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Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain.

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Abstract

We obtain a necessary and a sufficient condition expressed in terms of Wiener type tests involving the parabolic $W_{q'}^{2,1}$ -capacity, where $q' = \frac{q}{q-1}$, for the existence of large solutions to equation $\partial_t u - \Delta u + u^q = 0$ in non-cylindrical domain, where $q > 1$. Also, we provide a sufficient condition associated with equation $\partial_t u - \Delta u + e^u - 1 = 0$. Besides, we apply our results to equation: $\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0$ for $a, b > 0$, $1 < p < 2$ and $q > 1$.

Keywords. Bessel capacities; Hausdorff capacities; parabolic boundary; Riesz potential; maximal solutions.

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1 Introduction

The aim of this paper is to study large solutions to nonlinear parabolic equations in an arbitrary bounded open set $O \subset \mathbb{R}^{N+1}$, $N \geq 2$. These are solutions $u \in C^{2,1}(O)$ of equations

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u &= 0 && \text{in } O, \\ \lim_{\delta \rightarrow 0} \inf_{O \cap Q_\delta(x,t)} u &= \infty && \text{for all } (x,t) \in \partial_p O, \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} \partial_t u - \Delta u + \text{sign}(u)(e^{|u|} - 1) &= 0 && \text{in } O, \\ \lim_{\delta \rightarrow 0} \inf_{O \cap Q_\delta(x,t)} u &= \infty && \text{for all } (x,t) \in \partial_p O, \end{aligned} \quad (1.2)$$

where $q > 1$ and $\partial_p O$ is the parabolic boundary of O , i.e, the set all points $X = (x, t) \in \partial O$ such that the intersection of the cylinder $Q_\delta(x, t) := B_\delta(x) \times (t - \delta^2, t)$ with O^c is not empty for any $\delta > 0$. By the maximal principle for parabolic equations we can assume that all solutions of (1.1) and (1.2) are positive. Hence we can consider only positive solutions of preceding equations.

In [14], we studied the existence and the uniqueness of solution of general equations in a cylindrical domain,

$$\begin{aligned} \partial_t u - \Delta u + f(u) &= 0 && \text{in } \Omega \times (0, \infty), \\ u &= \infty && \text{in } \partial_p(\Omega \times (0, \infty)), \end{aligned} \quad (1.3)$$

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where Ω is a bounded open set in \mathbb{R}^N and f is a continuous real-valued function, nondecreasing on \mathbb{R} such that $f(0) \geq 0$ and $f(a) > 0$ for some $a > 0$. In order to obtain the existence of a maximal solution of $\partial_t u - \Delta u + f(u) = 0$ in $\Omega \times (0, \infty)$ we need to assume that

$$(i) \quad \int_a^\infty \left(\int_0^s f(\tau) d\tau \right)^{-\frac{1}{2}} ds < \infty$$

$$(ii) \quad \int_a^\infty (f(s))^{-1} ds < \infty.$$
(1.4)

Note that, condition (i) due to Keller-Osserman condition, is also a necessary and sufficient for the existence of a maximal solution to

$$-\Delta u + f(u) = 0 \quad \text{in } \Omega$$
(1.5)

Condition (ii) is a necessary and sufficient for the existence of a solution of the ODE

$$\varphi' + f(\varphi) = 0 \quad \text{in } (0, \infty).$$
(1.6)

This solution tends to ∞ at 0. In [14], it is shown that if for any $m \in \mathbb{R}$ there exist $L = L(m) > 0$ such that

$$\text{for any } x, y \geq m \Rightarrow f(x + y) \geq f(x) + f(y) - L,$$

and if (1.5) has a large solution, then (1.3) admits a solution.

It is not always true that the maximal solution to (1.5) is a large solution. However, if f satisfies

$$\int_1^\infty s^{-2(N-1)/(N-2)} f(s) ds < \infty \quad \text{if } N \geq 3$$

or

$$\inf \left\{ a \geq 0 \int_0^\infty f(s) e^{-as} ds < \infty \right\} \quad \text{if } N = 2.$$

then (1.5) has a large solution for any bounded domain Ω .

When $f(u) = u^q$, $q > 1$ and $N \geq 3$, the first above condition is satisfied if and only if $q < q_c := \frac{N}{N-2}$, this is called *the sub-critical case*. When $q \geq q_c$, a necessary and sufficient condition for the existence of large solution of (1.5) expressed in term of Wiener test, is

$$\int_0^1 \frac{\text{Cap}_{2,q'}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } x \in \partial\Omega.$$
(1.7)

In the case $q = 2$ it is obtained by probabilistic methods by Dhersin and Le Gall [4] and in the general case by Labutin [6]. Here, $q' = \frac{q}{q-1}$ and $\text{Cap}_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$.

In [10] we obtain sufficient conditions when $f(u) = e^u - 1$, involving the the Hausdorff \mathcal{H}_1^{N-2} -capacity in \mathbb{R}^N , namely,

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } x \in \partial\Omega.$$
(1.8)

In this article we give a necessary and a sufficient condition for the existence of solutions to (1.1) in a bounded non-cylindrical domain $O \subset \mathbb{R}^{N+1}$, expressed in terms of a Wiener test based upon the parabolic $W_{q'}^{2,1}$ -capacity in \mathbb{R}^{N+1} . We also give a sufficient condition

associated (1.2) where the parabolic $W_q^{2,1}$ -capacity is replaced the parabolic Hausdorff \mathcal{PH}_ρ^N -capacity. These capacities are defined as follows: if $K \subset \mathbb{R}^{N+1}$ is compact set, we set

$$\text{Cap}_{2,1,q'}(K) = \inf\{\|\varphi\|_{W_{q'}^{2,1}(\mathbb{R}^{N+1})}^q : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } K\},$$

where

$$\|\varphi\|_{W_{q'}^{2,1}(\mathbb{R}^{N+1})} = \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^{q'}(\mathbb{R}^{N+1})} + \|\nabla \varphi\|_{L^{q'}(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\dots,N} \left\| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\|_{L^{q'}(\mathbb{R}^{N+1})}.$$

and for Suslin set $E \subset \mathbb{R}^{N+1}$,

$$\text{Cap}_{2,1,q'}(E) = \sup\{\text{Cap}_{2,1,q'}(D) : D \subset E, D \text{ compact}\}.$$

Thanks to a result due to Richard and Bagby, [2], the capacities $\text{Cap}_{2,1,p}$ and $\text{Cap}_{\mathcal{G}_2,p}$ are equivalent in the sense that, for any Suslin set $K \subset \mathbb{R}^{N+1}$, there holds

$$C^{-1} \text{Cap}_{2,1,q'}(K) \leq \text{Cap}_{\mathcal{G}_2,q'}(K) \leq C \text{Cap}_{2,1,p}(K)$$

for some $C = C(N, q)$, where $\text{Cap}_{\mathcal{G}_2,q'}$ is the parabolic Bessel \mathcal{G}_2 -capacity, see [11]. For $E \subset \mathbb{R}^{N+1}$, we define $\mathcal{PH}_\rho^N(E)$ by

$$\mathcal{PH}_\rho^N(E) = \inf \left\{ \sum_j r_j^N : E \subset \bigcup B_{r_j}(x_j) \times (t_j - r_j^2, t_j + r_j^2), r_j \leq \rho \right\}.$$

It is easy to see that, for $0 < \sigma \leq \rho$ and $E \subset \mathbb{R}^{N+1}$, there holds

$$\mathcal{PH}_\rho^N(E) \leq \mathcal{PH}_\sigma^N(E) \leq C(N) \left(\frac{\rho}{\sigma} \right)^2 \mathcal{PH}_\rho^N(E). \quad (1.9)$$

Now we are ready to state the main two results of this paper.

Theorem 1.1 *Let $N \geq 2$ and $q \geq q_* := \frac{N+2}{N}$. Then*

(i) *The equation*

$$\partial_t u - \Delta u + u^q = 0 \text{ in } O \quad (1.10)$$

admits a large solution if for any $(x, t) \in \partial_p O$

$$\sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2)))}{r_k^N} = \infty, \quad (1.11)$$

where $r_k = 4^{-k}$, and $N \geq 3$ when $q = q_$.*

(ii) *If equation (1.10) is a large solution, then*

$$\int_0^1 \frac{\text{Cap}_{2,1,q'}(O^c \cap Q_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} = \infty \quad (1.12)$$

for any $(x, t) \in \partial_p O$, where $Q_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t)$.

Theorem 1.2 *Let $N \geq 2$. The equation*

$$\partial_t u - \Delta u + e^u - 1 = 0 \text{ in } O \quad (1.13)$$

admits a large solution if

$$\sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2)))}{r_k^N} = \infty, \quad (1.14)$$

for any $(x, t) \in \partial_p O$, with $r_k = 4^{-k}$.

From properties of the $W_{q'}^{2,1}$ -capacity and the \mathcal{PH}_1^N -capacity, relation (1.11) holds if $q > q_*$ and

$$\sum_{k=1}^{\infty} r_k^{-N} |O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2))|^{1 - \frac{2q'}{N+2}} = \infty.$$

Similarly, (1.14) is true if

$$\sum_{k=1}^{\infty} r_k^{-N} |O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2))|^{\frac{N}{N+2}} = \infty.$$

As a consequence of Theorem 1.1 we derive a sufficient condition for the existence of large solution of some viscous Hamilton-Jacobi parabolic equations.

Theorem 1.3 *Let $q_1 > 1$. If there exists a large solution $v \in C^{2,1}(O)$ of*

$$\partial_t v - \Delta v + v^{q_1} = 0 \quad \text{in } O.$$

Then, for any $a, b > 0$, $1 < q < q_1$ and $1 < p < \frac{2q_1}{q_1+1}$, problem

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p + bu^q &= 0 & \text{in } O, \\ u &= \infty & \text{on } \partial_p O, \end{aligned} \tag{1.15}$$

admits a solution $u \in C^{2,1}(O)$ which satisfies

$$u(x, t) \geq C \min \left\{ a^{-\frac{1}{p-1}} R^{-\frac{2-p}{p-1} + \frac{2}{\alpha(q_1-1)}}, b^{-\frac{1}{q-1}} R^{-\frac{2}{q-1} + \frac{2}{\alpha(q_1-1)}} \right\} (v(x, t))^{\frac{1}{\alpha}}$$

for all $(x, t) \in O$ where $R > 0$ is such that $O \subset \tilde{Q}_R(x_0, t_0)$, $C = C(N, p, q, q_1) > 0$ and $\alpha = \max \left\{ \frac{2(p-1)}{(q_1-1)(2-p)}, \frac{q-1}{q_1-1} \right\} \in (0, 1)$.

2 Preliminaries

Throughout the paper, we denote $Q_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t]$ and $\tilde{Q}_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t + \rho^2)$ for $(x, t) \in \mathbb{R}^{N+1}$, $\rho > 0$ and $r_k = 4^{-k}$ for all $k \in \mathbb{Z}$. We also denote $A \lesssim (\gtrsim) B$ if $A \leq (\geq) CB$ for some C depending on some structural constants, $A \asymp B$ if $A \lesssim B \lesssim A$.

Definition 2.1 *Let $R \in (0, \infty]$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, the set of positive Radon measures in \mathbb{R}^{N+1} . We define R -truncated Riesz parabolic potential \mathbb{I}_2 of μ by*

$$\mathbb{I}_2^R[\mu](x, t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} \quad \text{for all } (x, t) \in \mathbb{R}^{N+1},$$

and the R -truncated fractional maximal parabolic potential of μ by

$$\mathbb{M}_\alpha^R[\mu](x, t) = \sup_{0 < \rho < R} \frac{\mu(\tilde{Q}_\rho(x, t))}{\rho^{N+2-\alpha}} \quad \text{for all } (x, t) \in \mathbb{R}^{N+1}.$$

We recall two results in [11].

Theorem 2.2 *Let $R > 0$, K be a compact set in \mathbb{R}^{N+1} . There exists $\mu := \mu_K \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with compact support in K such that*

$$\mu(K) \asymp \text{Cap}_{2,1,q'}(K) \asymp \int_{\mathbb{R}^{N+1}} (\mathbb{I}_2^{2R}[\mu])^q dx dt$$

where the constants of equivalence depend on N and R . The measure μ_K is called the capacitary measure of K

Theorem 2.3 Let $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ and $R > 0$. There exist positive constants C_1, C_2 such that

$$\int_Q \exp(C_1 \mathbb{I}_2^R[\mu_Q]) \leq C_2,$$

for all $Q = \tilde{Q}_r(y, s) \subset \mathbb{R}^{N+1}$, $r > 0$ such that $\|\mathbb{M}_2^R[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} dxdt \leq 1$.

It is easy to see that Frostman's Lemma in [13, Th. 3.4.27], which is at the basis of the dual definition of Hausdorff capacities with doubling weight, is valid for the parabolic Hausdorff \mathcal{PH}_ρ^N -capacity version. Therefore there holds

Theorem 2.4 There holds

$$\sup \{ \mu(K) : \mu \in \mathfrak{M}^+(\mathbb{R}^{N+1}), \text{supp}(\mu) \subset K, \|\mathbb{M}_2^R[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \} \asymp \mathcal{PH}_\rho^N(K)$$

for any compact set $K \subset \mathbb{R}^{N+1}$, where equivalent constant depends on N

For our purpose, we need the some results about the behavior of the capacity with respect to dilations.

Proposition 2.5 Let K be a compact set, $K \subset \overline{\tilde{Q}_{100}(0, 0)}$ and $1 < p < \frac{N+2}{2}$. Then

$$\text{Cap}_{2,1,p}(K) \gtrsim |K|^{1-\frac{2p}{N+2}}, \text{Cap}_{2,1,\frac{N+2}{2}}(K) \gtrsim \left(\log \left(\frac{|\tilde{Q}_{200}(0, 0)|}{|K|} \right) \right)^{-\frac{N}{2}}, \quad (2.1)$$

and

$$\text{Cap}_{2,1,p}(K_\rho) \asymp \rho^{N+2-2p} \text{Cap}_{2,1,p}(K), \quad (2.2)$$

$$\frac{1}{\text{Cap}_{2,1,\frac{N+2}{2}}(K_\rho)} \asymp \frac{1}{\text{Cap}_{2,1,\frac{N+2}{2}}(K)} + (\log(2/\rho))^{N/2} \quad (2.3)$$

for any $0 < \rho < 1$, where $K_\rho = \{(\rho x, \rho^2 t) : (x, t) \in K\}$.

Proposition 2.6 Let $K \subset \overline{\tilde{Q}_1(0, 0)}$ be a compact set and $1 < p \leq (N+2)/2$. Then, there exists a function $\varphi \in C_c^\infty(\tilde{Q}_2(0, 0))$, $0 \leq \varphi \leq 1$ and $\varphi|_D = 1$ for some open set $D \supset K$ such that

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\nabla\varphi|^p + |\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \text{Cap}_{2,1,p}(K). \quad (2.4)$$

We will give proofs of the above two propositions in the Appendix.

It is well know that there exists a semigroup $e^{t\Delta}$ corresponding to equation

$$\begin{aligned} \partial_t u - \Delta u &= \mu & \text{in } \tilde{Q}_R(0, 0), \\ u &= 0 & \text{on } \partial_p \tilde{Q}_R(0, 0) \end{aligned} \quad (2.5)$$

with $\mu \in C^\infty(B_R(0) \times (0, R^2))$, i.e, we can write a solution u of (2.5) as follows

$$u(x, t) = \int_0^t (e^{(t-s)\Delta} \mu)(x, s) ds \quad \text{for all } (x, t) \in \tilde{Q}_R(0, 0).$$

We denote by \mathbb{H} the heat kernel:

$$\mathbb{H}(x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}} \chi_{t>0}.$$

We have

$$|u(x, t)| \leq (\mathbb{H} * \mu)(x, t) \quad \text{for all } (x, t) \in \tilde{Q}_R(0, 0).$$

In [11] we show that

$$|(\mathbb{H} * \mu)|(x, t) \leq C_1(N) \mathbb{I}_2^{2R}[|\mu|](x, t) \quad \text{for all } (x, t) \in \tilde{Q}_R(0, 0).$$

Here μ is extended by 0 in $(\tilde{Q}_R(0, 0))^c$. Thus,

$$\left| \int_0^t \left(e^{(t-s)\Delta} \mu \right) (x, s) ds \right| \leq C_1(N) \mathbb{I}_2^{2R}[|\mu|](x, t) \quad \text{for all } (x, t) \in \tilde{Q}_R(0, 0). \quad (2.6)$$

Moreover, we also prove in [11], that if $\mu \geq 0$ then for $(x, t) \in \tilde{Q}_R(0, 0)$ and $B_\rho(x) \subset B_R(0)$,

$$\int_0^t \left(e^{(t-s)\Delta} \mu \right) (x, s) ds \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N}, \quad (2.7)$$

with $\rho_k = 4^{-k}\rho$.

It is easy to see that estimates (2.6) and (2.7) also holds for any bounded Radon measure μ in $\tilde{Q}_R(0, 0)$. The following result is proved in [3] and [8], also see [11].

Theorem 2.7 *Let $q > 1$, $R > 0$ and μ be bounded Radon measure in $\tilde{Q}_R(0, 0)$.*

(i) *If μ is absolutely continuous with respect to $Cap_{2,1,q'}$ in $\tilde{Q}_R(0, 0)$, then there exists a unique weak solution u to equations*

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u &= \mu && \text{in } \tilde{Q}_R(0, 0), \\ u &= 0 && \text{on } \partial_p \tilde{Q}_R(0, 0). \end{aligned}$$

(ii) *If $\exp(C_1(N) \mathbb{I}_2^{2R}[|\mu|]) \in L^1(\tilde{Q}_R(0, 0))$ then there exists a unique weak solution v to equations*

$$\begin{aligned} \partial_t v - \Delta v + \text{sign}(v)(e^{|v|} - 1) &= \mu && \text{in } \tilde{Q}_R(0, 0), \\ v &= 0 && \text{on } \partial_p \tilde{Q}_R(0, 0). \end{aligned}$$

where the constant $C_1(N)$ is the one of inequality (2.6).

From estimates (2.6) and (2.7) and using comparison principle we get the estimates from below of the solutions u and v obtained in Theorem 2.7.

Proposition 2.8 *If $\mu \geq 0$ then the functions u and v of the previous theorem are nonnegative and satisfy*

$$u(x, t) \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N)^{q+1} \mathbb{I}_2^{2R} \left[(\mathbb{I}_2^{2R}[\mu])^q \right] (x, t) \quad (2.8)$$

and

$$v(x, t) \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N) \mathbb{I}_2^{2R} \left[\exp(C_1(N) \mathbb{I}_2^{2R}[\mu]) - 1 \right] (x, t). \quad (2.9)$$

for any $(x, t) \in \tilde{Q}_R(0, 0)$ and $B_\rho(x) \subset B_R(0)$ and $\rho_k = 4^{-k}\rho$.

3 Maximal solutions

In this section we assume that O is a arbitrary, non-cylindrical and bounded open set in \mathbb{R}^{N+1} and $q > 1$. We will prove the existence of a maximal solution of

$$\partial_t u - \Delta u + u^q = 0 \quad (3.1)$$

in O . We also get analogous result where u^q is replace by $e^u - 1$.

It is easy to see that if u satisfies (3.1) in $\tilde{Q}_r(0, 0)$ ($Q_r(0, 0)$) then $u_a(x, t) = a^{-2/(q-1)}u(ax, a^2t)$ satisfies (3.1) in $\tilde{Q}_{r/a}(0, 0)$ ($Q_{r/a}(0, 0)$) for any $a > 0$.

If $X = (x, t) \in O$, the parabolic distance from X to the parabolic boundary $\partial_p O$ of O is defined by

$$d(X, \partial_p O) = \inf_{\substack{(y, s) \in \partial_p O \\ s \leq t}} \max\{|x - y|, (t - s)^{\frac{1}{2}}\}.$$

It is easy to see that there exists $C = C(N, q) > 0$ such that the function V defined by

$$V(x, t) = C \left((\rho^2 + t)^{-\frac{1}{q-1}} + \left(\frac{\rho^2 - |x|^2}{\rho} \right)^{-\frac{2}{q-1}} \right) \text{ in } B_\rho(0) \times (-\rho^2, 0)$$

satisfies

$$\partial_t V - \Delta V + V^q \geq 0 \text{ in } B_\rho(0) \times (-\rho^2, 0) \quad (3.2)$$

Proposition 3.1 *There exists a maximal solution $u \in C^{2,1}(O)$ of (3.1) and it satisfies*

$$u(x, t) \leq C(d((x, t), \partial_p O))^{-\frac{2}{q-1}} \text{ for all } (x, t) \in O. \quad (3.3)$$

for some $C = C(N, q)$

Proof. Let \mathcal{D}_k , $k \in \mathbb{Z}$ be the collection of all the dyadic parabolic cubes (abridged p -cubes) of the form

$$\{(x_1, \dots, x_N, t) : m_j 2^{-k} \leq x_j \leq (m_j + 1)2^{-k}, j = 1, \dots, N, m_{N+1} 4^{-k} \leq t \leq (m_{N+1} + 1)4^{-k}\}$$

where $m_j \in \mathbb{Z}$. The following properties hold,

- a. for each integer k , \mathcal{D}_k is a partition of \mathbb{R}^{N+1} and all p -cubes in \mathcal{D}_k have the same sidelengths.
- b. if the interiors of two p -cubes Q in \mathcal{D}_{k_1} and P in \mathcal{D}_{k_2} , denoted $\overset{\circ}{Q}, \overset{\circ}{P}$, have nonempty intersection then either Q is contained in P or P contains Q .
- c. Each Q in \mathcal{D}_k is union of 2^{N+2} p -cubes in \mathcal{D}_{k+1} with disjoint interiors.

Let $k_0 \in \mathbb{N}$ be such that $Q \subset D$ for some $Q \in \mathcal{D}_{k_0}$. Set $O_k = \bigcup_{\substack{Q \in \mathcal{D}_k \\ Q \subset O}} Q \quad \forall k \geq k_0$, we

have $O_k \subset O_{k+1}$ and $O = \bigcup_{k \geq k_0} O_k = \bigcup_{k \geq k_0} \overset{\circ}{O}_k$. More precisely, there exist real numbers $a_1, a_2, \dots, a_{n(k)}$ and open sets $\Omega_1, \Omega_2, \dots, \Omega_{n(k)}$ in \mathbb{R}^N such that

$$a_i < a_i + 4^{-k} \leq a_{i+1} < a_{i+1} + 4^k \text{ for } i = 1, \dots, n(k) - 1$$

and

$$\overset{\circ}{O}_k = \bigcup_{i=1}^{n(k)-1} (\Omega_i \times (a_i, a_i + 4^{-k})) \bigcup (\Omega_{n(k)} \times (a_{n(k)}, a_{n(k)} + 4^{-k})).$$

For $k \geq k_0$, we will show that there exist a solution $u_k \in C^{2,1}(\overset{\circ}{O}_k)$ to problem

$$\begin{aligned} \partial_t u_k - \Delta u_k + u_k^q &= 0 & \text{in } \overset{\circ}{O}_k, \\ u_k(x, t) &\rightarrow \infty & \text{as } d((x, t), \partial_p \overset{\circ}{O}_k) \rightarrow 0. \end{aligned} \quad (3.4)$$

Indeed, by [5, 7] for $m > 0$ one can find nonnegative solutions $v_i \in C^{2,1}(\Omega_i \times (a_i, a_i + 4^{-k})) \cap C(\bar{\Omega}_i \times [a_i, a_i + 4^{-k}])$ for $i = 1, \dots, n(k)$ to equations

$$\begin{aligned} \partial_t v_1 - \Delta v_1 + v_1^q &= 0 && \text{in } \Omega_1 \times (a_1, a_1 + 4^{-k}), \\ v_1(x, t) &= m && \text{on } \partial\Omega_1 \times (a_1, a_1 + 4^{-k}), \\ v_1(x, t_1) &= m && \text{in } \Omega_1, \end{aligned}$$

and

$$\begin{aligned} \partial_t v_i - \Delta v_i + v_i^q &= 0 && \text{in } \Omega_i \times (a_i, a_i + 4^{-k}), \\ v_i(x, t) &= m && \text{on } \partial\Omega_i \times (a_i, a_i + 4^{-k}), \\ v_i(x, t_i) &= \begin{cases} m & \text{in } \Omega_i \\ m\chi_{\Omega_i \setminus \Omega_{i-1}}(x) + v_{i-1}(x, a_{i-1} + 4^{-k})\chi_{\Omega_{i-1}}(x) & \text{if } a_i > a_{i-1} + 4^{-k}, \\ & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly,

$$u_{k,m} = v_i \text{ in } \Omega_i \times (a_i, a_i + 4^{-k}] \text{ for } i = 1, \dots, n(k)$$

is a solution in $C^{2,1}(\mathring{O}_k) \cap C(O_k)$ to equation

$$\begin{cases} \partial_t u_{k,m} - \Delta u_{k,m} + u_{k,m}^q = 0 & \text{in } \mathring{O}_k, \\ u_{k,m} = m & \text{on } \partial_p \mathring{O}_k. \end{cases}$$

Moreover, for $(x, t) \in \mathring{O}_k$, we can see that $B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t) \subset \mathring{O}_k$ where $d = d((x, t), \partial_p \mathring{O}_k)$. From (3.2), we verify that

$$U(y, s) = V(y - x, s - t) = C \left((\rho^2 + s - t)^{-\frac{1}{q-1}} + \left(\frac{\rho^2 - |x - y|^2}{\rho} \right)^{-\frac{2}{q-1}} \right)$$

with $\rho = d/2$, satisfies

$$\partial_t U - \Delta U + U^q \geq 0 \text{ in } B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t). \quad (3.5)$$

Applying the comparison principle we get

$$u_{k,m}(y, s) \leq U(y, s) \text{ in } B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t],$$

which implies

$$u_{k,m}(x, t) \leq C \left(d((x, t), \partial_p \mathring{O}_k) \right)^{-\frac{2}{q-1}} \text{ for all } (x, t) \in \mathring{O}_k. \quad (3.6)$$

From this, we also obtain uniform local bounds for $\{u_{k,m}\}_m$. By standard regularity theory see [5, 7], $\{u_{k,m}\}_m$ is uniformly locally bounded in $C^{2,1}$. Hence, up to a subsequence, $u_{k,m} \rightarrow u_k \in C_{\text{loc}}^{1,0}(\mathring{O}_k)$. Passing the limit, we derive that u_k is a weak solution of (3.4) in \mathring{O}_k , which satisfies $u_k(x, t) \rightarrow \infty$ as $d((x, t), \partial_p \mathring{O}_k) \rightarrow 0$ and

$$u_k(x, t) \leq C \left(d((x, t), \partial_p \mathring{O}_k) \right)^{-\frac{2}{q-1}} \text{ for all } (x, t) \in \mathring{O}_k.$$

Let $m > 0$ and $k \geq k_0$. Since $u_{k+1,m} \leq m$ in O_k and $O_k \subset O_{k+1}$, it follows by the comparison principle applied to $u_{k+1,m}$ and $u_{k,m}$ in the sub-domains $\Omega_1 \times (a_1, a_1 + 4^{-k})$, $\Omega_2 \times (a_2, a_2 + 4^{-k})$, ..., $\Omega_{n(k)} \times (a_{n(k)}, a_{n(k)} + 4^{-k})$ of \mathring{O}_k to obtain at end that $u_{k+1,m} \leq u_{k,m}$ in \mathring{O}_k , and thus $u_{k+1} \leq u_k$ in \mathring{O}_k . In particular, $\{u_k\}_k$ is uniformly locally bounded in L_{loc}^∞ . We use the same compactness property as above to obtain that $u_k \rightarrow u$ where u is a solution of (3.1) and satisfies (3.3). By construction u is the maximal solution. \blacksquare

Remark 3.2 Let $R \geq 2r \geq 2$, K be a compact subset in $\overline{\tilde{Q}_r(0,0)}$. Arguing as one can easily it is clear that there exists a maximal solution of

$$\begin{aligned} \partial_t u - \Delta u + u^q &= 0 & \text{in } \tilde{Q}_R(0,0) \setminus K, \\ u &= 0 & \text{on } \partial_p \tilde{Q}_R(0,0), \end{aligned} \quad (3.7)$$

which satisfies

$$u(x, t) \leq C(d((x, t), \partial_p(\tilde{Q}_R(0,0) \setminus K)))^{-\frac{2}{q-1}} \quad \forall (x, t) \in \tilde{Q}_R(0,0) \setminus K, \quad (3.8)$$

for some $C = C(N, q)$. Furthermore, assume K_1, K_2, \dots, K_m are compact subsets in $\overline{\tilde{Q}_r(0,0)}$ and $K = K_1 \cup \dots \cup K_m$. Let u, u_1, \dots, u_m be the maximal solutions of (3.7) in $\tilde{Q}_R(0,0) \setminus K, \tilde{Q}_R(0,0) \setminus K_1, \tilde{Q}_R(0,0) \setminus K_2, \dots, \tilde{Q}_R(0,0) \setminus K_m$, respectively, then

$$u \leq \sum_{j=1}^m u_j \quad \text{in } \tilde{Q}_R \setminus K. \quad (3.9)$$

Remark 3.3 If the equation (3.1) admits a large solution for some $q > 1$ then for any $1 < q_1 < q$, equation

$$\partial_t u - \Delta u + u^{q_1} = 0 \quad \text{in } O \quad (3.10)$$

admits also a large solution.

Indeed, assume that u is a large solution of (3.1) and v is the maximal solution of (3.10). Take $R > 0$ such that $O \subset B_R(0) \times (-R^2, R^2)$, then the function V defined by

$$V(x, t) = (q-1)^{-\frac{1}{q-1}} (2R^2 + t)^{-\frac{1}{q-1}},$$

satisfies (3.1). It follows for all $(x, t) \in O$

$$u(x, t) \geq \inf_{(y,s) \in O} V(x, t) \geq (q-1)^{-\frac{1}{q-1}} R^{-\frac{2}{q-1}} =: a_0.$$

Thus, $\tilde{u} = a_0^{-\frac{q-q_1}{q_1-1}} u$ is a subsolution of (3.10). Therefore $v \geq a_0^{-\frac{q-q_1}{q_1-1}} u$ in O , thus v is a large solution.

Remark 3.4 (Sub-critical case) Assume that $1 < q < q_*$. One easily see that the function

$$U(x, t) = \frac{C}{t^{\frac{1}{q-1}}} e^{\frac{|x|^2}{4t}} \chi_{t>0} \quad (3.11)$$

is a subsolution of (3.1) in $\mathbb{R}^{N+1} \setminus \{(0,0)\}$, where $C = \left(\frac{2}{q-1} - \frac{N}{2}\right)^{\frac{1}{q-1}}$.

Therefore, the maximal solutions u of (3.1) in O verify

$$u(x, t) \geq C \frac{1}{(t-s)^{\frac{1}{q-1}}} e^{\frac{|x-y|^2}{4(t-s)}} \chi_{t>s}, \quad (3.12)$$

for all $(x, t) \in O$ and $(y, s) \in \partial_p O$.

Remark 3.5 Note that if $u \in C^{2,1}(O)$ is a solution of (3.1) for some $q > 1$ then, for $a, b > 0$ and $1 < p \leq 2$, $v = b^{-\frac{1}{q-1}} u$ is a super-solution of

$$\partial_t v - \Delta v + a|\nabla v|^p + bv^q = 0 \quad \text{in } O. \quad (3.13)$$

Thus, we can apply the argument of the previous proof, with equation (3.13) replaced by (3.1), to deduce that there exists a maximal solution $v \in C^{2,1}(O)$ of (3.13) satisfying

$$v(x, t) \leq Cb^{-\frac{1}{q-1}}(d((x, t), \partial_p O))^{-\frac{2}{q-1}} \quad \text{for all } (x, t) \in O.$$

Furthermore, if $1 < q < q_*$, $q = \frac{2p}{p+1}$, $a, b > 0$ then the function U in Remark 3.4 is a subsolution of (3.13) in $\mathbb{R}^{N+1} \setminus \{(0, 0)\}$, for some $C = C(N, p, q, a, b)$. Therefore, we conclude that every maximal solution of $v \in C^{2,1}(O)$ of (3.13) satisfy

$$u(x, t) \geq C \frac{1}{(t-s)^{\frac{1}{q-1}}} e^{\frac{|x-y|^2}{4(t-s)}} \chi_{t>s} \quad (3.14)$$

for all $(x, t) \in O$ and $(y, s) \in \partial_p O$.

Next, we consider the following equation

$$\partial_t u - \Delta u + e^u - 1 = 0. \quad (3.15)$$

It is easy to see that the two functions

$$V_1(t) = -\log\left(\frac{t + \rho^2}{1 + \rho^2}\right) \quad \text{and} \quad V_2(x) = C_1 - 2 \log\left(\frac{\rho^2 - |x|^2}{\rho}\right)$$

satisfy

$$V_1' + e^{V_1} - 1 \geq 0 \quad \text{in } (-\rho^2, 0]$$

and

$$-\Delta V_2 + e^{V_2} - 1 \geq 0 \quad \text{in } B_\rho(0)$$

for some $C = C(N)$. Using $e^a + e^b \leq e^{a+b} - 1$ for $a, b \geq 0$, we obtain that $V_1 + V_2$ is a supersolution of equation (3.15) in $B_\rho(0) \times (-\rho^2, 0]$. By the same argument as in Proposition 3.1 and the estimate of the above supersolution, we obtain

Proposition 3.6 *There exists a maximal solution $u \in C^{2,1}(O)$ of*

$$\partial_t u - \Delta u + e^u - 1 = 0 \quad \text{in } O \quad (3.16)$$

and it satisfies

$$u(x, t) \leq C - \log\left(\frac{(d((x, t), \partial_p O))^3}{4 + (d((x, t), \partial_p O))^2}\right) \quad \text{for all } (x, t) \in O, \quad (3.17)$$

for some $C = C(N)$.

The next three propositions will be useful to prove Theorem 1.1-(ii).

Proposition 3.7 *Let $K \subset \overline{\tilde{Q}_1(0, 0)}$ be a compact set and $q > 1$, $R \geq 100$. Let u be a solution of (3.7) in $\tilde{Q}_R(0, 0) \setminus K$ and φ as in Proposition 2.6 with $p = q'$. Set $\xi = (1 - \varphi)^{2q'}$. Then,*

$$\int_{\tilde{Q}_R(0, 0)} u (|\Delta \xi| + |\nabla \xi| + |\partial_t \xi|) dx dt \lesssim \text{Cap}_{2,1,q'}(K) \quad (3.18)$$

and

$$u(x, t) \lesssim \text{Cap}_{2,1,q'}(K) + R^{-\frac{2}{q-1}} \quad \text{for any } (x, t) \in \tilde{Q}_{R/5}(0, 0) \setminus \tilde{Q}_2(0, 0), \quad (3.19)$$

$$\int_{\tilde{Q}_2(0, 0)} u \xi dx dt \lesssim \text{Cap}_{2,1,q'}(K) + R^{-\frac{2}{q-1}} \quad (3.20)$$

where constants in above inequalities only depend on N, q .

Proof. *Step 1.* First, we need to show that

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \text{Cap}_{2,1,q'}(K). \quad (3.21)$$

Actually, using by parts integration and the Green formula, one has

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u_R^q \xi dxdt &= - \int_{\tilde{Q}_R(0,0)} \partial_t u \xi dxdt + \int_{\tilde{Q}_R(0,0)} (\Delta u) \xi dxdt \\ &= \int_{\tilde{Q}_R(0,0)} u \partial_t \xi dxdt + \int_{\tilde{Q}_R(0,0)} u \Delta \xi dxdt + \int_{-R^2}^{R^2} \int_{\partial B_R(0)} \left(\xi \frac{\partial u}{\partial \nu} - u \frac{\partial \xi}{\partial \nu} \right) dS dt \end{aligned}$$

where ν is the outer normal unit vector on $\partial B_R(0)$. Clearly,

$$\frac{\partial u}{\partial \nu} \leq 0 \quad \text{and} \quad \frac{\partial \xi}{\partial \nu} = 0 \quad \text{on} \quad \partial B_R(0).$$

Thus,

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt &\leq \int_{\tilde{Q}_R(0,0)} u |\partial_t \xi| dxdt + \int_{\tilde{Q}_R(0,0)} u |\Delta \xi| dxdt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u(1-\varphi)^{2q'-1} |\partial_t \varphi| dxdt + 2q'(2q'-1) \int_{\tilde{Q}_R(0,0)} u(1-\varphi)^{2q'-2} |\nabla \varphi|^2 dxdt \\ &\quad + 2q' \int_{\tilde{Q}_R(0,0)} u(1-\varphi)^{2q'-1} |\Delta \varphi| dxdt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\partial_t \varphi| dxdt + 2q'(2q'-1) \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\nabla \varphi|^2 dxdt \\ &\quad + 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\Delta \varphi| dxdt. \end{aligned} \quad (3.22)$$

In the last inequality, we have used the fact that $(1-\phi)^{2q'-1} \leq (1-\phi)^{2q'-1} = \xi^{1/q}$. Hence, by Hölder inequality,

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt &\lesssim \int_{\tilde{Q}_R(0,0)} |\partial_t \varphi|^{q'} dxdt + \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt \\ &\quad + \int_{\tilde{Q}_R(0,0)} |\Delta \varphi|^{q'} dxdt. \end{aligned}$$

By the Gagliardo-Nirenberg inequality,

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt &\lesssim \|\varphi\|_{L^\infty(\tilde{Q}_R(0,0))}^{q'} \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt \\ &\lesssim \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt. \end{aligned}$$

Hence, we find

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \int_{\tilde{Q}_R(0,0)} (|\partial_t \varphi|^{q'} + |D^2 \varphi|^{q'}) dxdt$$

and derive (3.21) from (2.4). In view of (3.22), we also obtain

$$\int_{\tilde{Q}_R(0,0)} u (|\Delta \xi| + |\partial_t \xi|) dxdt, \lesssim \text{Cap}_{2,1,q'}(K)$$

and

$$\int_{\tilde{Q}_R(0,0)} u|\nabla\xi|dxdt, \lesssim \text{Cap}_{2,1,q'}(K),$$

since

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u|\nabla\xi|dxdt &= 2q' \int_{\tilde{Q}_R(0,0)} u\xi^{(2q'-1)/2q'} |\nabla\varphi|dxdt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u\xi^{1/q} |\nabla\varphi|dxdt \\ &\lesssim \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt + \int_{\tilde{Q}_R(0,0)} |\nabla\varphi|^{q'} dxdt. \end{aligned}$$

It yields (3.18).

Step 2. Let η be a cut off function on $\tilde{Q}_{R/4}(0,0)$ with respect to $\tilde{Q}_{R/3}(0,0)$ such that $|\partial_t\eta| + |D^2\eta| \lesssim R^{-2}$ and $|\nabla\eta| \lesssim R^{-1}$. We have

$$\partial_t(\eta\xi u) - \Delta(\eta\xi u) = F \in C_c(\tilde{Q}_{R/3}(0,0)).$$

Hence, we can write

$$(\eta\xi u)(x,t) = \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} F(y,s) ds dy \quad \forall (x,t) \in \mathbb{R}^{N+1}.$$

Now, we fix $(x,t) \in \tilde{Q}_{R/5}(0,0) \setminus \tilde{Q}_2(0,0)$. Since $\text{supp}\{|\nabla\eta|\} \cap \text{supp}\{|\nabla\xi|\} = \emptyset$ and

$$\begin{aligned} F &= \eta\xi(\partial_t u - \Delta u) + 2(\eta\nabla\xi + \xi\nabla\eta)\nabla u + (\xi\partial_t\eta + \eta\partial_t\xi + 2\nabla\eta\nabla\xi + \Delta\eta\xi + \eta\Delta\xi)u \\ &\leq 2(\eta\nabla\xi + \xi\nabla\eta)\nabla u + (\xi\partial_t\eta + \eta\partial_t\xi + \xi\Delta\eta + \eta\Delta\xi)u, \end{aligned}$$

there holds

$$\begin{aligned} u(x,t) &= (\eta\xi u)(x,t) \leq 2 \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta\nabla\xi + \xi\nabla\eta)\nabla u ds dy \\ &\quad + \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta\partial_t\xi + \eta\Delta\xi) u ds dy \\ &\quad + \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} (\partial_t\eta\xi + \xi\Delta\eta) u ds dy. \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By parts integration

$$\begin{aligned} I_1 &= -2(4\pi)^{-N/2} \int_{-\infty}^t \int_{\mathbb{R}^N} \frac{(x-y)}{2(t-s)^{(N+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta\nabla\xi + \xi\nabla\eta) u dy ds \\ &\quad - 2(4\pi)^{-N/2} \int_{-\infty}^t \int_{\mathbb{R}^N} \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\xi\Delta\eta + \eta\Delta\xi) u dy ds. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} &\lesssim \left(\max\{|x-y|, |t-s|^{1/2}\}\right)^{-N}, \\ \left| \frac{(x-y)}{2(t-s)^{(N+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \right| &\lesssim \left(\max\{|x-y|, |t-s|^{1/2}\}\right)^{-N-1}, \end{aligned}$$

and

$$\begin{aligned} \max\{|x-y|, |t-s|^{1/2}\} &\gtrsim 1 \quad \forall (y, s) \in \text{supp}\{|D^\alpha \xi|\} \cup \text{supp}\{|\partial_t \xi|\}, \\ \max\{|x-y|, |t-s|^{1/2}\} &\gtrsim R \quad \forall (y, s) \in \text{supp}\{|D^\alpha \eta|\} \cup \text{supp}\{|\partial_t \eta|\} \quad \forall |\alpha| \geq 1. \end{aligned}$$

We deduce

$$\begin{aligned} I_1 &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N-1} (\eta|\nabla \xi| + \xi|\nabla \eta|) u \, dyds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\xi|\Delta \eta| + \eta|\Delta \xi|) u \, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\nabla \xi| + |\Delta \xi|) u \, dyds + \int_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} (R^{-N-1}|\nabla \eta| + R^{-N}|\Delta \eta|) u \, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\nabla \xi| + |\Delta \xi|) u \, dyds + \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u, \\ I_2 &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (|\partial_t \xi| + |\Delta \xi|) u \, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\Delta \xi|) u \, dyds, \end{aligned}$$

and

$$\begin{aligned} I_3 &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (|\partial_t \eta| + |\Delta \eta|) u \, dyds \\ &\lesssim \int_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} R^{-N} (|\partial_t \eta| + |\Delta \eta|) u \, dyds \\ &\lesssim \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u. \end{aligned}$$

Hence,

$$u(x, t) \leq I_1 + I_2 + I_3 \lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\nabla \xi| + |\Delta \xi|) u \, dyds + \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u.$$

Combining this with (3.18) and (3.8), we obtain (3.19).

Step 3. Let θ be a cut off function on $\tilde{Q}_3(0,0)$ with respect to $\tilde{Q}_4(0,0)$. As above, we have for any $(x, t) \in \mathbb{R}^{N+1}$

$$\begin{aligned} (\theta \xi u)(x, t) &\lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N-1} (\theta|\nabla \xi| + \xi|\nabla \theta|) u \, dyds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\theta|\Delta \xi| + \xi|\Delta \theta|) u \, dyds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\theta|\partial_t \xi| + \theta|\Delta \xi|) u \, dyds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\xi|\partial_t \theta| + \xi|\Delta \theta|) u \, dyds. \end{aligned}$$

Hence, by Fubini theorem,

$$\begin{aligned} \int_{\tilde{Q}_2(0,0)} \eta u \, dxdt &= \int_{\tilde{Q}_2(0,0)} \theta \eta u \, dxdt \\ &\lesssim A \int_{\mathbb{R}^{N+1}} (\theta|\nabla \xi| + \xi|\nabla \theta| + \theta|\Delta \xi| + \xi|\Delta \theta| + \theta|\partial_t \xi| + \xi|\partial_t \theta|) u \, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\nabla \xi| + |\Delta \xi|) u \, dyds + \sup_{\tilde{Q}_4(0,0) \setminus \tilde{Q}_3(0,0)} u \end{aligned}$$

where

$$A = \sup_{(y,s) \in \tilde{Q}_4(0,0)} \int_{\tilde{Q}_2(0,0)} ((\max\{|x-y|, |t-s|^{1/2}\})^{-N} + (\max\{|x-y|, |t-s|^{1/2}\})^{-N-1}) dx dt$$

Therefore we obtain (3.20) from (3.18) and (3.19). \blacksquare

Proposition 3.8 *Let $K \subset \{(x, t) : \varepsilon < \max\{|x|, |t|^{1/2}\} < 1\}$ be a compact set, $0 < \varepsilon < 1$ and let u be the maximal solution of (3.7) in $\tilde{Q}_R(0, 0) \setminus K$ with $R \geq 100$. Then*

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=-2}^{j_\varepsilon-2} \frac{\text{Cap}_{2,1,q'}(K \cap \tilde{Q}_{\rho_j}(0,0))}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if } q > q_*, \quad (3.23)$$

and

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=0}^{j_\varepsilon} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if } q = q_*, \quad (3.24)$$

where $\rho_j = 2^{-j}$, $K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : (x, t) \in K \cap \tilde{Q}_{\rho_{j-2}}(0, 0)\}$ and $j_\varepsilon \in \mathbb{N}$ is such that $\rho_{j_\varepsilon} \leq \varepsilon < \rho_{j_\varepsilon-1}$.

Proof. For $j \in \mathbb{N}$, we define $S_j = \{x : \rho_j \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-1}\}$. Fix any $1 \leq j \leq j_\varepsilon$. We cover S_j by $L = L(N) \in \mathbb{N}^*$ closed cylinders

$$\overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}, \quad k = 1, \dots, L(N)$$

where $(x_{k,j}, t_{k,j}) \in S_j$.

For $k = 1, \dots, L(N)$, let $u_j, u_{k,j}$ be the maximal solutions of (3.7) where K is replaced by $K \cap S_j$ and $K \cap \overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}$, respectively. Clearly the function $\tilde{u}_{k,j}$ defined by

$$\tilde{u}_{k,j}(x, t) = \rho_{j+3}^{\frac{2}{q-1}} u_{k,j}(\rho_{j+3}x + x_{k,j}, \rho_{j+3}^2t + t_{k,j})$$

is the maximal solution of (3.7) when $(K, \tilde{Q}_R(0, 0))$ is replaced by $(K_{k,j}, \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}, -t_{k,j}))$, with

$$K_{k,j} = \{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y, s) \in -(x_{k,j}, t_{k,j}) + K \cap \overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}\} \subset \tilde{Q}_1(0, 0).$$

Let $\bar{u}_{k,j}$ be the maximal solution of (3.7) with $(K, \tilde{Q}_R(0, 0))$ replaced by $(K_{k,j}, \tilde{Q}_{2R/\rho_{j+3}}(0, 0))$. Since $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}, -t_{k,j}) \subset \tilde{Q}_{2R/\rho_{j+3}}(0, 0)$, thus using the comparison principle as in the proof of Proposition 3.1 we obtain $\tilde{u}_{k,j} \leq \bar{u}_{k,j}$ in $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}, -t_{k,j}) \setminus K_{k,j}$ and thus

$$\tilde{u}_{k,j}(x, t) \lesssim \text{Cap}_{2,1,q'}(K_{k,j}) + (R/\rho_{j+3})^{-\frac{2}{q-1}},$$

for any $(x, t) \in (\tilde{Q}_{R/(5\rho_{j+3})}(0, 0) \cap \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}, -t_{k,j})) \setminus \tilde{Q}_2(0, 0) = D$.

Fix $(x_0, t_0) \in \tilde{Q}_{\varepsilon/4}(0, 0)$. Clearly, $((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}^2) \in D$, hence

$$u_{k,j}(x_0, t_0) = \rho_{j+3}^{-\frac{2}{q-1}} \tilde{u}_{k,j}((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}^2) \lesssim \frac{\text{Cap}_{2,1,q'}(K_{k,j})}{\rho_j^{\frac{2}{q-1}}} + R^{-\frac{2}{q-1}}.$$

Therefore, using (3.9) in Remark (3.2) and the fact that

$$\text{Cap}_{2,1,q'}(K_{k,j}) = \text{Cap}_{2,1,q'}(K_{k,j} + (x_{k,j}/\rho_{j+3}, t_{k,j}/\rho_{j+3}^2)) \leq \text{Cap}_{2,1,q'}(K_j),$$

we derive

$$\begin{aligned} u(x_0, t_0) &\leq \sum_{j=1}^{j_\varepsilon} u_j(x_0, t_0) \leq \sum_{j=1}^{j_\varepsilon} \sum_{k=1}^{L(N)} u_{k,j}(x_0, t_0) \\ &\lesssim \sum_{j=0}^{j_\varepsilon} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^{\frac{2}{q-1}}} + j_\varepsilon R^{-\frac{2}{q-1}}, \end{aligned}$$

which yields (3.24). If $q > q_*$, then by (2.2) in Proposition (2.5), we have

$$\text{Cap}_{2,1,q'}(K_j) \lesssim \rho_{j+3}^{-N-2+2q'} \text{Cap}_{2,1,q'}(K \cap \tilde{Q}_{\rho_{j-2}}(0,0)),$$

which implies (3.23). ■

Proposition 3.9 *Let K, u, ξ be in Lemma 3.7. For any compact set K_0 in $\overline{\tilde{Q}_1(0,0)}$ with positive measure $|K_0|$, there exists $\varepsilon = \varepsilon(N, q, |K_0|) > 0$ such that*

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow \inf_{K_0} u \lesssim \int_{\tilde{Q}_2(0,0)} u \xi dx dt.$$

where the constant in the inequality \lesssim depends on K_0 . In particular,

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow \inf_{K_0} u \lesssim \text{Cap}_{2,1,q'}(K) + R^{-\frac{2}{q-1}}. \quad (3.25)$$

Proof. It is enough to assert that there is $\varepsilon > 0$ such that

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_1| \geq 1/2 |K_0| \quad (3.26)$$

where $K_1 = \{(x, t) \in K_0 : \xi(x, t) \geq 1/2\}$. By (2.1) in Proposition (2.5), we have

$$\begin{aligned} |K_0 \setminus K_1|^{1-\frac{2q'}{N+2}} &\lesssim \text{Cap}_{2,1,q'}(K_0 \setminus K_1) \quad \text{if } q > q_*, \text{ and} \\ \left(\log \left(\frac{|\tilde{Q}_{100}(0,0)|}{|K_0 \setminus K_1|} \right) \right)^{-\frac{N}{2}} &\lesssim \text{Cap}_{2,1,q'}(K_0 \setminus K_1) \quad \text{if } q = q_*. \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{Cap}_{2,1,q'}(K_0 \setminus K_1) &= \text{Cap}_{2,1,q'}(\{K_0 : \varphi > 1 - (1/2)^{1/(2q')}\}) \\ &\leq (1 - (1/2)^{1/(2q')})^{-q'} \int_{\mathbb{R}^{N+1}} \left(|D^2 \varphi|^{q'} + |\nabla \varphi|^{q'} + |\varphi|^{q'} + |\partial_t \varphi|^{q'} \right) dx dt \\ &\lesssim \text{Cap}_{2,1,q'}(K) \end{aligned}$$

So, one can find $\varepsilon = \varepsilon(N, q, |K_0|) > 0$ such that

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_0 \setminus K_1| \leq 1/2 |K_0|.$$

This implies (3.26). ■

4 Large solutions

In the first part of this section, we prove theorem 1.1-(ii), then we prove theorems 1.1-(i) and 1.2, at end we consider a parabolic viscous Hamilton-Jacobi equation.

4.1 Proof of Theorem 1.1-(ii)

Let $R_0 \geq 4$ such that $O \subset\subset \tilde{Q}_{R_0}(0,0)$. Assume that the equation (1.10) is a large solution u . Take any $(x,t) \in \partial_p O$. We will to prove that (1.12) holds. We can assume $(x,t) = (0,0)$. Set $K = \tilde{Q}_{2R_0}(0,0) \setminus O$ and define

$$\begin{aligned} T_j &= \{x : \rho_{j+1} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_j, t \leq 0\}, \\ \tilde{T}_j &= \{x : \rho_{j+3} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-2}, t \leq 0\}. \end{aligned}$$

Here $\rho_j = 2^{-j}$. For $j \geq 3$, let u_1, u_2, u_3, u_4 be the maximal solutions of (3.7) when K is replaced by $K \cap \overline{Q_{\rho_{j+3}}}(0,0)$, $K \cap \tilde{T}_j$, $(K \cap \overline{Q_1}(0,0)) \setminus Q_{\rho_{j-2}}(0,0)$ and $K \setminus Q_1(0,0)$ respectively and $R \geq 100R_0$. From (3.9) in Remark (3.2), we can assert that

$$u \leq u_1 + u_2 + u_3 + u_4 \quad \text{in } O \cap \{(x,t) \in \mathbb{R}^{N+1} : t \leq 0\}.$$

Thus,

$$\inf_{T_j} u \leq \|u_1\|_{L^\infty(T_j)} + \|u_3\|_{L^\infty(T_j)} + \|u_4\|_{L^\infty(T_j)} + \inf_{T_j} u_2. \quad (4.1)$$

Case 1: $q > q_*$. By (3.8) in Remark 3.2,

$$\|u_4\|_{L^\infty(T_j)} \lesssim 1. \quad (4.2)$$

By Proposition 3.8,

$$\|u_3\|_{L^\infty(T_j)} \lesssim \sum_{i=-2}^{j-4} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + jR^{-\frac{2}{q-1}}. \quad (4.3)$$

Since $(x,t) \mapsto \bar{u}_1(x,t) = \rho_{j+3}^{2/(q-1)} u_1(\rho_{j+3}x, \rho_{j+3}^2 t)$ is the maximal solution of (3.7) when $(K, \tilde{Q}_R(0,0))$ is replaced by $(\{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y,s) \in K \cap \overline{Q_{\rho_{j+3}}}(0,0)\}, \tilde{Q}_{R/\rho_{j+3}}(0,0))$, we derive, thanks to (3.19) in Proposition 3.7 and (2.2) in Proposition 2.5,

$$\|\bar{u}_1\|_{L^\infty(T_{-3})} \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j+2}}(0,0))}{\rho_j^{N+2-2q'}} + (R/\rho_{j+3})^{-\frac{2}{q-1}},$$

from which follows

$$\|u_1\|_{L^\infty(T_j)} \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j+2}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}}. \quad (4.4)$$

Since, $(x,t) \mapsto \bar{u}_2(x,t) = \rho_{j-2}^{2/(q-1)} u_2(\rho_{j-2}x, \rho_{j-2}^2 t)$ is the maximal solution of (3.7) when the couple $(K, \tilde{Q}_R(0,0))$ is replaced by $(\{(y/\rho_{j-2}, s/\rho_{j-2}^2) : (y,s) \in K \cap \tilde{T}_j\}, \tilde{Q}_{R/\rho_{j-2}}(0,0))$, Proposition 3.9 and relation (2.2) in Proposition 2.5 yield

$$\frac{\text{Cap}_{2,1,q'}(K \cap \tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \Rightarrow \inf_{T_2} \bar{u}_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap \tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}} + (R/\rho_{j-2})^{-\frac{2}{q-1}},$$

which implies

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^N} + R^{-\frac{2}{q-1}}. \quad (4.5)$$

for some $\varepsilon = \varepsilon(N, q) > 0$.

First, we assume that there exists $J \in \mathbb{N}$, $J \geq 10$ such that

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \quad \forall j \geq J.$$

Then, from (4.1) and (4.2), (4.3), (4.4), (4.5) we have

$$\inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + jR^{-\frac{2}{q-1}} + 1,$$

for any $j \geq J$. Letting $R \rightarrow \infty$,

$$\inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + 1.$$

Since $\inf_{T_j} u \rightarrow \infty$ as $j \rightarrow \infty$, we get

$$\sum_{i=0}^{\infty} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} = \infty$$

which implies that (1.12) holds with $(x, t) = (0, 0)$.

Alternatively, assume that for infinitely many j

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} > \varepsilon$$

Then,

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^N} > \rho_{j-2}^{2-2q'} \varepsilon \rightarrow \infty \quad \text{when } j \rightarrow \infty.$$

We also achieve that (1.12) holds with $(x, t) = (0, 0)$. Therefore, case $q > q_*$ proved.

Case 2: $q = q_*$. Similarly to **Case 1**, we have: for $j \geq 5$

$$\|u_4\|_{L^\infty(T_j)} \lesssim 1, \tag{4.6}$$

$$\|u_3\|_{L^\infty(T_j)} \lesssim \sum_{i=0}^{j-2} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_i^N} + jR^{-\frac{2}{q-1}}, \tag{4.7}$$

$$\|u_1\|_{L^\infty(T_j)} \lesssim \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{4.8}$$

$$\text{Cap}_{2,1,q'}(K_j) \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{4.9}$$

where $K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : (x, t) \in K \cap Q_{\rho_{j-3}}(0,0)\}$ and $\varepsilon = \varepsilon(N) > 0$.

Note that, from (2.2) in Proposition 2.5 we have

$$\frac{1}{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))} \leq \frac{c}{\text{Cap}_{2,1,q'}(K_j)} + cj^{N/2}$$

for any $j \geq 4$ where $c = c(N)$. If there are infinitely many $j \geq 4$ such that

$$\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) > \frac{1}{2cj^{N/2}},$$

then (1.12) holds with $(x, t) = (0, 0)$ since

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-3}^N} > \frac{2^{j-3}}{2cj^{N/2}} \rightarrow \infty \quad \text{when } j \rightarrow \infty.$$

Now, we assume that there exists $J \geq 5$ such that

$$\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \frac{1}{2c^j N^{j/2}}.$$

Then,

$$\text{Cap}_{2,1,q'}(K_j) \leq 2c \text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \quad \forall j \geq J.$$

This leads to

$$\text{Cap}_{2,1,q'}(K_j) \leq 2c \text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \varepsilon \quad \forall j \geq J' + J,$$

for some $J' = J'(N, q)$. Hence, from (4.6)-(4.9) we have, for any $j \geq J' + J + 3$,

$$\begin{aligned} \|u_4\|_{L^\infty(T_j)} &\lesssim 1, \\ \|u_3\|_{L^\infty(T_j)} &\lesssim \sum_{i=J'+J+1}^{j-2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{i-3}}(0,0))}{\rho_i^N} + C(J' + J) + jR^{-\frac{2}{q-1}}, \\ \|u_1\|_{L^\infty(T_j)} &\lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}}, \\ \inf_{T_j} u_2 &\lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}}, \end{aligned}$$

where $C(J' + J) = \sum_{i=0}^{J'+J} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_i^N}$.

Consequently, from (4.1) we derive

$$\inf_{T_j} u \lesssim \sum_{i=0}^j \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + C(J' + J) + 1 + jR^{-\frac{2}{q-1}} \quad \forall j \geq J' + J + 3.$$

Letting $R \rightarrow \infty$ and $j \rightarrow \infty$ we obtain

$$\sum_{i=0}^{\infty} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} = \infty,$$

i.e (1.12) holds with $(x, t) = (0, 0)$. This completes the proof of Theorem 1.1-(ii).

4.2 Proof of Theorem 1.1-(i) and Theorem 1.2

Fix $(x_0, t_0) \in \partial_p O$. We can assume that $(x_0, t_0) = 0$. Let $\delta \in (0, 1/100)$. For $(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O$, we set

$$\begin{aligned} M_k &= O^c \cap \left(\overline{B_{r_{k+2}}(y_0)} \times [s_0 - (73 + \frac{1}{2})r_{k+2}^2, s_0 - (70 + \frac{1}{2})r_{k+2}^2] \right) \quad \text{and} \\ S_k &= \{(x, t) : r_{k-1} \leq \max\{|x - y_0|, |t - s_0|^{\frac{1}{2}}\} < r_k\} \quad \text{for } k = 1, 2, \dots \end{aligned}$$

where $r_k = 4^{-k}$. Note that $M_k = \emptyset$ for k large enough and $M_k \subset S_k$ for all k . Let $R_0 \geq 4$ such that $O \subset\subset \tilde{Q}_{R_0}(0,0)$. By Theorem 2.2 and 2.4 and estimate (1.9) there exist two sequences $\{\mu_k\}_k$ and $\{\nu_k\}_k$ of nonnegative Radon measures such that

$$\text{supp}(\mu_k) \subset M_k, \quad \text{supp}(\nu_k) \subset M_k \quad \text{and} \quad (4.10)$$

$$\mu_k(M_k) \asymp \text{Cap}_{2,1,q'}(M_k) \asymp \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^{2R_0}[\mu_k] \right)^q dx dt \quad \text{and} \quad (4.11)$$

$$\nu_k(M_k) \asymp \mathcal{PH}_1^N(M_k), \quad \|\mathbb{M}_1^{2R_0}[\nu_k]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \quad \text{for } k = 1, 2, \dots \quad (4.12)$$

where equivalent constants depend on N, q, R_0 .

Take $\varepsilon > 0$ such that $\exp\left(C_1 \varepsilon \mathbb{I}_2^{2R_0} [\sum_{k=1}^{\infty} \nu_k]\right) \in L^1(\tilde{Q}_{R_0}(0, 0))$ where the constant $C_1 = C_1(N)$ is the one of inequality (2.6). By Theorem 2.7 and Proposition 2.8, there exist two nonnegative solutions U_1, U_2 of problems

$$\begin{aligned} \partial_t U_1 - \Delta U_1 + U_1^q &= \varepsilon \sum_{k=1}^{\infty} \mu_k && \text{in } \tilde{Q}_{R_0}(0, 0), \\ U_1 &= 0 && \text{on } \partial_p \tilde{Q}_{R_0}(0, 0). \end{aligned}$$

and

$$\begin{aligned} \partial_t U_2 - \Delta U_2 + e^{U_2} - 1 &= \varepsilon \sum_{k=1}^{\infty} \nu_k && \text{in } \tilde{Q}_{R_0}(0, 0), \\ U_2 &= 0 && \text{on } \partial_p \tilde{Q}_{R_0}(0, 0), \end{aligned}$$

respectively which satisfy

$$\begin{aligned} U_1(y_0, z_0) &\gtrsim \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon \frac{\mu_k(B_{\frac{r_i}{8}}(y_0) \times (s_0 - \frac{37}{128}r_i^2, s_0 - \frac{37}{128}r_i^2))}{r_i^N} \\ &\quad - \mathbb{I}_2^{2R_0} \left[\left(\mathbb{I}_2^{2R_0} [\varepsilon \sum_{k=1}^{\infty} \mu_k] \right)^q \right] (y_0, s_0) =: A \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} U_2(y_0, z_0) &\gtrsim \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon \frac{\nu_k(B_{\frac{r_i}{8}}(y_0) \times (s_0 - \frac{37}{128}r_i^2, s_0 - \frac{37}{128}r_i^2))}{r_i^N} \\ &\quad - \mathbb{I}_2^{2R_0} \left[\exp\left(C_1 \mathbb{I}_2^{2R_0} [\varepsilon \sum_{k=1}^{\infty} \nu_k]\right) - 1 \right] (y_0, s_0) =: B \end{aligned} \quad (4.14)$$

and $U_1, U_2 \in C^{2,1}(O)$.

Let u_1, u_2 be the maximal solutions of equations (3.1) and (3.16) respectively.

We have $u_1(y_0, s_0) \geq U_1(y_0, s_0)$ and $u_2(y_0, s_0) \geq U_2(y_0, s_0)$.

Now, we claim that

$$A \gtrsim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N} \quad (4.15)$$

and

$$B \gtrsim -c_1(R_0) + \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k)}{r_k^N}. \quad (4.16)$$

Proof of assertion (4.15). From (4.11) we have

$$A \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N} - \varepsilon^q A_0 \quad (4.17)$$

with

$$A_0 = \mathbb{I}_2^{2R_0} \left[\left(\mathbb{I}_2^{2R_0} [\sum_{k=1}^{\infty} \mu_k] \right)^q \right] (y_0, s_0).$$

Take $i_0 \in \mathbb{Z}$ such that $r_{i_0+1} < \max\{2R_0, 1\} \leq r_{i_0}$. We have

$$\begin{aligned}
A_0 &\lesssim \sum_{i=i_0}^{\infty} r_i^{-N} \int_{\tilde{Q}_{r_i}(y_0, s_0)} \left(\mathbb{I}_2^{2R_0} \left[\sum_{k=1}^{\infty} \mu_k \right] \right)^q dxdt \\
&= \sum_{i=i_0}^{\infty} \sum_{j=i}^{\infty} r_i^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} \left[\sum_{k=1}^{\infty} \mu_k \right] \right)^q dxdt \\
&= \sum_{j=k_0}^{\infty} \sum_{i=i_0}^j r_i^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} \left[\sum_{k=1}^{\infty} \mu_k \right] \right)^q dxdt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} \left[\sum_{k=1}^{\infty} \mu_k \right] \right)^q dxdt.
\end{aligned}$$

Here we used $\sum_{i=i_0}^j r_i^{-N} \leq \frac{4}{3} r_j^{-N}$ for all j in the last inequality.

Setting $\mu_k \equiv 0$ for all $i_0 - 1 \leq k \leq 0$, the previous inequality becomes

$$\begin{aligned}
A_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} \left[\mu_j + \sum_{k=i_0-1}^{j-1} \mu_k + \sum_{k=j+1}^{\infty} \mu_k \right] \right)^q dxdt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0} [\mu_j] \right)^q dxdt \\
&\quad + \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \|\mathbb{I}_2^{2R_0} [\mu_k]\|_{L^\infty(S_j)} \right)^q \\
&\quad + \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=j+1}^{\infty} \|\mathbb{I}_2^{2R_0} [\mu_k]\|_{L^\infty(S_j)} \right)^q \\
&= A_1 + A_2 + A_3. \tag{4.18}
\end{aligned}$$

Using (4.11) we obtain

$$A_1 \leq \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N}. \tag{4.19}$$

Next, using (4.10) we have for any $(x, t) \in S_j$ if $k \geq j+1$,

$$\mathbb{I}_2^{2R_0} [\mu_k](x, t) = \int_{r_{j+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N} \tag{4.20}$$

and if $k \leq j-1$

$$\mathbb{I}_2^{2R_0} [\mu_k](x, t) = \int_{r_{k+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}. \tag{4.21}$$

Thus,

$$A_2 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \quad \text{and} \quad A_3 \lesssim \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q.$$

Noticing that $(a+b)^q - a^q \leq q(a+b)^{q-1}b$ for any $a, b \geq 0$, we get

$$\begin{aligned}
& (1-4^{-2}) \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
&= \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q - \sum_{j=i_0+1}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-2} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
&\leq \sum_{j=i_0}^{\infty} q r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
& (1-4^{2-Nq}) \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q \\
&\leq \sum_{j=i_0}^{\infty} q r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Thus,

$$\begin{aligned}
A_2 + A_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N} \\
&\quad + \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Since $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^{N+2-2q'}$ if $q > q_*$ and $\mu_k(\mathbb{R}^{N+1}) \lesssim \min\{k^{-\frac{1}{q-1}}, 1\}$ if $q = q_*$ for any k , we always assert that

$$\begin{aligned}
r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} &\lesssim 1 \quad \text{and} \\
r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} &\lesssim r_{j+1}^{-N} \quad \text{for any } j.
\end{aligned}$$

In the case $q = q_*$ we assume $N \geq 3$ in order to verify that

$$\sum_{j=1}^{\infty} \mu_j(\mathbb{R}^{N+1}) \lesssim \sum_{k=1}^{\infty} k^{-\frac{1}{q-1}} < \infty.$$

This leads to

$$A_2 + A_3 \lesssim \sum_{k=1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}.$$

Combining this with (4.19) and (4.18), we deduce

$$A_0 \lesssim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N}.$$

Consequently, we obtain (4.15) from (4.17), for ε small enough.

Proof of assertion (4.16). From (4.12) we get

$$B \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k)}{r_k^N} - B_0$$

where

$$B_0 = \mathbb{I}_2^{2R_0} \left[\exp \left(C_1 \mathbb{I}_2^{2R_0} \left[\varepsilon \sum_{k=1}^{\infty} \nu_k \right] \right) - 1 \right] (y_0, s_0).$$

We show that

$$B_0 \leq c(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \quad (4.22)$$

In fact, as above we have

$$B_0 \lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp \left(C_1 \varepsilon \mathbb{I}_2^{2R_0} \left[\sum_{k=1}^{\infty} \nu_k \right] \right) dx dt.$$

Thus,

$$\begin{aligned} B_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp \left(3C_1 \varepsilon \mathbb{I}_2^{2R_0} [\nu_j] \right) dx dt \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp \left(3C_1 \varepsilon \sum_{k=i_0-1}^{j-1} \|\mathbb{I}_2^{2R_0} [\nu_k]\|_{L^\infty(S_j)} \right) \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp \left(3C_1 \varepsilon \sum_{k=j+1}^{\infty} \|\mathbb{I}_2^{2R_0} [\nu_k]\|_{L^\infty(S_j)} \right) \\ &= B_1 + B_2 + B_3. \end{aligned} \quad (4.23)$$

Here we used an inequality $\exp(a + b + c) \leq \exp(3a) + \exp(3b) + \exp(3c)$ for all a, b, c . By Theorem 2.3, we have

$$\int_{S_j} \exp \left(3C_1 \varepsilon \mathbb{I}_2^{2R_0} [\nu_j] \right) dx dt \lesssim r_j^{N+2} \quad \text{for all } j,$$

for $\varepsilon > 0$ small enough. Hence,

$$B_1 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \lesssim (\max\{2R_0, 1\})^2. \quad (4.24)$$

Note that estimates (4.20) and (4.21) are also true with ν_k ; we deduce

$$\begin{aligned} B_2 + B_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \exp \left(c_2 \varepsilon \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right) \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp \left(c_2 \varepsilon \sum_{k=j+1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N} \right). \end{aligned}$$

From (4.12) we have $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^N$ for all k , therefore

$$\begin{aligned} B_2 + B_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \exp(c_3 \varepsilon(j - i_0)) + \sum_{j=i_0}^{\infty} r_j^2 \exp(c_3 \varepsilon) \\ &\lesssim \sum_{j=i_0}^{\infty} \exp(c_3 \varepsilon(j - i_0) - 4 \log(2)j) + r_{i_0}^2 \\ &\leq c_4(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \end{aligned}$$

Combining this with (4.24) and (4.23) we obtain (4.22).

This implies straightforwardly $\exp\left(C_1 \varepsilon \mathbb{1}_2^{2R_0} [\sum_{k=1}^{\infty} \nu_k]\right) \in L^1(\tilde{Q}_{R_0}(0, 0))$.

We conclude that for any $(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O$,

$$u_1(y_0, s_0) \gtrsim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1q'}(M_k(y_0, s_0))}{r_k^N}$$

and

$$u_2(y_0, s_0) \gtrsim -c_1(R_0) + \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k(y_0, s_0))}{r_k^N},$$

where $r_k = 4^{-k}$ and

$$M_k(y_0, s_0) = O^c \cap \left(\overline{B_{r_{k+2}}(y_0)} \times [s_0 - (73 + \frac{1}{2})r_{k+2}^2, s_0 - (70 + \frac{1}{2})r_{k+2}^2] \right).$$

Take $r_{k_\delta+4} \leq \delta < r_{k_\delta+3}$, we have for $1 \leq k \leq k_\delta$

$$\begin{aligned} M_k(y_0, s_0) &\supset O^c \cap \left(B_{r_{k+2}-\delta}(0) \times \left(\delta^2 - (73 + \frac{1}{2})r_{k+2}^2, -\delta^2 - (70 + \frac{1}{2})r_{k+2}^2 \right) \right) \\ &\supset O^c \cap (B_{r_{k+3}}(0) \times (-73r_{k+2}^2, -71r_{k+2}^2)) \\ &= O^c \cap (B_{r_{k+3}}(0) \times (-1168r_{k+3}^2, -1136r_{k+3}^2)). \end{aligned}$$

Finally

$$\begin{aligned} &\inf_{(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O} u_1(y_0, s_0) \\ &\gtrsim \sum_{k=4}^{k_\delta+3} \frac{\text{Cap}_{2,1,q'}(O^c \cap (B_{r_k}(0) \times (-1168r_k^2, -1136r_k^2)))}{r_k^N} \rightarrow \infty \quad \text{as } \delta \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} &\inf_{(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O} u_2(y_0, s_0) \gtrsim -c_1(R_0) \\ &+ \sum_{k=4}^{k_\delta+3} \frac{\mathcal{PH}_1^N(O^c \cap (B_{r_k}(0) \times (-1168r_k^2, -1136r_k^2)))}{r_k^N} \rightarrow \infty \quad \text{as } \delta \rightarrow 0. \end{aligned}$$

This completes the proof of Theorem 1.1-(i) and Theorem 1.2.

4.3 The viscous Hamilton-Jacobi parabolic equations

In this section we apply our previous result to the question of existence of a large solution of the following type of parabolic viscous Hamilton-Jacobi equation

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p + bu^q &= 0 && \text{in } O, \\ u &= \infty && \text{on } \partial_p O, \end{aligned} \tag{4.25}$$

where $a > 0, b > 0$ and $1 < p \leq 2, q \geq 1$. First, we show that such a large solution to (4.25) does not exist when $q = 1$. Equivalently namely, for $a > 0, b > 0$ and $p > 1$ there exists no function $u \in C^{2,1}(O)$ satisfying

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p &\geq -bu && \text{in } O, \\ u &= \infty && \text{on } \partial_p O. \end{aligned} \quad (4.26)$$

Indeed, assuming that such a function $u \in C^{2,1}(O)$, exists, we define

$$U(x, t) = u(x, t)e^{bt} - \frac{\varepsilon}{2}|x|^2,$$

for $\varepsilon > 0$ and denote by $(x_0, t_0) \in O \setminus \partial_p O$ the point where U achieves its minimum in O , i.e. $U(x_0, t_0) = \inf\{U(x, t) : (x, t) \in O\}$. Clearly, we have

$$\partial_t U(x_0, t_0) \leq 0, \quad \Delta U(x_0, t_0) \geq 0 \quad \text{and} \quad \nabla U(x_0, t_0) = 0.$$

Thus,

$$\partial_t u(x_0, t_0) \leq -bu(x_0, t_0), \quad -\Delta u(x_0, t_0) \leq -\varepsilon N e^{-bt_0} \quad \text{and} \quad a|\nabla u(x_0, t_0)|^p = a\varepsilon^p |x_0|^p e^{-pb t_0},$$

from which follows

$$\begin{aligned} \partial_t u(x_0, t_0) - \Delta u(x_0, t_0) + a|\nabla u(x_0, t_0)|^p &\leq -bu(x_0, t_0) + \varepsilon e^{-bt_0} \left(-N + a\varepsilon^{p-1} |x_0|^p e^{-(p-1)bt_0} \right) \\ &< -bu(x_0, t_0) \end{aligned}$$

for ε small enough, we obtain contradiction.

Proof of Theorem 1.3. By Remark 3.3, we have

$$\inf\{v(x, t); (x, t) \in O\} \geq (q_1 - 1)^{-\frac{1}{q_1-1}} R^{-\frac{2}{q_1-1}}.$$

Take $V = \lambda v^{\frac{1}{\alpha}} \in C^{2,1}(O)$ for $\lambda > 0$. Thus $v = \lambda^{-\alpha} V^\alpha$,

$$\inf\{V(x, t); (x, t) \in O\} > 0\} \geq \lambda(q_1 - 1)^{-\frac{1}{\alpha(q_1-1)}} R^{-\frac{2}{\alpha(q_1-1)}},$$

and

$$\partial_t v - \Delta v + v^{q_1} = \alpha \lambda^{-\alpha} V^{\alpha-1} \partial_t V - \alpha \lambda^{-\alpha} V^{\alpha-1} \Delta V + \alpha(1-\alpha) \lambda^{-\alpha} V^{\alpha-1} \frac{|\nabla V|^2}{V} + \lambda^{-\alpha q_1} V^{\alpha q_1}.$$

This leads to

$$\partial_t V - \Delta V + (1-\alpha) \frac{|\nabla V|^2}{V} + \alpha^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} = 0 \quad \text{in } O.$$

Using Hölder's inequality,

$$\begin{aligned} (1-\alpha) \frac{|\nabla V|^2}{V} + (2\alpha)^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} &\geq c_1 |\nabla V|^p \lambda^{-\frac{\alpha(q_1-1)(2-p)}{2}} V^{-\frac{\alpha(q_1-1)(2-p)}{2} - (p-1)} \\ &\geq c_2 |\nabla V|^p \lambda^{-(p-1)} R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}} \end{aligned}$$

and

$$(2\alpha)^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} \geq c_3 \lambda^{-(q-1)} R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}} V^q.$$

Clearly, if we choose

$$\lambda = \min\left\{c_2^{\frac{1}{p-1}}, c_3^{\frac{1}{q-1}}\right\} \min\left\{a^{-\frac{1}{p-1}} R^{-\frac{2-p}{p-1} + \frac{2}{\alpha(q_1-1)}}, b^{-\frac{1}{q-1}} R^{-\frac{2}{q-1} + \frac{2}{\alpha(q_1-1)}}\right\}$$

then

$$\begin{aligned} c_2 \lambda^{-(p-1)} R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}} &\geq a, \\ c_3 \lambda^{-(q-1)} R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}} &\geq b, \end{aligned}$$

it follows

$$\partial_t V - \Delta V + a|\nabla V|^p + bV^q \leq 0 \quad \text{in } O$$

By Remark 3.5, there exists a maximal solution $u \in C^{2,1}(O)$ of

$$\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0 \quad \text{in } O$$

Therefore, $u \geq V = \lambda v^{\frac{1}{\alpha}}$ and u is a large solution of (4.25). This is complete the proof of Theorem. \blacksquare

5 Appendix

Proof of Proposition 2.5. First we have the following equivalence,

$$\int_{\mathbb{R}^{N+1}} (\mathbb{I}_2^1[\mu](x, t))^{(N+2)/N} dx dt \asymp \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t). \quad (5.1)$$

In fact, we have for $\rho_j = 2^{-j}$, $j \in \mathbb{Z}$,

$$\begin{aligned} \sum_{j=1}^{\infty} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{2/N} d\mu(x, t) &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \sum_{j=0}^{\infty} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{2/N} d\mu(x, t). \end{aligned}$$

Note that for any $j \in \mathbb{Z}$

$$\begin{aligned} \rho_j^{-N-2} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j+1}}(x, t)))^{(N+2)/N} dx dt &\lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{2/N} d\mu(x, t) \\ &\lesssim \rho_j^{-N-2} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j-1}}(x, t)))^{(N+2)/N} dx dt. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{j=2}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{(N+2)/N} dx dt &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \sum_{j=-1}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{(N+2)/N} dx dt. \end{aligned}$$

This yields

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} (\mathbb{M}_2^{1/4}[\mu](x, t))^{(N+2)/N} dx dt &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \int_{\mathbb{R}^{N+1}} (\mathbb{I}_2^4[\mu](x, t))^{(N+2)/N} dx dt. \end{aligned}$$

By [11, Theorem 4.2],

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{M}_2^{1/4}[\mu](x, t) \right)^{(N+2)/N} dxdt \asymp \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^4[\mu](x, t) \right)^{(N+2)/N} dxdt,$$

thus we obtain (5.1).

Now we come back proof of proposition. The first inequality in (2.1) was proved in [11]. We now prove the second inequality. By Theorem 2.4 there is $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$, $\text{supp}(\mu) \subset K$ such that

$$\|\mathbb{M}_2^2[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \quad \text{and} \quad \mu(K) \asymp \mathcal{P}\mathcal{H}_2^N(K) \gtrsim |K|^{N/(N+2)}. \quad (5.2)$$

Thanks to (5.1), we have for $\delta = \min\{1, (\mu(K))^{1/N}\}$

$$\begin{aligned} \|\mathbb{I}_2^1[\mu]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} &\asymp \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \delta^2 \int_{\mathbb{R}^{N+1}} d\mu(x, t) + \log(1/\delta) \left(\int_{\mathbb{R}^{N+1}} d\mu(x, t) \right)^{(N+2)/N} \\ &\lesssim (\mu(K))^{(N+2)/N} (1 + \log_+((\mu(K))^{-1})) \\ &\lesssim (\mu(K))^{(N+2)/N} \log \left(\frac{|\tilde{Q}_{200}(0, 0)|}{|K|} \right). \end{aligned}$$

Set $\tilde{\mu} = \left(\log \left(\frac{|\tilde{Q}_{200}(0, 0)|}{|K|} \right) \right)^{-N/(N+2)} \mu / \mu(K)$, then $\|\mathbb{I}_2^1[\tilde{\mu}]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1$.

It is well known that

$$\text{Cap}_{2,1, \frac{N+2}{2}}(K) \asymp \sup\{(\omega(K))^{(N+2)/2} : \omega \in \mathfrak{M}^+(K), \|\mathbb{I}_2^1[\omega]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1\} \quad (5.3)$$

see [11, Section 4]. This gives the second inequality in (2.1).

It is easy to prove (2.2) from its definition. Moreover, (5.3) implies that

$$\frac{1}{\text{Cap}_{2,1, \frac{N+2}{2}}(K)^{2/N}} \asymp \inf\{\|\mathbb{I}_2^1[\omega]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} : \omega \in \mathfrak{M}^+(K), \omega(K) = 1\}$$

We deduce from (5.1) that

$$\frac{1}{\text{Cap}_{2,1, \frac{N+2}{2}}(K)^{2/N}} \asymp \inf \left\{ \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) : \omega \in \mathfrak{M}^+(K), \omega(K) = 1 \right\}. \quad (5.4)$$

As in [6, proof of Lemma 2.2], it is easy to derive (2.3) from (5.4). \blacksquare

Proof of Proposition 2.6. Thanks to the Poincaré inequality, it is enough to show that there exists $\varphi \in C_c^\infty(\tilde{Q}_2(0, 0))$ such that $0 \leq \varphi \leq 1$, with $\varphi = 1$ in an open neighborhood of K and

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \text{Cap}_{2,1,p}(K). \quad (5.5)$$

By definition, one can find $0 \leq \phi \in S(\mathbb{R}^{N+1})$, $\phi \geq 1$ in a neighborhood of K such that

$$\int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\phi_t|^p) dxdt \leq 2\text{Cap}_{2,1,p}(K).$$

Let η be a cut off function on $\tilde{Q}_1(0, 0)$ with respect to $\tilde{Q}_{3/2}(0, 0)$ and $H \in C^\infty(\mathbb{R})$ such that

$0 \leq H(t) \leq t^+$, $|t||H''(t)| \lesssim 1$ for all $t \in \mathbb{R}$, $H(t) = 0$ for $t \leq 1/4$ and $H(t) = 1$ for $t \geq 3/4$.

We claim that

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\phi_t|^p) dxdt \quad (5.6)$$

where $\varphi = \eta H(\phi)$. Indeed, we have

$$|D^2\varphi| \lesssim |D^2\eta|H(\phi) + |\nabla\eta||H'(\phi)||\nabla\phi| + \eta|H''(\phi)||\nabla\phi|^2 + \eta|H''(\phi)||D^2\phi|$$

and

$$|\partial_t\varphi| \lesssim |\partial_t\eta|H(\phi) + \eta|H'(\phi)||\phi_t|, \quad H(\phi) \leq \phi, \quad \phi|H''(\phi)| \lesssim 1.$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt &\lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\phi_t|^p) dxdt \\ &\quad + \int_{\mathbb{R}^{N+1}} \frac{|\nabla\phi|^{2p}}{\phi^p} dxdt. \end{aligned}$$

This implies (5.6) since, according to [1], one has

$$\int_{\mathbb{R}^N} \frac{|\nabla\phi(t)|^{2p}}{\phi(t)^p} dx \lesssim \int_{\mathbb{R}^N} |D^2\phi(t)|^p dx \quad \forall t \in \mathbb{R}.$$

■

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