty}{\sim} J(x, \bar{a})T + v(x) + L.$$

*Proof.* The proof is a straightforward consequence of the two previous Lemmas above and the application of Theorem 4.5. □

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