



HAL
open science

Large time behaviour of mild solutions of Hamilton-Jacobi-Bellman equations in infinite dimension by a probabilistic approach

Ying Hu, Pierre-Yves Madec, Adrien Richou

► **To cite this version:**

Ying Hu, Pierre-Yves Madec, Adrien Richou. Large time behaviour of mild solutions of Hamilton-Jacobi-Bellman equations in infinite dimension by a probabilistic approach. 2014. hal-01006336v1

HAL Id: hal-01006336

<https://hal.science/hal-01006336v1>

Preprint submitted on 16 Jun 2014 (v1), last revised 15 Jan 2015 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Large time behaviour of mild solutions of Hamilton-Jacobi-Bellman equations in infinite dimension by a probabilistic approach

Ying Hu* Pierre-Yves Madec† Adrien Richou‡

June 15, 2014

Abstract

We study the large time behaviour of mild solutions of HJB equations in infinite dimension by a purely probabilistic approach. For that purpose, we show that the solution of a BSDE in finite horizon T taken at initial time behaves like a linear term in T shifted with the solution of the associated EBSDE taken at initial time. Moreover we give an explicit speed of convergence, which seems to appear very rarely in literature.

1 Introduction

We are concerned with the large time behaviour of solutions of the Cauchy problem in an infinite dimensional real Hilbert space H :

$$\begin{cases} u_t(t, x) = \mathcal{L}u(t, x) + f(x, \nabla u(t, x)G), & \forall (t, x) \in \mathbb{R}_+ \times H, \\ u(0, x) = g(x), & \forall x \in H, \end{cases}$$

where $u : \mathbb{R}_+ \times H \rightarrow \mathbb{R}$ is the unknown function and \mathcal{L} is the formal generator of the Kolmogorov semigroup \mathcal{P}_t of an H -valued random process solution of the following Ornstein-Uhlenbeck stochastic differential equation :

$$\begin{cases} dX_t = AX_t dt + GdW_t, & t \in \mathbb{R}_+, \\ X_0 = x, & x \in H, \end{cases}$$

where W is a Wiener process with values in another real Hilbert space Ξ , assumed to be separable. We recall that (formally), $\forall h : H \rightarrow \mathbb{R}$,

$$(\mathcal{L}h)(x) = \frac{1}{2} \text{Tr}(GG^* \nabla^2 h(x)) + \langle Ax, \nabla h(x) \rangle .$$

Our method is purely probabilistic, which can be described as follows.

First, let (v, λ) be the solution of the ergodic PDE:

$$\mathcal{L}v + f(x, \nabla v(x)G) = 0, \quad \forall x \in H.$$

*IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France (ying.hu@univ-rennes1.fr)

†IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France (pierre-yves.madec@univ-rennes1.fr)

‡Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France. CNRS, IMB, UMR 5251, F-33400 Talence, France. INRIA, Equipe ALEA, F-33400 Talence, France. (adrien.richou@math.u-bordeaux1.fr)

Then we have the following probabilistic representation. Let $(Y^{T,x}, Z^{T,x})$ be solution of the BSDE:

$$\begin{cases} dY_s^{T,x} = -f(X_s^x, Z_s^{T,x})ds + Z_s^{T,x}dW_s \\ Y_T^{T,x} = g(X_T^x), \end{cases}$$

and (Y, Z, λ) be solution of the EBSDE:

$$dY_s = -(f(X_s^x, Z_s^x) - \lambda)ds + Z_s^x dW_s.$$

Then

$$\begin{cases} Y_s^{T,x} = u(T, X_s^x), \\ Y_s^x = v(X_s^x). \end{cases}$$

Finally, due to Girsanov transformations and the use of an important coupling estimate result, we deduce

$$Y_0^{T,x} - \lambda T - Y_0^x \rightarrow L,$$

i.e.

$$u(T, x) - \lambda T - v(x) \rightarrow L.$$

Our method is not only purely probabilistic, but also gives a speed of convergence :

$$|u(T, x) - \lambda T - v(x) - L| \leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T}.$$

The constant μ appearing above is the polynomial growth power of $g(\cdot)$ and $f(\cdot, 0)$ and $\hat{\eta}$ is linked to the dissipative constant of A .

Large time behaviour of solutions has been studied for various types of HJB equations of second order; see, e.g., [1], [6], [8] and [9]. In [1], a result in finite dimension is stated under periodic assumptions for f and a periodic and Lipschitz assumption for g . Furthermore, they assume that $f(x, z)$ is of linear growth in z and bounded in x . In [6], some results are stated in finite dimensional framework, under locally Hölder conditions for the coefficients. More precisely, they assume that $f(x, z) = H_1(z) - H_2(x)$ with H_1 Lipschitz and under locally Hölder conditions for H_2 and g . They also treat the case of H_1 locally Lipschitz but consequently need to assume that H_2 and g are Lipschitz. Furthermore, they only treat the Laplacian case, namely they assume that $G = I_d$. No result on speed of convergence is given in this paper. In [8], the authors deal with the problem in finite dimension. They also only treat the Laplacian case and assume that $f(x, z)$ is convex and of quadratic growth in z and of polynomial growth in x . No result on speed of convergence is given in this paper. As far as we know, the explicit speed convergence only appears in [9], but in finite dimension and under periodic assumptions for $f(\cdot, z)$ and $g(\cdot)$. Furthermore, they assume that linear growth property in z for f with derivatives in z of first and second order uniformly bounded.

Roughly speaking, we will assume that $G : H \rightarrow H$ is invertible and bounded operator, $g : H \rightarrow \mathbb{R}$ continuous with polynomial growth and $f : H \times \Xi^*$ continuous, with polynomial growth in the first variable and Lipschitz in the second variable.

2 Notations

We introduce some notations; let E, F be real separable Hilbert spaces. The norm and the scalar product will be denoted by $|\cdot|$, $\langle \cdot, \cdot \rangle$, with subscripts if needed. $L(E, F)$ is the space of

linear bounded operators $E \rightarrow F$, with the operator norm, which is denoted by $|\cdot|_{L(E,F)}$. The domain of a linear (unbounded) operator A is denoted by $D(A)$. $L_2(E, F)$ denotes the space of Hilbert-Schmidt operators from E to F , endowed with the Hilbert-Schmidt norm, which is denoted by $|\cdot|_{L_2(E,F)}$.

Given $\phi \in B_b(E)$, the space of bounded and measurable functions $\phi : E \rightarrow \mathbb{R}$, we denote by $\|\phi\|_0 = \sup_{x \in E} |\phi(x)|$.

We say that a function $F : E \rightarrow F$ belongs to the class $\mathcal{G}^1(E, F)$ if it is continuous, has a Gâteaux differential $\nabla F(x) \in L(E, F)$ at any point $x \in E$, and for every $k \in E$, the mapping $x \mapsto \nabla F(x)k$ is continuous from E to F (i.e. $x \mapsto \nabla F(x)$) is continuous from E to $L(E, F)$ if the latter space is endowed with the strong operator topology). In connection with stochastic equations, the space \mathcal{G}^1 has been introduced in [5], to which we refer the reader for further properties.

Given a real and separable Hilbert space K and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration \mathcal{F}_t , we consider the following classes of stochastic processes.

1. $L^p_{\mathcal{P}}(\Omega, \mathcal{C}([0, T]; K))$, $p \in [1, \infty)$, $T > 0$, is the space of predictable processes Y with continuous paths on $[0, T]$ such that

$$\|Y\|_{L^p_{\mathcal{P}}(\Omega, \mathcal{C}([0, T]; K))} = \mathbb{E} \sup_{t \in [0, T]} |Y_t|_K^p < \infty.$$

2. $L^p_{\mathcal{P}}(\Omega, L^2([0, T]; K))$, $p \in [1, \infty)$, $T > 0$, is the space of predictable processes Y on $[0, T]$ such that

$$\|Y\|_{L^p_{\mathcal{P}}(\Omega, L^2([0, T]; K))} = \mathbb{E} \left(\int_0^T |Y_t|_K^2 dt \right)^{p/2} < \infty.$$

3. $L^2_{\mathcal{P}, \text{loc}}(\Omega, L^2([0, \infty); K))$ is the space of predictable processes Y on $[0, \infty)$ which belong to the space $L^2_{\mathcal{P}}(\Omega, L^2([0, T]; K))$ for every $T > 0$.

In the following, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a cylindrical Wiener process denoted by $(W_t)_{t \geq 0}$ with values in Ξ , a real and separable Hilbert space. \mathcal{F}_t denotes the natural filtration generated by $(W_s)_{s \leq t}$ and augmented with \mathbb{P} -null sets of \mathcal{F} . H denotes a real and separable Hilbert space in which the SDE will take values.

3 Preliminaries

We will need some result about the solution of the SDE when a perturbation term F is in the drift.

3.1 The perturbed forward SDE

Let us consider the following mild stochastic differential equation for an unknown process $(X_t)_{t \geq 0}$ with values in H :

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}F(s, X_s)ds + \int_0^t e^{(t-s)A}G(X_s^x)dW_s, \quad \forall t \geq 0, \quad \mathbb{P} - a.s. \quad (3.1)$$

Let us introduce the following hypothesis for the non-degenerate case:

Hypothesis 3.1. 1. A is an unbounded operator $A : D(A) \subset H \rightarrow H$, with $D(A)$ dense in H . We assume that A is dissipative and generates a stable C_0 -semigroup $\{e^{tA}\}_{t \geq 0}$. By this we mean that there exist a constant $\eta > 0$ and $M > 0$ such that

$$\langle Ax, x \rangle \leq -\eta|x|^2, \quad \forall x \in D(A); \quad |e^{tA}|_{L(H,H)} \leq Me^{-\eta t}, \quad \forall t \geq 0.$$

2. For all $s > 0$, e^{sA} is a Hilbert-Schmidt operator. Moreover $|e^{sA}|_{L_2(H,H)} \leq Ms^{-\gamma}$ and $\gamma \in [0, 1/2)$.
3. $F : \mathbb{R}_+ \times H \rightarrow H$ is bounded and measurable.
4. G is an operator in $L(\Xi, H)$.
5. G is invertible. We denote by G^{-1} its bounded inverse (Banach's Theorem).

Remark 3.1. Note that under the previous set of hypotheses, we immediately get that :

$$\begin{aligned} |e^{sA}G|_{L_2(\Xi,H)}^2 &\leq |G|_{L(\Xi,H)}^2 |e^{sA}|_{L_2(H,H)}^2 \\ &\leq |G|_{L(\Xi,H)}^2 e^{s/2A} e^{s/2A} |_{L_2(H,H)}^2 \\ &\leq |G|_{L(\Xi,H)}^2 e^{-\eta s} \left(\frac{s}{2}\right)^{-2\gamma}, \end{aligned}$$

which shows that for every $s > 0$ and $x \in H$, $e^{sA}G \in L_2(H, H)$.

Lemma 3.2. Assume that Hypothesis 3.1 (only points (1.)-(4.)) hold and that F is bounded and Lipschitz in x . Then for every $p \in [2, \infty)$, there exists a unique process $X^x \in L^p_{\mathcal{F}}(\Omega, \mathcal{C}([0, T]; H))$ solution of (3.1). Moreover,

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_t^x|^p \leq C(1 + |x|)^p, \quad (3.2)$$

for some constant C depending only on p, γ, M and $\sup_{t \geq 0} \sup_{x \in H} |F(t, x)|$ but independent of $T > 0$.

If F is only bounded and measurable, then the solution to equation 3.1 still exists but in the martingale sense. By this we mean, see [2], that there exists a new \mathcal{F} -Wiener process $(\widehat{W}^x)_{t \geq 0}$ with respect to a new probability measure $\widehat{\mathbb{P}}$ (absolutely continuous with respect to \mathbb{P}), and an \mathcal{F} -adapted process \widehat{X}^x with continuous trajectories for which (3.1) holds with W replaced by \widehat{W} . Moreover (3.2) still holds (with respect to new probability). Finally such a martingale solution is unique in law.

Proof. For the first part of the Lemma, see [5]. The end of the Lemma is a simple consequence of the Girsanov Theorem. \square

We define formally the Kolmogorov semigroup associated to Eq. (3.1) as follow : $\forall \phi : H \rightarrow \mathbb{R}$

$$\mathcal{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x).$$

Lemma 3.3 (Basic Coupling Estimates). Assume that Hypothesis 3.1 and that F is bounded and Lipschitz. Then there exist $\hat{c} > 0$ and $\hat{\eta} > 0$ such that for all $\phi : H \rightarrow \mathbb{R}$ measurable with polynomial growth (i.e. $\exists C, \mu > 0$ such that $\forall x \in H, |\phi(x)| \leq C(1 + |x|^\mu)$), $\forall x, y \in H$,

$$|\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leq \hat{c}(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}t}. \quad (3.3)$$

We stress the fact that \hat{c} and $\hat{\eta}$ depend on F only through $\sup_{t \geq 0} \sup_{x \in H} |F(t, x)|$.

Proof. In the same manner as in the proof of Theorem 2.4 in [3], we obtain, for every $x, y \in H$,

$$\mathbb{P}(X_t^x \neq X_t^y) \leq \hat{c}(1 + |x|^2 + |y|^2)e^{-\hat{\eta}t}.$$

Now write, for every $x, y \in H$ and $\phi : H \rightarrow \mathbb{R}$ measurable and such that $\forall x \in H, |\phi(x)| \leq C(1 + |x|^\mu)$

$$\begin{aligned} |\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| &\leq \sqrt{\mathbb{E}(|\phi(X_t^x) - \phi(X_t^y)|^2)}\mathbb{P}(X_t^x \neq X_t^y) \\ &\leq C(1 + |x|^\mu + |y|^\mu)(1 + |x|^2 + |y|^2)e^{-\hat{\eta}t} \\ &\leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}t}. \end{aligned}$$

□

Corollary 3.4. *Relation (3.3) can be extended to the case in which F is only bounded measurable and for all $t \geq 0$, there exists a uniformly bounded sequence of Lipschitz functions in $(F_n(t, \cdot))_{n \geq 1}$ (i.e. $\forall t \geq 0, \forall n \in \mathbb{N}, F_n(t, \cdot)$ is Lipschitz and $\sup_n \sup_t \sup_x |F_n(t, x)| < +\infty$) such that*

$$\lim_n F_n(t, x) = F(t, x), \quad \forall t \geq 0, \forall x \in H.$$

Clearly in this case in the definition of $\mathcal{P}_t[\phi]$ the mean value is taken with respect to the new probability $\hat{\mathbb{P}}$.

Proof. See Corollary 2.5 in [3].

□

We will need to apply the lemma above to some functions with particular form.

Lemma 3.5. *Let $\zeta, \zeta' : \mathbb{R}_+ \times H \rightarrow \Xi^*$ such that for all $s \geq 0$, $\zeta(s, \cdot)$ and $\zeta'(s, \cdot)$ are weakly* continuous with polynomial growth. We define*

$$\Upsilon(x) = \begin{cases} \frac{\psi(x, \zeta(s, x)) - \psi(x, \zeta'(s, x))}{|\zeta(s, x) - \zeta'(s, x)|^2} (\zeta(s, x) - \zeta'(s, x)), & \text{if } \zeta(s, x) \neq \zeta'(s, x), \\ 0, & \text{if } \zeta(s, x) = \zeta'(s, x). \end{cases}$$

There exists a uniformly bounded sequence of Lipschitz functions $(\Upsilon_n(s, \cdot))_{n \geq 1}$ (i.e. $\forall n, \Upsilon_n(s, \cdot)$ is Lipschitz and $\sup_s \sup_n \sup_x |\Upsilon_n(s, x)| < \infty$) such that

$$\lim_n \Upsilon_n(s, x) = \Upsilon(s, x), \quad \forall s \geq 0, \forall x \in H.$$

Proof. See the proof of Lemma 3.5 in [3].

□

3.2 The BSDE

Let us fix $T > 0$ and let us consider the following BSDE in finite horizon for an unknown process $(Y_t^{T,x}, Z_t^{T,x})_{t \geq 0}$ with values in $\mathbb{R} \times \Xi^*$:

$$Y_t^{T,x} = \xi^T + \int_t^T f(X_s^x, Z_s^{T,x}) ds - \int_t^T Z_s^{T,x} dW_s, \quad \forall t \in [0, T], \quad (3.4)$$

where $(X_t^x)_{t \geq 0}$ is the solution of (3.1).

We assume the following :

Hypothesis 3.2. There exist $l > 0$, $\mu \geq 0$ such that the function $f : H \times \Xi^* \rightarrow \mathbb{R}$ and ξ^T satisfy :

1. $F : H \rightarrow H$ is a Lipschitz, bounded and belongs to the class \mathcal{G}^1 ,
2. ξ^T is a H valued random variable \mathcal{F}_T measurable and there exists $p \geq 0$ such that $|\xi^T| \leq C(1 + \sup_{0 \leq t \leq T} |X_t^x|^p)$,
3. $\forall x \in H, \forall z, z' \in \Xi^*, |f(x, z) - f(x, z')| \leq l|z - z'|$,
4. $\forall x \in H, |f(x, 0)| \leq C(1 + |x|^\mu)$.

Lemma 3.6. Assume that Hypotheses (3.1) and (3.2) hold true then there exists a unique solution in $L^2_{\mathcal{F}}(\Omega, \mathcal{C}([0, T]; H)) \times L^2_{\mathcal{F}}(\Omega, L^2([0, T]; \Xi^*))$ for all $p \geq 2$ to the BSDE (3.4).

See [5], Proposition 4.3.

3.3 The EBSDE

Let us consider the following ergodic BSDE (EBSDE for short in the what follows) for an unknown process $(Y_t, Z_t, \lambda)_{t \geq 0}$ with values in $\mathbb{R} \times \Xi^* \times \mathbb{R}$:

$$Y_t^x = Y_T^x + \int_t^T f(X_s^x, Z_s^x) ds - \int_t^T Z_s^x dW_s, \quad \forall T > 0, \forall t \in [0, T]. \quad (3.5)$$

Hypothesis 3.3. There exist $l > 0$, $\mu \geq 0$ such that the functions $F : H \rightarrow H$ and $f : H \times \Xi^* \rightarrow \mathbb{R}$ satisfy :

1. $F : H \rightarrow H$ is a Lipschitz bounded function and belongs to the class \mathcal{G}^1 ,
2. $\forall x \in H, \forall z, z' \in \Xi^*, |f(x, z) - f(x, z')| \leq l|z - z'|$,
3. $\forall x \in H, |f(x, 0)| \leq C(1 + |x|^\mu)$.

Lemma 3.7 (Existence). Assume that Hypotheses (3.1) and (3.3) hold true then there exists a solution $(Y^x, Z^x, \lambda) \in L^2_{\mathcal{F}}(\Omega, \mathcal{C}([0, T]; H)) \times L^2_{\mathcal{F}}(\Omega, L^2([0, T]; \Xi^*)) \times \mathbb{R}$, for all $T > 0$ to the EBSDE (3.5). Moreover there exists $v : H \rightarrow \mathbb{R}$ of class \mathcal{G}^1 such that, for all $x, x' \in H$, for all $t \geq 0$:

$$\begin{aligned} Y_t^x &= v(X_t^x) \quad \text{and} \quad Z_t^x = \nabla v(X_t^x)G, \\ v(0) &= 0, \\ |v(x) - v(x')| &\leq C(1 + |x|^{2+\mu} + |x'|^{2+\mu}), \\ |\nabla v(x)| &\leq C(1 + |x|^{2+\mu}). \end{aligned}$$

Proof. This can be proved in the same way as in [3]. □

In the following, we denote by $(Y^{T,x}, Z^{T,x})$ the solution of (3.4) given by Lemma 3.6 and by (Y^x, Z^x, λ) the solution of (3.4) given by Lemma 3.7.

4 Large time behaviour of solutions

4.1 Path-dependent framework

Theorem 4.1. *Assume that Hypotheses 3.1, 3.2 and 3.3 hold true. Then, $\forall T \geq 0$:*

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leq \frac{C(1 + |x|^{2+\mu})}{T}. \quad (4.1)$$

In particular,

$$\frac{Y_0^{T,x}}{T} \xrightarrow{T \rightarrow +\infty} \lambda,$$

uniformly in all compact set of H .

Proof. For all $x \in H$, $T > 0$:

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leq \left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| + \left| \frac{Y_0^x}{T} \right|.$$

We have :

$$\begin{aligned} Y_0^{T,x} - Y_0^x - \lambda T &= \xi^T - v(X_T^x) + \int_0^T f(X_s^x, Z_s^{T,x}) - f(X_s^x, Z_s^x) ds - \int_0^T Z_s^{T,x} - Z_s^x dW_s \\ &= \xi^T - v(X_T^x) + \int_0^T (Z_s^{T,x} - Z_s) \beta_s^T ds - \int_0^T Z_s^{T,x} - Z_s dW_s, \end{aligned}$$

where

$$\beta_s^T = \begin{cases} \frac{(f(X_s^x, Z_s^{T,x}) - f(X_s^x, Z_s^x))(Z_s^{T,x} - Z_s^x)^*}{|Z_s^{T,x} - Z_s^x|^2}, & \text{if } Z_s^{T,x} - Z_s^x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

The process β_s^T is progressively measurable and bounded by l therefore we can apply Girsanov Theorem to obtain that there exist a probability measure \mathbb{Q}^T under which $\widetilde{W}_t^T = -\int_0^t \beta_s^T ds + W_t$ is a Wiener process. We recall that if we define $M_T = \exp\left(\int_0^T \beta_s^T dW_s - \frac{1}{2} \int_0^T |\beta_s^T|^2 ds\right)$, the following formula holds : $d\mathbb{Q}^T = M_T d\mathbb{P}$.

Taking the expectation with respect to \mathbb{Q}^T we get

$$Y_0^{T,x} - Y_0^x - \lambda T = \mathbb{E}^{\mathbb{Q}^T}(\xi^T - v(X_T^x)). \quad (4.2)$$

Thus

$$\left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| \leq C \frac{1 + \mathbb{E}^{\mathbb{Q}^T}[\sup_{0 \leq t \leq T} |X_t^x|^\mu]}{T} + C \frac{1 + \mathbb{E}^{\mathbb{Q}^T}(|X_T^x|^{2+\mu})}{T}.$$

The process $(X_t^x)_{t \geq 0}$ is the mild solution of

$$\begin{cases} dX_t^x = AX_t^x dt + F(X_t^x) dt + G\beta_t^T \mathbf{1}_{t < T} dt + Gd\widetilde{W}_t^T, & t \geq 0, \\ X_0^x = x. \end{cases}$$

Thus, by Lemma (3.2), there exists a constant C which does not depend on time such that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^T} \left[\sup_{0 \leq t \leq T} |X_t^x|^\mu \right] &\leq C(1 + |x|^\mu), \\ \mathbb{E}^{\mathbb{Q}^T} \left[\sup_{0 \leq t \leq T} |X_t^x|^2 \right] &\leq C(1 + |x|^{2+\mu}),\end{aligned}$$

therefore

$$\left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| \leq \frac{C(1 + |x|^{2+\mu})}{T}. \quad (4.3)$$

Finally note that

$$\left| \frac{Y_0^x}{T} \right| \leq \frac{C(1 + |x|^{2+\mu})}{T},$$

which gives the result. \square

4.2 Markovian framework

For the Markovian framework, we replace Hypothesis 3.2 by the following

Hypothesis 4.1. There exist $l > 0$, $\mu \geq 0$ such that the function $f : H \times \Xi^* \rightarrow \mathbb{R}$ and ξ^T satisfy :

1. $F \equiv 0$,
2. $\xi^T = g(X_T)$ for a continuous function $g : H \rightarrow H$ with polynomial growth, i.e. there exists $\mu > 0$ such that $|g(x)| \leq C(1 + |x|^\mu)$,
3. $\forall x \in H, \forall z, z' \in \Xi^*, |f(x, z) - f(x, z')| \leq l|z - z'|$,
4. $f : H \times \Xi^* \rightarrow \mathbb{R}$ is continuous and of polynomial growth, i.e. $\forall x \in H, |f(x, 0)| \leq C(1 + |x|^\mu)$.

Theorem 4.2. Assume that Hypotheses 3.1, 3.3 and 4.1 hold true. Then there exists $L \in \mathbb{R}$ such that,

$$\forall x \in H, \quad Y_0^{T,x} - \lambda T - Y_0^x \xrightarrow{T \rightarrow +\infty} L.$$

Furthermore the following speed of convergence holds

$$|Y_0^{T,x} - \lambda T - v(x) - L| \leq C(1 + |x|^2 + |x|^\mu)(1 + |x|^2)e^{-\hat{\eta}T}.$$

Proof. Let us denote by $(X^{t,x}, Y^{T,t,x}, Z^{T,t,x})$ the mild solution of

$$\begin{cases} dX_s = AX_s ds + GdW_s, & s \geq t \\ dY_s = -f(X_s, Z_s) ds + Z_s dW_s, & s \in [t, T] \\ X_t = x \in H, \\ Y_T = g(X_T). \end{cases}$$

Now we define

$$\begin{aligned}u_T(t, x) &:= Y_t^{T,t,x} \\ w_T(t, x) &:= u_T(t, x) - \lambda(T - t) - v(x).\end{aligned}$$

We recall that $Y_s^{T,t,x} = u_T(s, X_s^{t,x})$ and that $Y_s^x = v(X_s^x)$, where v is defined in the Lemma 3.5. Remark that taking $t = 0$ implies $Y_s^{T,0,x} = Y_s^{T,x}$. We recall that for all $T, S > 0$, u_T is the unique mild solution of

$$\begin{cases} \frac{\partial u_T(t,x)}{\partial t} + \mathcal{L}u(t,x) + f(x, \nabla u(t,x)G) = 0, & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u_T(T,x) = g(x), & \forall x \in H, \end{cases}$$

and that u_{T+S} is the unique mild solution of

$$\begin{cases} \frac{\partial u_{T+S}(t,x)}{\partial t} + \mathcal{L}u(t,x) + f(x, \nabla u(t,x)G) = 0, & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u_{T+S}(T+S,x) = g(x), & \forall x \in H, \end{cases}$$

This implies for all $x \in H$,

$$u_T(0,x) = u_{T+S}(S,x).$$

and then,

$$w_T(0,x) = w_{T+S}(S,x) \tag{4.4}$$

We are going to need some estimates on w_T given in the following Lemma.

Lemma 4.3. *Under the hypothesis of the Theorem 4.2, there exists $C > 0$ such that $\forall x, y \in H, \forall T > 0$,*

$$\begin{aligned} |w_T(0,x)| &\leq C(1 + |x|^{2+\mu}), \\ |\nabla_x w_T(0,x)| &\leq \frac{Ce^{C\varepsilon}}{\sqrt{\varepsilon}}(1 + |x|^{2+\mu}), \forall 0 < \varepsilon \leq T, \\ |w_T(0,x) - w_T(0,y)| &\leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}. \end{aligned}$$

Proof of Lemma 4.3. The first inequality of the Lemma is a direct application of estimate (4.3). Indeed, $\forall x \in H, \forall T > 0$,

$$\begin{aligned} |w_T(0,x)| &= |u_T(0,x) - \lambda T - v(x)| \\ &= |Y_0^{T,x} - Y_0^x - \lambda T| \\ &\leq C(1 + |x|^{2+\mu}), \end{aligned} \tag{4.5}$$

which gives the first inequality of Lemma.

Now, let us establish the gradient estimate. First remark that the inequality (4.5) still holds for $w_T(t,x)$, i.e. $\forall x \in H, \forall T > t$,

$$|w_T(t,x)| \leq C(1 + |x|^{2+\mu}). \tag{4.6}$$

The process $(w_T(s, X_s^{t,x}))_{t \leq s \leq T}$ satisfies the following equation, for all $t \leq s \leq T$,

$$\begin{aligned} w_T(s, X_s^{t,x}) &= w_T(T, X_T^{t,x}) + \int_s^T (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_s^{t,x}, Z_s^{t,x}))ds \\ &\quad - \int_s^T (Z_r^{T,t,x} - Z_r^{t,x})dW_r. \end{aligned}$$

Now remark that for all $t < T' \leq T$ the following equation hold by unicity of solutions, $\forall t < s \leq T'$

$$\begin{aligned} w_T(s, X_s^{t,x}) &= w_T(T', X_{T'}^{t,x}) + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x}))dr \\ &\quad - \int_s^{T'} (Z_r^{T,t,x} - Z_r^{t,x})dW_r \\ &= w_{T-T'}(0, X_{T'}^{t,x}) + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x} - Z_r^x + Z_r^x) - f(X_r^{t,x}, Z_r^{t,x}))dr \\ &\quad - \int_t^{T'} (Z_r^{T,t,x} - Z_r^x)dW_r. \end{aligned}$$

where we have used the equality (4.4) for the second line.

And we recall that (see [4] Theorem 4.2 and [3] Theorem 3.8), $\forall x, h \in H, \forall s \in [t, T]$

$$\begin{aligned} Z_s^{T,t,x} &= \nabla_x u_T(s, X_s^{t,x})G, \\ Z_s^x &= \nabla_x v(X_s^{t,x})G. \end{aligned}$$

Then :

$$Z_r^{T,t,x} - Z_r^{t,x} = \nabla_x w_T(r, X_r^{t,x})G.$$

Thus applying the Bismut-Elworthy formula (see [4], Theorem 4.2), $\forall x, h \in H, \forall t < T$:

$$\begin{aligned} \nabla_x w_T(t, x)h &= \mathbb{E} \int_t^T [f(X_s^{t,x}, \nabla_x w_T(r, X_r^{t,x})G - Z_s^{t,x}) - f(X_s^{t,x}, Z_s^{t,x})]U^h(s, t, x)ds \\ &\quad + \mathbb{E} [w_{T-T'}(0, X_{T'}^{t,x})]U^h(T', t, x), \end{aligned}$$

where, $\forall 0 \leq s \leq T, \forall x \in H$,

$$U^h(s, t, x) = \frac{1}{s-t} \int_t^s \langle G^{-1} \nabla_x X_u^{t,x} h, d\widetilde{W}_u \rangle.$$

Let us recall that

$$\nabla_x X_s^{t,x} h = e^{(s-t)A} h,$$

then,

$$\begin{aligned} \mathbb{E}|U^h(s, t, x)|^2 &= \frac{1}{|s-t|^2} \int_t^s |G^{-1} \nabla_x X(u, t, x)h|^2 du \\ &\leq \frac{C|h|^2}{s-t}, \end{aligned}$$

where C is independant on t, s and x .

Then, $\forall x, h \in H, \forall t < T$, using inequality (4.6)

$$|\nabla_x w_T(t, x)h| \leq C \int_t^{T'} \mathbb{E} \frac{|\nabla_x w_T(s, X_s^{t,x})h|}{\sqrt{s-t}} ds + C \frac{(1 + |x|^{2+\mu})|h|}{\sqrt{T'-t}}.$$

We define

$$\varphi(s) = \sup_x \sup_h \frac{|\nabla_x w_T(t, x)h|}{(1 + |x|^{2+\mu})|h|},$$

then

$$|\nabla_x w_T(t, x)h| \leq C \int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}} \mathbb{E}(1 + |X_s^{t,x}|^{2+\mu})|h| ds + C \frac{(1 + |x|^{2+\mu})|h|}{\sqrt{T'-t}}$$

which leads to

$$\frac{|\nabla_x w_T(t, x)h|}{(1 + |x|^{2+\mu})|h|} \leq C \int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}} ds + \frac{C}{\sqrt{T'-t}}.$$

Taking the supremum over x and h , we have

$$\varphi(t) \leq C \int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}} ds + \frac{C}{\sqrt{T'-t}}.$$

Then by Lemma 7.1.1 in [7] :

$$|\nabla_x w_T(t, x)h| \leq C|h| \frac{(1 + |x|^{2+\mu})}{\sqrt{T'-t}} e^{C(T'-t)},$$

which gives us the second estimate of Lemma for $t = 0$:

$$|\nabla_x w_T(0, x)h| \leq C|h| \frac{(1 + |x|^2 + |x|^\mu)}{\sqrt{T'}} e^{CT'}.$$

For the third inequality of Lemma, we have by equation (4.2), $\forall x \in H, \forall T > 0$:

$$\begin{aligned} w_T(0, x) &= \mathbb{E}^{\mathbb{Q}^T}(g(X_T^x) - v(X_T^x)) \\ &= \mathbb{E}(g(U_T^x) - v(U_T^x)), \end{aligned}$$

where U^x is the mild solution of the following equation defined $\forall t \in \mathbb{R}$:

$$dU_t^x = [AU_t^x + F(U_t^x) + \beta_t^T \mathbf{1}_{t < T}] dt + G dW_t, \quad U_0^x = x.$$

Therefore, $\forall x \in H \forall T > 0$:

$$|w_T(0, x) - w_T(0, y)| = |\mathbb{E}(g(U_T^x) + v(U_T^x)) - \mathbb{E}(g(U_T^y) + v(U_T^y))|$$

Remark now that in the Markovian framework, $(\beta_t^T \mathbf{1}_{t \leq T})_{t \geq 0}$ is Markovian. Furthermore β_s^T is uniformly bounded in T by l therefore we can apply the Corollary 3.4, we obtain

$$|w_T(0, x) - w_T(0, y)| \leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}. \quad (4.7)$$

□

Now, let us come back to the proof of the Theorem. The first estimate of Lemma 4.3 allow us to construct, by a diagonal procedure, a sequence $(T_i)_i \nearrow +\infty$ such that for a function w defined on a countable dense subset of H (denoted by D) : $(w : D \rightarrow \mathbb{R})$ the following hold

$$\forall x \in D, \lim_{i \rightarrow +\infty} w_{T_i}(0, x) = w(x).$$

Then, it is possible to extend this function to the whole H thanks to the second inequality of Lemma 4.3.

Now, let us show that $w : H \rightarrow \mathbb{R}$ is constant. We have, by the third inequality of Lemma 4.3, for all $x, y \in H$ and $T > 0$,

$$|w_T(0, x) - w_T(0, y)| \leq C(1 + |x|^2 + |x|^\mu + |y|^2 + |y|^\mu)(1 + |x|^2 + |y|^2)e^{-\hat{\eta}T}.$$

Applying the previous inequality with $T = T_i$ and taking the limit in i show us that $x \mapsto w(x)$ is a constant function, namely there exists $L_1 \in \mathbb{R}$ (independent of x) such that : $\forall x \in H$

$$\lim_i w_{T_i}(0, x) = L_1.$$

Now remark that $\{w_T(0, \cdot); T > 1\}$ is a relatively compact subspace of the space of continuous functions $H \rightarrow \mathbb{R}$ for the uniform distance thanks to the two first inequalities of Lemma 4.3. Therefore, if we show that $\{w_T(0, \cdot); T > 1\}$ admits only one accumulation point, it will implies that

$$\lim_{T \rightarrow +\infty} w_T(0, x) = L_1.$$

Now we claim that the accumulation point is unique. Let us assume that there exists another subsequence $(T'_i)_{i \in \mathbb{N}}$ such that $\lim_i w_{T'_i}(0, x) = L_2$ (note that it must be a constant by the third inequality of Lemma 4.3).

Let us write, $\forall x \in H, \forall T, S > 0$:

$$\begin{aligned} w_{T+S}(0, x) &= Y_0^{T+S, x} - \lambda(T+S) - Y_0^x \\ &= Y_S^{T+S, x} - \lambda T - Y_S^x + \int_0^S (f(X_r^x, Z_r^{T+S, x}) - f(X_r^x, Z_r^x)) dr \\ &\quad - \int_0^S (Z_r^{T+S, x} - Z_r^x) dW_r \\ &= Y_S^{T+S, x} - \lambda T - Y_S^x + \int_0^S (Z_r^{T+S, x} - Z_r^x) d\widetilde{W}_r, \end{aligned}$$

where

$$\beta_s^{T+S} = \begin{cases} \frac{(f(X_s^x, Z_s^{T+S, x}) - f(X_s^x, Z_s^x))(Z_s^{T+S, x} - Z_s^x)^*}{|Z_s^{T+S, x} - Z_s^x|^2}, & \text{if } Z_s^{T+S, x} - Z_s^x \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\widetilde{W}_t^{T+S} = - \int_0^t \beta_s^{T+S} ds + W_t.$$

Taking the expectation with respect to the probability \mathbb{Q}^{T+S} under which \widetilde{W}^{T+S} is a Brownian motion we get (using equality (4.4) for the third line):

$$\begin{aligned} w_{T+S}(0, x) &= \mathbb{E}^{\mathbb{Q}^{T+S}}(Y_S^{T+S, x} - \lambda T - Y_S^x) \\ &= \mathbb{E}^{\mathbb{Q}^{T+S}}(w_{T+S}(S, X_S^x)) \\ &= \mathbb{E}^{\mathbb{Q}^{T+S}}(w_T(0, X_S^x)) \\ &= \mathbb{E}(w_T(0, U_S^x)) \end{aligned} \tag{4.8}$$

where U^x is the mild solution of the following equation defined $\forall t \in \mathbb{R}_+$:

$$dU_t^x = [AU_t^x + F(U_t^x) + \beta_t^{T+S} \mathbf{1}_{t < T+S}]dt + GdW_t, \quad U_0^x = x.$$

This implies, substituting T by T'_i and S by $T_i - T'_i$, (up to a subsequence for $(T_i)_{i \in \mathbb{N}}$ such that $T_i > T'_i$)

$$w_{T_i}(0, x) = \mathbb{E}(w_{T'_i}(0, U_{T_i - T'_i}^x))$$

We have:

$$\lim_i w_{T_i}(0, x) = L_1.$$

Furthermore,

$$\begin{aligned} |w_{T_i}(0, U_{T_i - T'_i}^x) - L_2| &\leq |w_{T'_i}(0, U_{T_i - T'_i}) - w_{T'_i}(0, x)| + |w_{T'_i}(0, x) - L_2| \\ &\leq C(1 + |x|^{2+\mu} + |U_{T_i - T'_i}|^{2+\mu})e^{-\hat{\eta}T'_i} + |w_{T'_i}(0, x) - L_2|, \end{aligned}$$

thanks to the third inequality of Lemma 4.3. Now remark that $\forall p > 0, \forall \varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(|U_{T_i - T'_i}|^p e^{-p\hat{\eta}T'_i} > \varepsilon) &\leq \mathbb{E}[|U_{T_i - T'_i}|^p e^{-p\hat{\eta}T'_i}] / \varepsilon \\ &\leq C \frac{e^{-p\hat{\eta}T'_i}}{\varepsilon}, \end{aligned}$$

which shows that

$$|w_{T'_i}(0, U_{T_i - T'_i}) - L_2| \xrightarrow{i \rightarrow +\infty} 0, \quad \mathbb{P} - a.s.$$

Then, by the dominated convergence theorem :

$$\lim_i \mathbb{E}(w_{T'_i}(0, U_{T_i - T'_i})) = L_2.$$

Therefore $L_1 = L_2$, which as mentioned before, implies that

$$\lim_{T \rightarrow +\infty} w_T(0, x) = L_1.$$

Now we prove that this convergence holds with an explicit speed of convergence. Let us write, $\forall x \in H, \forall T > 0$,

$$\begin{aligned} |w_T(0, x) - L| &= \lim_{V \rightarrow +\infty} |w_T(0, x) - w_V(0, x)| \\ &= \lim_{V \rightarrow +\infty} |w_T(0, x) - \mathbb{E}(w_T(0, U_{V-T}^x))| \end{aligned}$$

thanks to equality (4.8), where U^x is the mild solution of the following equation defined $\forall t \in \mathbb{R}_+$:

$$dU_t^x = [AU_t^x + F(U_t^x) + \beta_t^V \mathbf{1}_{t < V}]dt + GdW_t, \quad U_0^x = x.$$

Now, thanks to the third estimate in Lemma 4.3, one have,

$$\begin{aligned} |w_T(0, x) - L| &\leq \lim_{V \rightarrow +\infty} C \mathbb{E} (1 + |x|^{2+\mu} + |U_{V-T}^x|^{2+\mu}) e^{-\hat{\eta}T} \\ &\leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T}. \end{aligned}$$

□

References

- [1] G. Barles and P. E. Souganidis. Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations. *SIAM J. Math. Anal.*, 32(6):1311–1323 (electronic), 2001.
- [2] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
- [3] Arnaud Debussche, Ying Hu, and Gianmario Tessitore. Ergodic BSDEs under weak dissipative assumptions. *Stochastic Process. Appl.*, 121(3):407–426, 2011.
- [4] Marco Fuhrman and Gianmario Tessitore. The Bismut-Elworthy formula for backward SDEs and applications to nonlinear Kolmogorov equations and control in infinite dimensional spaces. *Stoch. Stoch. Rep.*, 74(1-2):429–464, 2002.
- [5] Marco Fuhrman and Gianmario Tessitore. Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. *Ann. Probab.*, 30(3):1397–1465, 2002.
- [6] Yasuhiro Fujita, Hitoshi Ishii, and Paola Loreti. Asymptotic solutions of Hamilton-Jacobi equations in Euclidean n space. *Indiana Univ. Math. J.*, 55(5):1671–1700, 2006.
- [7] Daniel Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.
- [8] Naoyuki Ichihara and Shuenn-Jyi Sheu. Large time behavior of solutions of Hamilton-Jacobi-Bellman equations with quadratic nonlinearity in gradients. *SIAM J. Math. Anal.*, 45(1):279–306, 2013.
- [9] Gawtum Namah and Jean-Michel Roquejoffre. Convergence to periodic fronts in a class of semilinear parabolic equations. *NoDEA Nonlinear Differential Equations Appl.*, 4(4):521–536, 1997.