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# Large time behaviour of mild solutions of Hamilton-Jacobi-Bellman equations in infinite dimension by a probabilistic approach

Ying Hu\*      Pierre-Yves Madec†      Adrien Richou‡

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## Abstract

We study the large time behaviour of mild solutions of HJB equations in infinite dimension by a purely probabilistic approach. For that purpose, we show that the solution of a BSDE in finite horizon  $T$  taken at initial time behaves like a linear term in  $T$  shifted with the solution of the associated EBSDE taken at initial time. Moreover we give an explicit speed of convergence, which seems to appear very rarely in literature.

## 1 Introduction

We are concerned with the large time behaviour of solutions of the Cauchy problem in an infinite dimensional real Hilbert space  $H$  :

$$\begin{cases} u_t(t, x) = \mathcal{L}u(t, x) + f(x, \nabla u(t, x)G), & \forall (t, x) \in \mathbb{R}_+ \times H, \\ u(0, x) = g(x), & \forall x \in H, \end{cases}$$

where  $u : \mathbb{R}_+ \times H \rightarrow \mathbb{R}$  is the unknown function and  $\mathcal{L}$  is the formal generator of the Kolmogorov semigroup  $\mathcal{P}_t$  of an  $H$ -valued random process solution of the following Ornstein-Uhlenbeck stochastic differential equation :

$$\begin{cases} dX_t = AX_t dt + GdW_t, & t \in \mathbb{R}_+, \\ X_0 = x, & x \in H, \end{cases}$$

where  $W$  is a Wiener process with values in another real Hilbert space  $\Xi$ , assumed to be separable. We recall that (formally),  $\forall h : H \rightarrow \mathbb{R}$ ,

$$(\mathcal{L}h)(x) = \frac{1}{2} \text{Tr}(GG^* \nabla^2 h(x)) + \langle Ax, \nabla h(x) \rangle .$$

Our method is purely probabilistic, which can be described as follows.

First, let  $(v, \lambda)$  be the solution of the ergodic PDE:

$$\mathcal{L}v + f(x, \nabla v(x)G) = 0, \quad \forall x \in H.$$

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Then we have the following probabilistic representation. Let  $(Y^{T,x}, Z^{T,x})$  be solution of the BSDE:

$$\begin{cases} dY_s^{T,x} = -f(X_s^x, Z_s^{T,x})ds + Z_s^{T,x}dW_s \\ Y_T^{T,x} = g(X_T^x), \end{cases}$$

and  $(Y, Z, \lambda)$  be solution of the EBSDE:

$$dY_s = -(f(X_s^x, Z_s^x) - \lambda)ds + Z_s^x dW_s.$$

Then

$$\begin{cases} Y_s^{T,x} = u(T, X_s^x), \\ Y_s^x = v(X_s^x). \end{cases}$$

Finally, due to Girsanov transformations and the use of an important coupling estimate result, we deduce

$$Y_0^{T,x} - \lambda T - Y_0^x \rightarrow L,$$

i.e.

$$u(T, x) - \lambda T - v(x) \rightarrow L.$$

Our method is not only purely probabilistic, but also gives a speed of convergence :

$$|u(T, x) - \lambda T - v(x) - L| \leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T}.$$

The constant  $\mu$  appearing above is the polynomial growth power of  $g(\cdot)$  and  $f(\cdot, 0)$  and  $\hat{\eta}$  is linked to the dissipative constant of  $A$ .

Large time behaviour of solutions has been studied for various types of HJB equations of second order; see, e.g., [1], [6], [8] and [9]. In [1], a result in finite dimension is stated under periodic assumptions for  $f$  and a periodic and Lipschitz assumption for  $g$ . Furthermore, they assume that  $f(x, z)$  is of linear growth in  $z$  and bounded in  $x$ . In [6], some results are stated in finite dimensional framework, under locally Hölder conditions for the coefficients. More precisely, they assume that  $f(x, z) = H_1(z) - H_2(x)$  with  $H_1$  Lipschitz and under locally Hölder conditions for  $H_2$  and  $g$ . They also treat the case of  $H_1$  locally Lipschitz but consequently need to assume that  $H_2$  and  $g$  are Lipschitz. Furthermore, they only treat the Laplacian case, namely they assume that  $G = I_d$ . No result on speed of convergence is given in this paper. In [8], the authors deal with the problem in finite dimension. They also only treat the Laplacian case and assume that  $f(x, z)$  is convex and of quadratic growth in  $z$  and of polynomial growth in  $x$ . No result on speed of convergence is given in this paper. As far as we know, the explicit speed convergence only appears in [9], but in finite dimension and under periodic assumptions for  $f(\cdot, z)$  and  $g(\cdot)$ . Furthermore, they assume that linear growth property in  $z$  for  $f$  with derivatives in  $z$  of first and second order uniformly bounded.

Roughly speaking, we will assume that  $G : H \rightarrow H$  is invertible and bounded operator,  $g : H \rightarrow \mathbb{R}$  continuous with polynomial growth and  $f : H \times \Xi^*$  continuous, with polynomial growth in the first variable and Lipschitz in the second variable.

## 2 Notations

We introduce some notations; let  $E, F$  be real separable Hilbert spaces. The norm and the scalar product will be denoted by  $|\cdot|$ ,  $\langle \cdot, \cdot \rangle$ , with subscripts if needed.  $L(E, F)$  is the space of

linear bounded operators  $E \rightarrow F$ , with the operator norm, which is denoted by  $|\cdot|_{L(E,F)}$ . The domain of a linear (unbounded) operator  $A$  is denoted by  $D(A)$ .  $L_2(E, F)$  denotes the space of Hilbert-Schmidt operators from  $E$  to  $F$ , endowed with the Hilbert-Schmidt norm, which is denoted by  $|\cdot|_{L_2(E,F)}$ .

Given  $\phi \in B_b(E)$ , the space of bounded and measurable functions  $\phi : E \rightarrow \mathbb{R}$ , we denote by  $\|\phi\|_0 = \sup_{x \in E} |\phi(x)|$ .

We say that a function  $F : E \rightarrow F$  belongs to the class  $\mathcal{G}^1(E, F)$  if it is continuous, has a Gâteaux differential  $\nabla F(x) \in L(E, F)$  at any point  $x \in E$ , and for every  $k \in E$ , the mapping  $x \mapsto \nabla F(x)k$  is continuous from  $E$  to  $F$  (i.e.  $x \mapsto \nabla F(x)$  ) is continuous from  $E$  to  $L(E, F)$  if the latter space is endowed with the strong operator topology). In connection with stochastic equations, the space  $\mathcal{G}^1$  has been introduced in [5], to which we refer the reader for further properties.

Given a real and separable Hilbert space  $K$  and a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a filtration  $\mathcal{F}_t$ , we consider the following classes of stochastic processes.

1.  $L^p_{\mathcal{P}}(\Omega, \mathcal{C}([0, T]; K))$ ,  $p \in [1, \infty)$ ,  $T > 0$ , is the space of predictable processes  $Y$  with continuous paths on  $[0, T]$  such that

$$\|Y\|_{L^p_{\mathcal{P}}(\Omega, \mathcal{C}([0, T]; K))} = \mathbb{E} \sup_{t \in [0, T]} |Y_t|_K^p < \infty.$$

2.  $L^p_{\mathcal{P}}(\Omega, L^2([0, T]; K))$ ,  $p \in [1, \infty)$ ,  $T > 0$ , is the space of predictable processes  $Y$  on  $[0, T]$  such that

$$\|Y\|_{L^p_{\mathcal{P}}(\Omega, L^2([0, T]; K))} = \mathbb{E} \left( \int_0^T |Y_t|_K^2 dt \right)^{p/2} < \infty.$$

3.  $L^2_{\mathcal{P}, \text{loc}}(\Omega, L^2([0, \infty); K))$  is the space of predictable processes  $Y$  on  $[0, \infty)$  which belong to the space  $L^2_{\mathcal{P}}(\Omega, L^2([0, T]; K))$  for every  $T > 0$ .

In the following, we consider a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a cylindrical Wiener process denoted by  $(W_t)_{t \geq 0}$  with values in  $\Xi$ , a real and separable Hilbert space.  $\mathcal{F}_t$  denotes the natural filtration generated by  $(W_s)_{s \leq t}$  and augmented with  $\mathbb{P}$ -null sets of  $\mathcal{F}$ .  $H$  denotes a real and separable Hilbert space in which the SDE will take values.

### 3 Preliminaries

We will need some result about the solution of the SDE when a perturbation term  $F$  is in the drift.

#### 3.1 The perturbed forward SDE

Let us consider the following mild stochastic differential equation for an unknown process  $(X_t)_{t \geq 0}$  with values in  $H$  :

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}F(s, X_s)ds + \int_0^t e^{(t-s)A}G(X_s^x)dW_s, \quad \forall t \geq 0, \quad \mathbb{P} - a.s. \quad (3.1)$$

Let us introduce the following hypothesis for the non-degenerate case:

**Hypothesis 3.1.** 1.  $A$  is an unbounded operator  $A : D(A) \subset H \rightarrow H$ , with  $D(A)$  dense in  $H$ . We assume that  $A$  is dissipative and generates a stable  $C_0$ -semigroup  $\{e^{tA}\}_{t \geq 0}$ . By this we mean that there exist a constant  $\eta > 0$  and  $M > 0$  such that

$$\langle Ax, x \rangle \leq -\eta|x|^2, \quad \forall x \in D(A); \quad |e^{tA}|_{L(H,H)} \leq Me^{-\eta t}, \quad \forall t \geq 0.$$

2. For all  $s > 0$ ,  $e^{sA}$  is a Hilbert-Schmidt operator. Moreover  $|e^{sA}|_{L_2(H,H)} \leq Ms^{-\gamma}$  and  $\gamma \in [0, 1/2)$ .
3.  $F : \mathbb{R}_+ \times H \rightarrow H$  is bounded and measurable.
4.  $G$  is an operator in  $L(\Xi, H)$ .
5.  $G$  is invertible. We denote by  $G^{-1}$  its bounded inverse (Banach's Theorem).

**Remark 3.1.** Note that under the previous set of hypotheses, we immediately get that :

$$\begin{aligned} |e^{sA}G|_{L_2(\Xi,H)}^2 &\leq |G|_{L(\Xi,H)}^2 |e^{sA}|_{L_2(H,H)}^2 \\ &\leq |G|_{L(\Xi,H)}^2 e^{s/2A} e^{s/2A} |_{L_2(H,H)}^2 \\ &\leq |G|_{L(\Xi,H)}^2 e^{-\eta s} \left(\frac{s}{2}\right)^{-2\gamma}, \end{aligned}$$

which shows that for every  $s > 0$  and  $x \in H$ ,  $e^{sA}G \in L_2(H, H)$ .

**Lemma 3.2.** Assume that Hypothesis 3.1 (only points (1.)-(4.)) hold and that  $F$  is bounded and Lipschitz in  $x$ . Then for every  $p \in [2, \infty)$ , there exists a unique process  $X^x \in L^p_{\mathcal{F}}(\Omega, \mathcal{C}([0, T]; H))$  solution of (3.1). Moreover,

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_t^x|^p \leq C(1 + |x|^p), \quad (3.2)$$

for some constant  $C$  depending only on  $p, \gamma, M$  and  $\sup_{t \geq 0} \sup_{x \in H} |F(t, x)|$  but independent of  $T > 0$ .

If  $F$  is only bounded and measurable, then the solution to equation 3.1 still exists but in the martingale sense. By this we mean, see [2], that there exists a new  $\mathcal{F}$ -Wiener process  $(\widehat{W}^x)_{t \geq 0}$  with respect to a new probability measure  $\widehat{\mathbb{P}}$  (absolutely continuous with respect to  $\mathbb{P}$ ), and an  $\mathcal{F}$ -adapted process  $\widehat{X}^x$  with continuous trajectories for which (3.1) holds with  $W$  replaced by  $\widehat{W}$ . Moreover (3.2) still holds (with respect to new probability). Finally such a martingale solution is unique in law.

*Proof.* For the first part of the Lemma, see [5]. The end of the Lemma is a simple consequence of the Girsanov Theorem.  $\square$

We define formally the Kolmogorov semigroup associated to Eq. (3.1) as follow :  $\forall \phi : H \rightarrow \mathbb{R}$

$$\mathcal{P}_t[\phi](x) = \mathbb{E}\phi(X_t^x).$$

**Lemma 3.3** (Basic Coupling Estimates). Assume that Hypothesis 3.1 and that  $F$  is bounded and Lipschitz. Then there exist  $\hat{c} > 0$  and  $\hat{\eta} > 0$  such that for all  $\phi : H \rightarrow \mathbb{R}$  measurable with polynomial growth (i.e.  $\exists C, \mu > 0$  such that  $\forall x \in H, |\phi(x)| \leq C(1 + |x|^\mu)$ ),  $\forall x, y \in H$ ,

$$|\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| \leq \hat{c}(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}t}. \quad (3.3)$$

We stress the fact that  $\hat{c}$  and  $\hat{\eta}$  depend on  $F$  only through  $\sup_{t \geq 0} \sup_{x \in H} |F(t, x)|$ .

*Proof.* In the same manner as in the proof of Theorem 2.4 in [3], we obtain, for every  $x, y \in H$ ,

$$\mathbb{P}(X_t^x \neq X_t^y) \leq \hat{c}(1 + |x|^2 + |y|^2)e^{-\hat{\eta}t}.$$

Now write, for every  $x, y \in H$  and  $\phi : H \rightarrow \mathbb{R}$  measurable and such that  $\forall x \in H, |\phi(x)| \leq C(1 + |x|^\mu)$

$$\begin{aligned} |\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](y)| &\leq \sqrt{\mathbb{E}(|\phi(X_t^x) - \phi(X_t^y)|^2)}\mathbb{P}(X_t^x \neq X_t^y) \\ &\leq C(1 + |x|^\mu + |y|^\mu)(1 + |x|^2 + |y|^2)e^{-\hat{\eta}t} \\ &\leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}t}. \end{aligned}$$

□

**Corollary 3.4.** *Relation (3.3) can be extended to the case in which  $F$  is only bounded measurable and for all  $t \geq 0$ , there exists a uniformly bounded sequence of Lipschitz functions in  $(F_n(t, \cdot))_{n \geq 1}$  (i.e.  $\forall t \geq 0, \forall n \in \mathbb{N}, F_n(t, \cdot)$  is Lipschitz and  $\sup_n \sup_t \sup_x |F_n(t, x)| < +\infty$ ) such that*

$$\lim_n F_n(t, x) = F(t, x), \quad \forall t \geq 0, \forall x \in H.$$

Clearly in this case in the definition of  $\mathcal{P}_t[\phi]$  the mean value is taken with respect to the new probability  $\hat{\mathbb{P}}$ .

*Proof.* See Corollary 2.5 in [3].

□

We will need to apply the lemma above to some functions with particular form.

**Lemma 3.5.** *Let  $\zeta, \zeta' : \mathbb{R}_+ \times H \rightarrow \Xi^*$  such that for all  $s \geq 0$ ,  $\zeta(s, \cdot)$  and  $\zeta'(s, \cdot)$  are weakly\* continuous with polynomial growth. We define*

$$\Upsilon(x) = \begin{cases} \frac{\psi(x, \zeta(s, x)) - \psi(x, \zeta'(s, x))}{|\zeta(s, x) - \zeta'(s, x)|^2} (\zeta(s, x) - \zeta'(s, x)), & \text{if } \zeta(s, x) \neq \zeta'(s, x), \\ 0, & \text{if } \zeta(s, x) = \zeta'(s, x). \end{cases}$$

There exists a uniformly bounded sequence of Lipschitz functions  $(\Upsilon_n(s, \cdot))_{n \geq 1}$  (i.e.  $\forall n, \Upsilon_n(s, \cdot)$  is Lipschitz and  $\sup_s \sup_n \sup_x |\Upsilon_n(s, x)| < \infty$ ) such that

$$\lim_n \Upsilon_n(s, x) = \Upsilon(s, x), \quad \forall s \geq 0, \forall x \in H.$$

*Proof.* See the proof of Lemma 3.5 in [3].

□

## 3.2 The BSDE

Let us fix  $T > 0$  and let us consider the following BSDE in finite horizon for an unknown process  $(Y_t^{T,x}, Z_t^{T,x})_{t \geq 0}$  with values in  $\mathbb{R} \times \Xi^*$  :

$$Y_t^{T,x} = \xi^T + \int_t^T f(X_s^x, Z_s^{T,x}) ds - \int_t^T Z_s^{T,x} dW_s, \quad \forall t \in [0, T], \quad (3.4)$$

where  $(X_t^x)_{t \geq 0}$  is the solution of (3.1).

We assume the following :

**Hypothesis 3.2.** There exist  $l > 0$ ,  $\mu \geq 0$  such that the function  $f : H \times \Xi^* \rightarrow \mathbb{R}$  and  $\xi^T$  satisfy :

1.  $F : H \rightarrow H$  is a Lipschitz, bounded and belongs to the class  $\mathcal{G}^1$ ,
2.  $\xi^T$  is a  $H$  valued random variable  $\mathcal{F}_T$  measurable and there exists  $p \geq 0$  such that  $|\xi^T| \leq C(1 + \sup_{0 \leq t \leq T} |X_t^x|^p)$ ,
3.  $\forall x \in H, \forall z, z' \in \Xi^*, |f(x, z) - f(x, z')| \leq l|z - z'|$ ,
4.  $\forall x \in H, |f(x, 0)| \leq C(1 + |x|^\mu)$ .

**Lemma 3.6.** Assume that Hypotheses (3.1) and (3.2) hold true then there exists a unique solution in  $L^2_{\mathcal{F}}(\Omega, \mathcal{C}([0, T]; H)) \times L^2_{\mathcal{F}}(\Omega, L^2([0, T]; \Xi^*))$  for all  $p \geq 2$  to the BSDE (3.4).

See [5], Proposition 4.3.

### 3.3 The EBSDE

Let us consider the following ergodic BSDE (EBSDE for short in the what follows) for an unknown process  $(Y_t, Z_t, \lambda)_{t \geq 0}$  with values in  $\mathbb{R} \times \Xi^* \times \mathbb{R}$  :

$$Y_t^x = Y_T^x + \int_t^T f(X_s^x, Z_s^x) ds - \int_t^T Z_s^x dW_s, \quad \forall T > 0, \forall t \in [0, T]. \quad (3.5)$$

**Hypothesis 3.3.** There exist  $l > 0$ ,  $\mu \geq 0$  such that the functions  $F : H \rightarrow H$  and  $f : H \times \Xi^* \rightarrow \mathbb{R}$  satisfy :

1.  $F : H \rightarrow H$  is a Lipschitz bounded function and belongs to the class  $\mathcal{G}^1$ ,
2.  $\forall x \in H, \forall z, z' \in \Xi^*, |f(x, z) - f(x, z')| \leq l|z - z'|$ ,
3.  $\forall x \in H, |f(x, 0)| \leq C(1 + |x|^\mu)$ .

**Lemma 3.7 (Existence).** Assume that Hypotheses (3.1) and (3.3) hold true then there exists a solution  $(Y^x, Z^x, \lambda) \in L^2_{\mathcal{F}}(\Omega, \mathcal{C}([0, T]; H)) \times L^2_{\mathcal{F}}(\Omega, L^2([0, T]; \Xi^*)) \times \mathbb{R}$ , for all  $T > 0$  to the EBSDE (3.5). Moreover there exists  $v : H \rightarrow \mathbb{R}$  of class  $\mathcal{G}^1$  such that, for all  $x, x' \in H$ , for all  $t \geq 0$  :

$$\begin{aligned} Y_t^x &= v(X_t^x) \quad \text{and} \quad Z_t^x = \nabla v(X_t^x)G, \\ v(0) &= 0, \\ |v(x) - v(x')| &\leq C(1 + |x|^{2+\mu} + |x'|^{2+\mu}), \\ |\nabla v(x)| &\leq C(1 + |x|^{2+\mu}). \end{aligned}$$

*Proof.* This can be proved in the same way as in [3]. □

In the following, we denote by  $(Y^{T,x}, Z^{T,x})$  the solution of (3.4) given by Lemma 3.6 and by  $(Y^x, Z^x, \lambda)$  the solution of (3.4) given by Lemma 3.7.

## 4 Large time behaviour of solutions

### 4.1 Path-dependent framework

**Theorem 4.1.** *Assume that Hypotheses 3.1, 3.2 and 3.3 hold true. Then,  $\forall T \geq 0$  :*

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leq \frac{C(1 + |x|^{2+\mu})}{T}. \quad (4.1)$$

In particular,

$$\frac{Y_0^{T,x}}{T} \xrightarrow{T \rightarrow +\infty} \lambda,$$

uniformly in all compact set of  $H$ .

*Proof.* For all  $x \in H$ ,  $T > 0$ :

$$\left| \frac{Y_0^{T,x}}{T} - \lambda \right| \leq \left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| + \left| \frac{Y_0^x}{T} \right|.$$

We have :

$$\begin{aligned} Y_0^{T,x} - Y_0^x - \lambda T &= \xi^T - v(X_T^x) + \int_0^T f(X_s^x, Z_s^{T,x}) - f(X_s^x, Z_s^x) ds - \int_0^T Z_s^{T,x} - Z_s^x dW_s \\ &= \xi^T - v(X_T^x) + \int_0^T (Z_s^{T,x} - Z_s) \beta_s^T ds - \int_0^T Z_s^{T,x} - Z_s dW_s, \end{aligned}$$

where

$$\beta_s^T = \begin{cases} \frac{(f(X_s^x, Z_s^{T,x}) - f(X_s^x, Z_s^x))(Z_s^{T,x} - Z_s^x)^*}{|Z_s^{T,x} - Z_s^x|^2}, & \text{if } Z_s^{T,x} - Z_s^x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

The process  $\beta_s^T$  is progressively measurable and bounded by  $l$  therefore we can apply Girsanov Theorem to obtain that there exist a probability measure  $\mathbb{Q}^T$  under which  $\widetilde{W}_t^T = -\int_0^t \beta_s^T ds + W_t$  is a Wiener process. We recall that if we define  $M_T = \exp\left(\int_0^T \beta_s^T dW_s - \frac{1}{2} \int_0^T |\beta_s^T|^2 ds\right)$ , the following formula holds :  $d\mathbb{Q}^T = M_T d\mathbb{P}$ .

Taking the expectation with respect to  $\mathbb{Q}^T$  we get

$$Y_0^{T,x} - Y_0^x - \lambda T = \mathbb{E}^{\mathbb{Q}^T}(\xi^T - v(X_T^x)). \quad (4.2)$$

Thus

$$\left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| \leq C \frac{1 + \mathbb{E}^{\mathbb{Q}^T}[\sup_{0 \leq t \leq T} |X_t^x|^\mu]}{T} + C \frac{1 + \mathbb{E}^{\mathbb{Q}^T}(|X_T^x|^{2+\mu})}{T}.$$

The process  $(X_t^x)_{t \geq 0}$  is the mild solution of

$$\begin{cases} dX_t^x = AX_t^x dt + F(X_t^x) dt + G\beta_t^T \mathbf{1}_{t < T} dt + Gd\widetilde{W}_t^T, & t \geq 0, \\ X_0^x = x. \end{cases}$$

Thus, by Lemma (3.2), there exists a constant  $C$  which does not depend on time such that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^T} \left[ \sup_{0 \leq t \leq T} |X_t^x|^\mu \right] &\leq C(1 + |x|^\mu), \\ \mathbb{E}^{\mathbb{Q}^T} \left[ \sup_{0 \leq t \leq T} |X_t^x|^2 \right] &\leq C(1 + |x|^{2+\mu}),\end{aligned}$$

therefore

$$\left| \frac{Y_0^{T,x} - Y_0^x - \lambda T}{T} \right| \leq \frac{C(1 + |x|^{2+\mu})}{T}. \quad (4.3)$$

Finally note that

$$\left| \frac{Y_0^x}{T} \right| \leq \frac{C(1 + |x|^{2+\mu})}{T},$$

which gives the result.  $\square$

## 4.2 Markovian framework

For the Markovian framework, we replace Hypothesis 3.2 by the following

**Hypothesis 4.1.** There exist  $l > 0$ ,  $\mu \geq 0$  such that the function  $f : H \times \Xi^* \rightarrow \mathbb{R}$  and  $\xi^T$  satisfy :

1.  $F \equiv 0$ ,
2.  $\xi^T = g(X_T)$  for a continuous function  $g : H \rightarrow H$  with polynomial growth, i.e. there exists  $\mu > 0$  such that  $|g(x)| \leq C(1 + |x|^\mu)$ ,
3.  $\forall x \in H, \forall z, z' \in \Xi^*, |f(x, z) - f(x, z')| \leq l|z - z'|$ ,
4.  $f : H \times \Xi^* \rightarrow \mathbb{R}$  is continuous and of polynomial growth, i.e.  $\forall x \in H, |f(x, 0)| \leq C(1 + |x|^\mu)$ .

**Theorem 4.2.** Assume that Hypotheses 3.1, 3.3 and 4.1 hold true. Then there exists  $L \in \mathbb{R}$  such that,

$$\forall x \in H, \quad Y_0^{T,x} - \lambda T - Y_0^x \xrightarrow{T \rightarrow +\infty} L.$$

Furthermore the following speed of convergence holds

$$|Y_0^{T,x} - \lambda T - v(x) - L| \leq C(1 + |x|^2 + |x|^\mu)(1 + |x|^2)e^{-\hat{\eta}T}.$$

*Proof.* Let us denote by  $(X^{t,x}, Y^{T,t,x}, Z^{T,t,x})$  the mild solution of

$$\begin{cases} dX_s = AX_s ds + GdW_s, & s \geq t \\ dY_s = -f(X_s, Z_s) ds + Z_s dW_s, & s \in [t, T] \\ X_t = x \in H, \\ Y_T = g(X_T). \end{cases}$$

Now we define

$$\begin{aligned}u_T(t, x) &:= Y_t^{T,t,x} \\ w_T(t, x) &:= u_T(t, x) - \lambda(T - t) - v(x).\end{aligned}$$

We recall that  $Y_s^{T,t,x} = u_T(s, X_s^{t,x})$  and that  $Y_s^x = v(X_s^x)$ , where  $v$  is defined in the Lemma 3.5. Remark that taking  $t = 0$  implies  $Y_s^{T,0,x} = Y_s^{T,x}$ . We recall that for all  $T, S > 0$ ,  $u_T$  is the unique mild solution of

$$\begin{cases} \frac{\partial u_T(t,x)}{\partial t} + \mathcal{L}u(t,x) + f(x, \nabla u(t,x)G) = 0, & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u_T(T,x) = g(x), & \forall x \in H, \end{cases}$$

and that  $u_{T+S}$  is the unique mild solution of

$$\begin{cases} \frac{\partial u_{T+S}(t,x)}{\partial t} + \mathcal{L}u(t,x) + f(x, \nabla u(t,x)G) = 0, & \forall (t,x) \in \mathbb{R}_+ \times H, \\ u_{T+S}(T+S,x) = g(x), & \forall x \in H, \end{cases}$$

This implies for all  $x \in H$ ,

$$u_T(0,x) = u_{T+S}(S,x).$$

and then,

$$w_T(0,x) = w_{T+S}(S,x) \tag{4.4}$$

We are going to need some estimates on  $w_T$  given in the following Lemma.

**Lemma 4.3.** *Under the hypothesis of the Theorem 4.2, there exists  $C > 0$  such that  $\forall x, y \in H, \forall T > 0$ ,*

$$\begin{aligned} |w_T(0,x)| &\leq C(1 + |x|^{2+\mu}), \\ |\nabla_x w_T(0,x)| &\leq \frac{Ce^{C\varepsilon}}{\sqrt{\varepsilon}}(1 + |x|^{2+\mu}), \forall 0 < \varepsilon \leq T, \\ |w_T(0,x) - w_T(0,y)| &\leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}. \end{aligned}$$

*Proof of Lemma 4.3.* The first inequality of the Lemma is a direct application of estimate (4.3). Indeed,  $\forall x \in H, \forall T > 0$ ,

$$\begin{aligned} |w_T(0,x)| &= |u_T(0,x) - \lambda T - v(x)| \\ &= |Y_0^{T,x} - Y_0^x - \lambda T| \\ &\leq C(1 + |x|^{2+\mu}), \end{aligned} \tag{4.5}$$

which gives the first inequality of Lemma.

Now, let us establish the gradient estimate. First remark that the inequality (4.5) still holds for  $w_T(t,x)$ , i.e.  $\forall x \in H, \forall T > t$ ,

$$|w_T(t,x)| \leq C(1 + |x|^{2+\mu}). \tag{4.6}$$

The process  $(w_T(s, X_s^{t,x}))_{t \leq s \leq T}$  satisfies the following equation, for all  $t \leq s \leq T$ ,

$$\begin{aligned} w_T(s, X_s^{t,x}) &= w_T(T, X_T^{t,x}) + \int_s^T (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_s^{t,x}, Z_s^{t,x}))ds \\ &\quad - \int_s^T (Z_r^{T,t,x} - Z_r^{t,x})dW_r. \end{aligned}$$

Now remark that for all  $t < T' \leq T$  the following equation hold by unicity of solutions,  $\forall t < s \leq T'$

$$\begin{aligned} w_T(s, X_s^{t,x}) &= w_T(T', X_{T'}^{t,x}) + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x}) - f(X_r^{t,x}, Z_r^{t,x}))dr \\ &\quad - \int_s^{T'} (Z_r^{T,t,x} - Z_r^{t,x})dW_r \\ &= w_{T-T'}(0, X_{T'}^{t,x}) + \int_s^{T'} (f(X_r^{t,x}, Z_r^{T,t,x} - Z_r^x + Z_r^x) - f(X_r^{t,x}, Z_r^{t,x}))dr \\ &\quad - \int_t^{T'} (Z_r^{T,t,x} - Z_r^x)dW_r. \end{aligned}$$

where we have used the equality (4.4) for the second line.

And we recall that (see [4] Theorem 4.2 and [3] Theorem 3.8),  $\forall x, h \in H, \forall s \in [t, T]$

$$\begin{aligned} Z_s^{T,t,x} &= \nabla_x u_T(s, X_s^{t,x})G, \\ Z_s^x &= \nabla_x v(X_s^{t,x})G. \end{aligned}$$

Then :

$$Z_r^{T,t,x} - Z_r^{t,x} = \nabla_x w_T(r, X_r^{t,x})G.$$

Thus applying the Bismut-Elworthy formula (see [4], Theorem 4.2),  $\forall x, h \in H, \forall t < T$  :

$$\begin{aligned} \nabla_x w_T(t, x)h &= \mathbb{E} \int_t^T [f(X_s^{t,x}, \nabla_x w_T(r, X_r^{t,x})G - Z_s^{t,x}) - f(X_s^{t,x}, Z_s^{t,x})]U^h(s, t, x)ds \\ &\quad + \mathbb{E} [w_{T-T'}(0, X_{T'}^{t,x})]U^h(T', t, x), \end{aligned}$$

where,  $\forall 0 \leq s \leq T, \forall x \in H$ ,

$$U^h(s, t, x) = \frac{1}{s-t} \int_t^s \langle G^{-1} \nabla_x X_u^{t,x} h, d\widetilde{W}_u \rangle.$$

Let us recall that

$$\nabla_x X_s^{t,x} h = e^{(s-t)A} h,$$

then,

$$\begin{aligned} \mathbb{E}|U^h(s, t, x)|^2 &= \frac{1}{|s-t|^2} \int_t^s |G^{-1} \nabla_x X(u, t, x)h|^2 du \\ &\leq \frac{C|h|^2}{s-t}, \end{aligned}$$

where  $C$  is independant on  $t, s$  and  $x$ .

Then,  $\forall x, h \in H, \forall t < T$ , using inequality (4.6)

$$|\nabla_x w_T(t, x)h| \leq C \int_t^{T'} \mathbb{E} \frac{|\nabla_x w_T(s, X_s^{t,x})h|}{\sqrt{s-t}} ds + C \frac{(1 + |x|^{2+\mu})|h|}{\sqrt{T'-t}}.$$

We define

$$\varphi(s) = \sup_x \sup_h \frac{|\nabla_x w_T(t, x)h|}{(1 + |x|^{2+\mu})|h|},$$

then

$$|\nabla_x w_T(t, x)h| \leq C \int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}} \mathbb{E}(1 + |X_s^{t,x}|^{2+\mu})|h| ds + C \frac{(1 + |x|^{2+\mu})|h|}{\sqrt{T'-t}}$$

which leads to

$$\frac{|\nabla_x w_T(t, x)h|}{(1 + |x|^{2+\mu})|h|} \leq C \int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}} ds + \frac{C}{\sqrt{T'-t}}.$$

Taking the supremum over  $x$  and  $h$ , we have

$$\varphi(t) \leq C \int_t^{T'} \frac{\varphi(s)}{\sqrt{s-t}} ds + \frac{C}{\sqrt{T'-t}}.$$

Then by Lemma 7.1.1 in [7] :

$$|\nabla_x w_T(t, x)h| \leq C|h| \frac{(1 + |x|^{2+\mu})}{\sqrt{T'-t}} e^{C(T'-t)},$$

which gives us the second estimate of Lemma for  $t = 0$ :

$$|\nabla_x w_T(0, x)h| \leq C|h| \frac{(1 + |x|^2 + |x|^\mu)}{\sqrt{T'}} e^{CT'}.$$

For the third inequality of Lemma, we have by equation (4.2),  $\forall x \in H, \forall T > 0$  :

$$\begin{aligned} w_T(0, x) &= \mathbb{E}^{\mathbb{Q}^T}(g(X_T^x) - v(X_T^x)) \\ &= \mathbb{E}(g(U_T^x) - v(U_T^x)), \end{aligned}$$

where  $U^x$  is the mild solution of the following equation defined  $\forall t \in \mathbb{R}$  :

$$dU_t^x = [AU_t^x + F(U_t^x) + \beta_t^T \mathbf{1}_{t < T}] dt + G dW_t, \quad U_0^x = x.$$

Therefore,  $\forall x \in H \forall T > 0$  :

$$|w_T(0, x) - w_T(0, y)| = |\mathbb{E}(g(U_T^x) + v(U_T^x)) - \mathbb{E}(g(U_T^y) + v(U_T^y))|$$

Remark now that in the Markovian framework,  $(\beta_t^T \mathbf{1}_{t \leq T})_{t \geq 0}$  is Markovian. Furthermore  $\beta_s^T$  is uniformly bounded in  $T$  by  $l$  therefore we can apply the Corollary 3.4, we obtain

$$|w_T(0, x) - w_T(0, y)| \leq C(1 + |x|^{2+\mu} + |y|^{2+\mu})e^{-\hat{\eta}T}. \quad (4.7)$$

□

Now, let us come back to the proof of the Theorem. The first estimate of Lemma 4.3 allow us to construct, by a diagonal procedure, a sequence  $(T_i)_i \nearrow +\infty$  such that for a function  $w$  defined on a countable dense subset of  $H$  (denoted by  $D$ ) :  $(w : D \rightarrow \mathbb{R})$  the following hold

$$\forall x \in D, \lim_{i \rightarrow +\infty} w_{T_i}(0, x) = w(x).$$

Then, it is possible to extend this function to the whole  $H$  thanks to the second inequality of Lemma 4.3.

Now, let us show that  $w : H \rightarrow \mathbb{R}$  is constant. We have, by the third inequality of Lemma 4.3, for all  $x, y \in H$  and  $T > 0$ ,

$$|w_T(0, x) - w_T(0, y)| \leq C(1 + |x|^2 + |x|^\mu + |y|^2 + |y|^\mu)(1 + |x|^2 + |y|^2)e^{-\hat{\eta}T}.$$

Applying the previous inequality with  $T = T_i$  and taking the limit in  $i$  show us that  $x \mapsto w(x)$  is a constant function, namely there exists  $L_1 \in \mathbb{R}$  (independent of  $x$ ) such that :  $\forall x \in H$

$$\lim_i w_{T_i}(0, x) = L_1.$$

Now remark that  $\{w_T(0, \cdot); T > 1\}$  is a relatively compact subspace of the space of continuous functions  $H \rightarrow \mathbb{R}$  for the uniform distance thanks to the two first inequalities of Lemma 4.3. Therefore, if we show that  $\{w_T(0, \cdot); T > 1\}$  admits only one accumulation point, it will implies that

$$\lim_{T \rightarrow +\infty} w_T(0, x) = L_1.$$

Now we claim that the accumulation point is unique. Let us assume that there exists another subsequence  $(T'_i)_{i \in \mathbb{N}}$  such that  $\lim_i w_{T'_i}(0, x) = L_2$  (note that it must be a constant by the third inequality of Lemma 4.3).

Let us write,  $\forall x \in H, \forall T, S > 0$  :

$$\begin{aligned} w_{T+S}(0, x) &= Y_0^{T+S, x} - \lambda(T+S) - Y_0^x \\ &= Y_S^{T+S, x} - \lambda T - Y_S^x + \int_0^S (f(X_r^x, Z_r^{T+S, x}) - f(X_r^x, Z_r^x)) dr \\ &\quad - \int_0^S (Z_r^{T+S, x} - Z_r^x) dW_r \\ &= Y_S^{T+S, x} - \lambda T - Y_S^x + \int_0^S (Z_r^{T+S, x} - Z_r^x) d\widetilde{W}_r, \end{aligned}$$

where

$$\beta_s^{T+S} = \begin{cases} \frac{(f(X_s^x, Z_s^{T+S, x}) - f(X_s^x, Z_s^x))(Z_s^{T+S, x} - Z_s^x)^*}{|Z_s^{T+S, x} - Z_s^x|^2}, & \text{if } Z_s^{T+S, x} - Z_s^x \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\widetilde{W}_t^{T+S} = - \int_0^t \beta_s^{T+S} ds + W_t.$$

Taking the expectation with respect to the probability  $\mathbb{Q}^{T+S}$  under which  $\widetilde{W}^{T+S}$  is a Brownian motion we get (using equality (4.4) for the third line):

$$\begin{aligned} w_{T+S}(0, x) &= \mathbb{E}^{\mathbb{Q}^{T+S}}(Y_S^{T+S, x} - \lambda T - Y_S^x) \\ &= \mathbb{E}^{\mathbb{Q}^{T+S}}(w_{T+S}(S, X_S^x)) \\ &= \mathbb{E}^{\mathbb{Q}^{T+S}}(w_T(0, X_S^x)) \\ &= \mathbb{E}(w_T(0, U_S^x)) \end{aligned} \tag{4.8}$$

where  $U^x$  is the mild solution of the following equation defined  $\forall t \in \mathbb{R}_+$  :

$$dU_t^x = [AU_t^x + F(U_t^x) + \beta_t^{T+S} \mathbf{1}_{t < T+S}]dt + GdW_t, \quad U_0^x = x.$$

This implies, substituting  $T$  by  $T'_i$  and  $S$  by  $T_i - T'_i$ , (up to a subsequence for  $(T_i)_{i \in \mathbb{N}}$  such that  $T_i > T'_i$ )

$$w_{T_i}(0, x) = \mathbb{E}(w_{T'_i}(0, U_{T_i - T'_i}^x))$$

We have:

$$\lim_i w_{T_i}(0, x) = L_1.$$

Furthermore,

$$\begin{aligned} |w_{T_i}(0, U_{T_i - T'_i}^x) - L_2| &\leq |w_{T'_i}(0, U_{T_i - T'_i}) - w_{T'_i}(0, x)| + |w_{T'_i}(0, x) - L_2| \\ &\leq C(1 + |x|^{2+\mu} + |U_{T_i - T'_i}|^{2+\mu})e^{-\hat{\eta}T'_i} + |w_{T'_i}(0, x) - L_2|, \end{aligned}$$

thanks to the third inequality of Lemma 4.3. Now remark that  $\forall p > 0, \forall \varepsilon > 0$ ,

$$\begin{aligned} \mathbb{P}(|U_{T_i - T'_i}|^p e^{-p\hat{\eta}T'_i} > \varepsilon) &\leq \mathbb{E}[|U_{T_i - T'_i}|^p e^{-p\hat{\eta}T'_i}] / \varepsilon \\ &\leq C \frac{e^{-p\hat{\eta}T'_i}}{\varepsilon}, \end{aligned}$$

which shows that

$$|w_{T'_i}(0, U_{T_i - T'_i}) - L_2| \xrightarrow{i \rightarrow +\infty} 0, \quad \mathbb{P} - a.s.$$

Then, by the dominated convergence theorem :

$$\lim_i \mathbb{E}(w_{T'_i}(0, U_{T_i - T'_i})) = L_2.$$

Therefore  $L_1 = L_2$ , which as mentioned before, implies that

$$\lim_{T \rightarrow +\infty} w_T(0, x) = L_1.$$

Now we prove that this convergence holds with an explicit speed of convergence. Let us write,  $\forall x \in H, \forall T > 0$ ,

$$\begin{aligned} |w_T(0, x) - L| &= \lim_{V \rightarrow +\infty} |w_T(0, x) - w_V(0, x)| \\ &= \lim_{V \rightarrow +\infty} |w_T(0, x) - \mathbb{E}(w_T(0, U_{V-T}^x))| \end{aligned}$$

thanks to equality (4.8), where  $U^x$  is the mild solution of the following equation defined  $\forall t \in \mathbb{R}_+$  :

$$dU_t^x = [AU_t^x + F(U_t^x) + \beta_t^V \mathbf{1}_{t < V}]dt + GdW_t, \quad U_0^x = x.$$

Now, thanks to the third estimate in Lemma 4.3, one have,

$$\begin{aligned} |w_T(0, x) - L| &\leq \lim_{V \rightarrow +\infty} C \mathbb{E} (1 + |x|^{2+\mu} + |U_{V-T}^x|^{2+\mu}) e^{-\hat{\eta}T} \\ &\leq C(1 + |x|^{2+\mu})e^{-\hat{\eta}T}. \end{aligned}$$

□

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