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# Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain.

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## Abstract

We obtain a necessary and a sufficient condition expressed in terms of Wiener type tests involving the parabolic  $W_{q'}^{2,1}$ -capacity, where  $q' = \frac{q}{q-1}$ , for the existence of large solutions to equation  $\partial_t u - \Delta u + u^q = 0$  in non-cylindrical domain, where  $q > 1$ . Also, we provide a sufficient condition associated with equation  $\partial_t u - \Delta u + e^u - 1 = 0$ . Besides, we apply our results to equation:  $\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0$  for  $a, b > 0$ ,  $1 < p < 2$  and  $q > 1$ .

*Keywords.* Bessel capacities; Hausdorff capacities; parabolic boundary; Riesz potential; maximal solutions.

*2010 Mathematics Subject Classification.* 35K58, 28A12, 46E35.

## 1 Introduction

The aim of this paper is to study large solutions to nonlinear parabolic equations in an arbitrary bounded open set  $O \subset \mathbb{R}^{N+1}$ ,  $N \geq 2$ . These are solutions  $u \in C^{2,1}(O)$  of equations

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u &= 0 && \text{in } O, \\ \lim_{\delta \rightarrow 0} \inf_{O \cap Q_\delta(x,t)} u &= \infty && \text{for all } (x,t) \in \partial_p O, \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} \partial_t u - \Delta u + \text{sign}(u)(e^{|u|} - 1) &= 0 && \text{in } O, \\ \lim_{\delta \rightarrow 0} \inf_{O \cap Q_\delta(x,t)} u &= \infty && \text{for all } (x,t) \in \partial_p O, \end{aligned} \quad (1.2)$$

where  $q > 1$  and  $\partial_p O$  is the parabolic boundary of  $O$ , i.e, the set all points  $X = (x, t) \in \partial O$  such that the intersection of the cylinder  $Q_\delta(x, t) := B_\delta(x) \times (t - \delta^2, t)$  with  $O^c$  is not empty for any  $\delta > 0$ . By the maximal principle for parabolic equations we can assume that all solutions of (1.1) and (1.2) are positive. Hence we can consider only positive solutions of preceding equations.

In [14], we studied the existence and the uniqueness of solution of general equations in a cylindrical domain,

$$\begin{aligned} \partial_t u - \Delta u + f(u) &= 0 && \text{in } \Omega \times (0, \infty), \\ u &= \infty && \text{in } \partial_p(\Omega \times (0, \infty)), \end{aligned} \quad (1.3)$$

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where  $\Omega$  is a bounded open set in  $\mathbb{R}^N$  and  $f$  is a continuous real-valued function, nondecreasing on  $\mathbb{R}$  such that  $f(0) \geq 0$  and  $f(a) > 0$  for some  $a > 0$ . In order to obtain the existence of a maximal solution of  $\partial_t u - \Delta u + f(u) = 0$  in  $\Omega \times (0, \infty)$  we need to assume that

$$(i) \quad \int_a^\infty \left( \int_0^s f(\tau) d\tau \right)^{-\frac{1}{2}} ds < \infty$$

$$(ii) \quad \int_a^\infty (f(s))^{-1} ds < \infty.$$
(1.4)

Note that, condition (i) due to Keller-Osserman condition, is also a necessary and sufficient for the existence of a maximal solution to

$$-\Delta u + f(u) = 0 \quad \text{in } \Omega$$
(1.5)

Condition (ii) is a necessary and sufficient for the existence of a solution of the ODE

$$\varphi' + f(\varphi) = 0 \quad \text{in } (0, \infty).$$
(1.6)

This solution tends to  $\infty$  at 0. In [14], it is shown that if for any  $m \in \mathbb{R}$  there exist  $L = L(m) > 0$  such that

$$\text{for any } x, y \geq m \Rightarrow f(x + y) \geq f(x) + f(y) - L,$$

and if (1.5) has a large solution, then (1.3) admits a solution.

It is not always true that the maximal solution to (1.5) is a large solution. However, if  $f$  satisfies

$$\int_1^\infty s^{-2(N-1)/(N-2)} f(s) ds < \infty \quad \text{if } N \geq 3$$

or

$$\inf \left\{ a \geq 0 \int_0^\infty f(s) e^{-as} ds < \infty \right\} \quad \text{if } N = 2.$$

then (1.5) has a large solution for any bounded domain  $\Omega$ .

When  $f(u) = u^q$ ,  $q > 1$  and  $N \geq 3$ , the first above condition is satisfied if and only if  $q < q_c := \frac{N}{N-2}$ , this is called *the sub-critical case*. When  $q \geq q_c$ , a necessary and sufficient condition for the existence of large solution of (1.5) expressed in term of Wiener test, is

$$\int_0^1 \frac{\text{Cap}_{2,q'}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } x \in \partial\Omega.$$
(1.7)

In the case  $q = 2$  it is obtained by probabilistic methods by Dhersin and Le Gall [4] and in the general case by Labutin [6]. Here,  $q' = \frac{q}{q-1}$  and  $\text{Cap}_{2,q'}$  is the capacity associated to the Sobolev space  $W^{2,q'}(\mathbb{R}^N)$ .

In [10] we obtain sufficient conditions when  $f(u) = e^u - 1$ , involving the the Hausdorff  $\mathcal{H}_1^{N-2}$ -capacity in  $\mathbb{R}^N$ , namely,

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } x \in \partial\Omega.$$
(1.8)

In this article we give a necessary and a sufficient condition for the existence of solutions to (1.1) in a bounded non-cylindrical domain  $O \subset \mathbb{R}^{N+1}$ , expressed in terms of a Wiener test based upon the parabolic  $W_{q'}^{2,1}$ -capacity in  $\mathbb{R}^{N+1}$ . We also give a sufficient condition

associated (1.2) where the parabolic  $W_q^{2,1}$ -capacity is replaced the parabolic Hausdorff  $\mathcal{PH}_\rho^N$ -capacity. These capacities are defined as follows: if  $K \subset \mathbb{R}^{N+1}$  is compact set, we set

$$\text{Cap}_{2,1,q'}(K) = \inf\{\|\varphi\|_{W_{q'}^{2,1}(\mathbb{R}^{N+1})}^q : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } K\},$$

where

$$\|\varphi\|_{W_{q'}^{2,1}(\mathbb{R}^{N+1})} = \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^{q'}(\mathbb{R}^{N+1})} + \|\nabla \varphi\|_{L^{q'}(\mathbb{R}^{N+1})} + \sum_{i,j=1,2,\dots,N} \left\| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\|_{L^{q'}(\mathbb{R}^{N+1})}.$$

and for Suslin set  $E \subset \mathbb{R}^{N+1}$ ,

$$\text{Cap}_{2,1,q'}(E) = \sup\{\text{Cap}_{2,1,q'}(D) : D \subset E, D \text{ compact}\}.$$

Thanks to a result due to Richard and Bagby, [2], the capacities  $\text{Cap}_{2,1,p}$  and  $\text{Cap}_{\mathcal{G}_2,p}$  are equivalent in the sense that, for any Suslin set  $K \subset \mathbb{R}^{N+1}$ , there holds

$$C^{-1} \text{Cap}_{2,1,q'}(K) \leq \text{Cap}_{\mathcal{G}_2,q'}(K) \leq C \text{Cap}_{2,1,p}(K)$$

for some  $C = C(N, q)$ , where  $\text{Cap}_{\mathcal{G}_2,q'}$  is the parabolic Bessel  $\mathcal{G}_2$ -capacity, see [11]. For  $E \subset \mathbb{R}^{N+1}$ , we define  $\mathcal{PH}_\rho^N(E)$  by

$$\mathcal{PH}_\rho^N(E) = \inf \left\{ \sum_j r_j^N : E \subset \bigcup B_{r_j}(x_j) \times (t_j - r_j^2, t_j + r_j^2), r_j \leq \rho \right\}.$$

It is easy to see that, for  $0 < \sigma \leq \rho$  and  $E \subset \mathbb{R}^{N+1}$ , there holds

$$\mathcal{PH}_\rho^N(E) \leq \mathcal{PH}_\sigma^N(E) \leq C(N) \left( \frac{\rho}{\sigma} \right)^2 \mathcal{PH}_\rho^N(E). \quad (1.9)$$

Now we are ready to state the main two results of this paper.

**Theorem 1.1** *Let  $N \geq 2$  and  $q \geq q_* := \frac{N+2}{N}$ . Then*

(i) *The equation*

$$\partial_t u - \Delta u + u^q = 0 \text{ in } O \quad (1.10)$$

*admits a large solution if for any  $(x, t) \in \partial_p O$*

$$\sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2)))}{r_k^N} = \infty, \quad (1.11)$$

*where  $r_k = 4^{-k}$ , and  $N \geq 3$  when  $q = q_*$ .*

(ii) *If equation (1.10) is a large solution, then*

$$\int_0^1 \frac{\text{Cap}_{2,1,q'}(O^c \cap Q_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} = \infty \quad (1.12)$$

*for any  $(x, t) \in \partial_p O$ , where  $Q_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t)$ .*

**Theorem 1.2** *Let  $N \geq 2$ . The equation*

$$\partial_t u - \Delta u + e^u - 1 = 0 \text{ in } O \quad (1.13)$$

*admits a large solution if*

$$\sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2)))}{r_k^N} = \infty, \quad (1.14)$$

*for any  $(x, t) \in \partial_p O$ , with  $r_k = 4^{-k}$ .*

From properties of the  $W_{q'}^{2,1}$ -capacity and the  $\mathcal{PH}_1^N$ -capacity, relation (1.11) holds if  $q > q_*$  and

$$\sum_{k=1}^{\infty} r_k^{-N} |O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2))|^{1 - \frac{2q'}{N+2}} = \infty.$$

Similarly, (1.14) is true if

$$\sum_{k=1}^{\infty} r_k^{-N} |O^c \cap (B_{r_k}(x) \times (t - 1168r_k^2, t - 1136r_k^2))|^{\frac{N}{N+2}} = \infty.$$

As a consequence of Theorem 1.1 we derive a sufficient condition for the existence of large solution of some viscous Hamilton-Jacobi parabolic equations.

**Theorem 1.3** *Let  $q_1 > 1$ . If there exists a large solution  $v \in C^{2,1}(O)$  of*

$$\partial_t v - \Delta v + v^{q_1} = 0 \quad \text{in } O.$$

*Then, for any  $a, b > 0$ ,  $1 < q < q_1$  and  $1 < p < \frac{2q_1}{q_1+1}$ , problem*

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p + bu^q &= 0 & \text{in } O, \\ u &= \infty & \text{on } \partial_p O, \end{aligned} \tag{1.15}$$

*admits a solution  $u \in C^{2,1}(O)$  which satisfies*

$$u(x, t) \geq C \min \left\{ a^{-\frac{1}{p-1}} R^{-\frac{2-p}{p-1} + \frac{2}{\alpha(q_1-1)}}, b^{-\frac{1}{q-1}} R^{-\frac{2}{q-1} + \frac{2}{\alpha(q_1-1)}} \right\} (v(x, t))^{\frac{1}{\alpha}}$$

*for all  $(x, t) \in O$  where  $R > 0$  is such that  $O \subset \tilde{Q}_R(x_0, t_0)$ ,  $C = C(N, p, q, q_1) > 0$  and  $\alpha = \max \left\{ \frac{2(p-1)}{(q_1-1)(2-p)}, \frac{q-1}{q_1-1} \right\} \in (0, 1)$ .*

## 2 Preliminaries

Throughout the paper, we denote  $Q_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t]$  and  $\tilde{Q}_\rho(x, t) = B_\rho(x) \times (t - \rho^2, t + \rho^2)$  for  $(x, t) \in \mathbb{R}^{N+1}$ ,  $\rho > 0$  and  $r_k = 4^{-k}$  for all  $k \in \mathbb{Z}$ . We also denote  $A \lesssim (\gtrsim) B$  if  $A \leq (\geq) CB$  for some  $C$  depending on some structural constants,  $A \asymp B$  if  $A \lesssim B \lesssim A$ .

**Definition 2.1** *Let  $R \in (0, \infty]$  and  $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ , the set of positive Radon measures in  $\mathbb{R}^{N+1}$ . We define  $R$ -truncated Riesz parabolic potential  $\mathbb{I}_2$  of  $\mu$  by*

$$\mathbb{I}_2^R[\mu](x, t) = \int_0^R \frac{\mu(\tilde{Q}_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} \quad \text{for all } (x, t) \in \mathbb{R}^{N+1},$$

*and the  $R$ -truncated fractional maximal parabolic potential of  $\mu$  by*

$$\mathbb{M}_\alpha^R[\mu](x, t) = \sup_{0 < \rho < R} \frac{\mu(\tilde{Q}_\rho(x, t))}{\rho^{N+2-\alpha}} \quad \text{for all } (x, t) \in \mathbb{R}^{N+1}.$$

We recall two results in [11].

**Theorem 2.2** *Let  $R > 0$ ,  $K$  be a compact set in  $\mathbb{R}^{N+1}$ . There exists  $\mu := \mu_K \in \mathfrak{M}^+(\mathbb{R}^{N+1})$  with compact support in  $K$  such that*

$$\mu(K) \asymp \text{Cap}_{2,1,q'}(K) \asymp \int_{\mathbb{R}^{N+1}} (\mathbb{I}_2^{2R}[\mu])^q dx dt$$

*where the constants of equivalence depend on  $N$  and  $R$ . The measure  $\mu_K$  is called the capacity measure of  $K$*

**Theorem 2.3** Let  $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$  and  $R > 0$ . There exist positive constants  $C_1, C_2$  such that

$$\int_Q \exp(C_1 \mathbb{I}_2^R[\mu_Q]) \leq C_2,$$

for all  $Q = \tilde{Q}_r(y, s) \subset \mathbb{R}^{N+1}$ ,  $r > 0$  such that  $\|\mathbb{M}_2^R[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} dxdt \leq 1$ .

It is easy to see that Frostman's Lemma in [13, Th. 3.4.27], which is at the basis of the dual definition of Hausdorff capacities with doubling weight, is valid for the parabolic Hausdorff  $\mathcal{PH}_\rho^N$ -capacity version. Therefore there holds

**Theorem 2.4** There holds

$$\sup \{ \mu(K) : \mu \in \mathfrak{M}^+(\mathbb{R}^{N+1}), \text{supp}(\mu) \subset K, \|\mathbb{M}_2^R[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \} \asymp \mathcal{PH}_\rho^N(K)$$

for any compact set  $K \subset \mathbb{R}^{N+1}$ , where equivalent constant depends on  $N$

For our purpose, we need the some results about the behavior of the capacity with respect to dilations.

**Proposition 2.5** Let  $K$  be a compact set,  $K \subset \overline{\tilde{Q}_{100}(0, 0)}$  and  $1 < p < \frac{N+2}{2}$ . Then

$$\text{Cap}_{2,1,p}(K) \gtrsim |K|^{1-\frac{2p}{N+2}}, \text{Cap}_{2,1,\frac{N+2}{2}}(K) \gtrsim \left( \log \left( \frac{|\tilde{Q}_{200}(0, 0)|}{|K|} \right) \right)^{-\frac{N}{2}}, \quad (2.1)$$

and

$$\text{Cap}_{2,1,p}(K_\rho) \asymp \rho^{N+2-2p} \text{Cap}_{2,1,p}(K), \quad (2.2)$$

$$\frac{1}{\text{Cap}_{2,1,\frac{N+2}{2}}(K_\rho)} \asymp \frac{1}{\text{Cap}_{2,1,\frac{N+2}{2}}(K)} + (\log(2/\rho))^{N/2} \quad (2.3)$$

for any  $0 < \rho < 1$ , where  $K_\rho = \{(\rho x, \rho^2 t) : (x, t) \in K\}$ .

**Proposition 2.6** Let  $K \subset \overline{\tilde{Q}_1(0, 0)}$  be a compact set and  $1 < p \leq (N+2)/2$ . Then, there exists a function  $\varphi \in C_c^\infty(\tilde{Q}_2(0, 0))$ ,  $0 \leq \varphi \leq 1$  and  $\varphi|_D = 1$  for some open set  $D \supset K$  such that

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\nabla\varphi|^p + |\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \text{Cap}_{2,1,p}(K). \quad (2.4)$$

We will give proofs of the above two propositions in the Appendix.

It is well know that there exists a semigroup  $e^{t\Delta}$  corresponding to equation

$$\begin{aligned} \partial_t u - \Delta u &= \mu & \text{in } \tilde{Q}_R(0, 0), \\ u &= 0 & \text{on } \partial_p \tilde{Q}_R(0, 0) \end{aligned} \quad (2.5)$$

with  $\mu \in C^\infty(B_R(0) \times (0, R^2))$ , i.e, we can write a solution  $u$  of (2.5) as follows

$$u(x, t) = \int_0^t (e^{(t-s)\Delta} \mu)(x, s) ds \quad \text{for all } (x, t) \in \tilde{Q}_R(0, 0).$$

We denote by  $\mathbb{H}$  the heat kernel:

$$\mathbb{H}(x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}} \chi_{t>0}.$$

We have

$$|u(x, t)| \leq (\mathbb{H} * \mu)(x, t) \quad \text{for all } (x, t) \in \tilde{Q}_R(0, 0).$$

In [11] we show that

$$|(\mathbb{H} * \mu)|(x, t) \leq C_1(N) \mathbb{I}_2^{2R}[|\mu|](x, t) \quad \text{for all } (x, t) \in \tilde{Q}_R(0, 0).$$

Here  $\mu$  is extended by 0 in  $(\tilde{Q}_R(0, 0))^c$ . Thus,

$$\left| \int_0^t \left( e^{(t-s)\Delta} \mu \right) (x, s) ds \right| \leq C_1(N) \mathbb{I}_2^{2R}[|\mu|](x, t) \quad \text{for all } (x, t) \in \tilde{Q}_R(0, 0). \quad (2.6)$$

Moreover, we also prove in [11], that if  $\mu \geq 0$  then for  $(x, t) \in \tilde{Q}_R(0, 0)$  and  $B_\rho(x) \subset B_R(0)$ ,

$$\int_0^t \left( e^{(t-s)\Delta} \mu \right) (x, s) ds \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N}, \quad (2.7)$$

with  $\rho_k = 4^{-k}\rho$ .

It is easy to see that estimates (2.6) and (2.7) also holds for any bounded Radon measure  $\mu$  in  $\tilde{Q}_R(0, 0)$ . The following result is proved in [3] and [8], also see [11].

**Theorem 2.7** *Let  $q > 1$ ,  $R > 0$  and  $\mu$  be bounded Radon measure in  $\tilde{Q}_R(0, 0)$ .*

(i) *If  $\mu$  is absolutely continuous with respect to  $\text{Cap}_{2,1,q}$  in  $\tilde{Q}_R(0, 0)$ , then there exists a unique weak solution  $u$  to equations*

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u &= \mu && \text{in } \tilde{Q}_R(0, 0), \\ u &= 0 && \text{on } \partial_p \tilde{Q}_R(0, 0). \end{aligned}$$

(ii) *If  $\exp(C_1(N) \mathbb{I}_2^{2R}[|\mu|]) \in L^1(\tilde{Q}_R(0, 0))$  then there exists a unique weak solution  $v$  to equations*

$$\begin{aligned} \partial_t v - \Delta v + \text{sign}(v)(e^{|v|} - 1) &= \mu && \text{in } \tilde{Q}_R(0, 0), \\ v &= 0 && \text{on } \partial_p \tilde{Q}_R(0, 0). \end{aligned}$$

where the constant  $C_1(N)$  is the one of inequality (2.6).

From estimates (2.6) and (2.7) and using comparison principle we get the estimates from below of the solutions  $u$  and  $v$  obtained in Theorem 2.7.

**Proposition 2.8** *If  $\mu \geq 0$  then the functions  $u$  and  $v$  of the previous theorem are nonnegative and satisfy*

$$u(x, t) \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N)^{q+1} \mathbb{I}_2^{2R} \left[ (\mathbb{I}_2^{2R}[\mu])^q \right] (x, t) \quad (2.8)$$

and

$$v(x, t) \geq C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N) \mathbb{I}_2^{2R} \left[ \exp(C_1(N) \mathbb{I}_2^{2R}[\mu]) - 1 \right] (x, t). \quad (2.9)$$

for any  $(x, t) \in \tilde{Q}_R(0, 0)$  and  $B_\rho(x) \subset B_R(0)$  and  $\rho_k = 4^{-k}\rho$ .

### 3 Maximal solutions

In this section we assume that  $O$  is a arbitrary, non-cylindrical and bounded open set in  $\mathbb{R}^{N+1}$  and  $q > 1$ . We will prove the existence of a maximal solution of

$$\partial_t u - \Delta u + u^q = 0 \quad (3.1)$$

in  $O$ . We also get analogous result where  $u^q$  is replace by  $e^u - 1$ .

It is easy to see that if  $u$  satisfies (3.1) in  $\tilde{Q}_r(0, 0)$  ( $Q_r(0, 0)$ ) then  $u_a(x, t) = a^{-2/(q-1)}u(ax, a^2t)$  satisfies (3.1) in  $\tilde{Q}_{r/a}(0, 0)$  ( $Q_{r/a}(0, 0)$ ) for any  $a > 0$ .

If  $X = (x, t) \in O$ , the parabolic distance from  $X$  to the parabolic boundary  $\partial_p O$  of  $O$  is defined by

$$d(X, \partial_p O) = \inf_{\substack{(y, s) \in \partial_p O \\ s \leq t}} \max\{|x - y|, (t - s)^{\frac{1}{2}}\}.$$

It is easy to see that there exists  $C = C(N, q) > 0$  such that the function  $V$  defined by

$$V(x, t) = C \left( (\rho^2 + t)^{-\frac{1}{q-1}} + \left( \frac{\rho^2 - |x|^2}{\rho} \right)^{-\frac{2}{q-1}} \right) \quad \text{in } B_\rho(0) \times (-\rho^2, 0)$$

satisfies

$$\partial_t V - \Delta V + V^q \geq 0 \quad \text{in } B_\rho(0) \times (-\rho^2, 0) \quad (3.2)$$

**Proposition 3.1** *There exists a maximal solution  $u \in C^{2,1}(O)$  of (3.1) and it satisfies*

$$u(x, t) \leq C(d((x, t), \partial_p O))^{-\frac{2}{q-1}} \quad \text{for all } (x, t) \in O. \quad (3.3)$$

for some  $C = C(N, q)$

**Proof.** Let  $\mathcal{D}_k$ ,  $k \in \mathbb{Z}$  be the collection of all the dyadic parabolic cubes (abridged  $p$ -cubes) of the form

$$\{(x_1, \dots, x_N, t) : m_j 2^{-k} \leq x_j \leq (m_j + 1)2^{-k}, j = 1, \dots, N, m_{N+1} 4^{-k} \leq t \leq (m_{N+1} + 1)4^{-k}\}$$

where  $m_j \in \mathbb{Z}$ . The following properties hold,

- a. for each integer  $k$ ,  $\mathcal{D}_k$  is a partition of  $\mathbb{R}^{N+1}$  and all  $p$ -cubes in  $\mathcal{D}_k$  have the same sidelengths.
- b. if the interiors of two  $p$ -cubes  $Q$  in  $\mathcal{D}_{k_1}$  and  $P$  in  $\mathcal{D}_{k_2}$ , denoted  $\overset{\circ}{Q}, \overset{\circ}{P}$ , have nonempty intersection then either  $Q$  is contained in  $R$  or  $Q$  contains  $R$ .
- c. Each  $Q$  in  $\mathcal{D}_k$  is union of  $2^{N+2}$   $p$ -cubes in  $\mathcal{D}_{k+1}$  with disjoint interiors.

Let  $k_0 \in \mathbb{N}$  be such that  $Q \subset D$  for some  $Q \in \mathcal{D}_{k_0}$ . Set  $O_k = \bigcup_{\substack{Q \in \mathcal{D}_k \\ Q \subset O}} Q \quad \forall k \geq k_0$ , we

have  $O_k \subset O_{k+1}$  and  $O = \bigcup_{k \geq k_0} O_k = \bigcup_{k \geq k_0} \overset{\circ}{O}_k$ . More precisely, there exist real numbers  $a_1, a_2, \dots, a_{n(k)}$  and open sets  $\Omega_1, \Omega_2, \dots, \Omega_{n(k)}$  in  $\mathbb{R}^N$  such that

$$a_i < a_i + 4^{-k} \leq a_{i+1} < a_{i+1} + 4^k \quad \text{for } i = 1, \dots, n(k) - 1$$

and

$$\overset{\circ}{O}_k = \bigcup_{i=1}^{n(k)-1} (\Omega_i \times (a_i, a_i + 4^{-k})) \bigcup (\Omega_{n(k)} \times (a_{n(k)}, a_{n(k)} + 4^{-k})).$$

For  $k \geq k_0$ , we will show that there exist a solution  $u_k \in C^{2,1}(\overset{\circ}{O}_k)$  to problem

$$\begin{aligned} \partial_t u_k - \Delta u_k + u_k^q &= 0 & \text{in } \overset{\circ}{O}_k, \\ u_k(x, t) &\rightarrow \infty & \text{as } d((x, t), \partial_p \overset{\circ}{O}_k) \rightarrow 0. \end{aligned} \quad (3.4)$$



Indeed, by [5, 7] for  $m > 0$  one can find nonnegative solutions  $v_i \in C^{2,1}(\Omega_i \times (a_i, a_i + 4^{-k})) \cap C(\bar{\Omega}_i \times [a_i, a_i + 4^{-k}])$  for  $i = 1, \dots, n(k)$  to equations

$$\begin{aligned} \partial_t v_1 - \Delta v_1 + v_1^q &= 0 && \text{in } \Omega_1 \times (a_1, a_1 + 4^{-k}), \\ v_1(x, t) &= m && \text{on } \partial\Omega_1 \times (a_1, a_1 + 4^{-k}), \\ v_1(x, t_1) &= m && \text{in } \Omega_1, \end{aligned}$$

and

$$\begin{aligned} \partial_t v_i - \Delta v_i + v_i^q &= 0 && \text{in } \Omega_i \times (a_i, a_i + 4^{-k}), \\ v_i(x, t) &= m && \text{on } \partial\Omega_i \times (a_i, a_i + 4^{-k}), \\ v_i(x, t_i) &= \begin{cases} m & \text{in } \Omega_i \\ m\chi_{\Omega_i \setminus \Omega_{i-1}}(x) + v_{i-1}(x, a_{i-1} + 4^{-k})\chi_{\Omega_{i-1}}(x) & \text{if } a_i > a_{i-1} + 4^{-k}, \\ & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly,

$$u_{k,m} = v_i \text{ in } \Omega_i \times (a_i, a_i + 4^{-k}] \text{ for } i = 1, \dots, n(k)$$

is a solution in  $C^{2,1}(\mathring{O}_k) \cap C(O_k)$  to equation

$$\begin{cases} \partial_t u_{k,m} - \Delta u_{k,m} + u_{k,m}^q = 0 & \text{in } \mathring{O}_k, \\ u_{k,m} = m & \text{on } \partial_p \mathring{O}_k. \end{cases}$$

Moreover, for  $(x, t) \in \mathring{O}_k$ , we can see that  $B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t) \subset \mathring{O}_k$  where  $d = d((x, t), \partial_p \mathring{O}_k)$ . From (3.2), we verify that

$$U(y, s) = V(y - x, s - t) = C \left( (\rho^2 + s - t)^{-\frac{1}{q-1}} + \left( \frac{\rho^2 - |x - y|^2}{\rho} \right)^{-\frac{2}{q-1}} \right)$$

with  $\rho = d/2$ , satisfies

$$\partial_t U - \Delta U + U^q \geq 0 \text{ in } B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t). \quad (3.5)$$

Applying the comparison principle we get

$$u_{k,m}(y, s) \leq U(y, s) \text{ in } B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t],$$

which implies

$$u_{k,m}(x, t) \leq C \left( d((x, t), \partial_p \mathring{O}_k) \right)^{-\frac{2}{q-1}} \text{ for all } (x, t) \in \mathring{O}_k. \quad (3.6)$$

From this, we also obtain uniform local bounds for  $\{u_{k,m}\}_m$ . By standard regularity theory see [5, 7],  $\{u_{k,m}\}_m$  is uniformly locally bounded in  $C^{2,1}$ . Hence, up to a subsequence,  $u_{k,m} \rightarrow u_k \in C_{\text{loc}}^{1,0}(\mathring{O}_k)$ . Passing the limit, we derive that  $u_k$  is a weak solution of (3.4) in  $\mathring{O}_k$ , which satisfies  $u_k(x, t) \rightarrow \infty$  as  $d((x, t), \partial_p \mathring{O}_k) \rightarrow 0$  and

$$u_k(x, t) \leq C \left( d((x, t), \partial_p \mathring{O}_k) \right)^{-\frac{2}{q-1}} \text{ for all } (x, t) \in \mathring{O}_k.$$

Let  $m > 0$  and  $k \geq k_0$ . Since  $u_{k+1,m} \leq m$  in  $O_k$  and  $O_k \subset O_{k+1}$ , it follows by the comparison principle applied to  $u_{k+1,m}$  and  $u_{k,m}$  in the sub-domains  $\Omega_1 \times (a_1, a_1 + 4^{-k})$ ,  $\Omega_2 \times (a_2, a_2 + 4^{-k})$ , ...,  $\Omega_{n(k)} \times (a_{n(k)}, a_{n(k)} + 4^{-k})$  of  $\mathring{O}_k$  to obtain at end that  $u_{k+1,m} \leq u_{k,m}$  in  $\mathring{O}_k$ , and thus  $u_{k+1} \leq u_k$  in  $\mathring{O}_k$ . In particular,  $\{u_k\}_k$  is uniformly locally bounded in  $L_{\text{loc}}^\infty$ . We use the same compactness property as above to obtain that  $u_k \rightarrow u$  where  $u$  is a solution of (3.1) and satisfies (3.3). By construction  $u$  is the maximal solution.  $\blacksquare$

**Remark 3.2** Let  $R \geq 2r \geq 2$ ,  $K$  be a compact subset in  $\overline{\tilde{Q}_r(0,0)}$ . Arguing as one can easily it is clear that there exists a maximal solution of

$$\begin{aligned} \partial_t u - \Delta u + u^q &= 0 & \text{in } \tilde{Q}_R(0,0) \setminus K, \\ u &= 0 & \text{on } \partial_p \tilde{Q}_R(0,0), \end{aligned} \quad (3.7)$$

which satisfies

$$u(x, t) \leq C(d((x, t), \partial_p(\tilde{Q}_R(0,0) \setminus K)))^{-\frac{2}{q-1}} \quad \forall (x, t) \in \tilde{Q}_R(0,0) \setminus K, \quad (3.8)$$

for some  $C = C(N, q)$ . Furthermore, assume  $K_1, K_2, \dots, K_m$  are compact subsets in  $\overline{\tilde{Q}_r(0,0)}$  and  $K = K_1 \cup \dots \cup K_m$ . Let  $u, u_1, \dots, u_m$  be the maximal solutions of (3.7) in  $\tilde{Q}_R(0,0) \setminus K, \tilde{Q}_R(0,0) \setminus K_1, \tilde{Q}_R(0,0) \setminus K_2, \dots, \tilde{Q}_R(0,0) \setminus K_m$ , respectively, then

$$u \leq \sum_{j=1}^m u_j \quad \text{in } \tilde{Q}_R \setminus K. \quad (3.9)$$

**Remark 3.3** If the equation (3.1) admits a large solution for some  $q > 1$  then for any  $1 < q_1 < q$ , equation

$$\partial_t u - \Delta u + u^{q_1} = 0 \quad \text{in } O \quad (3.10)$$

admits also a large solution.

Indeed, assume that  $u$  is a large solution of (3.1) and  $v$  is the maximal solution of (3.10). Take  $R > 0$  such that  $O \subset B_R(0) \times (-R^2, R^2)$ , then the function  $V$  defined by

$$V(x, t) = (q-1)^{-\frac{1}{q-1}} (2R^2 + t)^{-\frac{1}{q-1}},$$

satisfies (3.1). It follows for all  $(x, t) \in O$

$$u(x, t) \geq \inf_{(y,s) \in O} V(x, t) \geq (q-1)^{-\frac{1}{q-1}} R^{-\frac{2}{q-1}} =: a_0.$$

Thus,  $\tilde{u} = a_0^{-\frac{q-q_1}{q_1-1}} u$  is a subsolution of (3.10). Therefore  $v \geq a_0^{-\frac{q-q_1}{q_1-1}} u$  in  $O$ , thus  $v$  is a large solution.

**Remark 3.4 (Sub-critical case)** Assume that  $1 < q < q_*$ . One easily see that the function

$$U(x, t) = \frac{C}{t^{\frac{1}{q-1}}} e^{\frac{|x|^2}{4t}} \chi_{t>0} \quad (3.11)$$

is a subsolution of (3.1) in  $\mathbb{R}^{N+1} \setminus \{(0,0)\}$ , where  $C = \left(\frac{2}{q-1} - \frac{N}{2}\right)^{\frac{1}{q-1}}$ .

Therefore, the maximal solutions  $u$  of (3.1) in  $O$  verify

$$u(x, t) \geq C \frac{1}{(t-s)^{\frac{1}{q-1}}} e^{\frac{|x-y|^2}{4(t-s)}} \chi_{t>s}, \quad (3.12)$$

for all  $(x, t) \in O$  and  $(y, s) \in \partial_p O$ .

**Remark 3.5** Note that if  $u \in C^{2,1}(O)$  is a solution of (3.1) for some  $q > 1$  then, for  $a, b > 0$  and  $1 < p \leq 2$ ,  $v = b^{-\frac{1}{q-1}} u$  is a super-solution of

$$\partial_t v - \Delta v + a|\nabla v|^p + bv^q = 0 \quad \text{in } O. \quad (3.13)$$

Thus, we can apply the argument of the previous proof, with equation (3.13) replaced by (3.1), to deduce that there exists a maximal solution  $v \in C^{2,1}(O)$  of (3.13) satisfying

$$v(x, t) \leq Cb^{-\frac{1}{q-1}}(d((x, t), \partial_p O))^{-\frac{2}{q-1}} \quad \text{for all } (x, t) \in O.$$

Furthermore, if  $1 < q < q_*$ ,  $q = \frac{2p}{p+1}$ ,  $a, b > 0$  then the function  $U$  in Remark 3.4 is a subsolution of (3.13) in  $\mathbb{R}^{N+1} \setminus \{(0, 0)\}$ , for some  $C = C(N, p, q, a, b)$ . Therefore, we conclude that every maximal solution of  $v \in C^{2,1}(O)$  of (3.13) satisfy

$$u(x, t) \geq C \frac{1}{(t-s)^{\frac{1}{q-1}}} e^{\frac{|x-y|^2}{4(t-s)}} \chi_{t>s} \quad (3.14)$$

for all  $(x, t) \in O$  and  $(y, s) \in \partial_p O$ .

Next, we consider the following equation

$$\partial_t u - \Delta u + e^u - 1 = 0. \quad (3.15)$$

It is easy to see that the two functions

$$V_1(t) = -\log\left(\frac{t + \rho^2}{1 + \rho^2}\right) \quad \text{and} \quad V_2(x) = C_1 - 2 \log\left(\frac{\rho^2 - |x|^2}{\rho}\right)$$

satisfy

$$V_1' + e^{V_1} - 1 \geq 0 \quad \text{in } (-\rho^2, 0]$$

and

$$-\Delta V_2 + e^{V_2} - 1 \geq 0 \quad \text{in } B_\rho(0)$$

for some  $C = C(N)$ . Using  $e^a + e^b \leq e^{a+b} - 1$  for  $a, b \geq 0$ , we obtain that  $V_1 + V_2$  is a supersolution of equation (3.15) in  $B_\rho(0) \times (-\rho^2, 0]$ . By the same argument as in Proposition 3.1 and the estimate of the above supersolution, we obtain

**Proposition 3.6** *There exists a maximal solution  $u \in C^{2,1}(O)$  of*

$$\partial_t u - \Delta u + e^u - 1 = 0 \quad \text{in } O \quad (3.16)$$

and it satisfies

$$u(x, t) \leq C - \log\left(\frac{(d((x, t), \partial_p O))^3}{4 + (d((x, t), \partial_p O))^2}\right) \quad \text{for all } (x, t) \in O, \quad (3.17)$$

for some  $C = C(N)$ .

The next three propositions will be useful to prove Theorem 1.1-(ii).

**Proposition 3.7** *Let  $K \subset \overline{\tilde{Q}_1(0, 0)}$  be a compact set and  $q > 1$ ,  $R \geq 100$ . Let  $u$  be a solution of (3.7) in  $\tilde{Q}_R(0, 0) \setminus K$  and  $\varphi$  as in Proposition 2.6 with  $p = q'$ . Set  $\xi = (1 - \varphi)^{2q'}$ . Then,*

$$\int_{\tilde{Q}_R(0, 0)} u (|\Delta \xi| + |\nabla \xi| + |\partial_t \xi|) dx dt \lesssim \text{Cap}_{2,1,q'}(K) \quad (3.18)$$

and

$$u(x, t) \lesssim \text{Cap}_{2,1,q'}(K) + R^{-\frac{2}{q-1}} \quad \text{for any } (x, t) \in \tilde{Q}_{R/5}(0, 0) \setminus \tilde{Q}_2(0, 0), \quad (3.19)$$

$$\int_{\tilde{Q}_2(0, 0)} u \xi dx dt \lesssim \text{Cap}_{2,1,q'}(K) + R^{-\frac{2}{q-1}} \quad (3.20)$$

where constants in above inequalities only depend on  $N, q$ .

**Proof.** *Step 1.* First, we need to show that

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \text{Cap}_{2,1,q'}(K). \quad (3.21)$$

Actually, using by parts integration and the Green formula, one has

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u_R^q \xi dxdt &= - \int_{\tilde{Q}_R(0,0)} \partial_t u \xi dxdt + \int_{\tilde{Q}_R(0,0)} (\Delta u) \xi dxdt \\ &= \int_{\tilde{Q}_R(0,0)} u \partial_t \xi dxdt + \int_{\tilde{Q}_R(0,0)} u \Delta \xi dxdt + \int_{-R^2}^{R^2} \int_{\partial B_R(0)} \left( \xi \frac{\partial u}{\partial \nu} - u \frac{\partial \xi}{\partial \nu} \right) dS dt \end{aligned}$$

where  $\nu$  is the outer normal unit vector on  $\partial B_R(0)$ . Clearly,

$$\frac{\partial u}{\partial \nu} \leq 0 \quad \text{and} \quad \frac{\partial \xi}{\partial \nu} = 0 \quad \text{on} \quad \partial B_R(0).$$

Thus,

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt &\leq \int_{\tilde{Q}_R(0,0)} u |\partial_t \xi| dxdt + \int_{\tilde{Q}_R(0,0)} u |\Delta \xi| dxdt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u (1 - \varphi)^{2q'-1} |\partial_t \varphi| dxdt + 2q'(2q' - 1) \int_{\tilde{Q}_R(0,0)} u (1 - \varphi)^{2q'-2} |\nabla \varphi|^2 dxdt \\ &\quad + 2q' \int_{\tilde{Q}_R(0,0)} u (1 - \varphi)^{2q'-1} |\Delta \varphi| dxdt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\partial_t \varphi| dxdt + 2q'(2q' - 1) \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\nabla \varphi|^2 dxdt \\ &\quad + 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\Delta \varphi| dxdt. \end{aligned} \quad (3.22)$$

In the last inequality, we have used the fact that  $(1 - \phi)^{2q'-1} \leq (1 - \phi)^{2q'-1} = \xi^{1/q}$ . Hence, by Hölder inequality,

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt &\lesssim \int_{\tilde{Q}_R(0,0)} |\partial_t \varphi|^{q'} dxdt + \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt \\ &\quad + \int_{\tilde{Q}_R(0,0)} |\Delta \varphi|^{q'} dxdt. \end{aligned}$$

By the Gagliardo-Nirenberg inequality,

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt &\lesssim \|\varphi\|_{L^\infty(\tilde{Q}_R(0,0))}^{q'} \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt \\ &\lesssim \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt. \end{aligned}$$

Hence, we find

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \int_{\tilde{Q}_R(0,0)} (|\partial_t \varphi|^{q'} + |D^2 \varphi|^{q'}) dxdt$$

and derive (3.21) from (2.4). In view of (3.22), we also obtain

$$\int_{\tilde{Q}_R(0,0)} u (|\Delta \xi| + |\partial_t \xi|) dxdt, \lesssim \text{Cap}_{2,1,q'}(K)$$

and

$$\int_{\tilde{Q}_R(0,0)} u|\nabla\xi|dxdt, \lesssim \text{Cap}_{2,1,q'}(K),$$

since

$$\begin{aligned} \int_{\tilde{Q}_R(0,0)} u|\nabla\xi|dxdt &= 2q' \int_{\tilde{Q}_R(0,0)} u\xi^{(2q'-1)/2q'} |\nabla\varphi|dxdt \\ &\leq 2q' \int_{\tilde{Q}_R(0,0)} u\xi^{1/q} |\nabla\varphi|dxdt \\ &\lesssim \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt + \int_{\tilde{Q}_R(0,0)} |\nabla\varphi|^{q'} dxdt. \end{aligned}$$

It yields (3.18).

*Step 2.* Let  $\eta$  be a cut off function on  $\tilde{Q}_{R/4}(0,0)$  with respect to  $\tilde{Q}_{R/3}(0,0)$  such that  $|\partial_t\eta| + |D^2\eta| \lesssim R^{-2}$  and  $|\nabla\eta| \lesssim R^{-1}$ . We have

$$\partial_t(\eta\xi u) - \Delta(\eta\xi u) = F \in C_c(\tilde{Q}_{R/3}(0,0)).$$

Hence, we can write

$$(\eta\xi u)(x,t) = \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} F(y,s) ds dy \quad \forall (x,t) \in \mathbb{R}^{N+1}.$$

Now, we fix  $(x,t) \in \tilde{Q}_{R/5}(0,0) \setminus \tilde{Q}_2(0,0)$ . Since  $\text{supp}\{|\nabla\eta|\} \cap \text{supp}\{|\nabla\xi|\} = \emptyset$  and

$$\begin{aligned} F &= \eta\xi(\partial_t u - \Delta u) + 2(\eta\nabla\xi + \xi\nabla\eta)\nabla u + (\xi\partial_t\eta + \eta\partial_t\xi + 2\nabla\eta\nabla\xi + \Delta\eta\xi + \eta\Delta\xi)u \\ &\leq 2(\eta\nabla\xi + \xi\nabla\eta)\nabla u + (\xi\partial_t\eta + \eta\partial_t\xi + \xi\Delta\eta + \eta\Delta\xi)u, \end{aligned}$$

there holds

$$\begin{aligned} u(x,t) &= (\eta\xi u)(x,t) \leq 2 \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta\nabla\xi + \xi\nabla\eta)\nabla u ds dy \\ &\quad + \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta\partial_t\xi + \eta\Delta\xi) u ds dy \\ &\quad + \int_{\mathbb{R}^N} \int_{-\infty}^t \frac{1}{(4\pi(t-s))^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\partial_t\eta\xi + \xi\Delta\eta) u ds dy. \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By parts integration

$$\begin{aligned} I_1 &= -2(4\pi)^{-N/2} \int_{-\infty}^t \int_{\mathbb{R}^N} \frac{(x-y)}{2(t-s)^{(N+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\eta\nabla\xi + \xi\nabla\eta) u dy ds \\ &\quad - 2(4\pi)^{-N/2} \int_{-\infty}^t \int_{\mathbb{R}^N} \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} (\xi\Delta\eta + \eta\Delta\xi) u dy ds. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} &\lesssim \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N}, \\ \left| \frac{(x-y)}{2(t-s)^{(N+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \right| &\lesssim \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N-1}, \end{aligned}$$

and

$$\begin{aligned} \max\{|x-y|, |t-s|^{1/2}\} &\gtrsim 1 \quad \forall (y, s) \in \text{supp}\{|D^\alpha \xi|\} \cup \text{supp}\{|\partial_t \xi|\}, \\ \max\{|x-y|, |t-s|^{1/2}\} &\gtrsim R \quad \forall (y, s) \in \text{supp}\{|D^\alpha \eta|\} \cup \text{supp}\{|\partial_t \eta|\} \quad \forall |\alpha| \geq 1. \end{aligned}$$

We deduce

$$\begin{aligned} I_1 &\lesssim \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N-1} (\eta|\nabla \xi| + \xi|\nabla \eta|) u \, dyds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\xi|\Delta \eta| + \eta|\Delta \xi|) u \, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\nabla \xi| + |\Delta \xi|) u \, dyds + \int_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} (R^{-N-1}|\nabla \eta| + R^{-N}|\Delta \eta|) u \, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\nabla \xi| + |\Delta \xi|) u \, dyds + \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u, \\ I_2 &\lesssim \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (|\partial_t \xi| + |\Delta \xi|) u \, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\Delta \xi|) u \, dyds, \end{aligned}$$

and

$$\begin{aligned} I_3 &\lesssim \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (|\partial_t \eta| + |\Delta \eta|) u \, dyds \\ &\lesssim \int_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} R^{-N} (|\partial_t \eta| + |\Delta \eta|) u \, dyds \\ &\lesssim \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u. \end{aligned}$$

Hence,

$$u(x, t) \leq I_1 + I_2 + I_3 \lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\nabla \xi| + |\Delta \xi|) u \, dyds + \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u.$$

Combining this with (3.18) and (3.8), we obtain (3.19).

*Step 3.* Let  $\theta$  be a cut off function on  $\tilde{Q}_3(0,0)$  with respect to  $\tilde{Q}_4(0,0)$ . As above, we have for any  $(x, t) \in \mathbb{R}^{N+1}$

$$\begin{aligned} (\theta \xi u)(x, t) &\lesssim \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N-1} (\theta|\nabla \xi| + \xi|\nabla \theta|) u \, dyds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\theta|\Delta \xi| + \xi|\Delta \theta|) u \, dyds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\theta|\partial_t \xi| + \theta|\Delta \xi|) u \, dyds \\ &\quad + \int_{\mathbb{R}^{N+1}} \left( \max\{|x-y|, |t-s|^{1/2}\} \right)^{-N} (\xi|\partial_t \theta| + \xi|\Delta \theta|) u \, dyds. \end{aligned}$$

Hence, by Fubini theorem,

$$\begin{aligned} \int_{\tilde{Q}_2(0,0)} \eta u \, dxdt &= \int_{\tilde{Q}_2(0,0)} \theta \eta u \, dxdt \\ &\lesssim A \int_{\mathbb{R}^{N+1}} (\theta|\nabla \xi| + \xi|\nabla \theta| + \theta|\Delta \xi| + \xi|\Delta \theta| + \theta|\partial_t \xi| + \xi|\partial_t \theta|) u \, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\nabla \xi| + |\Delta \xi|) u \, dyds + \sup_{\tilde{Q}_4(0,0) \setminus \tilde{Q}_3(0,0)} u \end{aligned}$$

where

$$A = \sup_{(y,s) \in \tilde{Q}_4(0,0)} \int_{\tilde{Q}_2(0,0)} ((\max\{|x-y|, |t-s|^{1/2}\})^{-N} + (\max\{|x-y|, |t-s|^{1/2}\})^{-N-1}) dx dt$$

Therefore we obtain (3.20) from (3.18) and (3.19).  $\blacksquare$

**Proposition 3.8** *Let  $K \subset \{(x, t) : \varepsilon < \max\{|x|, |t|^{1/2}\} < 1\}$  be a compact set,  $0 < \varepsilon < 1$  and let  $u$  be the maximal solution of (3.7) in  $\tilde{Q}_R(0, 0) \setminus K$  with  $R \geq 100$ . Then*

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=-2}^{j_\varepsilon-2} \frac{\text{Cap}_{2,1,q'}(K \cap \tilde{Q}_{\rho_j}(0,0))}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if } q > q_*, \quad (3.23)$$

and

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=0}^{j_\varepsilon} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if } q = q_*, \quad (3.24)$$

where  $\rho_j = 2^{-j}$ ,  $K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : (x, t) \in K \cap \tilde{Q}_{\rho_{j-2}}(0, 0)\}$  and  $j_\varepsilon \in \mathbb{N}$  is such that  $\rho_{j_\varepsilon} \leq \varepsilon < \rho_{j_\varepsilon-1}$ .

**Proof.** For  $j \in \mathbb{N}$ , we define  $S_j = \{x : \rho_j \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-1}\}$ . Fix any  $1 \leq j \leq j_\varepsilon$ . We cover  $S_j$  by  $L = L(N) \in \mathbb{N}^*$  closed cylinders

$$\overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}, \quad k = 1, \dots, L(N)$$

where  $(x_{k,j}, t_{k,j}) \in S_j$ .

For  $k = 1, \dots, L(N)$ , let  $u_j, u_{k,j}$  be the maximal solutions of (3.7) where  $K$  is replaced by  $K \cap S_j$  and  $K \cap \overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}$ , respectively. Clearly the function  $\tilde{u}_{k,j}$  defined by

$$\tilde{u}_{k,j}(x, t) = \rho_{j+3}^{\frac{2}{q-1}} u_{k,j}(\rho_{j+3}x + x_{k,j}, \rho_{j+3}^2t + t_{k,j})$$

is the maximal solution of (3.7) when  $(K, \tilde{Q}_R(0, 0))$  is replaced by  $(K_{k,j}, \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}, -t_{k,j}))$ , with

$$K_{k,j} = \{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y, s) \in -(x_{k,j}, t_{k,j}) + K \cap \overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}\} \subset \tilde{Q}_1(0, 0).$$

Let  $\bar{u}_{k,j}$  be the maximal solution of (3.7) with  $(K, \tilde{Q}_R(0, 0))$  replaced by  $(K_{k,j}, \tilde{Q}_{2R/\rho_{j+3}}(0, 0))$ . Since  $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}, -t_{k,j}) \subset \tilde{Q}_{2R/\rho_{j+3}}(0, 0)$ , thus using the comparison principle as in the proof of Proposition 3.1 we obtain  $\tilde{u}_{k,j} \leq \bar{u}_{k,j}$  in  $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}, -t_{k,j}) \setminus K_{k,j}$  and thus

$$\tilde{u}_{k,j}(x, t) \lesssim \text{Cap}_{2,1,q'}(K_{k,j}) + (R/\rho_{j+3})^{-\frac{2}{q-1}},$$

for any  $(x, t) \in (\tilde{Q}_{R/(5\rho_{j+3})}(0, 0) \cap \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}, -t_{k,j})) \setminus \tilde{Q}_2(0, 0) = D$ .

Fix  $(x_0, t_0) \in \tilde{Q}_{\varepsilon/4}(0, 0)$ . Clearly,  $((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}^2) \in D$ , hence

$$u_{k,j}(x_0, t_0) = \rho_{j+3}^{-\frac{2}{q-1}} \tilde{u}_{k,j}((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}^2) \lesssim \frac{\text{Cap}_{2,1,q'}(K_{k,j})}{\rho_j^{\frac{2}{q-1}}} + R^{-\frac{2}{q-1}}.$$

Therefore, using (3.9) in Remark (3.2) and the fact that

$$\text{Cap}_{2,1,q'}(K_{k,j}) = \text{Cap}_{2,1,q'}(K_{k,j} + (x_{k,j}/\rho_{j+3}, t_{k,j}/\rho_{j+3}^2)) \leq \text{Cap}_{2,1,q'}(K_j),$$

we derive

$$\begin{aligned} u(x_0, t_0) &\leq \sum_{j=1}^{j_\varepsilon} u_j(x_0, t_0) \leq \sum_{j=1}^{j_\varepsilon} \sum_{k=1}^{L(N)} u_{k,j}(x_0, t_0) \\ &\lesssim \sum_{j=0}^{j_\varepsilon} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^{\frac{2}{q-1}}} + j_\varepsilon R^{-\frac{2}{q-1}}, \end{aligned}$$

which yields (3.24). If  $q > q_*$ , then by (2.2) in Proposition (2.5), we have

$$\text{Cap}_{2,1,q'}(K_j) \lesssim \rho_{j+3}^{-N-2+2q'} \text{Cap}_{2,1,q'}(K \cap \tilde{Q}_{\rho_{j-2}}(0,0)),$$

which implies (3.23). ■

**Proposition 3.9** *Let  $K, u, \xi$  be in Lemma 3.7. For any compact set  $K_0$  in  $\overline{\tilde{Q}_1(0,0)}$  with positive measure  $|K_0|$ , there exists  $\varepsilon = \varepsilon(N, q, |K_0|) > 0$  such that*

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow \inf_{K_0} u \lesssim \int_{\tilde{Q}_2(0,0)} u \xi dx dt.$$

where the constant in the inequality  $\lesssim$  depends on  $K_0$ . In particular,

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow \inf_{K_0} u \lesssim \text{Cap}_{2,1,q'}(K) + R^{-\frac{2}{q-1}}. \quad (3.25)$$

**Proof.** It is enough to assert that there is  $\varepsilon > 0$  such that

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_1| \geq 1/2 |K_0| \quad (3.26)$$

where  $K_1 = \{(x, t) \in K_0 : \xi(x, t) \geq 1/2\}$ . By (2.1) in Proposition (2.5), we have

$$\begin{aligned} |K_0 \setminus K_1|^{1-\frac{2q'}{N+2}} &\lesssim \text{Cap}_{2,1,q'}(K_0 \setminus K_1) \quad \text{if } q > q_*, \text{ and} \\ \left( \log \left( \frac{|\tilde{Q}_{100}(0,0)|}{|K_0 \setminus K_1|} \right) \right)^{-\frac{N}{2}} &\lesssim \text{Cap}_{2,1,q'}(K_0 \setminus K_1) \quad \text{if } q = q_*. \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{Cap}_{2,1,q'}(K_0 \setminus K_1) &= \text{Cap}_{2,1,q'}(\{K_0 : \varphi > 1 - (1/2)^{1/(2q')}\}) \\ &\leq (1 - (1/2)^{1/(2q')})^{-q'} \int_{\mathbb{R}^{N+1}} \left( |D^2 \varphi|^{q'} + |\nabla \varphi|^{q'} + |\varphi|^{q'} + |\partial_t \varphi|^{q'} \right) dx dt \\ &\lesssim \text{Cap}_{2,1,q'}(K) \end{aligned}$$

So, one can find  $\varepsilon = \varepsilon(N, q, |K_0|) > 0$  such that

$$\text{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_0 \setminus K_1| \leq 1/2 |K_0|.$$

This implies (3.26). ■

## 4 Large solutions

In the first part of this section, we prove theorem 1.1-(ii), then we prove theorems 1.1-(i) and 1.2, at end we consider a parabolic viscous Hamilton-Jacobi equation.



#### 4.1 Proof of Theorem 1.1-(ii)

Let  $R_0 \geq 4$  such that  $O \subset\subset \tilde{Q}_{R_0}(0,0)$ . Assume that the equation (1.10) is a large solution  $u$ . Take any  $(x,t) \in \partial_p O$ . We will to prove that (1.12) holds. We can assume  $(x,t) = (0,0)$ . Set  $K = \tilde{Q}_{2R_0}(0,0) \setminus O$  and define

$$\begin{aligned} T_j &= \{x : \rho_{j+1} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_j, t \leq 0\}, \\ \tilde{T}_j &= \{x : \rho_{j+3} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-2}, t \leq 0\}. \end{aligned}$$

Here  $\rho_j = 2^{-j}$ . For  $j \geq 3$ , let  $u_1, u_2, u_3, u_4$  be the maximal solutions of (3.7) when  $K$  is replaced by  $K \cap \overline{Q_{\rho_{j+3}}}(0,0)$ ,  $K \cap \tilde{T}_j$ ,  $(K \cap \overline{Q_1}(0,0)) \setminus Q_{\rho_{j-2}}(0,0)$  and  $K \setminus Q_1(0,0)$  respectively and  $R \geq 100R_0$ . From (3.9) in Remark (3.2), we can assert that

$$u \leq u_1 + u_2 + u_3 + u_4 \quad \text{in } O \cap \{(x,t) \in \mathbb{R}^{N+1} : t \leq 0\}.$$

Thus,

$$\inf_{T_j} u \leq \|u_1\|_{L^\infty(T_j)} + \|u_3\|_{L^\infty(T_j)} + \|u_4\|_{L^\infty(T_j)} + \inf_{T_j} u_2. \quad (4.1)$$

**Case 1:**  $q > q_*$ . By (3.8) in Remark 3.2,

$$\|u_4\|_{L^\infty(T_j)} \lesssim 1. \quad (4.2)$$

By Proposition 3.8,

$$\|u_3\|_{L^\infty(T_j)} \lesssim \sum_{i=-2}^{j-4} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + jR^{-\frac{2}{q-1}}. \quad (4.3)$$

Since  $(x,t) \mapsto \bar{u}_1(x,t) = \rho_{j+3}^{2/(q-1)} u_1(\rho_{j+3}x, \rho_{j+3}^2 t)$  is the maximal solution of (3.7) when  $(K, \tilde{Q}_R(0,0))$  is replaced by  $(\{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y,s) \in K \cap \overline{Q_{\rho_{j+3}}}(0,0)\}, \tilde{Q}_{R/\rho_{j+3}}(0,0))$ , we derive, thanks to (3.19) in Proposition 3.7 and (2.2) in Proposition 2.5,

$$\|\bar{u}_1\|_{L^\infty(T_{-3})} \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j+2}}(0,0))}{\rho_j^{N+2-2q'}} + (R/\rho_{j+3})^{-\frac{2}{q-1}},$$

from which follows

$$\|u_1\|_{L^\infty(T_j)} \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j+2}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}}. \quad (4.4)$$

Since,  $(x,t) \mapsto \bar{u}_2(x,t) = \rho_{j-2}^{2/(q-1)} u_2(\rho_{j-2}x, \rho_{j-2}^2 t)$  is the maximal solution of (3.7) when the couple  $(K, \tilde{Q}_R(0,0))$  is replaced by  $(\{(y/\rho_{j-2}, s/\rho_{j-2}^2) : (y,s) \in K \cap \tilde{T}_j\}, \tilde{Q}_{R/\rho_{j-2}}(0,0))$ , Proposition 3.9 and relation (2.2) in Proposition 2.5 yield

$$\frac{\text{Cap}_{2,1,q'}(K \cap \tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \Rightarrow \inf_{T_2} \bar{u}_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap \tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}} + (R/\rho_{j-2})^{-\frac{2}{q-1}},$$

which implies

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^N} + R^{-\frac{2}{q-1}}. \quad (4.5)$$

for some  $\varepsilon = \varepsilon(N, q) > 0$ .

First, we assume that there exists  $J \in \mathbb{N}$ ,  $J \geq 10$  such that

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} \leq \varepsilon \quad \forall j \geq J.$$

Then, from (4.1) and (4.2), (4.3), (4.4), (4.5) we have

$$\inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + jR^{-\frac{2}{q-1}} + 1,$$

for any  $j \geq J$ . Letting  $R \rightarrow \infty$ ,

$$\inf_{T_j} u \lesssim \sum_{i=-2}^{j+2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + 1.$$

Since  $\inf_{T_j} u \rightarrow \infty$  as  $j \rightarrow \infty$ , we get

$$\sum_{i=0}^{\infty} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} = \infty$$

which implies that (1.12) holds with  $(x, t) = (0, 0)$ .

Alternatively, assume that for infinitely many  $j$

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}} > \varepsilon$$

Then,

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^N} > \rho_{j-2}^{2-2q'} \varepsilon \rightarrow \infty \quad \text{when } j \rightarrow \infty.$$

We also achieve that (1.12) holds with  $(x, t) = (0, 0)$ . Therefore, case  $q > q_*$  proved.

**Case 2:**  $q = q_*$ . Similarly to **Case 1**, we have: for  $j \geq 5$

$$\|u_4\|_{L^\infty(T_j)} \lesssim 1, \tag{4.6}$$

$$\|u_3\|_{L^\infty(T_j)} \lesssim \sum_{i=0}^{j-2} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_i^N} + jR^{-\frac{2}{q-1}}, \tag{4.7}$$

$$\|u_1\|_{L^\infty(T_j)} \lesssim \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{4.8}$$

$$\text{Cap}_{2,1,q'}(K_j) \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{4.9}$$

where  $K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : (x, t) \in K \cap Q_{\rho_{j-3}}(0,0)\}$  and  $\varepsilon = \varepsilon(N) > 0$ . Note that, from (2.2) in Proposition 2.5 we have

$$\frac{1}{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))} \leq \frac{c}{\text{Cap}_{2,1,q'}(K_j)} + cj^{N/2}$$

for any  $j \geq 4$  where  $c = c(N)$ . If there are infinitely many  $j \geq 4$  such that

$$\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) > \frac{1}{2cj^{N/2}},$$

then (1.12) holds with  $(x, t) = (0, 0)$  since

$$\frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-3}^N} > \frac{2^{j-3}}{2cj^{N/2}} \rightarrow \infty \quad \text{when } j \rightarrow \infty.$$

Now, we assume that there exists  $J \geq 5$  such that

$$\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \frac{1}{2c j^{N/2}}.$$

Then,

$$\text{Cap}_{2,1,q'}(K_j) \leq 2c \text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \quad \forall j \geq J.$$

This leads to

$$\text{Cap}_{2,1,q'}(K_j) \leq 2c \text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0)) \leq \varepsilon \quad \forall j \geq J' + J,$$

for some  $J' = J'(N, q)$ . Hence, from (4.6)-(4.9) we have, for any  $j \geq J' + J + 3$ ,

$$\begin{aligned} \|u_4\|_{L^\infty(T_j)} &\lesssim 1, \\ \|u_3\|_{L^\infty(T_j)} &\lesssim \sum_{i=J'+J+1}^{j-2} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{i-3}}(0,0))}{\rho_i^N} + C(J' + J) + jR^{-\frac{2}{q-1}}, \\ \|u_1\|_{L^\infty(T_j)} &\lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}}, \\ \inf_{T_j} u_2 &\lesssim \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}}, \end{aligned}$$

where  $C(J' + J) = \sum_{i=0}^{J'+J} \frac{\text{Cap}_{2,1,q'}(K_j)}{\rho_i^N}$ .

Consequently, from (4.1) we derive

$$\inf_{T_j} u \lesssim \sum_{i=0}^j \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + C(J' + J) + 1 + jR^{-\frac{2}{q-1}} \quad \forall j \geq J' + J + 3.$$

Letting  $R \rightarrow \infty$  and  $j \rightarrow \infty$  we obtain

$$\sum_{i=0}^{\infty} \frac{\text{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} = \infty,$$

i.e (1.12) holds with  $(x, t) = (0, 0)$ . This completes the proof of Theorem 1.1-(ii).

## 4.2 Proof of Theorem 1.1-(i) and Theorem 1.2

Fix  $(x_0, t_0) \in \partial_p O$ . We can assume that  $(x_0, t_0) = 0$ . Let  $\delta \in (0, 1/100)$ . For  $(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O$ , we set

$$\begin{aligned} M_k &= O^c \cap \left( \overline{B_{r_{k+2}}(y_0)} \times [s_0 - (73 + \frac{1}{2})r_{k+2}^2, s_0 - (70 + \frac{1}{2})r_{k+2}^2] \right) \quad \text{and} \\ S_k &= \{(x, t) : r_{k-1} \leq \max\{|x - y_0|, |t - s_0|^{\frac{1}{2}}\} < r_k\} \quad \text{for } k = 1, 2, \dots \end{aligned}$$

where  $r_k = 4^{-k}$ . Note that  $M_k = \emptyset$  for  $k$  large enough and  $M_k \subset S_k$  for all  $k$ . Let  $R_0 \geq 4$  such that  $O \subset\subset \tilde{Q}_{R_0}(0,0)$ . By Theorem 2.2 and 2.4 and estimate (1.9) there exist two sequences  $\{\mu_k\}_k$  and  $\{\nu_k\}_k$  of nonnegative Radon measures such that

$$\text{supp}(\mu_k) \subset M_k, \quad \text{supp}(\nu_k) \subset M_k \quad \text{and} \quad (4.10)$$

$$\mu_k(M_k) \asymp \text{Cap}_{2,1,q'}(M_k) \asymp \int_{\mathbb{R}^{N+1}} \left( \mathbb{I}_2^{2R_0}[\mu_k] \right)^q dx dt \quad \text{and} \quad (4.11)$$

$$\nu_k(M_k) \asymp \mathcal{PH}_1^N(M_k), \quad \|\mathbb{M}_1^{2R_0}[\nu_k]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \quad \text{for } k = 1, 2, \dots \quad (4.12)$$

where equivalent constants depend on  $N, q, R_0$ .

Take  $\varepsilon > 0$  such that  $\exp\left(C_1 \varepsilon \mathbb{I}_2^{2R_0} [\sum_{k=1}^{\infty} \nu_k]\right) \in L^1(\tilde{Q}_{R_0}(0, 0))$  where the constant  $C_1 = C_1(N)$  is the one of inequality (2.6). By Theorem 2.7 and Proposition 2.8, there exist two nonnegative solutions  $U_1, U_2$  of problems

$$\begin{aligned} \partial_t U_1 - \Delta U_1 + U_1^q &= \varepsilon \sum_{k=1}^{\infty} \mu_k && \text{in } \tilde{Q}_{R_0}(0, 0), \\ U_1 &= 0 && \text{on } \partial_p \tilde{Q}_{R_0}(0, 0). \end{aligned}$$

and

$$\begin{aligned} \partial_t U_2 - \Delta U_2 + e^{U_2} - 1 &= \varepsilon \sum_{k=1}^{\infty} \nu_k && \text{in } \tilde{Q}_{R_0}(0, 0), \\ U_2 &= 0 && \text{on } \partial_p \tilde{Q}_{R_0}(0, 0), \end{aligned}$$

respectively which satisfy

$$\begin{aligned} U_1(y_0, z_0) &\gtrsim \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon \frac{\mu_k(B_{r_i/8}(y_0) \times (s_0 - \frac{37}{128}r_i^2, s_0 - \frac{37}{128}r_i^2))}{r_i^N} \\ &\quad - \mathbb{I}_2^{2R_0} \left[ \left( \mathbb{I}_2^{2R_0} [\varepsilon \sum_{k=1}^{\infty} \mu_k] \right)^q \right] (y_0, s_0) =: A \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} U_2(y_0, z_0) &\gtrsim \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} \varepsilon \frac{\nu_k(B_{r_i/8}(y_0) \times (s_0 - \frac{37}{128}r_i^2, s_0 - \frac{37}{128}r_i^2))}{r_i^N} \\ &\quad - \mathbb{I}_2^{2R_0} \left[ \exp\left(C_1 \mathbb{I}_2^{2R_0} [\varepsilon \sum_{k=1}^{\infty} \nu_k]\right) - 1 \right] (y_0, s_0) =: B \end{aligned} \quad (4.14)$$

and  $U_1, U_2 \in C^{2,1}(O)$ .

Let  $u_1, u_2$  be the maximal solutions of equations (3.1) and (3.16) respectively.

We have  $u_1(y_0, s_0) \geq U_1(y_0, s_0)$  and  $u_2(y_0, s_0) \geq U_2(y_0, s_0)$ .

Now, we claim that

$$A \gtrsim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N} \quad (4.15)$$

and

$$B \gtrsim -c_1(R_0) + \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k)}{r_k^N}. \quad (4.16)$$

**Proof of assertion (4.15).** From (4.11) we have

$$A \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N} - \varepsilon^q A_0 \quad (4.17)$$

with

$$A_0 = \mathbb{I}_2^{2R_0} \left[ \left( \mathbb{I}_2^{2R_0} [\sum_{k=1}^{\infty} \mu_k] \right)^q \right] (y_0, s_0).$$

Take  $i_0 \in \mathbb{Z}$  such that  $r_{i_0+1} < \max\{2R_0, 1\} \leq r_{i_0}$ . We have

$$\begin{aligned}
A_0 &\lesssim \sum_{i=i_0}^{\infty} r_i^{-N} \int_{\tilde{Q}_{r_i}(y_0, s_0)} \left( \mathbb{I}_2^{2R_0} \left[ \sum_{k=1}^{\infty} \mu_k \right] \right)^q dxdt \\
&= \sum_{i=i_0}^{\infty} \sum_{j=i}^{\infty} r_i^{-N} \int_{S_j} \left( \mathbb{I}_2^{2R_0} \left[ \sum_{k=1}^{\infty} \mu_k \right] \right)^q dxdt \\
&= \sum_{j=k_0}^{\infty} \sum_{i=i_0}^j r_i^{-N} \int_{S_j} \left( \mathbb{I}_2^{2R_0} \left[ \sum_{k=1}^{\infty} \mu_k \right] \right)^q dxdt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left( \mathbb{I}_2^{2R_0} \left[ \sum_{k=1}^{\infty} \mu_k \right] \right)^q dxdt.
\end{aligned}$$

Here we used  $\sum_{i=i_0}^j r_i^{-N} \leq \frac{4}{3} r_j^{-N}$  for all  $j$  in the last inequality.

Setting  $\mu_k \equiv 0$  for all  $i_0 - 1 \leq k \leq 0$ , the previous inequality becomes

$$\begin{aligned}
A_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left( \mathbb{I}_2^{2R_0} \left[ \mu_j + \sum_{k=i_0-1}^{j-1} \mu_k + \sum_{k=j+1}^{\infty} \mu_k \right] \right)^q dxdt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left( \mathbb{I}_2^{2R_0} [\mu_j] \right)^q dxdt \\
&\quad + \sum_{j=i_0}^{\infty} r_j^2 \left( \sum_{k=i_0-1}^{j-1} \|\mathbb{I}_2^{2R_0} [\mu_k]\|_{L^\infty(S_j)} \right)^q \\
&\quad + \sum_{j=i_0}^{\infty} r_j^2 \left( \sum_{k=j+1}^{\infty} \|\mathbb{I}_2^{2R_0} [\mu_k]\|_{L^\infty(S_j)} \right)^q \\
&= A_1 + A_2 + A_3. \tag{4.18}
\end{aligned}$$

Using (4.11) we obtain

$$A_1 \leq \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N}. \tag{4.19}$$

Next, using (4.10) we have for any  $(x, t) \in S_j$  if  $k \geq j+1$ ,

$$\mathbb{I}_2^{2R_0} [\mu_k](x, t) = \int_{r_{j+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N} \tag{4.20}$$

and if  $k \leq j-1$

$$\mathbb{I}_2^{2R_0} [\mu_k](x, t) = \int_{r_{k+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x, t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}. \tag{4.21}$$

Thus,

$$A_2 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \quad \text{and} \quad A_3 \lesssim \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q.$$

Noticing that  $(a+b)^q - a^q \leq q(a+b)^{q-1}b$  for any  $a, b \geq 0$ , we get

$$\begin{aligned}
& (1-4^{-2}) \sum_{j=i_0}^{\infty} r_j^2 \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
&= \sum_{j=i_0}^{\infty} r_j^2 \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q - \sum_{j=i_0+1}^{\infty} r_j^2 \left( \sum_{k=i_0-1}^{j-2} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
&\leq \sum_{j=i_0}^{\infty} q r_j^2 \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
& (1-4^{2-Nq}) \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q \\
&\leq \sum_{j=i_0}^{\infty} q r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Thus,

$$\begin{aligned}
A_2 + A_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N} \\
&\quad + \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Since  $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^{N+2-2q'}$  if  $q > q_*$  and  $\mu_k(\mathbb{R}^{N+1}) \lesssim \min\{k^{-\frac{1}{q-1}}, 1\}$  if  $q = q_*$  for any  $k$ , we always assert that

$$\begin{aligned}
r_j^2 \left( \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} &\lesssim 1 \quad \text{and} \\
r_j^{2-Nq} \left( \sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} &\lesssim r_{j+1}^{-N} \quad \text{for any } j.
\end{aligned}$$

In the case  $q = q_*$  we assume  $N \geq 3$  in order to verify that

$$\sum_{j=1}^{\infty} \mu_j(\mathbb{R}^{N+1}) \lesssim \sum_{k=1}^{\infty} k^{-\frac{1}{q-1}} < \infty.$$

This leads to

$$A_2 + A_3 \lesssim \sum_{k=1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}.$$

Combining this with (4.19) and (4.18), we deduce

$$A_0 \lesssim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1,q'}(M_k)}{r_k^N}.$$

Consequently, we obtain (4.15) from (4.17), for  $\varepsilon$  small enough.

**Proof of assertion (4.16).** From (4.12) we get

$$B \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k)}{r_k^N} - B_0$$

where

$$B_0 = \mathbb{I}_2^{2R_0} \left[ \exp \left( C_1 \mathbb{I}_2^{2R_0} \left[ \varepsilon \sum_{k=1}^{\infty} \nu_k \right] \right) - 1 \right] (y_0, s_0).$$

We show that

$$B_0 \leq c(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \quad (4.22)$$

In fact, as above we have

$$B_0 \lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp \left( C_1 \varepsilon \mathbb{I}_2^{2R_0} \left[ \sum_{k=1}^{\infty} \nu_k \right] \right) dx dt.$$

Thus,

$$\begin{aligned} B_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp \left( 3C_1 \varepsilon \mathbb{I}_2^{2R_0} [\nu_j] \right) dx dt \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp \left( 3C_1 \varepsilon \sum_{k=i_0-1}^{j-1} \|\mathbb{I}_2^{2R_0} [\nu_k]\|_{L^\infty(S_j)} \right) \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp \left( 3C_1 \varepsilon \sum_{k=j+1}^{\infty} \|\mathbb{I}_2^{2R_0} [\nu_k]\|_{L^\infty(S_j)} \right) \\ &= B_1 + B_2 + B_3. \end{aligned} \quad (4.23)$$

Here we used an inequality  $\exp(a+b+c) \leq \exp(3a) + \exp(3b) + \exp(3c)$  for all  $a, b, c$ . By Theorem 2.3, we have

$$\int_{S_j} \exp \left( 3C_1 \varepsilon \mathbb{I}_2^{2R_0} [\nu_j] \right) dx dt \lesssim r_j^{N+2} \quad \text{for all } j,$$

for  $\varepsilon > 0$  small enough. Hence,

$$B_1 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \lesssim (\max\{2R_0, 1\})^2. \quad (4.24)$$

Note that estimates (4.20) and (4.21) are also true with  $\nu_k$ ; we deduce

$$\begin{aligned} B_2 + B_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \exp \left( c_2 \varepsilon \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right) \\ &\quad + \sum_{j=i_0}^{\infty} r_j^2 \exp \left( c_2 \varepsilon \sum_{k=j+1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N} \right). \end{aligned}$$

From (4.12) we have  $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^N$  for all  $k$ , therefore

$$\begin{aligned} B_2 + B_3 &\lesssim \sum_{j=i_0}^{\infty} r_j^2 \exp(c_3 \varepsilon(j - i_0)) + \sum_{j=i_0}^{\infty} r_j^2 \exp(c_3 \varepsilon) \\ &\lesssim \sum_{j=i_0}^{\infty} \exp(c_3 \varepsilon(j - i_0) - 4 \log(2)j) + r_{i_0}^2 \\ &\leq c_4(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \end{aligned}$$

Combining this with (4.24) and (4.23) we obtain (4.22).

This implies straightforwardly  $\exp\left(C_1 \varepsilon \mathbb{1}_2^{2R_0} [\sum_{k=1}^{\infty} \nu_k]\right) \in L^1(\tilde{Q}_{R_0}(0, 0))$ .

We conclude that for any  $(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O$ ,

$$u_1(y_0, s_0) \gtrsim \sum_{k=1}^{\infty} \frac{\text{Cap}_{2,1q'}(M_k(y_0, s_0))}{r_k^N}$$

and

$$u_2(y_0, s_0) \gtrsim -c_1(R_0) + \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k(y_0, s_0))}{r_k^N},$$

where  $r_k = 4^{-k}$  and

$$M_k(y_0, s_0) = O^c \cap \left( \overline{B_{r_{k+2}}(y_0)} \times [s_0 - (73 + \frac{1}{2})r_{k+2}^2, s_0 - (70 + \frac{1}{2})r_{k+2}^2] \right).$$

Take  $r_{k_\delta+4} \leq \delta < r_{k_\delta+3}$ , we have for  $1 \leq k \leq k_\delta$

$$\begin{aligned} M_k(y_0, s_0) &\supset O^c \cap \left( B_{r_{k+2}-\delta}(0) \times \left( \delta^2 - (73 + \frac{1}{2})r_{k+2}^2, -\delta^2 - (70 + \frac{1}{2})r_{k+2}^2 \right) \right) \\ &\supset O^c \cap (B_{r_{k+3}}(0) \times (-73r_{k+2}^2, -71r_{k+2}^2)) \\ &= O^c \cap (B_{r_{k+3}}(0) \times (-1168r_{k+3}^2, -1136r_{k+3}^2)). \end{aligned}$$

Finally

$$\begin{aligned} &\inf_{(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O} u_1(y_0, s_0) \\ &\gtrsim \sum_{k=4}^{k_\delta+3} \frac{\text{Cap}_{2,1,q'}(O^c \cap (B_{r_k}(0) \times (-1168r_k^2, -1136r_k^2)))}{r_k^N} \rightarrow \infty \quad \text{as } \delta \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} &\inf_{(y_0, s_0) \in (B_\delta(0) \times (-\delta^2, \delta^2)) \cap O} u_2(y_0, s_0) \gtrsim -c_1(R_0) \\ &+ \sum_{k=4}^{k_\delta+3} \frac{\mathcal{PH}_1^N(O^c \cap (B_{r_k}(0) \times (-1168r_k^2, -1136r_k^2)))}{r_k^N} \rightarrow \infty \quad \text{as } \delta \rightarrow 0. \end{aligned}$$

This completes the proof of Theorem 1.1-(i) and Theorem 1.2.

### 4.3 The viscous Hamilton-Jacobi parabolic equations

In this section we apply our previous result to the question of existence of a large solution of the following type of parabolic viscous Hamilton-Jacobi equation

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p + bu^q &= 0 && \text{in } O, \\ u &= \infty && \text{on } \partial_p O, \end{aligned} \tag{4.25}$$



where  $a > 0, b > 0$  and  $1 < p \leq 2, q \geq 1$ . First, we show that such a large solution to (4.25) does not exist when  $q = 1$ . Equivalently namely, for  $a > 0, b > 0$  and  $p > 1$  there exists no function  $u \in C^{2,1}(O)$  satisfying

$$\begin{aligned} \partial_t u - \Delta u + a|\nabla u|^p &\geq -bu && \text{in } O, \\ u &= \infty && \text{on } \partial_p O. \end{aligned} \quad (4.26)$$

Indeed, assuming that such a function  $u \in C^{2,1}(O)$ , exists, we define

$$U(x, t) = u(x, t)e^{bt} - \frac{\varepsilon}{2}|x|^2,$$

for  $\varepsilon > 0$  and denote by  $(x_0, t_0) \in O \setminus \partial_p O$  the point where  $U$  achieves its minimum in  $O$ , i.e.  $U(x_0, t_0) = \inf\{U(x, t) : (x, t) \in O\}$ . Clearly, we have

$$\partial_t U(x_0, t_0) \leq 0, \quad \Delta U(x_0, t_0) \geq 0 \quad \text{and} \quad \nabla U(x_0, t_0) = 0.$$

Thus,

$$\partial_t u(x_0, t_0) \leq -bu(x_0, t_0), \quad -\Delta u(x_0, t_0) \leq -\varepsilon N e^{-bt_0} \quad \text{and} \quad a|\nabla u(x_0, t_0)|^p = a\varepsilon^p |x_0|^p e^{-pb t_0},$$

from which follows

$$\begin{aligned} \partial_t u(x_0, t_0) - \Delta u(x_0, t_0) + a|\nabla u(x_0, t_0)|^p &\leq -bu(x_0, t_0) + \varepsilon e^{-bt_0} \left( -N + a\varepsilon^{p-1} |x_0|^p e^{-(p-1)bt_0} \right) \\ &< -bu(x_0, t_0) \end{aligned}$$

for  $\varepsilon$  small enough, we obtain contradiction.

**Proof of Theorem 1.3.** By Remark 3.3, we have

$$\inf\{v(x, t); (x, t) \in O\} \geq (q_1 - 1)^{-\frac{1}{q_1-1}} R^{-\frac{2}{q_1-1}}.$$

Take  $V = \lambda v^{\frac{1}{\alpha}} \in C^{2,1}(O)$  for  $\lambda > 0$ . Thus  $v = \lambda^{-\alpha} V^\alpha$ ,

$$\inf\{V(x, t); (x, t) \in O\} > 0\} \geq \lambda(q_1 - 1)^{-\frac{1}{\alpha(q_1-1)}} R^{-\frac{2}{\alpha(q_1-1)}},$$

and

$$\partial_t v - \Delta v + v^{q_1} = \alpha \lambda^{-\alpha} V^{\alpha-1} \partial_t V - \alpha \lambda^{-\alpha} V^{\alpha-1} \Delta V + \alpha(1-\alpha) \lambda^{-\alpha} V^{\alpha-1} \frac{|\nabla V|^2}{V} + \lambda^{-\alpha q_1} V^{\alpha q_1}.$$

This leads to

$$\partial_t V - \Delta V + (1-\alpha) \frac{|\nabla V|^2}{V} + \alpha^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} = 0 \quad \text{in } O.$$

Using Hölder's inequality,

$$\begin{aligned} (1-\alpha) \frac{|\nabla V|^2}{V} + (2\alpha)^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} &\geq c_1 |\nabla V|^p \lambda^{-\frac{\alpha(q_1-1)(2-p)}{2}} V^{-\frac{\alpha(q_1-1)(2-p)}{2} - (p-1)} \\ &\geq c_2 |\nabla V|^p \lambda^{-(p-1)} R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}} \end{aligned}$$

and

$$(2\alpha)^{-1} \lambda^{-\alpha(q_1-1)} V^{\alpha q_1 - \alpha + 1} \geq c_3 \lambda^{-(q-1)} R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}} V^q.$$

Clearly, if we choose

$$\lambda = \min\left\{c_2^{\frac{1}{p-1}}, c_3^{\frac{1}{q-1}}\right\} \min\left\{a^{-\frac{1}{p-1}} R^{-\frac{2-p}{p-1} + \frac{2}{\alpha(q_1-1)}}, b^{-\frac{1}{q-1}} R^{-\frac{2}{q-1} + \frac{2}{\alpha(q_1-1)}}\right\}$$

then

$$\begin{aligned} c_2 \lambda^{-(p-1)} R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}} &\geq a, \\ c_3 \lambda^{-(q-1)} R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}} &\geq b, \end{aligned}$$

it follows

$$\partial_t V - \Delta V + a|\nabla V|^p + bV^q \leq 0 \quad \text{in } O$$

By Remark 3.5, there exists a maximal solution  $u \in C^{2,1}(O)$  of

$$\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0 \quad \text{in } O$$

Therefore,  $u \geq V = \lambda v^{\frac{1}{\alpha}}$  and  $u$  is a large solution of (4.25). This is complete the proof of Theorem.  $\blacksquare$

## 5 Appendix

**Proof of Proposition 2.5.** First we have the following equivalence,

$$\int_{\mathbb{R}^{N+1}} (\mathbb{I}_2^1[\mu](x, t))^{(N+2)/N} dx dt \asymp \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t). \quad (5.1)$$

In fact, we have for  $\rho_j = 2^{-j}$ ,  $j \in \mathbb{Z}$ ,

$$\begin{aligned} \sum_{j=1}^{\infty} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{2/N} d\mu(x, t) &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \sum_{j=0}^{\infty} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{2/N} d\mu(x, t). \end{aligned}$$

Note that for any  $j \in \mathbb{Z}$

$$\begin{aligned} \rho_j^{-N-2} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j+1}}(x, t)))^{(N+2)/N} dx dt &\lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{2/N} d\mu(x, t) \\ &\lesssim \rho_j^{-N-2} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j-1}}(x, t)))^{(N+2)/N} dx dt. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{j=2}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{(N+2)/N} dx dt &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \sum_{j=-1}^{\infty} \rho_j^{-N} \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x, t)))^{(N+2)/N} dx dt. \end{aligned}$$

This yields

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} (\mathbb{M}_2^{1/4}[\mu](x, t))^{(N+2)/N} dx dt &\lesssim \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \int_{\mathbb{R}^{N+1}} (\mathbb{I}_2^4[\mu](x, t))^{(N+2)/N} dx dt. \end{aligned}$$

By [11, Theorem 4.2],

$$\int_{\mathbb{R}^{N+1}} \left( \mathbb{M}_2^{1/4}[\mu](x, t) \right)^{(N+2)/N} dxdt \asymp \int_{\mathbb{R}^{N+1}} \left( \mathbb{I}_2^4[\mu](x, t) \right)^{(N+2)/N} dxdt,$$

thus we obtain (5.1).

Now we come back proof of proposition. The first inequality in (2.1) was proved in [11]. We now prove the second inequality. By Theorem 2.4 there is  $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ ,  $\text{supp}(\mu) \subset K$  such that

$$\|\mathbb{M}_2^2[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \quad \text{and} \quad \mu(K) \asymp \mathcal{PH}_2^N(K) \gtrsim |K|^{N/(N+2)}. \quad (5.2)$$

Thanks to (5.1), we have for  $\delta = \min\{1, (\mu(K))^{1/N}\}$

$$\begin{aligned} \|\mathbb{I}_2^1[\mu]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} &\asymp \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) \\ &\lesssim \delta^2 \int_{\mathbb{R}^{N+1}} d\mu(x, t) + \log(1/\delta) \left( \int_{\mathbb{R}^{N+1}} d\mu(x, t) \right)^{(N+2)/N} \\ &\lesssim (\mu(K))^{(N+2)/N} (1 + \log_+((\mu(K))^{-1})) \\ &\lesssim (\mu(K))^{(N+2)/N} \log \left( \frac{|\tilde{Q}_{200}(0, 0)|}{|K|} \right). \end{aligned}$$

Set  $\tilde{\mu} = \left( \log \left( \frac{|\tilde{Q}_{200}(0, 0)|}{|K|} \right) \right)^{-N/(N+2)} \mu / \mu(K)$ , then  $\|\mathbb{I}_2^1[\tilde{\mu}]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1$ .

It is well known that

$$\text{Cap}_{2,1, \frac{N+2}{2}}(K) \asymp \sup\{(\omega(K))^{(N+2)/2} : \omega \in \mathfrak{M}^+(K), \|\mathbb{I}_2^1[\omega]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1\} \quad (5.3)$$

see [11, Section 4]. This gives the second inequality in (2.1).

It is easy to prove (2.2) from its definition. Moreover, (5.3) implies that

$$\frac{1}{\text{Cap}_{2,1, \frac{N+2}{2}}(K)^{2/N}} \asymp \inf\{\|\mathbb{I}_2^1[\omega]\|_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} : \omega \in \mathfrak{M}^+(K), \omega(K) = 1\}$$

We deduce from (5.1) that

$$\frac{1}{\text{Cap}_{2,1, \frac{N+2}{2}}(K)^{2/N}} \asymp \inf \left\{ \int_{\mathbb{R}^{N+1}} \int_0^1 (\mu(\tilde{Q}_r(x, t)))^{2/N} \frac{dr}{r} d\mu(x, t) : \omega \in \mathfrak{M}^+(K), \omega(K) = 1 \right\}. \quad (5.4)$$

As in [6, proof of Lemma 2.2], it is easy to derive (2.3) from (5.4).  $\blacksquare$

**Proof of Proposition 2.6.** Thanks to the Poincaré inequality, it is enough to show that there exists  $\varphi \in C_c^\infty(\tilde{Q}_2(0, 0))$  such that  $0 \leq \varphi \leq 1$ , with  $\varphi = 1$  in an open neighborhood of  $K$  and

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \text{Cap}_{2,1,p}(K). \quad (5.5)$$

By definition, one can find  $0 \leq \phi \in S(\mathbb{R}^{N+1})$ ,  $\phi \geq 1$  in a neighborhood of  $K$  such that

$$\int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\phi_t|^p) dxdt \leq 2\text{Cap}_{2,1,p}(K).$$

Let  $\eta$  be a cut off function on  $\tilde{Q}_1(0, 0)$  with respect to  $\tilde{Q}_{3/2}(0, 0)$  and  $H \in C^\infty(\mathbb{R})$  such that

$0 \leq H(t) \leq t^+$ ,  $|t||H''(t)| \lesssim 1$  for all  $t \in \mathbb{R}$ ,  $H(t) = 0$  for  $t \leq 1/4$  and  $H(t) = 1$  for  $t \geq 3/4$ .

We claim that

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\phi_t|^p) dxdt \quad (5.6)$$

where  $\varphi = \eta H(\phi)$ . Indeed, we have

$$|D^2\varphi| \lesssim |D^2\eta|H(\phi) + |\nabla\eta||H'(\phi)||\nabla\phi| + \eta|H''(\phi)||\nabla\phi|^2 + \eta|H''(\phi)||D^2\phi|$$

and

$$|\partial_t\varphi| \lesssim |\partial_t\eta|H(\phi) + \eta|H'(\phi)||\phi_t|, \quad H(\phi) \leq \phi, \quad \phi|H''(\phi)| \lesssim 1.$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt &\lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\phi_t|^p) dxdt \\ &\quad + \int_{\mathbb{R}^{N+1}} \frac{|\nabla\phi|^{2p}}{\phi^p} dxdt. \end{aligned}$$

This implies (5.6) since, according to [1], one has

$$\int_{\mathbb{R}^N} \frac{|\nabla\phi(t)|^{2p}}{\phi(t)^p} dx \lesssim \int_{\mathbb{R}^N} |D^2\phi(t)|^p dx \quad \forall t \in \mathbb{R}.$$

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