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# An Anisotropic Bimodal Energy for detection of thin tubes and its approximation with $\Gamma$ -convergence

David Vicente

September 23, 2014

## Abstract

This work is a contribution to the problem of detection of thin structures, namely *tubes*, in a 2D or 3D image. We introduce a *bimodal* model for the case where the histogram of the image has two main modes. This model involves an energy functional depending on a function and a riemannian metric. One of the term of this energy is the anisotropic perimeter associated to the dual metric. We perform an approximation of this functional and prove it  $\Gamma$ -converges to the original one.

## Introduction

Let  $n$  be the dimension and  $\Omega \subset \mathbb{R}^n$  be a domain. Let  $g : \Omega \rightarrow [0; 1]$  be an image with two modes 0 and 1. The analysis will consist in searching a pair  $(p, \mathbf{g})$ , where  $p : \Omega \rightarrow \{0, 1\}$  is a binary function and  $\mathbf{g} : \Omega \times \mathbb{R}^n \rightarrow [0; +\infty[$  a riemannian metric which minimizes the functional

$$E(p, \mathbf{g}) = \int_{\Omega} (p - g)^2 dx + \beta \int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1} + \gamma \|\mathbf{g}\|_{W^{1,r}(\Omega)},$$

where  $S_p$  is the jump set of  $p$ ,  $\nu_p$  is a normal unitary vector to  $S_p$  and  $\mathcal{H}^{n-1}$  is the  $(n - 1)$ -dimensional Hausdorff measure. By assumption, for any  $x \in \Omega$ , the function  $\mathbf{g}(x, \cdot)$  is a scalar product. So, we may associate a unique symmetric matrix denoted  $\mathbf{g}_x \in S_n(\mathbb{R})$ . In this way, the last term of  $E$  is the norm of the function  $x \rightarrow \mathbf{g}_x$  in the Sobolev space  $W^{1,r}(\Omega; S_n(\mathbb{R}))$ , the exponent  $r$  will be precised later. The parameters  $\beta$  and  $\gamma$  are weights to determine. In order to detect thin tubes, the riemannian metric  $\mathbf{g}$  must belong to a suitable subset  $\mathcal{G} \subset S_n(\mathbb{R})$ . More precisely, any  $\mathbf{g} \in \mathcal{G}$  must take the form

$$\forall x \in \Omega, \quad \mathbf{g}_x = \text{Id}_n + \mu {}^t \mathbf{c}(x) \mathbf{c}(x),$$

where  $\mathbf{c} : \Omega \rightarrow \mathbb{R}^n$  is an unitary vector field. The parameter  $\mu$  is the *thickness* of the tubes.

In [1], it is proved that the second term of this functional is the anisotropic perimeter associated to the dual metric. So, with  $\mathbf{g}$  fixed, this functional inherits a lower semi-continuity property and it can be approximated in the sense of the  $\Gamma$ -convergence by an adapted family of functionals. We

generalize this work to the case where  $\mathbf{g}$  is also an unknown and takes the form defined above. More precisely, we introduce the following functional

$$E_\varepsilon(p, \mathbf{g}) = \int_{\Omega} (p - g)^2 dx + \beta \int_{\Omega} \left[ 9\varepsilon \mathbf{g}(x, \nabla p) + \frac{p^2(1-p)^2}{\varepsilon} \right] dx + \gamma \|\mathbf{g}\|_{W^{1,r}(\Omega)},$$

with  $p : \Omega \rightarrow [0; 1]$  a regular function. We prove that  $(E_\varepsilon)_{\varepsilon > 0}$  is an approximation of  $E$  when  $\varepsilon$  converges to  $0^+$ .

In section 1, we introduce the problem with the practical motivations and we give for the parameters  $\beta, \gamma, \mu$  a geometrical interpretation. In section 2, we recall some classical results and introduce the functional framework. Section 3 is devoted to the existence result of the minimizing problem. In section 4, we introduce the approximation process and prove the main result: the family  $(E_\varepsilon)_{\varepsilon > 0}$   $\Gamma$ -converges to  $E$ .

## 1 Presentation of the model

In what follows,  $n$  represents the spatial dimension of the image:  $n = 2$  for planar images and  $n = 3$  for 3-D images. We adopt the list of symbols:

- $x$  a scalar in  $\mathbb{R}$ ,
- $\mathbf{x} \cdot \mathbf{y}$  the usual scalar product of  $\mathbb{R}^n$ ,
- $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$  the euclidean norm of  $\mathbb{R}^n$ ,
- $\mathbb{S}^{n-1}$  the unit sphere of  $\mathbb{R}^n$ ,
- $B_r(x)$  the ball of  $\mathbb{R}^n$  with center  $x$  and radius  $r \geq 0$ ,
- $\|A\|$  a generic matricial norm in the space of  $n \times n$  matrices,
- $\text{sp}(A)$  the eigenvalues of  $A$  counted with their multiplicities.

### 1.1 Motivation

For the study of some diseases, it is interesting to focus on the blood status in a vessel network, especially on the volume of its microvasculature. To assess this, *in vivo* mice brain angiography is performed. This is based on the injection of a contrast medium and a MRI imaging process.

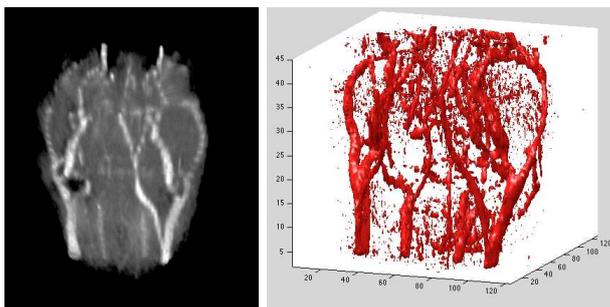


Figure 1.1: Mouse brain angiography and thresholding at 80% of the maximum intensity

The left image in Figure 1.1 is a planar projection of a 3-D image, the right image is the result of the manual segmentation with thresholding at

a level equal to 80% of the maximum intensity. The aim is to construct an automatic method to segment the network area corresponding to the blood while removing noise.

## 1.2 Geometric characterization of the problem

We give an heuristic way to introduce and motivate the model. We first present an isotropic model and show that it is not suitable for our problem. Then, we introduce an anisotropic term.

Let  $\Omega \subset \mathbb{R}^n$  be the domain of the image. We consider the following segmentation problem. Let  $\alpha > 0$  be the critical level of detection: if a set is with diameter lower than  $\alpha$  then it is considered as noise and has to be removed. Let  $\Gamma \subset \Omega$  be a curve with length  $\ell$  such that  $\alpha$  is negligible compared with  $\ell$ . We set  $T_{\ell,\alpha}$  the tubular neighborhood of  $\Gamma$  of the points of  $\Omega$  at distance lower than  $\alpha$  of  $\Gamma$ . Although the section of  $T_{\ell,\alpha}$  is critical we want to detect it because of its specific geometry. We set  $B_\alpha$  a ball with radius  $\alpha$ , it is considered as noise not because it has a critical diameter but because it has not the appropriate geometry of tubes.

Let  $I \subset \Omega$  be a generic set and we assume that we have the following disjoint decomposition (see figure 1.2)

$$I = T_{\ell,\alpha} \cup B_\alpha$$

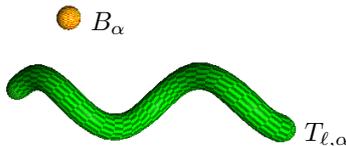


Figure 1.2: Decomposition of  $I$

The segmentation problem consists in combining two constraints. The first one is to remove  $B_\alpha$  type sets, because they have small radius and no tubular geometry. The second one is to detect the tubes  $T_{\ell,\alpha}$ .

For that, we propose an energy functional  $\mathcal{E}$  defined on the sets of  $\Omega$ . We say that a set  $F$  is a *better segmentation* than the set  $G$  if  $\mathcal{E}(F) < \mathcal{E}(G)$ . The functional  $\mathcal{E}$  is *adapted* to the problem if it satisfies the following conditions:

- i)  $\mathcal{E}(I \setminus B_\alpha) < \mathcal{E}(I)$ ,
- ii)  $\mathcal{E}(I) < \mathcal{E}(I \setminus T_{\ell,\alpha})$ .

Condition i) implies that removing  $B_\alpha$  provides a better segmentation than keeping it. Condition ii) implies that detecting  $T_{\ell,\alpha}$  gives a better segmentation than removing it.

We first consider an isotropic functional  $\mathcal{E}$  defined on the subsets of  $\Omega$  as:

$$\mathcal{E}(F) = \text{Vol}^n(F \Delta I) + \beta \text{Vol}^{n-1}(\partial F),$$

where  $\text{Vol}^n$  and  $\text{Vol}^{n-1}$  are respectively the volumes measure with dimension  $n$  and  $n - 1$ ,  $F \Delta I$  the symmetric difference of the sets  $F$  and  $I$ , the topological boundary of a set  $F$  is denoted  $\partial F$ . The parameter  $\beta > 0$  is a weight to tune. Many works have been devoted to this particular case. Let us mention contributions from Morel and Solimini [2]. We show that this model can not satisfy the constraints we imposed to our problem.

As  $T_{\ell,\alpha}$  and  $B_\alpha$  are disjoint, condition i) is equivalent to

$$\text{Vol}^n(B_\alpha) < \beta \text{Vol}^{n-1}(\partial B_\alpha). \quad (1.1)$$

The sets which minimize the ratio  $\text{Vol}^n/\text{Vol}^{n-1}$  are preferred by this energy. It is well known that, with the diameter fixed, the sets which minimize this ratio are the balls. Thus, if the inequality (1.1) is ensured, then any set with diameter lower than  $\alpha$  is removed. Taking the equivalences  $\text{Vol}^n(B_\alpha) \sim \alpha^n$  and  $\text{Vol}^{n-1}(\partial B_\alpha) \sim \alpha^{n-1}$ , this condition gives

$$\alpha < \beta. \quad (1.2)$$

Condition ii) is equivalent to

$$\beta \text{Vol}^{n-1}(\partial T_{\ell,\alpha}) < \text{Vol}^n(T_{\ell,\alpha}).$$

Taking the equivalences  $\text{Vol}^n(T_{\ell,\alpha}) \sim \ell \alpha^{n-1}$  and  $\text{Vol}^{n-1}(\partial T_{\ell,\alpha}) \sim \ell \alpha^{n-2}$ , this condition gives

$$\beta < \alpha. \quad (1.3)$$

Thus, conditions (1.2) and (1.3) are contradictory. Such a model is not adapted to this problem.

### 1.3 An anisotropic model

We rather introduce an energy term that involves a preference to sets *having a direction*, that is anisotropic sets. Let  $\mathbf{c} : \Omega \rightarrow \mathbb{S}^{n-1}$  be an unknown and unitary vector field that represents a direction in each point of the image.

Let  $x \in \partial F$  and  $\boldsymbol{\nu}_F(x)$  be a unit normal vector of the surface  $\partial F$  at  $x$ . We say that  $|\mathbf{c}(x) \cdot \boldsymbol{\nu}_F(x)|$  is the *action of the vector field  $\mathbf{c}$*  on  $\partial F$  at  $x$ . This term is zero if the field is tangent to the surface and it is maximum if it is orthogonal to the surface. We introduce the *total action* of  $\mathbf{c}$  on  $\partial F$ :

$$\text{Action}(F, \mathbf{c}) = \int_{\partial F} |\mathbf{c} \cdot \boldsymbol{\nu}_F| d\text{Vol}^{n-1}.$$

As  $\alpha \ll \ell$ , a field which minimizes the action of  $\mathbf{c}$  on  $\partial T_{\ell,\alpha}$  has to be tangent to  $\partial T_{\ell,\alpha}$  along the tube (see figure 1.3).

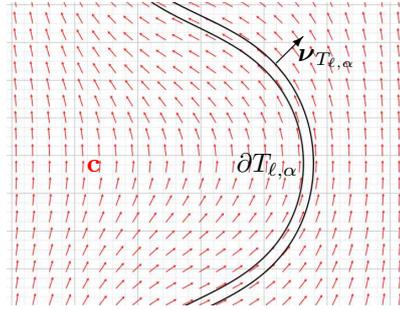


Figure 1.3: A field  $\mathbf{c}$  tangent to the edge of a tube  $T_{\ell,\alpha}$ .

Moreover, we introduce a regularization term defined on the vector field as

$$\text{Reg}(\mathbf{c}) = \int_{\Omega} \|D\mathbf{c}\|^r d\text{Vol}^n,$$

where  $\|\cdot\|$  is a pointwise matricial norm and we fix  $r > n$  (not necessary an integer) to ensure that the field is regular. Indeed, if  $r > n$  and  $\int_{\Omega} \|D\mathbf{c}\|^r < \infty$  then  $\mathbf{c}$  is continuous. The new expression of the energy is

$$\mathcal{E}(F, \mathbf{c}) = \text{Vol}^n(F \triangle A) + \underbrace{\beta [\text{Vol}^{n-1}(\partial F) + \mu \text{Action}(\partial F, \mathbf{c})]}_{\text{anisotropic term}} + \gamma \text{Reg}(\mathbf{c}), \quad (1.4)$$

where  $\beta$ ,  $\mu$  and  $\gamma$  are, as before, weights to tune. We have to verify the conditions i) and ii) of 1.2. We assume that  $T_{\ell, \alpha}$  is a linear and rigid tube of length  $\ell$  and section  $\alpha$ . Obviously, the best choice of  $\mathbf{c}$  is to choose it in the direction of the tube. Indeed, the action of the field on the tube is zero outside the two ends of the tube and the regularization is zero (see figure 1.4).

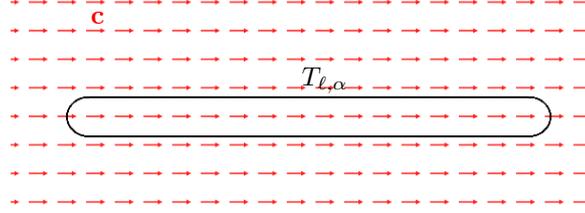


Figure 1.4: A tube  $T_{\ell, \alpha}$  and  $\mathbf{c}$  in the direction of the tube.

Using the following equivalences:

$$\text{Vol}^{n-1}(T_{\ell, \alpha}) \sim \ell \alpha^{n-1}, \quad \text{Vol}^{n-1}(\partial T_{\ell, \alpha}) \sim \ell \alpha^{n-2}, \quad \text{Action}(\partial T_{\ell, \alpha}, \mathbf{c}) \sim \alpha^{n-1}.$$

Condition ii) is equivalent to

$$\beta(\ell \alpha^{n-2} + \mu \alpha^{n-1}) < \ell \alpha^{n-1}. \quad (1.5)$$

For a ball  $B_{\alpha}$ , the field  $\mathbf{c}$  has to realize a compromise between its action on  $\partial B_{\alpha}$  and its regularization (see figure 1.5).

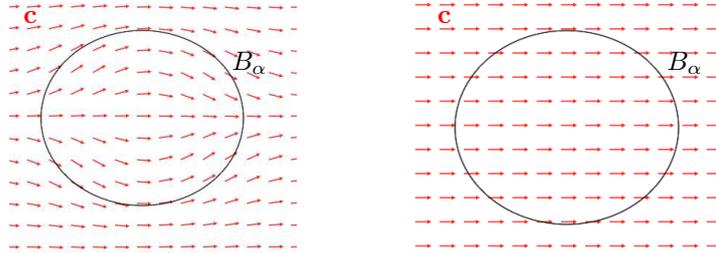


Figure 1.5: At left,  $\mathbf{c}$  minimizes the action, at right, the regularization

We make the homothetic change of variable between  $B_{\alpha}$  and  $B_1$ , a ball of radius 1. We denote  $\mathbf{c}_1 = \mathbf{c}(\alpha \cdot)$ . It gives:

$$\beta \mu \text{Action}(\partial B_{\alpha}, \mathbf{c}) + \gamma \text{Reg}(\mathbf{c}) = \beta \mu \alpha^{n-1} \text{Action}(\partial B_1, \mathbf{c}_1) + \gamma \alpha^{n-r} \text{Reg}(\mathbf{c}_1).$$

As  $\alpha$  is small and  $r > n$ , then if  $\mu \sim \gamma$  the parameter  $\mu\alpha^{n-1}$  is negligible with respect to  $\gamma\alpha^{n-r}$ . As a conclusion, the regularization is more important than the action for balls with small radius  $\alpha$ . The best choice for  $\mathbf{c}$  is a constant field. In this case the regularization is zero and the action is equal to the action on the tube with the same section. Then, condition i) is equivalent to

$$\alpha^n < \beta(\alpha^{n-1} + \mu\alpha^{n-1}). \quad (1.6)$$

The two conditions (1.5) and (1.6) are not contradictory anymore when  $\alpha$  is small and  $\alpha \ll \ell$ . For example, we can take

$$\begin{cases} \beta = \frac{\alpha}{2}, \\ \mu < \frac{\ell}{\alpha}, \\ \gamma = \mu. \end{cases}$$

## 1.4 Functional formulation

In the sequel, we are formulating  $\min \mathcal{E}$  (1.4) as a minimization problem of functions by connecting sets and functions via indicator functions. We define an image as a function  $g : \Omega \rightarrow [0; 1]$ . We assume that the domain  $\Omega \subset \mathbb{R}^n$  is Lipschitz-regular. The fundamental assumption of this model is that the histogram distribution of the image contains two main modes that we assume to be 0 and 1. Equivalently,  $g$  is almost equal to an indicator function. The unknown is a pair  $(p, \mathbf{c})$  where  $p : \Omega \rightarrow \{0; 1\}$  is a binary function and  $\mathbf{c} : \Omega \rightarrow \mathbb{S}^{n-1}$  is a unitary vector field which minimizes the energy

$$\int_{\Omega} (p - g)^2 dx + \beta \left[ \mathcal{H}^{n-1}(S_p) + \mu \int_{S_p} |\mathbf{c} \cdot \boldsymbol{\nu}_p| d\mathcal{H}^{n-1} \right] + \gamma \int_{\Omega} \|D\mathbf{c}\|^r dx, \quad (1.7)$$

where  $dx$  is the integration with respect to the  $n$ -dimensional Lebesgue measure,  $S_p$  is the jump set of  $p$ ,  $\boldsymbol{\nu}_p : S_p \rightarrow \mathbb{S}^{n-1}$  is a normal unit vector of  $S_p$  and  $d\mathcal{H}^{n-1}$  is the integration with respect to the  $(n-1)$ -dimensional Hausdorff measure. We can rewrite the functional defined in (1.7) as

$$\int_{\Omega} (p - g)^2 + \beta \int_{S_p} (1 + \mu |\mathbf{c} \cdot \boldsymbol{\nu}_p|) d\mathcal{H}^{n-1} + \gamma \int_{\Omega} \|D\mathbf{c}\|^r. \quad (1.8)$$

The second term of (1.8) corresponds to the anisotropic perimeter of  $S_p$  according to the metric  $\phi : \Omega \times \mathbb{R}^n \rightarrow [0; +\infty[$  defined as

$$\phi(x, \mathbf{v}) = |\mathbf{v}| + \mu |\mathbf{c}(x) \cdot \mathbf{v}|.$$

For more convenience in calculus, we will adopt the equivalent quadratic form

$$\int_{\Omega} (p - g)^2 + \beta \int_{S_p} \sqrt{1 + \mu^2 (\mathbf{c} \cdot \boldsymbol{\nu}_p)^2} d\mathcal{H}^{n-1} + \gamma \int_{\Omega} \|D\mathbf{c}\|^r. \quad (1.9)$$

It has an obvious invariance. Indeed, let  $\sigma : \Omega \rightarrow \{-1; 1\}$  be an arbitrary function, the functional (1.8) takes the same value for  $\mathbf{c}$  and  $\sigma\mathbf{c}$ . This invariance may generate numerical instability in the numerics. To remove it, we replace the unknown vector field  $\mathbf{c}$  by a riemannian metric which takes the form

$$\mathbf{g}(x, \mathbf{v}) = |\mathbf{v}|^2 + \mu^2 (\mathbf{c}(x) \cdot \mathbf{v})^2.$$

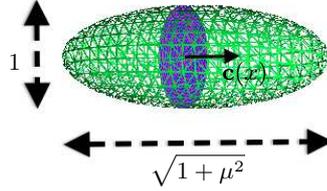
We introduce the definitive version of the functional as

$$E(p, \mathbf{g}) = \int_{\Omega} (p - g)^2 + \beta \int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1} + \gamma \|\mathbf{g}\|_{W^{1,r}(\Omega)}. \quad (1.10)$$

In [1], it is proved that  $\int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1}$  is the anisotropic perimeter associated to the dual metric denoted  $\mathbf{g}^0$ . We can explicitly calculate this metric

$$\mathbf{g}^0(x, \mathbf{v}) = |\mathbf{v}|^2 - \frac{\mu^2}{1 + \mu^2} (\mathbf{c}(x) \cdot \mathbf{v})^2.$$

The unit ball for this metric is an elongated ellipsoid in the direction of  $\mathbf{c}(x)$ .



For this metric, the points in direction of  $\mathbf{c}(x)$  are closer of  $x$  for the dual metric, than the points in the orthogonal directions. The ratio of the elongation is equal to  $\sqrt{1 + \mu^2}$ .

## 2 Functional framework

### 2.1 Sobolev spaces

We denote  $W^{1,2}(\Omega; [0; 1])$  the set of functions  $p$  which belong to the classical Sobolev space  $W^{1,2}(\Omega)$  such as  $p(x) \in [0; 1]$  a.e.  $x \in \Omega$ .

Let  $S_n(\mathbb{R})$  be the space of  $n \times n$  symmetric matrices. As  $S_n(\mathbb{R})$  is a vectorial space with finite dimension  $(n(n + 1)/2)$ , we may define the Sobolev space  $W^{1,r}(\Omega; S_n(\mathbb{R}))$ . For any riemannian metric  $\mathbf{g} : \Omega \times \mathbb{R}^n \rightarrow [0; +\infty[$ , we denote  $\mathbf{g}_x \in S_n(\mathbb{R})$  the symmetric matrix at point  $x \in \Omega$ , that is

$$\forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \mathbf{g}(x, \mathbf{v}) = (\mathbf{g}_x \mathbf{v}) \cdot \mathbf{v}.$$

Let  $\mathcal{G}$  be the subset of  $S_n(\mathbb{R})$  defined by

$$\mathcal{G} = \{\text{Id}_n + {}^t c c : c \in \mathbf{S}^{n-1}\}.$$

Obviously, any matrix which belongs to  $\mathcal{G}$  is symmetric definite positive, so any function defined in  $\Omega$  and taking its values in  $\mathcal{G}$  may be considered as a riemannian metric. We denote  $W^{1,r}(\Omega; \mathcal{G})$  the set of function  $\mathbf{g} \in W^{1,r}(\Omega; S_n(\mathbb{R}))$  such as, for almost every  $x \in \Omega$ , we have  $\mathbf{g}_x \in \mathcal{G}$ .

The coefficient  $r$  is determined according to the following classical theorem (see [3], for example).

**Theorem 2.1.** *Let  $\mathcal{C}(\Omega; S_n(\mathbb{R}))$  be the space of continuous functions defined on  $\Omega$  taking their values in  $S_n(\mathbb{R})$  endowed with the  $L^\infty$  norm. Let  $r > n$  and consider the Sobolev space  $W^{1,r}(\Omega; S_n(\mathbb{R}))$ , then the following compact embedding result holds*

$$W^{1,r}(\Omega; S_n(\mathbb{R})) \hookrightarrow \mathcal{C}(\Omega; S_n(\mathbb{R})).$$

The assumption  $r > n$  is motivated by the regularity of the metric and by what follows.

**Proposition 2.1.** *If  $r > n$ , then  $W^{1,r}(\Omega; \mathcal{G})$  is closed in  $W^{1,r}(\Omega; S_n(\mathbb{R}))$  for the weak topology associated to the Sobolev norm.*

To prove this proposition, we need the following lemma which will be useful throughout the article.

**Lemma 2.1.** *For  $G \in S_n(\mathbb{R})$ , we have*

- i)  $G \in \mathcal{G} \Rightarrow \forall \mathbf{v} \in \mathbb{R}^n, \quad |\mathbf{v}|^2 \leq (G\mathbf{v}) \cdot \mathbf{v} \leq 2|\mathbf{v}|^2,$
- ii)  $G \in \mathcal{G} \Leftrightarrow \text{sp}(G) = \{1; 1; 2\}.$

*Proof.* If  $G \in \mathcal{G}$  then there exists  $c \in \mathbb{S}^{n-1}$  such as  $G = \text{Id}_n + {}^tcc$  and Cauchy–Schwarz inequality gives  $|\mathbf{v}|^2 \leq (G\mathbf{v}) \cdot \mathbf{v} \leq 2|\mathbf{v}|^2$ . Moreover,  $c$  is an eigenvector associated with the eigenvalue 2 and the restriction of  $G$  to  $\{c\}^\perp$  is the identity. So, we have  $\text{sp}(G) = \{1; 1; 2\}$ .

Conversely, we assume that  $\text{sp}(G) = \{1; 1; 2\}$ . Let  $c$  be the unitary eigenvector associated to the eigenvalue 2. As  $G$  is symmetric then  $\{c\}^\perp$  is stable by  $G$  and necessary its restriction is the identity. We can conclude that  $G = \text{Id}_n + {}^tcc$ . □

Now, we can prove the proposition 2.1.

*Proof.* Let  $(\mathbf{g}^k)_k \subset W^{1,r}(\Omega; \mathcal{G})$  be a Cauchy sequence for the weak topology associated to  $W^{1,r}(\Omega; S_n(\mathbb{R}))$ . As  $r > n$ , the following inclusion is compact

$$W^{1,r}(\Omega; S_n(\mathbb{R})) \subset \mathcal{C}(\Omega; S_n(\mathbb{R})).$$

So,  $(\mathbf{g}^k)_k$  is also a Cauchy sequence for the  $L^\infty(\Omega; M_n(\mathbb{R}))$  norm. So, for  $x \in \Omega$  fixed, the sequence  $(\mathbf{g}_x^k)_k$  converges to a matrix  $\mathbf{g}_x$ . As the two characterizations of the lemma 2.1 are stable under the limit, then  $\mathbf{g}_x$  verifies this two conditions. This proves that  $\mathbf{g} \in \mathcal{G}$  and then  $W^{1,r}(\Omega; \mathcal{G})$  is closed in  $W^{1,r}(\Omega; M_n(\mathbb{R}))$ . □

For the need of many proofs, we need a density result for smooth functions. We recall the classical result (see [3]).

**Theorem 2.2.** *If  $1 \leq r' < +\infty$ , then  $\mathcal{C}^\infty \cap W^{1,r'}(\Omega)$  is dense in  $W^{1,r'}(\Omega)$  for the strong topology of  $W^{1,r'}(\Omega)$ .*

In our particular case, we need the following result.

**Proposition 2.2.** *The space  $\mathcal{C}^\infty \cap W^{1,r}(\Omega; \mathcal{G})$  is dense in  $W^{1,r}(\Omega; \mathcal{G})$  for the strong topology of  $W^{1,r}(\Omega; S_n(\mathbb{R}))$ .*

*Proof.* We give the outline of the proof which is quite abroad our subject.

1. For  $G \in \mathcal{G}$ , we can associate a unique vector space  $D \subset \mathbb{R}^n$  with dimension 1 which corresponds to the eigenspace associated to the eigenvalue 2.
2. The set of 1 dimensional space of  $\mathbb{R}^n$  is a smooth and compact manifold called the Grassmanian  $G_{1,n}(\mathbb{R})$ .
3.  $G_{1,n}(\mathbb{R})$  and  $\{\text{Id} + {}^tcc : c \in \mathbb{S}^{n-1}\}$  are compact  $\mathcal{C}^\infty$  diffeomorphic manifolds.
4. We may consider  $x \rightarrow \mathbf{g}_x$ , from  $\Omega$  to  $G_{1,n}(\mathbb{R})$ , as a continuous function of manifolds.

5. Using an atlas of the manifold  $G_{1,n}(\mathbb{R})$  and a smooth partition of unity, we may apply Meyers-Serrin theorem to the previous function in each map of the atlas.

□

## 2.2 Measure theory

Let  $\mathcal{C}_c(\Omega; \mathbb{R}^n)$  be the space of continuous functions with compact support in  $\Omega$  and taking their values in  $\mathbb{R}^n$ . We denote  $\mathcal{C}_0(\Omega; \mathbb{R}^n)$  the closure in the sup norm of  $\mathcal{C}_c(\Omega; \mathbb{R}^n)$ . Let  $\mathcal{M}(\Omega)$  be the space of Radon measures and  $\mathcal{M}(\Omega; \mathbb{R}^n)$  be the space of vectorial Radon measures over  $\Omega$ . For  $\lambda \in \mathcal{M}(\Omega; \mathbb{R}^n)$ , we denote

$$\|\lambda\|_{\mathcal{M}(\Omega; \mathbb{R}^n)} = \sup \left\{ \int_{\Omega} \varphi \cdot d\lambda : \varphi \in \mathcal{C}_c(\Omega; \mathbb{R}^n), \|\varphi\|_{L^\infty} \leq 1 \right\}.$$

This application is a norm and  $\mathcal{M}(\Omega; \mathbb{R}^n)$  is a Banach space. This topology is quite restrictive in our case, we introduce a weaker topology.

**Definition 2.1.** *Let  $\lambda \in \mathcal{M}(\Omega; \mathbb{R}^n)$ , the sequence  $(\lambda_k)_k \subset \mathcal{M}(\Omega; \mathbb{R}^n)$  weakly\* converges to  $\lambda$  if*

$$\lim_k \int_{\Omega} \varphi \cdot d\lambda_k = \int_{\Omega} \varphi \cdot d\lambda$$

for every  $\varphi \in \mathcal{C}_0(\Omega; \mathbb{R}^n)$ .

Endowed of this topology, the space  $\mathcal{M}(\Omega; \mathbb{R}^n)$  satisfies a compactness property.

**Theorem 2.3.** *If  $(\lambda_k)_k \subset \mathcal{M}(\Omega; \mathbb{R}^n)$  is a bounded sequence for the topology of the norm, then it has a weakly\* converging subsequence. Moreover, the norm is lower semicontinuous with respect to the weak\* convergence.*

Let  $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^+$  be a sublinear function with respect to the second variable, that is:

i)

$$\forall (x, \mathbf{v}_1, \mathbf{v}_2) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n, \quad \varphi(x, \mathbf{v}_1 + \mathbf{v}_2) \leq \varphi(x, \mathbf{v}_1) + \varphi(x, \mathbf{v}_2),$$

ii)

$$\forall (x, \mathbf{v}, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^+, \quad \varphi(x, t\mathbf{v}) = t\varphi(x, \mathbf{v}).$$

Suppose that  $\theta$  is a Radon measure and  $\lambda$  is a vectorial Radon measure on  $\Omega$ . According to Besicovitch derivation theorem (see [4])

$$\lim_{r \rightarrow 0} \frac{\lambda(B_r(x))}{\theta(B_r(x))}$$

exists and is finite for  $\theta$  almost every  $x$ , we denote by  $\frac{d\lambda}{d\theta}(x)$  this limit when it exists. We recall that  $\lambda$  is absolutely continuous with respect to  $\theta$  if  $\lambda(A) = 0$  whenever  $\theta(A) = 0$ . When this holds, we write  $\lambda \ll \theta$ . We consider the convex functional defined on the space  $\mathcal{M}(\Omega; \mathbb{R}^n)$  by

$$\Phi : \lambda \in \mathcal{M}(\Omega; \mathbb{R}^n) \mapsto \int_{\Omega} \varphi \left( x, \frac{d\lambda}{d\theta} \right) d\theta \quad (2.1)$$

where  $\theta$  is a positive measure such that  $\lambda \ll \theta$ . It is shown in [5] that the integral in (2.1) does not depend on the choice of  $\theta$ . For that reason, we will write it in the condensed form

$$\Phi(\lambda) = \int_{\Omega} \varphi(x, \lambda).$$

The functional  $\Phi$  has the following continuity properties which are proved in [6].

**Proposition 2.3.** *i) If  $\varphi$  is a lower semicontinuous on  $\Omega \times \mathbb{R}^n$ , then  $\Phi$  is lower semicontinuous on  $\mathcal{M}(\Omega; \mathbb{R}^n)$  for the topology introduced in 2.1.*

*ii) Assume that  $\varphi$  is continuous on  $\Omega \times \mathbb{R}^n$ . If  $(\lambda_k)_k$  weakly converges to  $\lambda$  and if, moreover,  $\int_{\Omega} |\lambda_k| \rightarrow \int_{\Omega} |\lambda|$ , then  $\Phi(\lambda_k)$  converges to  $\Phi(\lambda)$ .*

We give a variant of the coarea formula extended to the sublinear functionals which can be found in [7].

**Proposition 2.4.** *Let  $\Phi(x, s, v)$  a Borel function of  $\Omega \times \mathbb{R} \times \mathbb{R}^n$  which is sublinear in  $v$ . Let  $p$  be a Lipschitz continuous function on  $\Omega$  and denote, for  $t > 0$ ,  $S_t = \{x \in \Omega; p(x) < t\}$ . Then, for almost all  $t \in \mathbb{R}$ ,  $S_t$  belongs to  $BV(\Omega)$  and we have*

$$\int_{\Omega} \Phi(x, p, Dp) dx = \int_{\mathbb{R}} dt \int_{\Omega} \Phi(x, t, D\mathbf{1}_{S_t}).$$

### 2.3 Functions with bounded variation

A function  $u \in L^1(\Omega)$  is said to be with bounded variation if

$$\sup \left\{ \int_{\Omega} u \operatorname{div}(\varphi) : \varphi \in \mathcal{C}_0^1(\Omega; \mathbb{R}^n), \|\varphi\|_{L^\infty} \leq 1 \right\} < +\infty.$$

We denote  $TV(u)$  this upper bound and  $BV(\Omega)$  the set of such functions. The space  $BV(\Omega)$ , equipped with the following norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + TV(u)$$

is a Banach space. According to Riesz representation theorem, if  $u \in BV(\Omega)$  then  $Du \in \mathcal{M}(\Omega; \mathbb{R}^n)$ . The topology of the norm in  $BV(\Omega)$  is quite restrictive in our case, we consider a weaker one.

**Definition 2.2.** *A sequence  $(u_k)_k \subset BV(\Omega)$  weakly\* converges to  $u \in BV(\Omega)$  if  $(u_k)_k$  converges to  $u$  in  $L^1(\Omega)$  and  $Du_k$  weakly\* converges to  $Du$  in  $\mathcal{M}(\Omega; \mathbb{R}^n)$ .*

The space  $BV(\Omega)$  satisfies a compactness result for the weak\* convergence.

**Theorem 2.4.** *If  $(u_k)_k \subset BV(\Omega)$  is a bounded sequence for the topology of the norm, then it has a weakly\* converging subsequence.*

A criterium for weak\* convergence is stated in the following theorem.

**Theorem 2.5.** *Let  $(u_k)_k$  be a sequence of  $BV(\Omega)$ . Then  $(u_k)_k$  weakly\* converges to  $u \in BV(\Omega)$  if and only if  $(u_k)_k$  is bounded in  $BV(\Omega)$  and converges to  $u$  in  $L^1(\Omega)$ .*

## 2.4 Sets with finite perimeter

The following terminology is introduced in [8].

**Definition 2.3.** Let  $A \subset \Omega$  be a measurable set. A point  $x \in \Omega$  belongs to *measure theoretic boundary* of  $A$  if

$$\limsup_{r \rightarrow 0^+} \frac{\mathcal{L}^n(B(x, r) \cap A)}{r^n} > 0,$$

and

$$\liminf_{r \rightarrow 0^+} \frac{\mathcal{L}^n(B(x, r) \setminus A)}{r^n} > 0.$$

We denote  $\partial^* A$  the *measure theoretic boundary* of  $A$ .

**Definition 2.4.** A measurable set  $A \subset \Omega$  is said with *finite perimeter* if  $\mathbf{1}_A \in BV(\Omega)$ .

**Remark 2.1.** The concept of *measure theoretic boundary* generalizes the concept of *topological boundary* when  $A$  is regular. Indeed, if  $A \subset \Omega$  is an open whose topological boundary  $\partial A$  is a smooth and compact hypersurface, then we have

- i)  $\partial A = \partial^* A$ ,
- ii)  $TV(\mathbf{1}_A) = \mathcal{H}^{n-1}(\partial A)$ .

**Theorem 2.6.** Let  $A \subset \Omega$  be a set with finite perimeter. There exists a pairwise disjoint family of sets  $(S_i)_i$  and a set  $M \subset \Omega$  such as

- i) for all  $i$ ,  $S_i$  is a  $C^1$  and compact hypersurface of  $\Omega$ ,
- ii)  $\mathcal{H}^{n-1}(M) = 0$ ,
- iii)  $\partial^* A = M \cup \left(\bigcup_i S_i\right)$ .

**Theorem 2.7.** Let  $A \subset \Omega$  be a set with finite perimeter. The following generalized Gauss-Green formula holds: for  $\mathcal{H}^{n-1}$  almost every  $x \in \Omega$ , there exists  $\nu(x) \in \mathbb{S}^{n-1}$ , called the *inner normal vector* to  $A$  at  $x$ , such that for all  $\varphi \in C_c^1(\Omega; \mathbb{R}^n)$ ,

$$\int_{\Omega} \mathbf{1}_A \operatorname{div}(\varphi) dx = - \int_{\partial^* A \cap \Omega} \varphi \cdot \nu d\mathcal{H}^{n-1},$$

that is  $D\mathbf{1}_A = \nu \mathcal{H}^{n-1} \llcorner \partial^* A \cap \Omega$ .

**Definition 2.5.** Let  $A \subset \Omega$  be a set with finite perimeter and  $p = \mathbf{1}_A$ . With the theorems 2.6 and 2.7, we get  $S_p = \partial^* A$  and  $\nu_p = \nu$ .

**Proposition 2.5.** Let  $A \subset \Omega$  be a set with finite perimeter and  $p = \mathbf{1}_A$ . Then, we have  $\|Dp\|_{\mathcal{M}} = \mathcal{H}^{n-1}(S_p)$ .

**Remark 2.2.** If  $p = \mathbf{1}_A$  with  $A \subset \Omega$  an open subset whose boundary  $\partial A$  is a compact hypersurface with class  $C^1$ , then we have  $S_p = \partial A$ .

The following lemma is proved in [9]. It asserts that every set with bounded perimeter can be approximated by a sequence of smooth subsets of  $\mathbb{R}^n$ , all having the same volume inside  $\Omega$  and each of these boundaries satisfy a measure theoretic transversality condition with respect to  $\Omega$ .

**Lemma 2.2.** Let  $\Omega$  be an open, bounded subset of  $\mathbb{R}^n$  with Lipschitz continuous boundary, and let  $A$  be a measurable subset of  $\Omega$ . If  $A$  and  $\Omega \setminus A$  both contain a non-empty open ball, then there exists a sequence  $(A_k)_k$  of open bounded subsets of  $\mathbb{R}^n$  with smooth boundaries such that

i)

$$\lim_{k \rightarrow \infty} \mathcal{L}^n((A_k \cap \Omega) \triangle A) = 0, \quad \lim_{k \rightarrow \infty} \mathcal{H}^{n-1}(\partial A_k) = TV(\mathbf{1}_A);$$

ii)

$$\mathcal{L}^n(A_k \cap \Omega) = \mathcal{L}^n(A) \quad \text{for } k \text{ large enough};$$

iii)

$$\mathcal{H}^{n-1}(\partial A_k \cap \partial \Omega) = 0 \quad \text{for } k \text{ large enough}.$$

### 3 Existence result

In this section we prove that the function defined in (1.10) admits at least a minimizer in an appropriate functional space. In the following, for more simplicity in the notations, we assume that the parameters of the functionals are fixed at  $\beta = \mu = \gamma = 1$ .

#### 3.1 Specific domain of $E$

In this section we introduce the appropriate functional spaces which ensures the existence result of the minimizing problem.

$$\begin{aligned} \mathbb{B}(\Omega; [0, 1]) &= \{p \text{ measurable} : p(x) \in [0, 1] \text{ a.e. } x \in \Omega\}, \\ \text{BV}(\Omega; \{0, 1\}) &= \{p \in \text{BV}(\Omega) : p(x) \in \{0, 1\} \text{ a.e. } x \in \Omega\}, \\ \mathcal{G} &= \{\mathbf{g} : \exists \mathbf{c} : \Omega \rightarrow \mathbb{S}^{n-1}, \forall x \in \Omega, \mathbf{g}_x = \text{Id}_n + {}^t \mathbf{c}(x) \mathbf{c}(x)\}, \\ W^{1,r}(\Omega; \mathcal{G}) &= \mathcal{G} \cap W^{1,r}(\Omega; S_n(\mathbb{R})), \\ \mathcal{X} &= \mathbb{B}(\Omega; [0, 1]) \times W^{1,r}(\Omega; \mathcal{G}), \\ \mathcal{Y} &= \text{BV}(\Omega; \{0, 1\}) \times W^{1,r}(\Omega; \mathcal{G}), \end{aligned}$$

Let  $\mathcal{T}$  be the product topology on  $\mathcal{X}$  where:

- $\mathbb{B}(\Omega; [0, 1])$  is endowed with the almost everywhere convergence topology,
- $W^{1,r}(\Omega; \mathcal{G})$  is endowed with the weak topology associated to the Sobolev norm  $\|\cdot\|_{W^{1,r}(\Omega)}$ .

For a sequence  $((p_k, \mathbf{g}^k))_k$  which converges to  $(p, \mathbf{g})$  for this topology, we write  $(p_k, \mathbf{g}^k) \xrightarrow{\mathcal{T}} (p, \mathbf{g})$ . Since these spaces are metrizable, then  $(X, \mathcal{T})$  is also metrizable.

#### 3.2 Existence of minimizers

We recall

$$E(p, \mathbf{g}) = \int_{\Omega} (p - g)^2 dx + \int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1} + \|\mathbf{g}\|_{W^{1,r}(\Omega)},$$

for  $(p, \mathbf{g}) \in \mathcal{Y}$  and  $+\infty$  if  $(p, \mathbf{g}) \in \mathcal{X} \setminus \mathcal{Y}$ . We have the following minimization problem

$$(\mathcal{P}): \quad \text{Min} \{E(p, \mathbf{g}) : (p, \mathbf{g}) \in \mathcal{X}\}. \quad (3.1)$$

In this section, we prove that problem  $(\mathcal{P})$  admits at least one solution. We apply the *direct method* of calculus of variations. We exhibit a minimizing sequence which is compact for an appropriate topology. Then, we prove a lower semicontinuity result for  $E$  and conclude.

As  $E$  is bounded from below by 0, there exists a sequence  $(p_k, \mathbf{g}^k)_k \subset \mathcal{Y}$  such that  $(E(p_k, \mathbf{g}^k))_k$  converges to the minimum value of  $E$ . In the following theorem we prove that we can extract a converging sequence from  $(p_k, \mathbf{g}^k)_k$ .

**Theorem 3.1.** *Let  $(p_k, \mathbf{g}^k)_k \subset \mathcal{X}$  such that*

$$\exists M > 0, \quad \forall k, \quad E(p_k, \mathbf{g}^k) \leq M.$$

*Then, there exists a subsequence, still denoted  $(p_k, \mathbf{g}^k)_k$ , and  $(p, \mathbf{g}) \in \mathcal{Y}$  such that  $(p_k, \mathbf{g}^k) \xrightarrow{\mathcal{T}} (p, \mathbf{g})$ .*

*Proof.* As  $E(p_k, \mathbf{g}^k)$  is finite for any  $k$ , we have  $(p_k, \mathbf{g}^k)_k \subset \mathcal{Y}$ . We separate the arguments of the proof for the sequence  $(p_k)_k$  and  $(\mathbf{g}^k)_k$ .

**First Step:** Compactness result for  $(p_k)_k$ .

As  $p_k$  takes its values in  $[0; 1]$  and  $\Omega$  is bounded, then  $(p_k)_k$  is a bounded sequence of  $L^1(\Omega)$ . According to lemma 2.1, we have

$$\forall x \in \Omega, \quad 1 \leq \mathbf{g}^k(x, \nu_{p_k}).$$

The integration with respect to  $\mathcal{H}^{n-1} \llcorner S_{p_k}$  gives  $\mathcal{H}^{n-1}(S_{p_k}) \leq E(p_k, \mathbf{g}^k)$ . According to proposition 2.5, we have  $\|Dp_k\|_{\mathcal{M}} = \mathcal{H}^{n-1}(S_{p_k})$ , so  $(p_k)_k$  is a bounded sequence of  $\text{BV}(\Omega)$ . According to theorem 2.4, there exists a subsequence, still denoted  $(p_k)_k$ , and some  $p \in \text{BV}(\Omega)$  such that  $(p_k)_k$  weakly\* converges to  $p$ . According to theorem 2.5,  $(p_k)_k$  converges to  $p$  for the  $L^1(\Omega)$  norm. As  $p_k$  takes its values in  $\{0; 1\}$ , we deduce that  $p$  takes its values in  $\{0; 1\}$ .

**Second Step:** Compactness result for  $(\mathbf{g}^k)_k$ .

As  $\|\mathbf{g}^k\|_{W^{1,r}(\Omega; S_n(\mathbb{R}))} \leq E(p_k, \mathbf{g}^k)$ , then  $(\mathbf{g}^k)_k$  is a bounded sequence in  $W^{1,r}(\Omega; S_n(\mathbb{R}))$ . According to Banach-Alaoglu theorem, there exists a subsequence, still denoted  $(\mathbf{g}^k)_k$ , and  $\mathbf{g} \in W^{1,r}(\Omega; S_n(\mathbb{R}))$  such that  $(\mathbf{g}^k)_k$  weakly converges to  $\mathbf{g}$  in  $W^{1,r}(\Omega; S_n(\mathbb{R}))$ . According to proposition 2.1, we have  $\mathbf{g} \in W^{1,r}(\Omega; \mathcal{G})$ .  $\square$

**Theorem 3.2.** *The functional  $E : \mathcal{X} \rightarrow \mathbb{R}$  is lower semicontinuous for the topology  $\mathcal{T}$ .*

*Proof.* The lower semicontinuity of the terms  $p \rightarrow \int_{\Omega} (p - g)^2 dx$  and  $\mathbf{g} \rightarrow \|\mathbf{g}\|_{W^{1,r}(\Omega)}$  are well known results and the proof can be found for example in [3]. The remaining part of this result is the lowersemicontinuity of

$$(p, \mathbf{g}) \rightarrow \int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

We first prove the result with  $\mathbf{g}$  fixed and generalize without this assumption.

**First Step:** Let  $\mathbf{g} \in W^{1,r}(\Omega; \mathcal{G})$  be fixed and  $(p_k)_k \subset \text{BV}(\Omega; \{0; 1\})$  which weakly\* converges to  $p \in \text{BV}(\Omega; \{0; 1\})$ . Then, we have

$$\int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{S_{p_k}} \mathbf{g}(x, \nu_{p_k})^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

We define  $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$  as

$$\varphi(x, \mathbf{v}) = \mathbf{g}(x, \mathbf{v})^{\frac{1}{2}}.$$

As  $r > n$ , then we have  $W^{1,r}(\Omega) \subset \mathcal{C}(\Omega)$  and then  $\mathbf{g}$  is continuous. We deduce that  $\varphi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous as well.

According to theorem 2.7, we have  $Dp_k \ll \mathcal{H}^{n-1} \llcorner S_{p_k}$ ,  $Dp \ll \mathcal{H}^{n-1} \llcorner S_p$  and

$$\frac{d(Dp_k)}{d(\mathcal{H}^{n-1} \llcorner S_{p_k})} = \nu_{p_k} \mathbf{1}_{S_{p_k}}, \quad \frac{d(Dp)}{d(\mathcal{H}^{n-1} \llcorner S_p)} = \nu_p \mathbf{1}_{S_p}.$$

Moreover,  $\varphi$  is sublinear with respect to  $\mathbf{v}$ . According to proposition 2.3, we can conclude the proof of the *First Step*.

**Second Step:** Let  $(p_k, \mathbf{g}^k)_k \subset \mathcal{Y}$  such as  $(p_k)_k$  weakly\* converges to  $p \in \text{BV}(\Omega; \{0; 1\})$  and  $(\mathbf{g}^k)_k$  weakly converges to  $\mathbf{g} \in W^{1,r}(\Omega; \mathcal{G})$ . Then, we have

$$\int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{S_{p_k}} \mathbf{g}^k(x, \nu_{p_k})^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

Lemma 2.1 gives

$$\begin{aligned} \left| \int_{S_{p_k}} \mathbf{g}^k(x, \nu_{p_k})^{\frac{1}{2}} - \mathbf{g}(x, \nu_{p_k})^{\frac{1}{2}} d\mathcal{H}^{n-1} \right| &\leq \int_{S_{p_k}} \frac{|[(\mathbf{g}_x^k - \mathbf{g}_x) \nu_{p_k}] \cdot \nu_{p_k}|}{\mathbf{g}^k(x, \nu_{p_k})^{\frac{1}{2}} + \mathbf{g}(x, \nu_{p_k})^{\frac{1}{2}}} d\mathcal{H}^{n-1}, \\ &\leq \sup_{x \in \Omega} \|\mathbf{g}_x^k - \mathbf{g}_x\| \frac{\mathcal{H}^{n-1}(S_{p_k})}{2}. \end{aligned}$$

As  $(p_k)_k$  weakly\* converges to  $p$  in  $\text{BV}(\Omega)$  theorem 2.5 implies that  $(\mathcal{H}^{n-1}(S_{p_k}))_k$  is a bounded sequence. Moreover,  $(\mathbf{g}^k)_k$  weakly converges to  $\mathbf{g}$  in  $W^{1,r}(\Omega; S_n(\mathbb{R}))$  and, according to theorem 2.1, the inclusion  $W^{1,r}(\Omega; S_n(\mathbb{R})) \subset \mathcal{C}(\Omega; S_n(\mathbb{R}))$  is compact, it gives that  $(\sup \|\mathbf{g}_x^k - \mathbf{g}_x\|)_k$  converges to 0. So, we have the following limit

$$(\star)_k = \int_{S_{p_k}} \mathbf{g}^k(x, \nu_{p_k})^{\frac{1}{2}} d\mathcal{H}^{n-1} - \int_{S_{p_k}} \mathbf{g}(x, \nu_{p_k})^{\frac{1}{2}} d\mathcal{H}^{n-1} \rightarrow 0,$$

We decompose

$$\int_{S_{p_k}} \mathbf{g}^k(x, \nu_{p_k})^{\frac{1}{2}} d\mathcal{H}^{n-1} = (\star)_k + \int_{S_{p_k}} \mathbf{g}(x, \nu_{p_k})^{\frac{1}{2}} d\mathcal{H}^{n-1}$$

According to *First Step*, the lim inf in the previous expression gives

$$\int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{S_{p_k}} \mathbf{g}^k(x, \nu_{p_k})^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

□

We can now prove the existence of solutions for problem  $(\mathcal{P})$  (3.1).

**Theorem 3.3.** *The problem  $(\mathcal{P})$  admits at least one solution.*

*Proof.* Let  $(p_k, \mathbf{g}^k)_k$  be a minimizing sequence of  $E$ . According to theorem 3.1, there exists a subsequence, still denoted  $(p_k, \mathbf{g}^k)_k$  which converges to  $(p, \mathbf{g}) \in \mathcal{Y}$  for the topology  $\mathcal{T}$ . According to theorem 3.2, we have

$$E(p, \mathbf{g}) \leq \liminf_{k \rightarrow \infty} E(p_k, \mathbf{g}^k).$$

As  $(p_k, \mathbf{g}^k)_k$  is a minimizing sequence for  $E$ , we can conclude that  $(p, \mathbf{g})$  is a solution of  $(\mathcal{P})$ . □

## 4 Approximation process

In this section we give the main result: we introduce an approximated problem and prove a  $\Gamma$ -convergence result.

### 4.1 $\Gamma$ -convergence

We want to perform an approximation of the energy  $E$  more suitable for numeric applications. We will do that in the sense of the  $\Gamma$ -convergence. In this section, we give the definition. For more properties on this subject one refer to [10].

**Definition 4.1.** Let  $(\mathcal{X}, d)$  be a metrizable space,  $(E_k)_k$  a sequence of real-valued functions  $E_k : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ , and  $E : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ . The sequence  $(E_k)_k$   $\Gamma$ -converges to  $E$  at  $x \in \mathcal{X}$  if both the following conditions hold:

i) for all sequences  $(x_k)_k$  converging to  $x \in \mathcal{X}$ , one has

$$E(x) \leq \liminf_{k \rightarrow \infty} E_k(x_k), \quad (4.1)$$

ii) there exists a sequence  $(y_k)_k$  converging to  $x \in \mathcal{X}$  such that

$$E(x) \geq \limsup_{k \rightarrow \infty} E_k(y_k). \quad (4.2)$$

When i) and ii) hold for all  $x \in \mathcal{X}$ , we say that  $(E_k)_k$   $\Gamma$ -converges to  $E$  in  $(\mathcal{X}, d)$ .

The main interest of the  $\Gamma$ -convergence, in our case, is the following result.

**Theorem 4.1.** Let  $(E_k)_k$  be a sequence of functions which  $\Gamma$ -converges to  $E$  in  $(\mathcal{X}, d)$ . Let  $(x_k)_k$  be such that

$$\forall k, \quad E_k(x_k) \leq \inf_{x \in \mathcal{X}} E_k(x) + \varepsilon_k,$$

where  $\varepsilon_k > 0$  converges to 0. Assume that  $(x_k)_k$  is relatively compact; then every cluster point  $x$  of  $(x_k)_k$  is a minimizer of  $E$  and

$$\liminf_{k \rightarrow \infty} E_k(x_k) = E(x).$$

### 4.2 The main theorem

We introduce the functionals spaces for the approximation process.

$$\begin{aligned} W^{1,2}(\Omega; [0; 1]) &= \{p \in W^{1,2}(\Omega) : 0 \leq p(x) \leq 1 \text{ a.e. } x \in \Omega\}, \\ \mathcal{Z} &= W^{1,2}(\Omega; [0; 1]) \times W^{1,r}(\Omega; \mathcal{G}). \end{aligned}$$

Let  $H$ ,  $F$ ,  $F_\varepsilon$  and  $E_\varepsilon$  be the functions defined on  $\mathcal{X}$  and with values in  $[0; +\infty]$  as

$$\begin{aligned} H(p, \mathbf{g}) &= \int_{\Omega} (p - g)^2 dx + \|\mathbf{g}\|_{W^{1,r}(\Omega)}, \\ F(p, \mathbf{g}) &= \begin{cases} \int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1} & \text{if } (p, \mathbf{g}) \in \mathcal{Y}, \\ +\infty & \text{otherwise,} \end{cases} \\ F_\varepsilon(p, \mathbf{g}) &= \begin{cases} \int_{\Omega} \left[ 9\varepsilon \mathbf{g}(x, \nabla p) + \frac{p^2(1-p)^2}{\varepsilon} \right] dx & \text{if } (p, \mathbf{g}) \in \mathcal{Z}, \\ +\infty & \text{otherwise,} \end{cases} \\ E_\varepsilon &= H + F_\varepsilon. \end{aligned}$$

The following property shows that the domain  $\mathcal{Z} \subset \mathcal{X}$  is adapted for the approximation process.

**Proposition 4.1.** *Let  $(p_k, \mathbf{g}^k)_k \subset \mathcal{Z}$  be a sequence converging to  $(p, \mathbf{g}) \in \mathcal{X}$  for the topology  $\mathcal{T}$  and such that  $(E_{\varepsilon_k}(p_k, \mathbf{g}^k))_k$  is a bounded sequence. Then, we have  $(p, \mathbf{g}) \in \mathcal{Y}$ .*

*Proof.* It suffices to prove that  $p \in BV(\Omega)$ . According to lemma 2.1, we have  $|\nabla p_k|^2 \leq \mathbf{g}^k(x, \nabla p_k)$ , it gives

$$\int_{\Omega} \left[ 9\varepsilon_k |\nabla p_k|^2 + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right] dx \leq E_{\varepsilon_k}(p_k, \mathbf{g}^k). \quad (4.3)$$

We apply the inequality  $2ab \leq a^2 + b^2$  with  $a^2 = 9\varepsilon_k |\nabla p_k|^2$  and  $b^2 = \frac{p_k^2(1-p_k)^2}{\varepsilon_k}$

$$\int_{\Omega} |\nabla p_k| p_k (1-p_k) dx \leq E_{\varepsilon_k}(p_k, \mathbf{g}^k).$$

The left hand side of the inequality is the total variation of  $u_k = \frac{p_k^2}{2} - \frac{p_k^3}{3}$ , that is

$$\int_{\Omega} |\nabla u_k| dx \leq E_{\varepsilon_k}(p_k, \mathbf{g}^k).$$

As the right hand side is a bounded then  $(u_k)_k$  is a bounded sequence in  $BV(\Omega)$ . According to the theorems 2.4 and 2.5, there exists a subsequence which weakly\* converges and almost everywhere to  $u \in BV(\Omega)$ . By assumption,  $(p_k)_k$  converges almost everywhere to  $p$ , so by uniqueness of the limit  $u = p(1-p)$ . As  $p$  takes its values in  $\{0; 1\}$ , then  $u = \frac{p}{6}$  and  $p \in BV(\Omega)$ .  $\square$

The main result of this work is the following

**Theorem 4.2.** *Let  $(\varepsilon_k)_k$  be a sequence converging to  $0^+$ . Then, the sequence  $(E_{\varepsilon_k})_k$   $\Gamma$ -converges to  $E$  in  $\mathcal{X}$  for the topology  $\mathcal{T}$  introduced in 3.1.*

This results consists in proving two inequalities (4.1) and (4.2). The first inequality consists in the application of the method introduced in [11], while the second is specific to this problem.

### 4.3 The inequality for the lower $\Gamma$ -limit (4.1)

We now prove the first inequality (4.1). For any  $(p, \mathbf{g}) \in \mathcal{X}$ , we denote

$$E_-(p, \mathbf{g}) = \inf \left\{ \liminf_{k \rightarrow \infty} E_{\varepsilon_k}(p_k, \mathbf{g}^k) : ((p_k, \mathbf{g}^k))_k \subset \mathcal{Z}, (p_k, \mathbf{g}^k) \xrightarrow{\mathcal{T}} (p, \mathbf{g}) \right\}$$

We have to prove that  $E_- \geq E$  in  $\mathcal{X}$ . Let  $(p, \mathbf{g}) \in \mathcal{X}$  such that  $E_-(p, \mathbf{g}) < +\infty$ , several assumptions may be made.

- i) We have  $(p, \mathbf{g}) \in \mathcal{Y}$ . According to proposition 4.1,  $E_-(p, \mathbf{g}) < +\infty$  gives  $p \in BV(\Omega)$ .
- ii) There exists a sequence  $((p_k, \mathbf{g}^k))_k \subset \mathcal{Z}$  which converges to  $(p, \mathbf{g})$  for  $\mathcal{T}$  and  $E_{\varepsilon_k}(p_k, \mathbf{g}^k)$  converges to  $E_-(p, \mathbf{g})$ . Indeed, we can construct such a sequence by a diagonal extraction.
- iii) We have  $(p_k)_k \subset C^\infty \cap W^{1,2}(\Omega)$ . Indeed, according to Meyers-Serrin theorem, the space  $C^\infty \cap W^{1,2}(\Omega)$  is dense in  $W^{1,2}(\Omega)$ . So, by diagonal extraction we can construct such sequence.

As for the proof of theorem 3.2, we first prove an uniform convergence result for  $(\mathbf{g}^k)_k$  and then we calculate the limit with  $\mathbf{g}$  fixed.

**First Step: we have**

$$\varepsilon_k \left[ \int_{\Omega} \mathbf{g}^k(x, \nabla p_k) dx - \int_{\Omega} \mathbf{g}(x, \nabla p_k) dx \right] \rightarrow 0.$$

We have the following inequalities

$$\begin{aligned} |\mathbf{g}^k(x, \nabla p_k) - \mathbf{g}(x, \nabla p_k)| &= |(\mathbf{g}_x^k \nabla p_k) \cdot \nabla p_k - (\mathbf{g}_x \nabla p_k) \cdot \nabla p_k|, \\ &\leq |[(\mathbf{g}_x^k - \mathbf{g}_x) \nabla p_k] \cdot \nabla p_k|, \\ &\leq \|\mathbf{g}_x^k - \mathbf{g}_x\| |\nabla p_k|^2, \\ &\leq \sup_{x \in \Omega} \|\mathbf{g}_x^k - \mathbf{g}_x\| |\nabla p_k|^2. \end{aligned}$$

We denote  $\|\mathbf{g}^k - \mathbf{g}\|_{L^\infty} = \sup_{x \in \Omega} \|\mathbf{g}_x^k - \mathbf{g}_x\|$ . According to the previous inequalities, we have

$$\varepsilon_k \left| \int_{\Omega} \mathbf{g}^k(x, \nabla p_k) dx - \int_{\Omega} \mathbf{g}(x, \nabla p_k) dx \right| \leq \|\mathbf{g}^k - \mathbf{g}\|_{L^\infty} \varepsilon_k \int_{\Omega} |\nabla p_k|^2 dx. \quad (4.4)$$

According to inequality (4.3), the term  $\varepsilon_k \int_{\Omega} |\nabla p_k|^2$  is uniformly bounded with respect to  $k$ . Moreover,  $(\mathbf{g}^k)_k$  weakly converges to  $\mathbf{g}$  and the inclusion  $W^{1,r} \subset L^\infty$  is compact. It yields that  $(\mathbf{g}^k)_k$  converges to  $\mathbf{g}$  in  $L^\infty$ . It concludes the first step of the proof.

**Second Step: we have**

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}) \geq F(p, \mathbf{g}).$$

For any  $k \geq 0$ , the inequality  $a^2 + b^2 \geq 2ab$  gives

$$\int_{\Omega} \left[ 9\varepsilon_k \mathbf{g}(x, \nabla p_k) + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right] dx \geq \int_{\Omega} 6p_k(1-p_k) \mathbf{g}(x, \nabla p_k)^{\frac{1}{2}} dx.$$

Let  $\Phi : \Omega \times [0; 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^+$  be the function

$$\Phi(x, s, \mathbf{v}) = 6s(1-s) \mathbf{g}(x, \mathbf{v})^{\frac{1}{2}}.$$

This function is sublinear in  $\mathbf{v}$ . We denote  $S_t^k = \{x \in \Omega : p_k(x) < t\}$  Using the proposition 2.4, we can write

$$\int_{\Omega} 6p_k(1-p_k) \mathbf{g}(x, \nabla p_k)^{\frac{1}{2}} dx = \int_{\mathbb{R}} \int_{\Omega} 6t(1-t) \mathbf{g}(x, D\mathbf{1}_{S_t^k})^{\frac{1}{2}}.$$

Applying Fatou lemma and noting that  $D\mathbf{1}_{S_t}$  vanishes when  $t \notin [0; 1]$  gives

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}^k) \geq \int_0^1 6t(1-t) \liminf_{k \rightarrow \infty} \int_{\Omega} \mathbf{g}(x, D\mathbf{1}_{S_t^k})^{\frac{1}{2}}.$$

As the left hand side of this inequality is finished, for almost every  $t \in [0; 1]$  we have

$$\liminf_{k \rightarrow \infty} \int_{\Omega} \mathbf{g}(x, D\mathbf{1}_{S_t^k})^{\frac{1}{2}} < +\infty.$$

Lemma 2.1 gives

$$\int_{\Omega} |D\mathbf{1}_{S_t^k}| \leq \int_{\Omega} \mathbf{g}(x, D\mathbf{1}_{S_t^k})^{\frac{1}{2}},$$

so  $\|D\mathbf{1}_{S_t^k}\|_{\mathcal{M}}$  is bounded; this yields that  $(\mathbf{1}_{S_t^k})_k$  is weakly relatively compact in  $\text{BV}(\Omega)$ . We denote  $A = \{x \in \Omega; p(x) = 1\}$  and we compare  $\int_{\Omega} |p_k - p|$  and  $\int_{\Omega} |\mathbf{1}_{S_t^k} - \mathbf{1}_A|$ .

$$\begin{aligned}
\int_{\Omega} |p_k - p| dx &= \int_{\Omega} |p_k - \mathbf{1}_A| dx, \\
&\geq \int_{A \setminus S_t^k} |p_k - \mathbf{1}_A| dx + \int_{S_t^k \setminus A} |p_k - \mathbf{1}_A| dx, \\
&\geq (1-t) \int_{A \setminus S_t^k} |\mathbf{1}_{S_t^k} - \mathbf{1}_A| dx + t \int_{S_t^k \setminus A} |\mathbf{1}_{S_t^k} - \mathbf{1}_A| dx, \\
&\geq \min(t, 1-t) \int_{A \Delta S_t^k} |\mathbf{1}_{S_t^k} - \mathbf{1}_A| dx, \\
&\geq \min(t, 1-t) \int_{\Omega} |\mathbf{1}_{S_t^k} - \mathbf{1}_A| dx.
\end{aligned}$$

For any  $t \in ]0; 1[$ , the unique possible limit of  $(\mathbf{1}_{S_t^k})_k$  is  $\mathbf{1}_A$ . Thanks to proposition 2.3, we have

$$\liminf_{k \rightarrow \infty} \int_{\Omega} \left[ 9\varepsilon_k \mathbf{g}(x, \nabla p_k) + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right] dx \geq \int_{\Omega} \mathbf{g}(x, D\mathbf{1}_A)^{\frac{1}{2}}. \quad (4.5)$$

Theorem 2.6 and the notation of definition 2.7 give

$$\int_{\Omega} \mathbf{g}(x, D\mathbf{1}_A)^{\frac{1}{2}} = \int_{S_p} \mathbf{g}(x, \nu_p)^{\frac{1}{2}} d\mathcal{H}^{n-1}. \quad (4.6)$$

Relations (4.5) and (4.6) give the result of the Second Step.

**Third Step: we have**

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}^k) \geq F(p, \mathbf{g}).$$

We decompose

$$F_{\varepsilon_k}(p_k, \mathbf{g}^k) = (F_{\varepsilon_k}(p_k, \mathbf{g}^k) - F_{\varepsilon_k}(p_k, \mathbf{g})) + F_{\varepsilon_k}(p_k, \mathbf{g}).$$

According to the first step, the first term converges to 0 and according to the second step we can conclude that

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}^k) \geq F(p, \mathbf{g}).$$

**Conclusion**

As  $E_{\varepsilon_k} = H + F_{\varepsilon_k}$ , it gives

$$\liminf_{k \rightarrow \infty} E_{\varepsilon_k}(p_k, \mathbf{g}^k) \geq \liminf_{k \rightarrow \infty} H(p_k, \mathbf{g}^k) + \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}^k).$$

According to the third step, we have  $\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}^k) \geq F(p, \mathbf{g})$ . Moreover, as  $H$  is lower semicontinuous for the topology of  $\mathcal{X}$ , we have  $\liminf_{k \rightarrow \infty} H(p_k, \mathbf{g}^k) \geq H(p, \mathbf{g})$ . As  $E = F + H$ , it finishes the proof of the inequality for the lower  $\Gamma$ -limit.

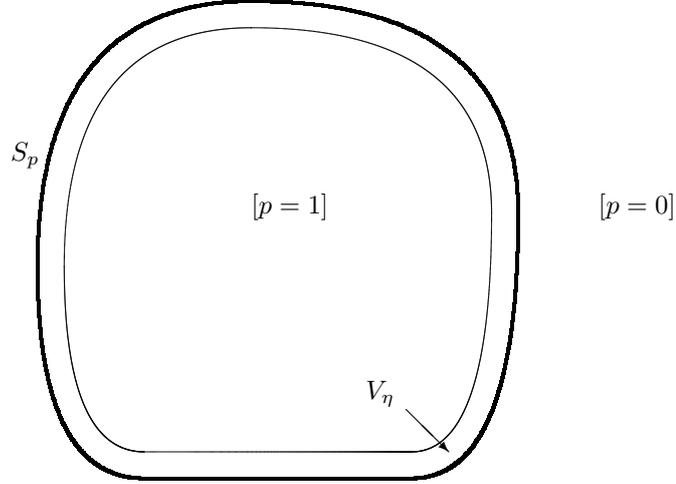


Figure 4.1: Partition of  $\Omega$  in three domains:  $[p = 0]$ ,  $[p = 1] \setminus V_\eta$  and  $V_\eta$ .

#### 4.4 The inequality for the *higher* $\Gamma$ -limit (4.2)

We may now prove the second part (4.2) of theorem 4.2.

*Proof.* We set  $(p, \mathbf{g}) \in \mathcal{X}$ . If  $p \notin \text{BV}(\Omega)$  then  $E(p, \mathbf{g}) = +\infty$ . So, we may assume that  $p$  belongs to  $\text{BV}(\Omega)$  and takes its values on  $\{0; 1\}$ , otherwise the result is ensured. Let  $(\varepsilon_k)_k$  be a sequence which converges to  $0^+$ . We construct a sequence of functions  $(p_k, \mathbf{g}^k)_k$  such that

$$\limsup_{k \rightarrow \infty} E_{\varepsilon_k}(p_k, \mathbf{g}^k) \leq E(p, \mathbf{g})$$

and  $(p_k, \mathbf{g}^k)_k$  converges to  $(p, \mathbf{g})$  for the topology  $\mathcal{T}$ . First, we construct it when  $S_p$  is a smooth surface and  $\mathbf{g}$  a smooth vector field. Then, we remove these assumptions and we use approximating results to prove it in the general setting.

**First step: we assume that  $S_p$  is a compact surface of class  $\mathcal{C}^2$  and  $\mathbf{g} \in \mathcal{C}^\infty \cap W^{1,r}(\Omega; \mathcal{G})$ .**

In this step, we set  $\mathbf{g}^k = \mathbf{g}$  for any  $k$ . Moreover, if  $(p_k)_k \subset W^{1,2}(\Omega; [0; 1])$  converges a.e. to  $p$ , then it converges for the  $L^1(\Omega)$  norm and  $(\int (p_k - p)^2 dx)_k$  converges to  $\int (p - p)^2 dx$ . So, it suffices to construct an appropriate sequence  $(p_k)_k$  which converges a.e. to  $p$  and such that  $\limsup F_{\varepsilon_k}(p_k, \mathbf{g}) \leq F(p, \mathbf{g})$ .

For  $\eta > 0$ , we introduce the following set  $V_\eta$  (see figure 4.1)

$$V_\eta = \{x \in [p = 1]: 0 < \text{dist}(x, S_p) < \eta\}.$$

Outside  $V_\eta$ , we define the function  $p_k$  as:

$$\begin{aligned} \forall x \in [p = 0], & \quad p_k(x) = 0, \\ \forall x \in [p = 1] \setminus V_\eta, & \quad p_k(x) = 1. \end{aligned}$$

The construction of  $p_k$  inside  $V_\eta$  will be precised. As we assume that  $S_p$  is a compact and of class  $\mathcal{C}^2$  surface, there exists  $\eta_0 > 0$  and a  $\mathcal{C}^1$ -diffeomorphism  $\phi : V_{\eta_0} \rightarrow S_p \times ]0; \eta_0[$  (see [9]), characterized by

$$\forall (\xi, t) \in S_p \times ]0; \eta_0[, \quad \phi(\xi + t\nu_p(\xi)) = (\xi, t).$$

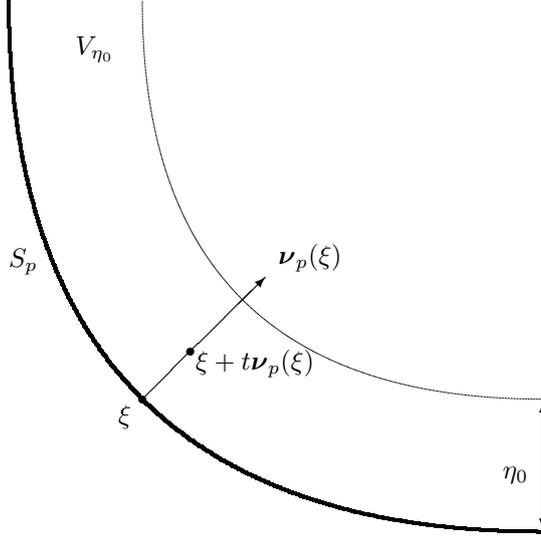


Figure 4.2: Slicing parametrization of  $V_{\eta_0}$ .

We denote  $\Sigma_\xi$  the slice

$$\Sigma_\xi = \{\xi + t\nu_p(\xi) : t \in [0; \eta_0]\},$$

We shall construct  $p_k$  slice by slice (see figure 4.2). Indeed,  $\phi : V_{\eta_0} \rightarrow S_p \times ]0; \eta_0[$  is a diffeomorphism, so it provides a complete construction of  $p_k$ . We denote by  $\chi_{k,\xi} : [0; \eta_0] \rightarrow \mathbb{R}$  the restriction of  $p_k$  to  $\Sigma_\xi$ . We introduce  $K$  defined on  $S_p \times [0; \eta_0]$  by

$$\forall (\xi, t) \in S_p \times [0; \eta_0], \quad K(\xi, t) = \mathbf{g}(\xi + t\nu_p(\xi), \nu_p(\xi))^{\frac{1}{2}}.$$

and  $\chi_{k,\xi}$  as the solution of the following differential equation

$$\begin{cases} \chi_{k,\xi}(0) = 0, \\ 3\sqrt{\varepsilon_k}K(\xi, t)\chi'_{k,\xi}(t) = \left[ \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{(\chi_{k,\xi}(t))^2(1-\chi_{k,\xi}(t))^2}{\varepsilon_k} \right]^{\frac{1}{2}} \end{cases} \quad \text{for } t \geq 0.$$

For  $t \geq 0$ , we have  $\chi'_{k,\xi}(t) \geq \frac{1}{3K(\xi, t)\varepsilon_k \sqrt{|\ln(\varepsilon_k)|}}$ . According to lemma 2.1, we have  $K(\xi, t) \leq \sqrt{2}$ . So, there exists a unique  $\eta_{k,\xi} > 0$  such that  $\chi_{k,\xi}(\eta_{k,\xi}) = 1$  and it satisfies

$$\sup_{\xi \in S_p} \eta_{k,\xi} \leq 3\sqrt{2}\varepsilon_k \sqrt{|\ln(\varepsilon_k)|}. \quad (4.7)$$

As  $\varepsilon_k \sqrt{|\ln(\varepsilon_k)|}$  converges to 0, then we can assume that  $\eta_{k,\xi} < \eta_0$  for any  $k$  and  $\xi \in S_p$ . Thus, we modify the definition of  $\chi_{k,\xi}$  as the solution of

the following equation

$$\begin{cases} \chi_{k,\xi}(0) = 0, \\ 3\sqrt{\varepsilon_k}K(\xi, t)\chi'_{k,\xi}(t) = \left[ \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{(\chi_{k,\xi}(t))^2(1-\chi_{k,\xi}(t))^2}{\varepsilon_k} \right]^{\frac{1}{2}} & \text{for } t \in ]0; \eta_{k,\xi}[, \\ \chi_{k,\xi}(t) = 1 & \text{for } t \in [\eta_{k,\xi}; \eta_0[. \end{cases} \quad (4.8)$$

We denote  $\eta_k = \sup \{\eta_{k,\xi} : \xi \in S_p\}$  and we define  $p_k$  as

$$\begin{cases} \forall x \in [p = 0], & p_k(x) = 0, \\ \forall (\xi, t) \in S_p \times ]0; \eta_0[, & p_k(\xi + t\nu_p(\xi)) = \chi_{k,\xi}(t), \\ \forall x \in [p = 1] \setminus V_{\eta_0}, & p_k(x) = 1. \end{cases} \quad (4.9)$$

According to (4.7), we have  $\eta_k \rightarrow 0$ , it implies  $p_k \rightarrow p$  almost everywhere.

With the definitions introduced in (4.8) and (4.9), we have to prove that  $\limsup F_{\varepsilon_k}(p_k, \mathbf{g}) \leq F(p, \mathbf{g})$ . In the sequel we take  $n = 3$  but the arguments are the same for  $n = 2$ . As  $S_p$  is a surface with class  $\mathcal{C}^2$ , there exists  $\mathbf{t}_1$  and  $\mathbf{t}_2$  two functions defined in  $S_p$  taking their values in the unit sphere  $\mathbb{S}^{n-1}$  and with class  $\mathcal{C}^1$  such that, for any  $\xi \in S_p$ , the vector triplet  $(\mathbf{t}_1(\xi), \mathbf{t}_2(\xi), \nu_p(\xi))$  is an orthonormal basis of  $\mathbb{R}^3$  (see figure 4.3).

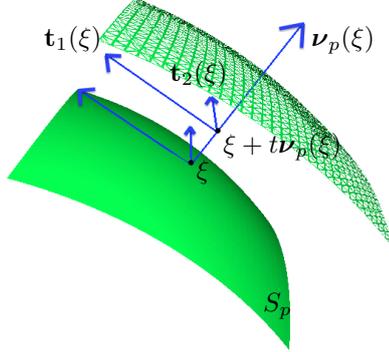


Figure 4.3: Moving basis  $(\mathbf{t}_1(\xi), \mathbf{t}_2(\xi), \nu_p(\xi))$  at  $\xi + t\nu_p(\xi)$ .

We need to prove the following lemma.

**Lemma 4.1.** *For any  $\varepsilon > 0$ ,  $\mathbf{v} \in \mathbb{R}^n$ ,  $x \in \Omega$  and  $\xi \in S_p$ , we have*

$$\mathbf{g}(x, \mathbf{v}) \leq (\mathbf{v} \cdot \nu_p)^2 (\mathbf{g}(x, \nu_p) + 2\varepsilon) + (\mathbf{v} \cdot \mathbf{t}_1)^2 \left(3 + \frac{1}{\varepsilon}\right) + (\mathbf{v} \cdot \mathbf{t}_2)^2 \left(3 + \frac{1}{\varepsilon}\right), \quad (4.10)$$

where  $\nu_p, \mathbf{t}_1$  and  $\mathbf{t}_2$  depends on  $\xi$ .

*Proof.* As  $\mathbf{g} \in \mathcal{G}$ , there exists  $\mathbf{c} : \Omega \rightarrow \mathbb{S}^{n-1}$  such that

$$\mathbf{g}(x, \mathbf{v}) = |\mathbf{v}|^2 + (\mathbf{c}(x) \cdot \mathbf{v})^2.$$

We can decompose

$$\begin{cases} |\mathbf{v}|^2 &= (\mathbf{v} \cdot \boldsymbol{\nu}_p)^2 + (\mathbf{v} \cdot \mathbf{t}_1)^2 + (\mathbf{v} \cdot \mathbf{t}_2)^2, \\ \mathbf{c}(x) \cdot \mathbf{v} &= (\mathbf{c}(x) \cdot \boldsymbol{\nu}_p)(\mathbf{v} \cdot \boldsymbol{\nu}_p) + (\mathbf{c}(x) \cdot \mathbf{t}_1)(\mathbf{v} \cdot \mathbf{t}_1) + (\mathbf{c}(x) \cdot \mathbf{t}_2)(\mathbf{v} \cdot \mathbf{t}_2), \end{cases}$$

We denote  $a = (\mathbf{c}(x) \cdot \boldsymbol{\nu}_p)(\mathbf{v} \cdot \boldsymbol{\nu}_p)$ ,  $b = (\mathbf{c}(x) \cdot \mathbf{t}_1)(\mathbf{v} \cdot \mathbf{t}_1)$  and  $c = (\mathbf{c}(x) \cdot \mathbf{t}_2)(\mathbf{v} \cdot \mathbf{t}_2)$ . Moreover, we have

$$\begin{aligned} (a + b + c)^2 &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc, \\ &\leq a^2 + b^2 + c^2 + \left( \varepsilon a^2 + \frac{b^2}{\varepsilon} \right) + \left( \varepsilon a^2 + \frac{c^2}{\varepsilon} \right) + (b^2 + c^2), \\ &\leq (1 + 2\varepsilon)a^2 + \left( 2 + \frac{1}{\varepsilon} \right) b^2 + \left( 2 + \frac{1}{\varepsilon} \right) c^2. \end{aligned}$$

We may introduce  $(\mathbf{c}(x) \cdot \mathbf{t}_1)^2 \leq 1$ ,  $(\mathbf{c}(x) \cdot \mathbf{t}_2)^2 \leq 1$  in the previous inequality, it gives the result of the lemma.  $\square$

If we apply lemma 4.1 in the definition of  $F_{\varepsilon_k}$ , we get

$$F_{\varepsilon_k}(p_k, \mathbf{g}) \leq (\star)_{k,0} + (\star)_{k,1} + (\star)_{k,2}, \quad (4.11)$$

where

$$(\star)_{k,0} = \int_{V_{\eta_k}} \left[ 9\varepsilon_k (\nabla p_k \cdot \boldsymbol{\nu}_p)^2 (\mathbf{g}(x, \boldsymbol{\nu}_p) + 2\varepsilon_k) + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right] dx, \quad (4.12)$$

$$(\star)_{k,1} = \int_{V_{\eta_k}} 9\varepsilon_k (\nabla p_k \cdot \mathbf{t}_1)^2 \left( 3 + \frac{1}{\varepsilon_k} \right) dx, \quad (4.13)$$

$$(\star)_{k,2} = \int_{V_{\eta_k}} 9\varepsilon_k (\nabla p_k \cdot \mathbf{t}_2)^2 \left( 3 + \frac{1}{\varepsilon_k} \right) dx. \quad (4.14)$$

We will prove the following assertions

$$\limsup_{k \rightarrow \infty} (\star)_{k,0} \leq \int_{S_p} \mathbf{g}(x, \boldsymbol{\nu}_p)^{\frac{1}{2}} d\mathcal{H}^{n-1}, \quad \lim_{k \rightarrow \infty} (\star)_{k,1} = 0, \quad \lim_{k \rightarrow \infty} (\star)_{k,2} = 0.$$

According to the decomposition (4.11), it is sufficient to conclude the *First Step*.

*Claim 1: We have the following inequality*

$$\limsup_{k \rightarrow \infty} (\star)_{k,0} \leq \int_{S_p} \mathbf{g}(x, \boldsymbol{\nu}_p)^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

Since

- $(\nabla p_k \cdot \boldsymbol{\nu}_p)^2 \leq |\nabla p_k|^2$ ,
- $|\nabla p_k|^2 \leq \mathbf{g}(x, \nabla p_k)$ , according to lemma 2.1,
- $\int_{\Omega} 9\varepsilon_k \mathbf{g}(x, \nabla p_k) dx \leq F_{\varepsilon_k}(p_k, \mathbf{g})$ ,
- $F_{\varepsilon_k}(p_k, \mathbf{g})$  is bounded,

then, we can conclude that  $\int_{V_{\eta_k}} 9\varepsilon_k (\nabla p_k \cdot \boldsymbol{\nu}_p)^2 dx$  is bounded. In particular,  $\varepsilon_k \int_{V_{\eta_k}} 9\varepsilon_k (\nabla p_k \cdot \boldsymbol{\nu}_p)^2 dx$  converges to 0. Then, for *Claim 1*, it suffices to prove

$$\limsup_{k \rightarrow \infty} \int_{V_{\eta_k}} \left[ 9\varepsilon_k (\nabla p_k \cdot \boldsymbol{\nu}_p)^2 \mathbf{g}(x, \boldsymbol{\nu}_p) + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right] dx \leq \int_{S_p} \mathbf{g}(x, \boldsymbol{\nu}_p)^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

Since

$$\begin{aligned} \frac{\partial p_k}{\partial \boldsymbol{\nu}_p(\xi)}(\xi + t\boldsymbol{\nu}_p(\xi)) &= \lim_{s \rightarrow 0} \frac{p_k(\xi + (s+t)\boldsymbol{\nu}_p(\xi)) - p_k(\xi + t\boldsymbol{\nu}_p(\xi))}{s}, \\ &= \lim_{s \rightarrow 0} \frac{\chi_{k,\xi}(s+t) - \chi_{k,\xi}(t)}{s}, \end{aligned}$$

then, for any  $(\xi, t) \in S_p \times ]0; \eta_k[$ , we have

$$\frac{\partial p_k}{\partial \boldsymbol{\nu}_p(\xi)}(\xi + t\boldsymbol{\nu}_p(\xi)) = \chi'_{k,\xi}(t).$$

This yields  $\nabla p_k \cdot \boldsymbol{\nu}_p = \chi'_{k,\xi}$ . According to the assumptions of regularity of  $S_p$ , as in [9], we may introduce the following change of variable:

$$\int_{S_p} \int_0^{\eta_0} \frac{dt \, d\mathcal{H}^2(\xi)}{\prod_{i=1}^2 (1 - \kappa_i(\xi)t)} = \int_{V_{\eta_0}} dx, \quad (4.15)$$

where  $\kappa_1(\xi), \kappa_2(\xi)$  are the principal curvatures of  $S_p$  at  $\xi$ . As  $S_p$  is a  $\mathcal{C}^2$  surface, then  $\kappa_1$  and  $\kappa_2$  are continuous on  $S_p$ . We denote

$$\Pi(\xi, t) = \frac{1}{\prod_{i=1}^2 (1 - \kappa_i(\xi)t)}.$$

This yields

$$\begin{aligned} (\star)_{k,0} &= \int_{S_p} \int_0^{\eta_k} \left( 9\varepsilon_k (\nabla p_k \cdot \boldsymbol{\nu}_p)^2 \mathbf{g}(x, \boldsymbol{\nu}_p) + \frac{p_k^2 (1 - p_k)^2}{\varepsilon_k} \right) \Pi \, dt \, d\mathcal{H}^2(\xi), \\ &= \int_{S_p} \int_0^{\eta_k} \left( 9\varepsilon_k (\chi'_{k,\xi})^2 K^2 + \frac{p_k^2 (1 - p_k)^2}{\varepsilon_k} \right) \Pi \, dt \, d\mathcal{H}^2(\xi). \end{aligned}$$

In these integrals we remove the dependance variables for the sake of simplicity:

$$\begin{aligned} x &= \xi + t\boldsymbol{\nu}_p(\xi), \quad \boldsymbol{\nu}_p = \boldsymbol{\nu}_p(\xi), \quad p_k = p_k(\xi + t\boldsymbol{\nu}_p(\xi)), \quad \Pi = \Pi(\xi, t), \\ \chi_{k,\xi} &= \chi_{k,\xi}(t), \quad K = K(\xi, t) \end{aligned}$$

and we set

$$a = 3\sqrt{\varepsilon_k} K \chi'_{k,\xi}, \quad b = \frac{\chi_{k,\xi}(1 - \chi_{k,\xi})}{\sqrt{\varepsilon_k}}.$$

With the construction of  $\chi_{k,\xi}$  in (4.8) we get  $a^2 = \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + b^2$  so that  $0 \leq b \leq a$  on  $[0; \eta_k]$  and

$$a^2 + b^2 \leq 2ab + \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|}.$$

This yields

$$\begin{aligned} (\star)_{k,0} &\leq \underbrace{\int_{S_p} \int_0^{\eta_k} 6K \chi'_{k,\xi} \chi_{k,\xi} (1 - \chi_{k,\xi}) \Pi \, dt \, d\mathcal{H}^2(\xi)}_{(\star)_{k,3}} \\ &\quad + \underbrace{\int_{S_p} \int_0^{\eta_k} \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} \Pi \, dt \, d\mathcal{H}^2(\xi)}_{(\star)_{k,4}}. \end{aligned}$$

The functions  $K$  and  $\Pi$  are uniformly bounded with respect to  $k$  in  $S_p \times ]0; \eta_0[$ . We denote  $M$  their upper bound. We have the following inequalities

$$\begin{aligned} (\star)_{k,4} &\leq M \int_S \int_0^{\eta_k} \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} dt \, d\mathcal{H}^2(\xi), \\ &\leq M \mathcal{H}^2(S_p) \frac{\eta_k}{\varepsilon_k |\ln(\varepsilon_k)|}. \end{aligned}$$

According to (4.7), we have  $\eta_k \leq 3\sqrt{2}\varepsilon_k \sqrt{|\ln(\varepsilon_k)|}$ . This yields

$$(\star)_{k,4} \leq \frac{3\sqrt{2}M\mathcal{H}^2(S_p)}{\sqrt{|\ln(\varepsilon_k)|}}.$$

and we have  $\lim_{k \rightarrow \infty} (\star)_{k,4} = 0$ . We denote

$$L(\xi, t) = K(\xi, t)\Pi(\xi, t).$$

We have the following decomposition

$$\begin{aligned} (\star)_{k,3} &= \underbrace{\int_{S_p} \int_0^{\eta_k} 6(L(\xi, t) - L(\xi, 0))\chi'_{k,\xi}(t)\chi_{k,\xi}(t)(1 - \chi_{k,\xi}(t))dt \, d\mathcal{H}^2(\xi)}_{(\star)_{k,5}} \\ &\quad + \underbrace{\int_{S_p} \int_0^{\eta_k} 6L(\xi, 0)\chi'_{k,\xi}(t)\chi_{k,\xi}(t)(1 - \chi_{k,\xi}(t))dt \, d\mathcal{H}^2(\xi)}_{(\star)_{k,6}}, \end{aligned}$$

and the following bound

$$(\star)_{k,5} \leq \left( \sup_{(\xi, t) \in S_p \times ]0; \eta_k[} (L(\xi, t) - L(\xi, 0)) \right) \int_{S_p} \int_0^{\eta_k} 6\chi'_{k,\xi}(t)\chi_{k,\xi}(t)(1 - \chi_{k,\xi}(t))dt \, d\mathcal{H}^2(\xi).$$

Since  $\chi_{k,\xi} \in W^{1,2}([0; \eta_k])$ , we may use the change of variable  $s = \chi_{k,\xi}(t)$  to obtain

$$\begin{aligned} (\star)_{k,5} &\leq 6 \left( \sup_{(\xi, t) \in S_p \times ]0; \eta_k[} (L(\xi, t) - L(\xi, 0)) \right) \int_{S_p} \int_{\chi_{k,\xi}(0)}^{\chi_{k,\xi}(\eta_k)} s(1-s)ds \, d\mathcal{H}^2(\xi), \\ &\leq \left( \sup_{(\xi, t) \in S_p \times ]0; \eta_k[} (L(\xi, t) - L(\xi, 0)) \right) \mathcal{H}^2(S_p). \end{aligned}$$

The surface  $S_p$  is compact and smooth and the function  $L$  is continuous. Then, the family  $(L(\cdot, t))_{t>0}$  uniformly converges to  $L(\cdot, 0)$  when  $t \rightarrow 0^+$ . We can deduce that  $\lim_{k \rightarrow \infty} (\star)_{k,5} = 0$ . Using the same change of variable  $s = \chi_{k,\xi}(t)$  in  $(\star)_{k,6}$  gives

$$\begin{aligned} (\star)_{k,6} &= 6 \int_{S_p} L(\xi, 0) \int_{\chi_{k,\xi}(0)}^{\chi_{k,\xi}(\eta_k)} s(1-s)ds \, d\mathcal{H}^2(\xi), \\ &= \int_{S_p} \mathbf{g}(\xi, \nu_p(\xi))^{\frac{1}{2}} d\mathcal{H}^2(\xi). \end{aligned}$$

To summarize, we have

$$\begin{cases} (\star)_{k,0} = (\star)_{k,4} + (\star)_{k,5} + (\star)_{k,6}, \\ \lim_{k \rightarrow \infty} (\star)_{k,4} = 0, \\ \lim_{k \rightarrow \infty} (\star)_{k,5} = 0, \\ (\star)_{k,6} = \int_{S_p} \mathbf{g}_\xi(\nu_p(\xi), \nu_p(\xi))^{\frac{1}{2}} d\mathcal{H}^2(\xi), \end{cases}$$

which conclude the proof of *Claim 1*.

*Claim 2: We have the following limits*

$$\lim_{k \rightarrow \infty} (\star)_{k,1} = 0, \quad \lim_{k \rightarrow \infty} (\star)_{k,2} = 0.$$

We prove the result for  $(\star)_{k,1}$ , the method for  $(\star)_{k,2}$  is the same. As  $S_p$  is a  $\mathcal{C}^2$  surface, the intersection of the affine plane  $P_1 = \xi + \text{Vect}(\mathbf{t}_1(\xi), \boldsymbol{\nu}_p(\xi))$  and  $S_p$  at the neighborhood of  $\xi \in S_p$  is a  $\mathcal{C}^2$ -planar curve. Let  $I$  be a neighborhood of 0 in  $\mathbb{R}$  and  $\gamma : I \rightarrow S_p$  be a local curvilinear parametrization of this curve such that

$$\begin{cases} \gamma(0) = \xi, \\ \gamma'(0) = \mathbf{t}_1(\xi), \\ \forall t \in I, \quad |\gamma'(t)| = 1. \end{cases}$$

As  $\boldsymbol{\nu}_p \circ \gamma(s)$  is orthogonal to  $\gamma'(s)$  for all  $s \in I$  and  $\gamma$  is a planar curve, there exists  $\tilde{\kappa}_1 : I \rightarrow \mathbb{R}$  such that

$$\frac{d(\boldsymbol{\nu}_p \circ \gamma)}{ds}(s) = -\tilde{\kappa}_1(\gamma(s))\mathbf{t}_1(\gamma(s)).$$

As  $\gamma$  is a curve of  $S_p$ ,  $\tilde{\kappa}_1$  is the sectional curvature of  $S_p$  in the direction of  $\mathbf{t}_1(\gamma(s))$ , we have  $|\tilde{\kappa}_1| \leq \max(|\kappa_1|, |\kappa_2|)$ .

We evaluate

$$\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t) = p_k(\gamma(s) + t\boldsymbol{\nu}_p(\gamma(s))) - p_k(\xi + t\boldsymbol{\nu}_p(\xi)).$$

So, we have the following asymptotic developpement at  $s = 0$

$$\gamma(s) + t\boldsymbol{\nu}_p(\gamma(s)) = \xi + t\boldsymbol{\nu}_p(\xi) + s(1 - \tilde{\kappa}_1(\xi)t)\mathbf{t}_1(\xi) + o(s),$$

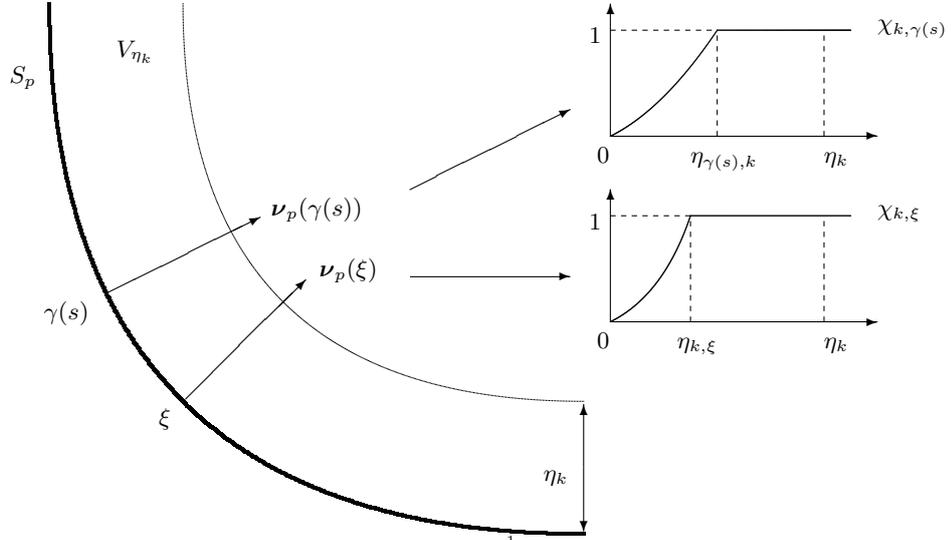
and we get

$$\lim_{s \rightarrow 0} \frac{\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)}{s(1 - \tilde{\kappa}_1(\xi)t)} = \nabla p_k(\xi + t\boldsymbol{\nu}_p(\xi)) \cdot \mathbf{t}_1(\xi). \quad (4.16)$$

We calculate the left hand side of (4.16). We recall the equations satisfied by  $\chi_{k,\gamma(s)}$  and  $\chi_{k,\xi}$  (see figure ??)

$$(E_1) : \begin{cases} \chi_{k,\xi}(0) = 0, \\ 3\sqrt{\varepsilon_k}K(\xi, t)\chi'_{k,\xi}(t) = \left[ \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{(\chi_{k,\xi}(t))^2 (1 - \chi_{k,\xi}(t))^2}{\varepsilon_k} \right]^{\frac{1}{2}} & \text{for } t \in ]0; \eta_{k,\xi}[, \\ \chi_{k,\xi}(t) = 1 & \text{for } t \geq \eta_{k,\xi}. \end{cases}$$

$$(E_2) : \begin{cases} \chi_{k,\gamma(s)}(0) = 0, \\ 3\sqrt{\varepsilon_k}K(\gamma(s), t)\chi'_{k,\gamma(s)}(t) = \left[ \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{(\chi_{k,\gamma(s)}(t))^2 (1 - \chi_{k,\gamma(s)}(t))^2}{\varepsilon_k} \right]^{\frac{1}{2}} & \text{for } t \in ]0; \eta_{\gamma(s),k}[, \\ \chi_{k,\gamma(s)}(t) = 1 & \text{for } t \geq \eta_{\gamma(s),k}. \end{cases}$$



We denote  $f_k(x) = \left[ \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{x^2(1-x)^2}{\varepsilon_k} \right]^{\frac{1}{2}}$  and  $Y_{k,s}(t) = \chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)$ . We calculate  $\frac{K(\xi, \cdot)}{K(\gamma(s), \cdot)}(E_2) - (E_1)$  and we denote  $\bar{\eta}_{k,s} = \min(\eta_{\gamma(s),k}, \eta_{\xi,k})$ . It comes

$$\begin{cases} Y_{k,s}(0) = 0, \\ 3\sqrt{\varepsilon_k}K(\xi, t)Y'_{k,s}(t) = \frac{K(\xi, t)}{K(\gamma(s), t)}f_k(\chi_{k,\gamma(s)}(t)) - f_k(\chi_{k,\xi}(t)) \text{ for } t \in ]0; \bar{\eta}_{k,s}[. \end{cases}$$

Lemma 2.1 gives

$$\forall (\xi, t) \in S_p \times ]0; \bar{\eta}_{k,s}[, \quad 1 \leq K(\xi, t) \leq \sqrt{2}. \quad (4.17)$$

As  $S_p$  is a  $\mathcal{C}^2$ -manifold and  $\mathbf{g} \in \mathcal{C}^\infty \cap W^{1,r}(\Omega; \mathcal{G})$ , then  $K$  is a  $\mathcal{C}^1$  function of class  $\mathcal{C}^1$  and there exists a constant  $\tau > 0$  such that

$$\forall (\xi, \xi', t) \in S_p \times ]0; \eta_k[, \quad |K(\xi, t) - K(\xi', t)| \leq \tau|\xi - \xi'| \quad (4.18)$$

Moreover, the study of  $f_k$  gives

$$\forall x \in [0; 1], \quad f_k(x) \leq \left[ \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{1}{16\varepsilon_k} \right]^{\frac{1}{2}} \quad |f'_k(x)| \leq \frac{1}{\sqrt{\varepsilon_k}}. \quad (4.19)$$

With

$$\begin{aligned} 3\sqrt{\varepsilon_k}K(\xi, t)Y'_{k,s}(t) &= \frac{K(\xi, t)}{K(\gamma(s), t)}f_k(\chi_{k,\gamma(s)}(t)) - f_k(\chi_{k,\xi}(t)), \\ &= \frac{K(\xi, t)}{K(\gamma(s), t)}(f_k(\chi_{k,\gamma(s)}(t)) - f_k(\chi_{k,\xi}(t))) \\ &\quad + f_k(\chi_{k,\xi}(t)) \left( \frac{K(\xi, t) - K(\gamma(s), t)}{K(\gamma(s), t)} \right) \end{aligned}$$

and (4.17), (4.18), (4.19), we get

$$3\sqrt{\varepsilon_k}Y'_{k,s}(t) \leq \frac{\sqrt{2}}{\sqrt{\varepsilon_k}}Y_{k,s}(t) + \tau s \sqrt{\frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{1}{16\varepsilon_k}}.$$

Thus,  $Y_{k,s}$  is a solution of the following differential ineqution

$$\begin{cases} Y_{k,s}(0) = 0, \\ Y'_{k,s}(t) \leq \frac{\sqrt{2}}{3\varepsilon_k}Y_{k,s}(t) + \frac{\tau s}{3\varepsilon_k} \sqrt{\frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16}} \text{ for } t \in ]0; \bar{\eta}_{k,s}[. \end{cases} \quad (4.20)$$

So, we have

$$Y_{k,s}(t) \leq \frac{\tau s}{\sqrt{2}} \sqrt{\frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16}} \left( \exp\left(\frac{\sqrt{2}t}{3\varepsilon_k}\right) - 1 \right). \quad (4.21)$$

The definition of  $Y_{k,s}$  gives

$$\frac{Y_{k,s}(t) - Y_{k,0}(t)}{s} = \frac{\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)}{s},$$

and inequality (4.21) implies

$$\forall t \in ]0; \bar{\eta}_{k,s}[, \quad \frac{\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)}{s} \leq \frac{\tau}{\sqrt{2}} \sqrt{\frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16}} \left( \exp\left(\frac{\sqrt{2}t}{3\varepsilon_k}\right) - 1 \right). \quad (4.22)$$

According to the continuous dependance of the solution of the equation (4.20) with respect to the parameter  $s$ , then  $\bar{\eta}_{k,s}$  converges to  $\eta_{k,\xi}$  when  $s$  converges to 0. So, the inequality (4.22) remains true in the neighborhood of any point  $t \in ]0; \eta_{k,\xi}[$ . With  $k, \xi$  and  $t \in ]0; \eta_{k,\xi}[$  fixed, we calculate the limit when  $s$  converges to 0, and we apply equality (4.16)

$$(1 - \tilde{\kappa}_1(\xi)t) \nabla p_k(\xi + t\nu_p(\xi)) \cdot \mathbf{t}_1(\xi) \leq \frac{\tau}{\sqrt{2}} \sqrt{\frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16}} \left( \exp\left(\frac{\sqrt{2}t}{3\varepsilon_k}\right) - 1 \right).$$

As  $\eta_k \rightarrow 0$  and  $\tilde{\kappa}_1$  is continuous, there exists  $r > 0$  such that

$$\forall (\xi, t) \in S_p \times ]0; \eta_k[, \quad r < (1 - \tilde{\kappa}_1(\xi)t).$$

This gives

$$(\nabla p_k(\xi + t\nu_p(\xi)) \cdot \mathbf{t}_1(\xi))^2 \leq \frac{\tau^2}{2r^2} \left( \frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16} \right) \left( \exp\left(\frac{\sqrt{2}t}{3\varepsilon_k}\right) - 1 \right)^2. \quad (4.23)$$

As  $\frac{1}{|\ln(\varepsilon_k)|} \rightarrow 0$ , there exists  $M > 0$  such that (4.23) becomes

$$(\nabla p_k(\xi + t\nu_p(\xi)) \cdot \mathbf{t}_1(\xi))^2 \leq M \exp\left(\frac{2\sqrt{2}t}{3\varepsilon_k}\right) + M.$$

As  $\frac{2\sqrt{2}}{3} \leq 1$ , we have

$$(\nabla p_k(\xi + t\nu_p(\xi)) \cdot \mathbf{t}_1(\xi))^2 \leq M \exp\left(\frac{t}{\varepsilon_k}\right) + M. \quad (4.24)$$

Introducing (4.24) in the definition of  $(\star)_{1,k}$  (4.12) gives

$$(\star)_{1,k} \leq \int_{S_p} \int_0^{\eta_k} 9\varepsilon_k \left( M \exp\left(\frac{t}{\varepsilon_k}\right) + M \right) \left[ 3 + \frac{1}{\varepsilon_k} \right] \Pi(\xi, t) dt d\mathcal{H}^2(\xi),$$

As  $\eta_k \rightarrow 0$ , the function  $\Pi$  is bounded and there exists a positive constant, still denoted  $M$ , such that

$$(\star)_{1,k} \leq \int_{S_p} \int_0^{\eta_k} \varepsilon_k \left( M \exp\left(\frac{t}{\varepsilon_k}\right) + M \right) M \left[ 3 + \frac{1}{\varepsilon_k} \right] dt d\mathcal{H}^2(\xi).$$

Thus, we have

$$(\star)_{1,k} \leq \left( \varepsilon_k^2 \left( \exp\left(\frac{\eta_k}{\varepsilon_k}\right) - 1 \right) + \varepsilon_k \eta_k \right) \left[ 3 + \frac{1}{\varepsilon_k} \right] M^2 \mathcal{H}^2(S_p).$$

According to (4.7)

$$\eta_k \leq 3\sqrt{2}\varepsilon_k \sqrt{|\ln(\varepsilon_k)|},$$

so, that

$$(\star)_{1,k} \leq \left( \varepsilon_k^2 \left[ \exp \left( 3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|} \right) - 1 \right] + 3\sqrt{2}\varepsilon_k^2 \sqrt{|\ln(\varepsilon_k)|} \right) \left[ 3 + \frac{1}{\varepsilon_k} \right] M^2 \mathcal{H}^2(S_p),$$

$$(\star)_{1,k} \leq \left[ \varepsilon_k \exp \left( 3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|} \right) - \varepsilon_k + 3\sqrt{2}\varepsilon_k \sqrt{|\ln(\varepsilon_k)|} \right] (1 + 3\varepsilon_k) M^2 \mathcal{H}^2(S_p),$$

$$(\star)_{1,k} \leq \left[ \exp \left( 3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|} + \ln(\varepsilon_k) \right) - \varepsilon_k + 3\sqrt{2}\varepsilon_k \sqrt{|\ln(\varepsilon_k)|} \right] M^2 \mathcal{H}^2(S_p).$$

As  $\varepsilon_k \rightarrow 0^+$  we have

$$\exp \left( 3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|} + \ln(\varepsilon_k) \right) \rightarrow 0^+, \quad \varepsilon_k \sqrt{|\ln(\varepsilon_k)|} \rightarrow 0^+.$$

We can conclude that  $(\star)_{1,k} \rightarrow 0$ .

**Second step: Assume that  $p \in \mathbf{BV}(\Omega)$ ,  $p$  takes its values in  $\{0, 1\}$  and  $\mathbf{g} \in \mathcal{C}^\infty \cap W^{1,r}(\Omega; \mathcal{G})$ .**

In this step, we still we set  $\mathbf{g}^k = \mathbf{g}$  for any  $k$ . For the same reason than in the previous step, it suffices to construct an appropriate sequence  $(p_k)_k$  which converges a.e. to  $p$  and such that  $\limsup F_{\varepsilon_k}(p_k, \mathbf{g}) \leq F(p, \mathbf{g})$ .

We denote  $A = p^{-1}(\{1\})$ . Let us first assume that  $A$  and  $\Omega \setminus A$  have nonempty interior. We can apply lemma 2.2. So, there exists a sequence  $(A_l)_l$  of open bounded subsets of  $\mathbb{R}^n$  with smooth boundaries such that

i)

$$\lim_{l \rightarrow \infty} \mathcal{L}^n((A_l \cap \Omega) \triangle A) = 0, \quad \lim_{l \rightarrow \infty} \mathcal{H}^{n-1}(\partial A_l) = \mathcal{H}^{n-1}(\partial A);$$

ii)

$$\mathcal{L}^n(A_l \cap \Omega) = \mathcal{L}^n(A) \quad \text{for } l \text{ large enough;}$$

iii)

$$\mathcal{H}^{n-1}(\partial A_l \cap \partial \Omega) = 0 \quad \text{for } l \text{ large enough;}$$

iv)

$$F(p_l, \mathbf{g}) \leq F(p, \mathbf{g}) + \frac{1}{l}, \quad (4.25)$$

where  $\mathcal{L}^n$  is the Lebesgue measure over  $\Omega$  and  $p_l = \mathbf{1}_{A_l \cap \Omega}$ . For (4.25) we use the fact that  $\|D\mathbf{1}_{A_l}\|_{\mathcal{M}} \rightarrow \|D\mathbf{1}_A\|_{\mathcal{M}}$  and proposition 2.3 ii). With i), ii) and iii), we can say that  $(p_l)_l$  is a bounded sequence of  $\mathbf{BV}(\Omega)$  which converges to  $p$  in  $L^1(\Omega)$ . According to theorem ??, there exists a subsequence, still denoted  $(p_l)_l$  which weakly\* converges to  $p$  in  $\mathbf{BV}(\Omega)$ . One can apply the result of the first step with  $p = p_l$ . So, there exists a sequence  $(p_{l,k})_k$  which weakly\* converges to  $p_l$  in  $\mathbf{BV}(\Omega)$  such that

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(p_{k,l}, \mathbf{g}) \leq F(p_l, \mathbf{g}). \quad (4.26)$$

With (4.25), (4.26) and a diagonal extraction there exists a sequence  $(p_k)_k$  which weakly\* converges to  $p$  such that

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}) \leq F(p, \mathbf{g}).$$

Let us remove the restriction that both  $A$  or  $\Omega \setminus A$  have non empty interior. First, we notice that if  $\mathcal{L}^n(A) = 0$  or  $\mathcal{L}^n(A) = \Omega$  the result is obvious by taking for all  $l$ ,  $A_l = \emptyset$  or  $A_l = \Omega$ . So, we may assume that  $0 < \mathcal{L}^n(A) < |\Omega|$ . There exists two points  $x_1, x_2$  such that

- $x_1 \in A$  and  $\forall r > 0$ ,  $\mathcal{L}^n(A \cap B(x_1, r)) > 0$ ,
- $x_2 \in \Omega \setminus A$  and  $\forall r > 0$ ,  $\mathcal{L}^n((\Omega \setminus A) \cap B(x_1, r)) > 0$ .

Consider the set  $A_{\theta_1, \theta_2} = (A \cup B(x_2, \theta_2)) \setminus B(x_1, \theta_1)$  and the function  $\Upsilon(\theta_1, \theta_2) = \mathcal{L}^n(A_{\theta_1, \theta_2})$ . As  $\Upsilon(0, \theta) > \mathcal{L}^n(A)$  and  $\Upsilon(\theta, 0) < \mathcal{L}^n(A)$  for any  $\theta > 0$ , there exists  $t \in ]0; 1[$  depending on  $\theta$  such that  $\Upsilon(t\theta, (1-t)\theta) = \mathcal{L}^n(A)$ . By construction,  $A_\theta$  and  $\Omega \setminus A_\theta$  have nonempty interior. The previous result gives the existence of  $(p_{\theta, k})_k \subset \text{BV}(\Omega; \{0; 1\})$  which weakly\* converges to  $p_\theta = \mathbf{1}_{A_\theta}$  in  $\text{BV}(\Omega)$  such that

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(p_{\theta, k}, \mathbf{g}) \leq F(p_\theta, \mathbf{g}). \quad (4.27)$$

Moreover,  $\mathcal{L}^n(A \triangle A_\theta)$  tends to 0 as  $\theta \rightarrow 0^+$ , and, using

$$\int_{S_{p_\theta}} \mathbf{g}(x, \nu_{p_\theta})^{\frac{1}{2}} \leq \int_{S_p} \mathbf{g}(x, \nu_{p_\theta})^{\frac{1}{2}} + \sqrt{2} \mathcal{H}^{n-1}(\partial B(x_1, \theta_1) \cup \partial B(x_2, \theta_2)),$$

we get

$$\limsup_{\theta \rightarrow 0^+} F(p_\theta, \mathbf{g}) \leq F(p, \mathbf{g}).$$

According to (4.27), with a diagonal extraction there exists a sequence  $(p_k)_k$  which weakly\* converges to  $p$  such that

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}) \leq F(p, \mathbf{g}).$$

**Last step: Assume that  $p \in \text{BV}(\Omega; \{0; 1\})$  and  $\mathbf{g} \in W^{1,r}(\Omega; \mathcal{G})$ .**

In this step we do not assume that  $(\mathbf{g}^k)_k$  is a constant sequence. Let  $(\mathbf{g}^l)_l$  as in proposition 2.2. As  $\mathbf{g}^l \in C^\infty \cap W^{1,r}_u(\Omega)$ , one apply the *Second step* of the proof, it gives

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}^l) \leq F(p, \mathbf{g}^l).$$

We have the following inequalities

$$\begin{aligned} |F(p, \mathbf{g}^l) - F(p, \mathbf{g})| &\leq \int_{S_p} |\mathbf{g}^l(x, \nu_p)^{\frac{1}{2}} - \mathbf{g}(x, \nu_p)^{\frac{1}{2}}| d\mathcal{H}^{n-1}, \\ &\leq \int_{S_p} \frac{|[(\mathbf{g}^l_x - \mathbf{g}_x) \nu_p] \cdot \nu_p|}{\mathbf{g}^l(x, \nu_p)^{\frac{1}{2}} + \mathbf{g}(x, \nu_p)^{\frac{1}{2}}} d\mathcal{H}^{n-1}, \\ &\leq \|\mathbf{g}^l - \mathbf{g}\|_{L^\infty} \frac{\mathcal{H}^{n-1}(S_p)}{2}. \end{aligned}$$

So, we deduce that  $(F(p, \mathbf{g}^l))_l$  converges to  $F(p, \mathbf{g})$ . With a diagonal extraction, we can conclude that there exists  $(p_k, \mathbf{g}^k)_k \subset \mathcal{Y}$  which converges for the topology  $\mathcal{T}$  to  $(p, \mathbf{g})$  such that

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(p_k, \mathbf{g}^k) \leq F(p, \mathbf{g}).$$

As  $(p_k)_k$  converges pointwise to  $p$  then  $(\int_\Omega (p_k - p)^2 dx)_k$  converges to  $\int_\Omega (p - p)^2 dx$ . Moreover, by construction  $(\|\mathbf{g}^k\|_{W^{1,r}})_k$  converges to  $\|\mathbf{g}\|_{W^{1,r}}$ . We conclude that

$$\limsup_{k \rightarrow \infty} E_{\varepsilon_k}(p_k, \cdot) \leq E(p, \mathbf{g}).$$

□

## 5 Conclusion

We have proved that the approximation process is suitable in the sense of  $\Gamma$ -convergence. We next use the approximated problem for numerical experimentation. The computation of  $p$  is done via a classical gradient descent method, while another strategy has to be developed for  $\mathbf{g}$ : this will be addressed in a future work.

On the other hand, the main hypothesis we did in this paper is the bimodality of histogram: this is quite restrictive for numerics. If this assumption is not ensured the previous model is not valid any longer and has to be modified: we will set a more general formulation that performs a similar segmentation without the binary constraint. Roughly speaking, we look for a pair  $(f, \mathbf{g})$  where  $f : \Omega \rightarrow [0; 1]$  is a function (not necessarily binary) and  $\mathbf{g}$  a riemannian metric. The corresponding energy to be minimized is :

$$\int_{\Omega} (f - g)^2 + \beta \int_{S_f} \mathbf{g}(x, \nu_f)^{\frac{1}{2}} d\mathcal{H}^{n-1} + \gamma \|\mathbf{g}\|_{W^{1,r}(\Omega)} + \rho \int_{\Omega \setminus S_f} |\nabla f|^2. \quad (5.1)$$

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