Γ-convergence of anisotropic vectorial functionals adapted to detection of thin tubes

David Vicente

Γ-convergence of anisotropic vectorial functionals adapted to detection of thin tubes

David Vicente

June 17, 2014

1 Introduction

This work is a contribution to the problem of detection of thin structures, namely *tubes*, in a 2D or 3D image. We adopt a geometrical point of view. As the concept of tube is geometric, it is therefore necessary to use mathematical tools to make the link between shapes, models and algorithms.

We introduce a variational model of Mumford-Shah type (see [1] for the seminal article). This approach generates a set of parameters whose role is essential but adjustment is not automatic. The mathematical model allows to develop strategies for tuning these parameters.

We introduce a *bimodal* model for the case where the histogram of the image has two main modes. This corresponds, for example, to the case of an angiography MRI.

Let n be the dimension and $\Omega \subset \mathbb{R}^n$ the domain of the image. The analysis will consist in searching a pair $(p, \mathbf{g})$, where $p : \Omega \to \{0, 1\}$ is a binary function and $\mathbf{g} : \Omega \times \mathbb{R}^n \to \mathbb{R}^+$ is such that for any $x \in \Omega$, $\mathbf{g}(x, \cdot)$ is a riemannian metric. This pair must minimize a functional E defined by:

$$E(p, \mathbf{g}) = \int_{S_p} \mathbf{g}(x, \nu_p(x))^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}(x),$$

where S_p is the jump set of p, ν_p is a normal unitary vector to S_p and $\mathcal{H}^{n-1}$ the $n-1$-dimensional Hausdorff measure. To detect thin tubes, we assume that the metric has a characteristic direction at each point of the domain. More precisely, the metric must take the form $\mathbf{g}(x, \mathbf{v}) = |\mathbf{v}|^2 + (\mathbf{c}(x) \cdot \mathbf{v})^2$, where $\mathbf{c}$ is an unitary vector field on Ω.

In [2], it is proved, in the general setting of Finsler metric, that this functional is the anisotropic perimeter associated to the dual metric if $\mathbf{g}$ is fixed. So, it inherits of lower semi-continuity property and it can be approximated in the sense of the Γ-convergence by the family of functionals $(E_\varepsilon)_{\varepsilon>0}$ defined by

$$E_\varepsilon(p) = \int_\Omega \varepsilon \mathbf{g}(x, \nabla p) + \frac{p^2(1-p)^2}{\varepsilon} \mathrm{d}x.$$

We generalize this work to the case where the metric $\mathbf{g}$ is also an unknown and takes the form defined above. More precisely, we will prove that the family $(F_\varepsilon)_{\varepsilon>0}$ defined by

$$F_\varepsilon(p, \mathbf{c}) = \int_\Omega 9\varepsilon \left[|\nabla p|^2 + (\mathbf{c} \cdot \nabla p)^2\right] + \frac{p^2(1-p)^2}{\varepsilon},$$

Γ-converges to

$$F(p,\mathbf{c}) = \int_{S_p} \sqrt{1+(\mathbf{c}\cdot\nu_p)^2}\mathrm{d}\mathcal{H}^{n-1}.$$

In section 2, we introduce the problem and the practical motivations. We recall some classical results and introduce the functional framework in section 3. Section 4 is devoted to the approximation process. We prove the main result, that is the Γ-convergence result.

from the practical point of view, of calculus of variation and introduce the functionals spaces used in the results, in sthe main result of approximation is given and proved.

2 Presentation of the model

In what follows, n represents the spatial dimension of the image: $n = 2$ for planar images and $n = 3$ for 3-D images. We adopt the list of symbols:

- $\mathbf{x} = (x_i)_{i=1\ldots n}$ a vector in $\mathbb{R}^n$,
- x a scalar in $\mathbb{R}$,
- $\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^n x_i y_i$ the usual scalar product of $\mathbb{R}^n$,
- $|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$ the euclidean norm of $\mathbb{R}^n$,
- $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$,
- $B_r(x)$ the ball of $\mathbb{R}^n$ with center x and radius $r \geq 0$,
- $\|A\|$ a generic matricial norm in $M_n(\mathbb{R})$,
- $\boldsymbol{\nu}(x)$ a normal vector of a surface at x in $\mathbb{R}^n$,
- $\mathcal{L}^n$ the n-dimensional Lebesgue measure of $\mathbb{R}^n$,
- $\mathcal{H}^k$ the k-dimensional Hausdorff measure of $\mathbb{R}^n$,
- $\mathcal{M}(\Omega;\mathbb{R}^n)$ the space of vector valued measures over Ω,
- $\mathcal{C}^k(\Omega)$ the space of k-differentiable functions defined on Ω,
- $\mathcal{C}_0^1(\Omega;\mathbb{R}^n)$ the space of differentiable functions defined on Ω with values in $\mathbb{R}^n$ and which tend to 0 at ∞,
- $\mathcal{C}_c(\Omega)$ the space of continuous functions defined on Ω with compact support,
- $A \vartriangle B$ the symetric difference of the sets: $(A \setminus B) \cup (B \setminus A)$,
- ∂A the topological boundary of $A \subset \mathbb{R}^n$.

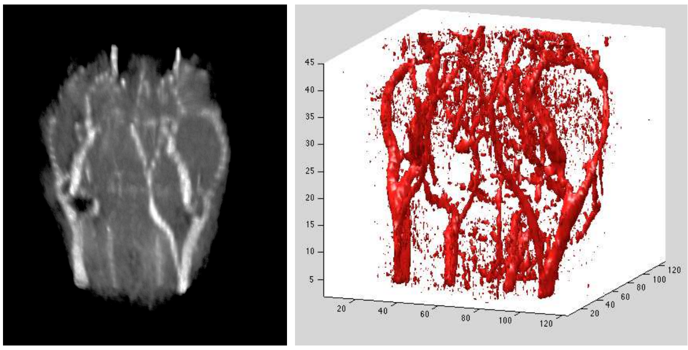

Figure 2.1: Mouse brain angiography and thresholding at 80% of the maximum intensity

2.1 Motivation of the problem

For the study of some diseases, it is interesting to focus on the blood status in a vessel network, especially on the volume of its microvasculature. To assess this, *in vivo* mice brain angiography is performed. This is based on the injection of a contrast medium and a MRI imaging process.

The left image in Figure 2.1 is a planar projection of a 3-D image, the right image is the result of the manual segmentation with thresholding at a level equal to 80% of the maximum intensity. The aim is to construct an automatic method to segment the network area corresponding to the blood while removing noise.

2.2 Geometric characterization of the problem

We give an heuristic way to introduce and motivate the model. We first present an isotropic model and show that it is not suitable for our problem. Then, we introduce an anisotropic term.

2.2.1 The generic problem

Let $\Omega \subset \mathbb{R}^n$ be the domain of the image. We consider the following segmentation problem. Let $\alpha > 0$ be the critical level of detection: if a set b_{α_0} is with diameter α_0 lower than α then it is considered as noise and has to be removed. Let $\Gamma \subset \Omega$ be a curve with length ℓ such that α is negligible compared with ℓ. We set $T_{\ell,\alpha}$ the tubular neighborhood of Γ of the points of Ω at distance lower than α of Γ. Although the section of $T_{\ell,\alpha}$ is near of the critical level we want to detect this set because of its specific geometry. We set B_α a ball with radius α, it is also considered as noise not because it has a critical diameter but because it has not the appropriate geometry of tubes.

Let $I \subset \Omega$ be a generic set and we assume that we have the following disjoint decomposition (see figure 2.2)

$$I = T_{\ell,\alpha} \cup B_\alpha \cup b_{\alpha_0}.$$

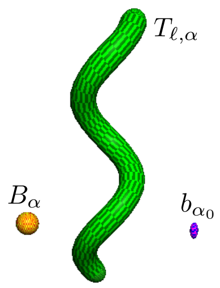

Figure 2.2: Decomposition of the image in three kinds of sets.

The segmentation problem consists in combining three constraints. The first one is to remove the sets of b_{α_0} type because their section is too small. The second one is to remove the B_α type sets, not because they have small radius, but because they are not tubes. The last one is to detect the tubes $T_{\ell,\alpha}$.

For that, we purpose an energy functional $\mathcal{E}$ defined on the sets of Ω. We say that a set F is a *better segmentation* than the set G if $\mathcal{E}(F) < \mathcal{E}(G)$. The functional $\mathcal{E}$ is *adapted* to the problem if it satisfies the three following conditions:

i) $\mathcal{E}(I \setminus b_{\alpha_0}) < \mathcal{E}(I)$,

ii) $\mathcal{E}(I \setminus B_\alpha) < \mathcal{E}(I)$,

iii) $\mathcal{E}(I) < \mathcal{E}(I \setminus T_{\ell,\alpha})$.

Conditions i) and ii) imply that removing b_{α_0} and B_α provides a better segmentation than keeping them. Condition iii) implies that detecting $T_{\ell,\alpha}$ gives a better segmentation than removing it.

2.2.2 An isotropic model

We first consider an isotropic functional $\mathcal{E}$ defined on the sets of $\mathcal{P}(\Omega)$ with finite volume and area as:

$$\mathcal{E}(F) = \text{Volume}(F \triangle I) + \beta \text{Area}(\partial F),$$

where $\beta > 0$ is a weight to tune. Many works have been devoted to this particular case. Let us mention contributions from Morel and Solimini [3]. We show that this model can not satisfy the three constraints we imposed to our problem.

As $T_{\ell,\alpha}$, B_α and b_{α_0} are disjoints, condition i) is equivalent to

$$\text{Volume}(b_{\alpha_0}) < \beta \text{Area}(\partial b_{\alpha_0}). \tag{2.1}$$

The sets which minimize the ratio Volume/Area are prefered by this energy. When the diameter is fixed, the sets which minimize this ratio are the balls. Thus, to ensure the inequality (2.1), it it sufficient to have

$$\text{Volume}(B_{\alpha_0}) < \beta \text{Area}(\partial B_{\alpha_0}).$$

Taking the equivalences Volume$(B_{\alpha_0}) \sim \alpha_0^n$ and Area$(\partial B_{\alpha_0}) \sim \alpha_0^{n-1}$, this condition gives

$$\alpha_0 < \beta.$$

For the same reasons, condition ii) gives

$$\alpha < \beta. \tag{2.2}$$

Condition iii) is equivalent to

$$\beta \text{Area}(\partial T_{\ell,\alpha}) < \text{Volume}(T_{\ell,\alpha}).$$

Taking the equivalences Volume$(T_{\ell,\alpha}) \sim \ell\alpha^{n-1}$ and Area$(T_{\ell,\alpha}) \sim \ell\alpha^{n-2}$, this condition gives

$$\beta < \alpha. \tag{2.3}$$

Thus, conditions (2.2) and (2.3) are contradictious. Such a model is not adapted to this problem.

2.2.3 An anisotropic model

We rather introduce an energy term that involves a preference to sets *having a direction*, that is anisotropic sets. Let $\mathbf{c} : \Omega \longrightarrow \mathbb{S}^{n-1}$ be an unknown and unitary vector field that represents a direction in each point of the image.

Let $x \in \partial F$ and $\nu_F(x)$ be a unit normal vector of the surface ∂F at x. We say that $|\mathbf{c}(x) \cdot \nu_F(x)|$ is the *action of the vector field* $\mathbf{c}$ on ∂F at x. This term is zero if the field is tangent to the surface and it is maximum if it is orthogonal to the surface. We introduce the *total action* of $\mathbf{c}$ on ∂F:

$$\text{Action}(F, \mathbf{c}) = \int_{\partial F} |\mathbf{c} \cdot \nu_F| \mathrm{d}\mathcal{H}^{n-1},$$

where $\mathcal{H}^{n-1}$ is the surface measure (namely the $n-1$-dimensional Hausdorff measure). As $\alpha \ll \ell$, a field which minimizes the action of $\mathbf{c}$ on $\partial T_{\ell,\alpha}$ has to be tangent to $\partial T_{\ell,\alpha}$ along the tube (see figure 2.3).

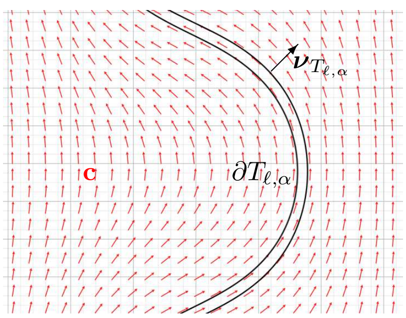

Figure 2.3: A field $\mathbf{c}$ tangent to the edge of a tube $T_{\ell,\alpha}$.

Moreover, we introduce a regularization term defined on the vector field as

$$\text{Regularization}(\mathbf{c}) = \int_{\Omega} \|D\mathbf{c}\|^{\lambda},$$

where $\|\cdot\|$ is a pointwise matricial norm and we fix $\lambda > n$ (not necessary an integer) to ensure that the field is regular. Indeed, if $\lambda > n$ and $\int_\Omega \|D\mathbf{c}\|^\lambda < \infty$ then $\mathbf{c}$ is continuous. The new expression of the energy is

$$\mathcal{E}(F,\mathbf{c}) = \underbrace{\text{Volume}(F \bigtriangleup A) + \beta\text{Area}(\partial F)}_{\text{isotropic terms}} + \underbrace{\mu\text{Action}(\partial F,\mathbf{c}) + \gamma\text{Regularization}(\mathbf{c})}_{\text{anisotropic terms}}, \tag{2.4}$$

where β, μ and γ are, as before, weights to tune. We have to verify the conditions i), ii) and iii) of 2.2.1. We assume that $T_{\ell,\alpha}$ is a linear and rigid tube of length ℓ and section α. Obviously, the best choice of $\mathbf{c}$ is to choose it in the direction of the tube. Indeed, the action of the field on the tube is zero outside the two ends of the tube and the regularization is zero (see figure 2.4).

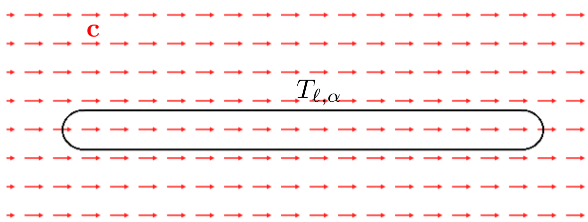

Figure 2.4: A linear tube $T_{\ell,\alpha}$ and $\mathbf{c}$ in the direction of the tube.

Using the following equivalences:

$$\text{Volume}(T_{\ell,\alpha}) \sim \ell\alpha^{n-1}, \quad \text{Area}(T_{\ell,\alpha}) \sim \ell\alpha^{n-2}, \quad \text{Action}(\partial T_{\ell,\alpha},\mathbf{c}) \sim \alpha^{n-1}.$$

Condition iii) is equivalent to

$$\beta\ell\alpha^{n-2} + \mu\alpha^{n-1} < \ell\alpha^{n-1}. \tag{2.5}$$

For a ball B_α, the field $\mathbf{c}$ has to realize a compromise between its action on ∂B_α and its regularization (see figure 2.5).

We make the homothetic change of variable between B_α and B_1, a ball of radius 1. We denote $\mathbf{c}_1 = \mathbf{c}(\alpha\cdot)$. It gives:

$$\mu\text{Action}(\partial B_\alpha,\mathbf{c}) + \gamma\text{Regularization}(\mathbf{c}) = \mu\alpha^{n-1}\text{Action}(\partial B_1,\mathbf{c}_1) + \gamma\alpha^{n-\lambda}\text{Regularization}(\mathbf{c}_1).$$

As α is small and $\lambda > n$, then if $\mu \sim \gamma$ the parameter $\mu\alpha^{n-1}$ is negligible with respect to $\gamma\alpha^{n-\lambda}$. As a conclusion, the regularization is more important than the action for balls with small radius α. The best choice for $\mathbf{c}$ is a constant field. In this case the regularization is zero and the action is equal to the action on the tube with the same section. Then, conditions i) and ii) are equivalent to

$$\alpha^n < \beta\alpha^{n-1} + \mu\alpha^{n-1}. \tag{2.6}$$

The three conditions (2.5) and (2.6) are not contradictious anymore when α is

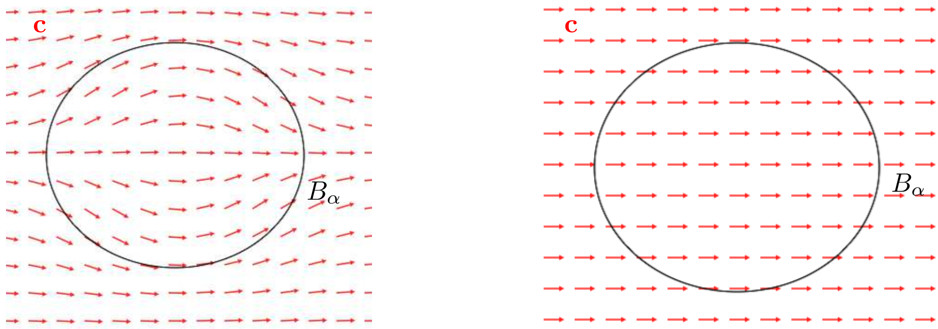

Figure 2.5: Two opposite situations: at left, $\mathbf{c}$ minimizes the action, at right, $\mathbf{c}$ minimizes the regularization.

small and $\alpha \ll \ell$. For example, we can take

$$\begin{cases} \mu = \gamma, \\ \beta < \frac{\alpha}{2}, \\ \mu < \frac{\ell}{2}, \\ \beta + \mu > \alpha. \end{cases}$$

As $\alpha \ll \ell$, the previous system admits solutions. Thus, this model is adapted to the problem.

2.3 Bimodal model

Let us give a functional formulation of the energy of (2.4). We define an image as a function $g : \Omega \to [0;1]$. We assume that the domain $\Omega \subset \mathbb{R}^n$ is Lipschitz-regular. The fondamental assumption of this model is that the histogram distribution of the image contains two main modes that we assume to be 0 and 1. Equivalently, g is almost equal to an indicator function. The unknown is a pair $(p, \mathbf{c})$ where $p : \Omega \to \{0;1\}$ is a binary function and $\mathbf{c} : \Omega \to \mathbb{S}^{n-1}$ is an unitary vector field which minimizes the energy

$$E(p, \mathbf{c}) = \int_\Omega (p-g)^2 + \beta \mathcal{H}^{n-1}(S_p) + \mu \int_{S_p} |\mathbf{c} \cdot \boldsymbol{\nu}_p| \, \mathrm{d}\mathcal{H}^{n-1} + \gamma \int_\Omega \|D\mathbf{c}\|^\lambda, \quad (2.7)$$

where S_p is the jump set of p. It is a $\mathcal{C}^1$-hypersurface of Ω and $\boldsymbol{\nu}_p : S_p \to \mathbb{S}^{n-1}$ is a normal unit vector of S_p. If p is the indicator function of a set $F \subset \Omega$ and F is a smooth hypersurface then S_p is equal to ∂F. For the general setting, its definition will be precised in section 3.3. The parameters β, μ and γ are weights to determine.

3 Functional framework

The following results are extracted from [4] for Radon measures and BV-spaces and [5] for functionals defined on measures spaces and Γ-convergence.

3.1 Radon measures and functionals defined on measure spaces

Definition 3.1. *Let $\mathcal{M}(\Omega)$ be the space of Radon measures and $\mathcal{M}(\Omega;\mathbb{R}^n)$ be the space of vectorial Radon measures.*

Theorem 3.1. *For $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^n)$, we denote*

$$\|\lambda\|_{\mathcal{M}(\Omega;\mathbb{R}^n)} = \sup\left\{\int_\Omega \varphi \cdot d\lambda : \varphi \in \mathcal{C}_c(\Omega;\mathbb{R}^n), \|\varphi\|_{L^\infty} \leq 1\right\}.$$

Then, this application is a norm on $\mathcal{M}(\Omega;\mathbb{R}^n)$.

Definition 3.2. *Let λ and θ be two measures on Ω. We say*

i) the ***restriction*** *of λ to the set $B \subset \Omega$ is the measure denoted $\lambda \llcorner B$ and defined by $\lambda \llcorner B(\cdot) = \lambda(B \cap \cdot)$,*

ii) λ is ***absolutely continuous with respect to*** *θ if $\lambda(A) = 0$ whenever $\theta(A) = 0$, when this holds, we write $\lambda \ll \theta$;*

iii) λ and θ are ***mutually singular*** *if there exists a set $B \subset \Omega$ such that $\lambda = \lambda \llcorner B$ and $\theta = \theta \llcorner (\Omega \setminus B)$, when this holds we write $\lambda \perp \theta$.*

The following result is also called Besicovitch's decomposition theorem.

Theorem 3.2. *Suppose that θ is a Radon measure and λ is a vectorial Radon measure on Ω. Then*

i) $\lim\limits_{r\to 0} \dfrac{\lambda(B_r(x))}{\theta(B_r(x))}$ exists and is finite for θ almost every x, we denote by $\frac{d\lambda}{d\theta}(x)$ this limit when it exists,

ii) $\lambda = \lambda_{ac} + \lambda_s$, where λ_{ac}, λ_s are vectorial Radon measures such that $\lambda_{ac} \ll \theta$ and $\lambda_s \perp \theta$,

iii) $\lambda_{ac}(B) = \int_B \frac{d\lambda}{d\theta}\, d\theta$.

Definition 3.3. *A sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega;\mathbb{R}^n)$ weakly converges to $\lambda \in \mathcal{M}(\Omega;\mathbb{R}^n)$ if we have*

$$\forall \varphi \in \mathcal{C}_c(\Omega;\mathbb{R}^n), \qquad \int_\Omega \varphi \cdot d\lambda_k \to \int_\Omega \varphi \cdot d\lambda.$$

Let $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}^+$ be a sublinear function with respect to the second variable, that is:

i)

$$\forall (x, \mathbf{v}_1, \mathbf{v}_2) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n, \qquad \varphi(x, \mathbf{v}_1 + \mathbf{v}_2) \leq \varphi(x, \mathbf{v}_1) + \varphi(x, \mathbf{v}_2),$$

ii)

$$\forall (x, \mathbf{v}, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^+, \qquad \varphi(x, t\mathbf{v}) = t\varphi(x, \mathbf{v}).$$

We consider the convex functional defined on the space $\mathcal{M}(\Omega;\mathbb{R}^n)$ by

$$\Phi : \lambda \in \mathcal{M}(\Omega;\mathbb{R}^n) \mapsto \int_\Omega \varphi\left(x, \frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\right) \mathrm{d}\theta \tag{3.1}$$

where θ is a positive measure such that $\lambda \ll \theta$. It is shown in [6] that the integral in (3.1) does not depend on the choice of θ. For that reason, we will write it in the condensed form

$$\Phi(\lambda) = \int_\Omega \varphi(x, \lambda).$$

The functional Φ has the following continuity properties which are proved in [7].

Proposition 3.1. *i) If φ is a lower semicontinuous on $\Omega \times \mathbb{R}^n$, then Φ is lower semicontinuous on $\mathcal{M}(\Omega;\mathbb{R}^n)$ for the topology introduced in 3.3.*

ii) Assume that φ is continuous on $\Omega \times \mathbb{R}^n$. If $(\lambda_k)_{k\in\mathbb{N}}$ weakly converges to λ and if, moreover, $\int_\Omega |\lambda_k| \to \int_\Omega |\lambda|$, then $\Phi(\lambda_k)$ converges to $\Phi(\lambda)$.

We give a variant of the coarea formula extended to the sublinear functionals which can be found in [8].

Proposition 3.2. *Let $\Phi(x, s, \boldsymbol{v})$ a Borel function of $\Omega\times\mathbb{R}\times\mathbb{R}^n$ which is sublinear in $\boldsymbol{v}$. Let p be a Lipschitz continuous function on Ω and denote, for $t > 0$, $S_t = \{x \in \Omega; p(x) < t\}$. Then, for almost all $t \in \mathbb{R}$, S_t belongs to $BV(\Omega)$ and we have*

$$\int_\Omega \Phi(x, p, Dp)dx = \int_\mathbb{R} dt \int_\Omega \Phi(x, t, D\mathbf{1}_{S_t}).$$

3.2 Functions with bounded variation

Definition 3.4. *A function $u \in L^1(\Omega)$ is said to be with **bounded variation** if*

$$\sup\left\{\int_\Omega u\,div(\varphi) \colon \varphi \in \mathcal{C}^1_0(\Omega;\mathbb{R}^n), \|\varphi\|_{L^\infty} \leq 1\right\} < +\infty.$$

We denote $TV(u)$ this upper bound and $BV(\Omega)$ the set of such functions.

Theorem 3.3. *Let be $u \in L^1(\Omega)$, then the derivative of u in the sense of the distributions is a bounded Radon measure. Moreover, we have $TV(u) = \|Du\|_{\mathcal{M}(\Omega;\mathbb{R}^n)}$.*

Theorem 3.4. *The space $BV(\Omega)$, equipped with the following norm*

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + TV(u)$$

is a Banach space.

Definition 3.5. *A sequence $(u_k)_{k\in\mathbb{N}}$ of $BV(\Omega)$ converges in the sense of the intermediate topology to $u \in BV(\Omega)$ if we have*

$$\begin{cases} u_k \to u & \text{in } L^1(\Omega), \\ \int_\Omega |Du_k| \longrightarrow \int_\Omega |Du|. \end{cases}$$

Theorem 3.5. *We have the two following compactness results*

i) the inclusion $BV(\Omega) \subset L^1(\Omega)$ is compact,

ii) if $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence of $BV(\Omega)$, then there exists a subsequence which intermediately converges to u.

3.3 Sets with finite perimeter

In the sequel, we are formulating $\min \mathcal{E}$ as a minimization problem of functions by connecting sets and functions via indicator functions. The following terminology is introduced in [4].

Definition 3.6. *Let $A \subset \Omega$ be a measurable set. A point $x \in \Omega$ belongs to* ***measure theoretic boundary*** *of A if*

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap A)}{r^n} > 0,$$

and

$$\liminf_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\setminus A)}{r^n} > 0.$$

We denote $\partial^* A$ the measure theoretic boundary of A .

Definition 3.7. *A measurable set $A \subset \Omega$ is said with* ***finite perimeter*** *if $\mathbf{1}_A \in BV(\Omega)$.*

Remark 3.1. *The concept of measure theoretic boundary generalizes the concept of topological boundary when A is regular. Indeed, if $A \subset \Omega$ is an open whose topological boundary ∂A is a smooth and compact hypersurface, then we have*

i) $\partial A = \partial^* A$,

ii) $TV(\mathbf{1}_A) = \mathcal{H}^{n-1}(\partial A)$.

Theorem 3.6. *Let $A \subset \Omega$ be a set with finite perimeter. There exists a pairwise disjoint family of sets $(S_i)_{i\in\mathbb{N}}$ and a set $M \subset \Omega$ such as*

i) for all $i \in \mathbb{N}$, S_i is a $\mathcal{C}^1$ and compact hypersurface of Ω,

ii) $\mathcal{H}^{n-1}(M) = 0$,

iii) $\partial^* A = M \cup \left(\bigcup_{i\in\mathbb{N}} S_i\right)$.

Theorem 3.7. *Let $A \subset \Omega$ be a set with finite perimeter. The following generalized Gauss-Green formula holds: for $\mathcal{H}^{n-1}$ almost every $x \in \Omega$, there exists $\boldsymbol{\nu}(x) \in \mathbb{S}^{n-1}$, called the* ***inner normal vector*** *to A at x, such that for all $\varphi \in \mathcal{C}^1_c(\Omega;\mathbb{R}^n)$,*

$$\int_\Omega \mathbf{1}_A\, div(\varphi)\, dx = -\int_{\partial * A\cap\Omega} \varphi\cdot\boldsymbol{\nu}\, d\mathcal{H}^{n-1},$$

that is $D\mathbf{1}_A = \boldsymbol{\nu}\mathcal{H}^{n-1} \llcorner \partial^* A\cap\Omega$.

Definition 3.8. *Let $A \subset \Omega$ be a set with finite perimeter and $p = \mathbf{1}_A$. With the theorems 3.6 and 3.7, we get $S_p = \partial^* A$ and $\boldsymbol{\nu}_p = \boldsymbol{\nu}$.*

Proposition 3.3. *Let $A \subset \Omega$ be a set with finite perimeter and $p = \mathbf{1}_A$. Then, we have $\|Dp\|_{\mathcal{M}} = \mathcal{H}^{n-1}(S_p)$.*

Remark 3.2. *If $p = \mathbf{1}_A$ with $A \subset \Omega$ an open subset whose boundary ∂A is a compact hypersurface with class $\mathcal{C}^1$, then we have $S_p = \partial A$.*

The following lemma is proved in [9]. It asserts that every set with bounded perimeter can be approximated by a sequence of smooth subsets of $\mathbb{R}^n$, all having the same volume inside Ω and each of these boundaries satisfy a measure theoretic transversality condition with respect to Ω.

Lemma 3.1. *Let Ω be an open, bounded subset of $\mathbb{R}^n$ with Lipschitz continuous boundary, and let A be a measurable subset of Ω. If A and $\Omega \setminus A$ both contain a non-empty open ball, then there exists a sequence $(A_k)_{k\in\mathbb{N}}$ of open bounded subsets of $\mathbb{R}^n$ with smooth boundaries such that*

i)

$$\lim_{k\to\infty} \mathcal{L}^n((A_k \cap \Omega) \triangle A) = 0, \qquad \lim_{k\to\infty} \mathcal{H}^{n-1}(\partial A_k) = TV(\mathbf{1}_A);$$

ii)

$$\mathcal{L}^n(A_k \cap \Omega) = \mathcal{L}^n(A) \qquad \textit{for } k \textit{ large enough};$$

iii)

$$\mathcal{H}^{n-1}(\partial A_k \cap \partial\Omega) = 0 \qquad \textit{for } k \textit{ large enough.}$$

3.4 Functional spaces

In this section we introduce the appropriate functional spaces for the approximation process.

$$\begin{array}{rcl}
\mathbb{B}(\Omega;[0,1]) & = & \{p \text{ measurable } : \ p(x) \in [0;1] \text{ a.e. } x \in \Omega\}\,, \\
\mathrm{BV}_p(\Omega) & = & \{p \in \mathrm{BV}(\Omega) \colon \ p(x) \in \{0;1\} \text{ a.e. } x \in \Omega\}\,, \\
W_p^{1,2}(\Omega) & = & \left\{p \in W^{1,2}(\Omega) \colon \ 0 \le p(x) \le 1 \text{ a.e. } x \in \Omega\right\}, \\
W_u^{1,\lambda}(\Omega;\mathbb{R}^n) & = & \left\{\mathbf{c} \in W^{1,\lambda}(\Omega;\mathbb{R}^n) \colon \ |\mathbf{c}(x)| = 1 \text{ a.e. } x \in \Omega\right\}, \\
\mathcal{X} & = & \mathbb{B}(\Omega;[0,1]) \times W_u^{1,\lambda}(\Omega), \\
\mathcal{Y} & = & \mathrm{BV}_p(\Omega) \times W_u^{1,\lambda}(\Omega;\mathbb{R}^n), \\
\mathcal{Z} & = & W_p^{1,2}(\Omega) \times W_u^{1,\lambda}(\Omega;\mathbb{R}^n).
\end{array}$$

Let $\mathcal{T}$ be the product topology on $\mathcal{X}$ where:

- $\mathbb{B}(\Omega;[0,1])$ is endowed with the almost everywhere convergence topology,
- $W_u^{1,\lambda}(\Omega)$ is endowed with the weak topology associated to its norm.

For a sequence $((p_k,\mathbf{c}_k))_{k\in\mathbb{N}}$ which converges to $(p,\mathbf{c})$ for this topology, we write $(p_k,\mathbf{c}_k) \xrightarrow{\mathcal{T}} (p,\mathbf{c})$. Since these spaces are metrizable, then $(X,\mathcal{T})$ is also metrizable.

3.5 Γ-convergence

We want to perform an approximation of the energy E more suitable for numerics applications. We will do that in the sense of the Γ-convergence. In this section, we give the definition and a useful property of continuity. For more properties on this subject one refer to [5], chapter 12.

Definition 3.9. *Let $(\mathcal{X}, d)$ be a metrizable space, $(E_k)_{k\in\mathbb{N}}$ a sequence of real-valued functions $E_k : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, and $E : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$. The sequence $(E_k)_{k\in\mathbb{N}}$ Γ-converges to E at $x \in \mathcal{X}$ if both the following conditions hold:*

i) for all sequences $(x_k)_{k\in\mathbb{N}}$ converging to $x \in \mathcal{X}$, one has

$$E(x) \leq \liminf_{k\to\infty} E_k(x_k), \tag{3.2}$$

ii) there exists a sequence $(y_k)_{k\in\mathbb{N}}$ converging to $x \in \mathcal{X}$ such that

$$E(x) \geq \limsup_{k\to\infty} E_k(y_k). \tag{3.3}$$

When i) and ii) hold for all $x \in \mathcal{X}$, we say that $(E_k)_{k\in\mathbb{N}}$ Γ-converges to E in $(\mathcal{X}, d)$.

The main interest of the Γ-convergence, in our case, is the following result.

Theorem 3.8. *Let $(E_k)_{k\in\mathbb{N}}$ be a sequence of functions which Γ-converges to E in $(\mathcal{X}, d)$. Let $(x_k)_{k\in\mathbb{N}}$ be such that*

$$\forall k \in \mathbb{N}, \quad E_k(x_k) \leq \inf_{x\in\mathcal{X}} E_k(x) + \varepsilon_k,$$

where $\varepsilon_k > 0$ converges to 0. Assume that $(x_k)_{k\in\mathbb{N}}$ is relatively compact; then every cluster point x of $(x_k)_{k\in\mathbb{N}}$ is a minimizer of E and

$$\liminf_{k\to\infty} E_k(x_k) = E(x).$$

Remark 3.3. *The previous theorem asserts that if x_k is an approximate solution of the minimizing problem of E_k, then it is also an approximated solution of the minimizing problem of E.*

4 Functional formulation of the problem

In this section we prove that the function defined in (2.7) admits at least a minimizer in an appropriate functional space. Then, we introduce an approximation of this problem and prove a convergence result.

4.1 Existence result

In order to simplify the notations, we assume that $\beta = \mu = \gamma = 1$. Thus, we denote

$$E(p, \mathbf{c}) = \int_\Omega (p - g)^2 + \mathcal{H}^{n-1}(S_p) + \int_{S_p} |\mathbf{c} \cdot \boldsymbol{\nu}_p| \mathrm{d}\mathcal{H}^{n-1} + \int_\Omega \|D\mathbf{c}\|^\lambda.$$

We have the following minimization problem

$$(\mathcal{P}): \qquad \operatorname{Min}\left\{E(p,\mathbf{c})\colon (p,\mathbf{c})\in\mathcal{Y}\right\}. \tag{4.1}$$

In this section we prove that problem $(\mathcal{P})$ admits at least one solution. We apply the *direct method* of calculus of variations. We exhibit a minimizing sequence which is compact for an appropriate topology. Then, we prove a lower semicontinuity result for E and conclude.

As E is bounded from below by 0, there exists a sequence $(p_k,\mathbf{c}_k)_{k\in\mathbb{N}}\subset\mathcal{Y}$ such that $(E(p_k,\mathbf{c}_k))_{k\in\mathbb{N}}$ converges to the minimum value of E. In the following theorem we prove that we can extract a converging sequence from $(p_k,\mathbf{c}_k)_{k\in\mathbb{N}}$.

Theorem 4.1. *Let $(p_k,\boldsymbol{c}_k)_{k\in\mathbb{N}}\subset\mathcal{Y}$ such that*

$$\exists M>0,\quad \forall k\in\mathbb{N},\quad E(p_k,\boldsymbol{c}_k)\leq M.$$

Then, there exists a subsequence, still denoted $(p_k,\boldsymbol{c}_k)_{k\in\mathbb{N}}$, and $(p,\boldsymbol{c})\in\mathcal{Y}$ such that $(p_k,\boldsymbol{c}_k)\xrightarrow{\mathcal{T}}(p,\boldsymbol{c})$.

Proof. As p_k takes its values in $[0;1]$ and Ω is bounded, then $(p_k)_{k\in\mathbb{N}}$ is a bounded sequence of $L^1(\Omega)$. According to proposition 3.3, we have $TV(p_k)=\mathcal{H}^{n-1}(S_{p_k})$. Moreover, we have $\mathcal{H}^{n-1}(S_{p_k})\leq E(p_k,\mathbf{c}_k)$, so $(TV(p_k))_{k\in\mathbb{N}}$ is bounded. We deduce that $(p_k)_{k\in\mathbb{N}}$ is a bounded sequence of $\mathrm{BV}(\Omega)$. According to theorem 3.5, there exists a subsequence, still denoted $(p_k)_{k\in\mathbb{N}}$, and some $p\in\mathrm{BV}(\Omega)$ such that $(p_k)_{k\in\mathbb{N}}$ converges to p intermediately. As p_k takes its values in $\{0;1\}$, we deduce that p takes its values in $\{0;1\}$.
As $\mathbf{c}_k$ takes its values in $\mathbb{S}^{n-1}$, and Ω is bounded, then $(\mathbf{c}_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\lambda(\Omega)$. Moreover, we have

$$\|D\mathbf{c}_k\|_{L^\lambda}\leq E(p_k,\mathbf{c}_k).$$

We deduce that $(\mathbf{c}_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,\lambda}(\Omega;\mathbb{R}^n)$. So, we can extract a sequence $(\mathbf{c}_k)_{k\in\mathbb{N}}$ and there exists $\mathbf{c}\in W^{1,\lambda}(\Omega;\mathbb{R}^n)$ such that $(\mathbf{c}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{c}$ weakly in $W^{1,\lambda}(\Omega;\mathbb{R}^n)$. As $\mathbf{c}_k$ takes its values in $\mathbb{S}^{n-1}$, we deduce that $\mathbf{c}$ takes its values too in $\mathbb{S}^{n-1}$. We can conclude that $(p,\mathbf{c})\in\mathcal{Y}$. □

Theorem 4.2. *The functional $E:\mathcal{Y}\to\mathbb{R}$ is lower semicontinuous for the product topology of the intermediate convergence on $BV(\Omega)$ and the weak-topology on $W^{1,\lambda}(\Omega;\mathbb{R}^n)$.*

Proof. The lower semicontinuity of the three terms $\int_\Omega(p-g)^2$, $\mathcal{H}^{n-1}(S_p)$ and $\int_\Omega\|D\mathbf{c}\|^\lambda$ are well known results and the proof can be found for example in [5]. The original part of this result is the lowersemicontinuity of $\int_{S_p}|\mathbf{c}\cdot\boldsymbol{\nu}_p|\mathrm{d}\mathcal{H}^{n-1}$. We first prove the result with $\mathbf{c}$ fixed and generalize without this assumption.

Claim 1: Let $\mathbf{c}\in W^{1,\lambda}_u(\Omega;\mathbb{R}^n)$ and $(p_k)_{k\in\mathbb{N}}\subset BV_p(\Omega)$ (see 3.4) which intermediately converges to $p\in BV_p(\Omega)$. Then, we have

$$\int_{S_p}|\boldsymbol{c}\cdot\boldsymbol{\nu}_p|d\mathcal{H}^{n-1}\leq\liminf_{k\to\infty}\int_{S_{p_k}}|\boldsymbol{c}\cdot\boldsymbol{\nu}_{p_k}|d\mathcal{H}^{n-1}.$$

We define $\varphi:\Omega\times\mathbb{R}^n\to\mathbb{R}$ as

$$\varphi(x,\mathbf{v})=|\mathbf{c}(x)\cdot\mathbf{v}|.$$

As $\lambda > n$, then we have $W^{1,\lambda}(\Omega) \subset \mathcal{C}^0(\Omega)$, so $\mathbf{c}$ is continuous. We deduce that $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is continuous as well.

According to theorem 3.7, we have $Dp_k \ll \mathcal{H}^{n-1} \llcorner S_{p_k}$, $Dp \ll \mathcal{H}^{n-1} \llcorner S_p$ and

$$\frac{\mathrm{d}(Dp_k)}{\mathrm{d}(\mathcal{H}^{n-1} \llcorner S_{p_k})} = \boldsymbol{\nu}_{p_k} \mathbf{1}_{S_{p_k}}, \qquad \frac{\mathrm{d}(Dp)}{\mathrm{d}(\mathcal{H}^{n-1} \llcorner S_p)} = \boldsymbol{\nu}_p \mathbf{1}_{S_p}.$$

Moreover, φ is sublinear with respect to $\mathbf{v}$. According to proposition 3.1, we can conclude the proof of *Claim 1*.

Claim 2: Let $(p_k, \boldsymbol{c}_k)_{k\in\mathbb{N}} \in \mathcal{Y}$ such as $(p_k)_{k\in\mathbb{N}}$ intermediately converges to $p \in BV_p(\Omega)$ and $(\boldsymbol{c}_k)_{k\in\mathbb{N}}$ weakly converges to $\boldsymbol{c} \in W_u^{1,\lambda}(\Omega;\mathbb{R}^n)$ in $W^{1,\lambda}(\Omega;\mathbb{R}^n)$. Then, we have

$$\int_{S_p} |\boldsymbol{c} \cdot \boldsymbol{\nu}_p| d\mathcal{H}^{n-1} \le \liminf_{k\to\infty} \int_{S_{p_k}} |\boldsymbol{c}_k \cdot \boldsymbol{\nu}_{p_k}| d\mathcal{H}^{n-1}.$$

Triangular inequality gives

$$\begin{aligned}
\left| \int_{S_{p_k}} |\mathbf{c}_k \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1} - \int_{S_{p_k}} |\mathbf{c} \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1} \right| &\le \int_{S_{p_k}} |(\mathbf{c}_k - \mathbf{c}) \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1}, \\
&\le \int_{S_{p_k}} |\mathbf{c}_k - \mathbf{c}| \mathrm{d}\mathcal{H}^{n-1}, \\
&\le \|\mathbf{c}_k - \mathbf{c}\|_{L^\infty} \mathcal{H}^{n-1}(S_{p_k}),
\end{aligned}$$

As $(p_k)_k$ intermediately converges to p in $\mathrm{BV}(\Omega)$ then $(\mathcal{H}^{n-1}(S_{p_k}))_{k\in\mathbb{N}}$ is a bounded sequence. As $\lambda > n$ then the inclusion $W^{1,\lambda}(\Omega) \subset \mathcal{C}^0(\Omega)$ is compact. As $(\mathbf{c}_k)_k$ weakly converges to $\mathbf{c}$ in $W^{1,\lambda}(\Omega)$ then $(\mathbf{c}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{c}$ in $L^\infty(\Omega)$. This two facts implie that

$$\|\mathbf{c}_k - \mathbf{c}\|_{L^\infty} \mathcal{H}^{n-1}(S_{p_k}) \to 0,$$

so we have

$$\int_{S_{p_k}} |\mathbf{c}_k \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1} - \int_{S_{p_k}} |\mathbf{c} \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1} \to 0.$$

Then, in the following expression:

$$\int_{S_{p_k}} |\mathbf{c}_k \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1} = \left(\int_{S_{p_k}} |\mathbf{c}_k \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1} - \int_{S_{p_k}} |\mathbf{c} \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1} \right) + \int_{S_{p_k}} |\mathbf{c} \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1}$$

the term between parentheses converges to 0. According to *Claim 1*, we can conclude that

$$\liminf_{k\to\infty} \int_{S_{p_k}} |\mathbf{c}_k \cdot \boldsymbol{\nu}_{p_k}| \mathrm{d}\mathcal{H}^{n-1} \ge \int_{S_p} |\mathbf{c} \cdot \boldsymbol{\nu}_p| \mathrm{d}\mathcal{H}^{n-1}.$$

$\square$

We can now prove the existence of solutions for problem $(\mathcal{P})$ (4.1).

Theorem 4.3. *Let $\lambda > n$, the problem $(\mathcal{P})$ admits at least one solution.*

Proof. Let $(p_k, \mathbf{c}_k)_{k\in\mathbb{N}}$ be a minimizing sequence of E. According to theorem 4.1, there exists a subsequence, still denoted $(p_k, \mathbf{c}_k)_{k\in\mathbb{N}}$, and $(p, \mathbf{c}) \in \mathcal{Y}$ such that $(p_k)_{k\in\mathbb{N}}$ converges to p in $L^1(\Omega)$ and $(\mathbf{c}_k)_{k\in\mathbb{N}}$ weakly converges to $\mathbf{c}$ in $W^{1,\lambda}(\Omega;\mathbb{R}^n)$. According to theorem 4.2 we have

$$E(p, \mathbf{c}) \leq \liminf_{k\to\infty} E(p_k, \mathbf{c}_k).$$

As $(p_k, \mathbf{c}_k)_{k\in\mathbb{N}}$ is a minimizing sequence of E, can conclude that $(p, \mathbf{c})$ is a solution of $(\mathcal{P})$. $\square$

4.2 Approximation process

In this section we give the main result: we introduce an approximated problem and prove a Γ-convergence result.

4.2.1 Motivations

Let us give an outlook of the approximation process and its proof. In [9], Modica and Mortola consider two functionals F^M_ε and F^M defined by

$$F^M_\varepsilon(p) = \int_\Omega 9\varepsilon|\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon},$$

$$F^M(p) = \mathcal{H}^{n-1}(S_p).$$

They prove the Γ-convergence of $(F^M_\varepsilon)_{\varepsilon>0}$ to F^M for ε converging to 0^+. In our case, we have to introduce the action of $\mathbf{c}$ on S_p. So, we consider the functional

$$F_\varepsilon(p, \mathbf{c}) = \int_\Omega 9\varepsilon\left[|\nabla p|^2 + (\mathbf{c}\cdot\nabla p)^2\right] + \frac{p^2(1-p)^2}{\varepsilon},$$

and prove that it Γ-converges to

$$F(p, \mathbf{c}) = \int_{S_p} \sqrt{1 + (\mathbf{c}\cdot\boldsymbol{\nu}_p)^2}\ \mathrm{d}\mathcal{H}^{n-1}.$$

The result is not a direct consequence of [9]. Indeed, the functional F_ε is not isotropic and the proof has to be adapted. Moreover, this expression is not exactly the same as the one we introduced in (2.7), in this case we had

$$\mathcal{H}^{n-1}(S_p) + \int_{S_p} |\mathbf{c}\cdot\boldsymbol{\nu}_p|\ \mathrm{d}\mathcal{H}^{n-1}.$$

According to the following equivalence

$$\forall (a,b) \in (\mathbb{R}^+)^2, \qquad \sqrt{a^2+b^2} \leq a + b \leq \sqrt{2}\sqrt{a^2+b^2},$$

the expression of E given by

$$E(p, \mathbf{c}) = \int_\Omega (p-g)^2 + \int_{S_p} \sqrt{\beta^2 + \mu^2(\mathbf{c}\cdot\boldsymbol{\nu}_p)^2}\ \mathrm{d}\mathcal{H}^{n-1} + \gamma\int_\Omega \|D\mathbf{c}\|^\lambda.$$

is equivalent to (2.7) from the minimizing point of view.

4.2.2 The main theorem

Let F, F_ε and G be the functions defined on $\mathcal{X}$ and with values in $\mathbb{R}\cup\{+\infty\}$ as

$$\begin{array}{rcl} G(p,\mathbf{c}) & = & \displaystyle\int_\Omega (p-g)^2 + \int_\Omega \|D\mathbf{c}\|^\lambda \text{ if } (p,\mathbf{c})\in\mathcal{X}, \\ F(p,\mathbf{c}) & = & \left\{ \begin{array}{l} \int_{S_p} \sqrt{1+(\mathbf{c}\cdot\boldsymbol{\nu}_p)^2}\mathrm{d}\mathcal{H}^{n-1} \text{ if } (p,\mathbf{c})\in\mathcal{Y}, \\ +\infty \text{ otherwise,} \end{array}\right. \\ F_\varepsilon(p,\mathbf{c}) & = & \left\{ \begin{array}{l} \int_\Omega 9\varepsilon\left[|\nabla p|^2+(\mathbf{c}\cdot\nabla p)^2\right] + \frac{p^2(1-p)^2}{\varepsilon} \text{ if } (p,\mathbf{c})\in\mathcal{Z}, \\ +\infty \text{ otherwise,} \end{array}\right. \\ E & = & G+F, \\ E_\varepsilon & = & G+F_\varepsilon. \end{array}$$

The main result of this work is the following

Theorem 4.4. *Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence converging to 0^+. Then, the sequence $(E_{\varepsilon_k})_{k\in\mathbb{N}}$ Γ-converges to E in $\mathcal{X}$ for the topology introduced in 3.4.*

This results consists in proving two inequalities (3.2) and (3.3). The first inequality consists in the application of the method introduced in [10], while the second is specific to this problem.

4.3 The inequality for the *lower* Γ-limit (3.2)

We now prove the first inequality (3.2). For any $(p,\mathbf{c})\in\mathcal{X}$, we denote

$$E_-(p,\mathbf{c}) = \inf\left\{\liminf_{k\to\infty} E_{\varepsilon_k}(p_k,\mathbf{c}_k)\colon\ ((p_k,\mathbf{c}_k))_{k\in\mathbb{N}}\subset\mathcal{Z},\quad (p_k,\mathbf{c}_k)\xrightarrow{\mathcal{T}}(p,\mathbf{c})\right\}$$

Let $(p,\mathbf{c})\in\mathcal{X}$, it suffices to prove that $E_-(p,\mathbf{c})\geq E(p,\mathbf{c})$. Without loss of generality, several assumptions can be made.

1. $E_-(p,\mathbf{c})<+\infty$, otherwise the result is ensured.

2. $(p,\mathbf{c})\in\mathcal{Y}$. Indeed, according to the first assumption, there exists $(p_k,\mathbf{c}_k)_{k\in\mathbb{N}}\subset\mathcal{Z}$ such that $(E_{\varepsilon_k}(p_k,\mathbf{c}_k))_{k\in\mathbb{N}}$ is bounded, so $(E_{\varepsilon_k}(p_k,\mathbf{c}_k))_{k\in\mathbb{N}}$ is bounded too. According to theorem (4.1), we can deduce that its limit belongs to $\mathcal{Y}$.

3. There exists a sequence $((p_k,\mathbf{c}_k))_{k\in\mathbb{N}}\subset\mathcal{Y}$ such that $((p_k,\mathbf{c}_k))_{k\in\mathbb{N}}$ converges to $(p,\mathbf{c})$ and $E_{\varepsilon_k}((p_k,\mathbf{c}_k),\Omega)$ converges to $E_-(p,\mathbf{c})$. Indeed, we can construct such a sequence by a diagonal extraction.

4. At last, $(p_k)_{k\in\mathbb{N}}\subset\mathcal{C}^\infty\cap W^{1,2}(\Omega)$. Indeed, according to Meyers-Serrin theorem, the space $\mathcal{C}^\infty\cap W^{1,2}(\Omega)$ is dense in $W^{1,2}(\Omega)$. Moreover, there exists C_ε such as for any $(p,q)\in\left(W^{1,2}(\Omega)\right)^2$, we have

   $$|E_\varepsilon(p,\mathbf{c})-E_\varepsilon(q,\mathbf{c})|\leq C_\varepsilon\left(|\|\nabla p\|_{L^2}^2-\|\nabla q\|_{L^2}^2|+|\langle\nabla p,\mathbf{c}\rangle_{L^2}-\langle\nabla q,\mathbf{c}\rangle_{L^2}|\right).$$

   So, by a diagonal extraction we may construct $(p_k)_{k\in\mathbb{N}}$.

First Step: we have

$$\varepsilon_k \left(\int_\Omega (\mathbf{c}_k \cdot \nabla p_k)^2 - \int_\Omega (\mathbf{c} \cdot \nabla p_k)^2 \right) \to 0.$$

As

$$|(\mathbf{c}_k \cdot \nabla p_k)^2 - (\mathbf{c} \cdot \nabla p_k)^2| \le 2\|\mathbf{c}_k - \mathbf{c}\|^2_{L^\infty} |\nabla p_k|^2,$$

then we have

$$\varepsilon_k \left| \int_\Omega (\mathbf{c}_k \cdot \nabla p_k)^2 - \int_\Omega (\mathbf{c} \cdot \nabla p_k)^2 \right| \le 2\|\mathbf{c}_k - \mathbf{c}\|^2_{L^\infty} \varepsilon_k \int_\Omega |\nabla p_k|^2.$$

As $\varepsilon_k \int_\Omega |\nabla p_k|^2 \le E_{\varepsilon_k}(p_k, \mathbf{c}_k)$, then the term $\varepsilon_k \int_\Omega |\nabla p_k|^2$ is uniformely bounded with respect to k. Moreover, $(\mathbf{c}_k)_{k\in\mathbb{N}}$ weakly converges to $\mathbf{c}$ and the inclusion $W^{1,\lambda} \subset L^\infty$ is compact. It yields that $(\mathbf{c}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{c}$ in L^∞. It concludes the first step of the proof.

Second Step: we have

$$\liminf_{k\to\infty} F_{\varepsilon_k}(p_k, \mathbf{c}) \ge F(p, \mathbf{c}).$$

For any $k \ge 0$, the inequality $A^2 + B^2 \ge 2AB$ gives

$$\int_\Omega \underbrace{9\varepsilon_k \left[|\nabla p_k|^2 + (\mathbf{c} \cdot \nabla p_k)^2\right]}_{A^2} + \underbrace{\frac{p_k^2(1-p_k)^2}{\varepsilon_k}}_{B^2} \ge \int_\Omega \underbrace{6p_k(1-p_k)\sqrt{|\nabla p_k|^2 + (\mathbf{c} \cdot \nabla p_k)^2}}_{2AB}.$$

Let $\Phi : \Omega \times [0;1] \times \mathbb{R}^n \to \mathbb{R}^+$ be the function

$$\Phi(x, s, \mathbf{v}) = 6s(1-s)\sqrt{|\mathbf{v}|^2 + (\mathbf{c}(x) \cdot \mathbf{v})^2}.$$

This function is sublinear in $\mathbf{v}$. We denote $S_t^k = \{x \in \Omega\colon p_k(x) < t\}$ Using the formula 3.2, we can write

$$\int_\Omega 6p_k(1-p_k)\sqrt{|\nabla p_k|^2 + (\mathbf{c} \cdot \nabla p_k)^2} = \int_\mathbb{R} \int_\Omega 6t(1-t)\sqrt{|D\mathbf{1}_{S_t^k}|^2 + (\mathbf{c}(x) \cdot D\mathbf{1}_{S_t^k})^2}.$$

Applying Fatou lemma and noting that $D\mathbf{1}_{S_t}$ vanishes when $t \notin [0;1]$ gives

$$\liminf_{k\to\infty} F_{\varepsilon_k}(p_k, \mathbf{c}_k) \ge \int_0^1 6t(1-t) \liminf_{k\to\infty} \int_\Omega \sqrt{|D\mathbf{1}_{S_t^k}|^2 + (\mathbf{c}(x) \cdot D\mathbf{1}_{S_t^k})^2}.$$

So, the sequence $\int_\Omega |D\mathbf{1}_{S_t^k}|$ is bounded; this yields that $(\mathbf{1}_{S_t^k})_{k\in\mathbb{N}}$ is weakly relatively compact in $\mathrm{BV}(\Omega)$. We denote $A = \{x \in \Omega\colon p(x) = 1\}$. We have

$$\begin{aligned}
\int_\Omega |p_k - p| &= \int_\Omega |p_k - \mathbf{1}_A|, \\
&\ge \int_{A\setminus S_t^k} |p_k - \mathbf{1}_A| + \int_{S_t^k \setminus A} |p_k - \mathbf{1}_A|, \\
&\ge (1-t)\int_{A\setminus S_t^k} |\mathbf{1}_{S_t^k} - \mathbf{1}_A| + t\int_{S_t^k\setminus A} |\mathbf{1}_{S_t^k} - \mathbf{1}_A|, \\
&\ge \min(t, 1-t)\int_{A\Delta S_t^k} |\mathbf{1}_{S_t^k} - \mathbf{1}_A|, \\
&\ge \min(t, 1-t)\int_\Omega |\mathbf{1}_{S_t^k} - \mathbf{1}_A|.
\end{aligned}$$

The unique possible limit of $(\mathbf{1}_{S_t^k})_{k\in\mathbb{N}}$ is $\mathbf{1}_A$. Then, thanks to proposition (3.1) we have

$$\liminf_{k\to\infty} \int_\Omega \sqrt{|D\mathbf{1}_{S_t^k}|^2 + (\mathbf{c}(x)\cdot D\mathbf{1}_{S_t^k})^2} \geq \int_{S_p} \sqrt{1+(\mathbf{c}\cdot\nu_p)^2}.$$

Third Step: we have

$$\liminf_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{c}_k) \geq F(p,\mathbf{c}).$$

We have the decomposition

$$F_{\varepsilon_k}(p_k,\mathbf{c}_k) = (F_{\varepsilon_k}(p_k,\mathbf{c}_k) - F_{\varepsilon_k}(p_k,\mathbf{c})) + F_{\varepsilon_k}(p_k,\mathbf{c}).$$

According to the first step, the first term converges to 0 and according to the second step we can conclude that

$$\liminf_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{c}_k) \geq F(p,\mathbf{c}).$$

Conclusion

As $E_{\varepsilon_k} = G + F_{\varepsilon_k}$, it gives

$$\liminf_{k\to\infty} E_{\varepsilon_k}(p_k,\mathbf{c}_k) \geq \liminf_{k\to\infty} G(p_k,\mathbf{c}_k) + \liminf_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{c}_k).$$

According to the third step, we have $\liminf F_{\varepsilon_k}(p_k,\mathbf{c}_k) \geq F(p,\mathbf{c})$. Moreover, as G is lower semicontinuous for the topology of $\mathcal{X}$, we have $\liminf G(p_k,\mathbf{c}_k) \geq G(p,\mathbf{c})$. As $E = F + G$, it finishes the proof of the inequality for the *lower* Γ-limit.

4.4 The inequality for the *higher* Γ-limit (3.3)

We may now prove the second part (3.3) of theorem 4.4.

Proof. We set $(p,\mathbf{c}) \in \mathcal{X}$. As $E_{\varepsilon_k}(p,\mathbf{c}) = +\infty$ if $p \notin \mathrm{BV}(\Omega)$, we can assume that p belongs to $\mathrm{BV}(\Omega)$ and takes its values on $\{0;1\}$, otherwise the result is ensured. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ a sequence which converges to 0^+. We construct a sequence of functions $(p_k,\mathbf{c}_k)_{k\in\mathbb{N}}$ such that

$$\limsup_{k\to\infty} E_{\varepsilon_k}(p_k,\mathbf{c}_k) \leq E(p,\mathbf{c})$$

and $(p_k,\mathbf{c}_k)_{k\in\mathbb{N}}$ converges to $(p,\mathbf{c})$ for the topology $\mathcal{T}$ (see page 10). First, we construct it when S_p is a smooth surface and $\mathbf{c}$ a smooth vector field. Then, we remove these assumptions and we use approximating results to prove it in the general setting.

First step: we assume that S_p is a compact surface of class $\mathcal{C}^2$ and $\mathbf{c} \in \mathcal{C}^\infty(\Omega) \cap W_u^{1,\lambda}(\Omega;\mathbb{R}^n)$.

We set $\mathbf{c}_k = \mathbf{c}$ for any $k \in \mathbb{N}$. Moreover, if $(p_k)_{k\in\mathbb{N}} \subset W_b^{1,2}(\Omega)$ converges a.e. to p, then it converges for the $L^1(\Omega)$ norm and $(\int (p_k - g)^2)_{k\in\mathbb{N}}$ converges to

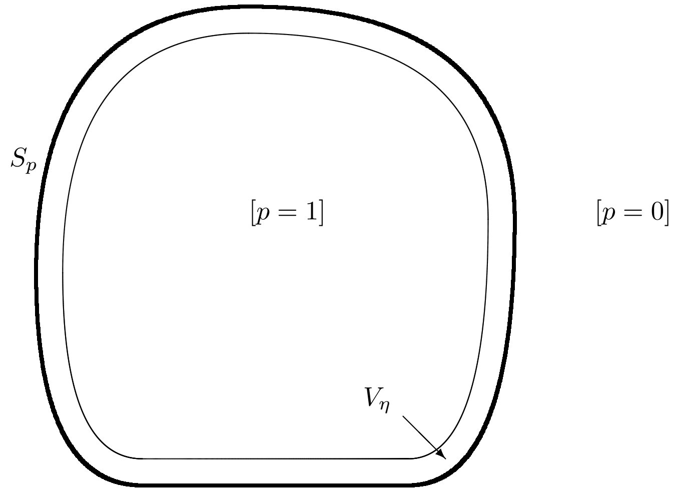

Figure 4.1: Partition of Ω in three domains: $[p=0]$, $[p=1]\setminus V_\eta$ and V_η.

$\int (p-g)^2$. So, it suffices to construct an appropriate sequence $(p_k)_{k\in\mathbb{N}}$ which converges a.e. to p and such that $\limsup F_{\varepsilon_k}(p_k,\mathbf{c}) \le F(p,\mathbf{c})$.

For $\eta > 0$, let V_η be the set $V_\eta = \{x \in [p=1] \colon 0 < \operatorname{dist}(x, S_p) < \eta\}$ (see figure 4.1).

Outside V_η, we define all the function p_k as:

$$\forall x \in [p=0], \qquad p_k(x) = 0,$$
$$\forall x \in [p=1]\setminus V_\eta, \qquad p_k(x) = 1.$$

The construction of p_k inside V_η will be precised. As we assume that S_p is a compact and $\mathcal{C}^2$-surface, there exists $\eta_0 > 0$ and a $\mathcal{C}^1$-diffeomorphism $\phi : V_{\eta_0} \to S_p\times]0;\eta_0[$ (see [9]), caracterized by

$$\forall (\xi,t) \in S_p\times]0;\eta_0[, \qquad \phi(\xi + t\boldsymbol{\nu}_p(\xi)) = (\xi,t).$$

We denote Σ_ξ the slice

$$\Sigma_\xi = \{\xi + t\boldsymbol{\nu}_p(\xi) \colon t \in [0;\eta_0]\},$$

We shall construct p_k slice by slice (see figure 4.2). Indeed, $\phi : V_{\eta_0} \to S_p\times]0;\eta_0[$ is a diffeomorphism, so up to a condition of regularity that we will prove in the following, it provides a complete construction of p_k. We denote by $\chi_{k,\xi} : [0;\eta_0] \to \mathbb{R}$ the restriction of p_k to Σ_ξ. We introduce K defined on $S_p\times[0;\eta_0]$ as

$$\forall (\xi,t) \in S_p\times[0;\eta_0], \qquad K(\xi,t) = \sqrt{1 + (\mathbf{c}(\xi + t\boldsymbol{\nu}_p(\xi)) \cdot \boldsymbol{\nu}_p(\xi))^2}.$$

and $\chi_{k,\xi}$ as the solution of the following differential equation

$$\left\{\begin{array}{rcl} \chi_{k,\xi}(0) &=& 0, \\ 3\sqrt{\varepsilon_k}K(\xi,t)\chi'_{k,\xi}(t) &=& \sqrt{\frac{1}{\varepsilon_k|\ln(\varepsilon_k)|} + \frac{(\chi_{k,\xi}(t))^2(1-\chi_{k,\xi}(t))^2}{\varepsilon_k}} \qquad \text{for } t \ge 0. \end{array}\right.$$

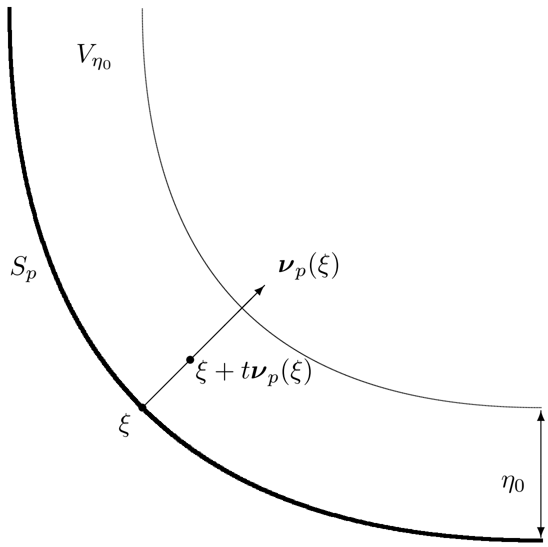

Figure 4.2: Slicing parametrization of V_{η_0}.

For $t \geq 0$, we have $\chi'_{k,\xi}(t) \geq \frac{1}{3K(\xi,t)\varepsilon_k\sqrt{|\ln(\varepsilon_k)|}}$. As $\mathbf{c}$ is an unitary vector field, we have $K(\xi,t) \leq \sqrt{2}$. So, there exists a unique $\eta_{k,\xi} > 0$ such that $\chi_{k,\xi}(\eta_{k,\xi}) = 1$ and

$$\sup_{\xi \in S_p} \eta_{k,\xi} \leq 3\sqrt{2}\varepsilon_k\sqrt{|\ln(\varepsilon_k)|}. \tag{4.2}$$

As $\varepsilon_k\sqrt{|\ln(\varepsilon_k)|}$ converges to 0, then we can assume that $\eta_{k,\xi} < \eta_0$ for any k and ξ. Thus, we change the definition of $\chi_{k,\xi}$ as the solution of the following equation

$$\left\{\begin{array}{rcll} \chi_{k,\xi}(0) & = & 0, & \\ 3\sqrt{\varepsilon_k}K(\xi,t)\chi'_{k,\xi}(t) & = & \sqrt{\frac{1}{\varepsilon_k|\ln(\varepsilon_k)|} + \frac{(\chi_{k,\xi}(t))^2(1-\chi_{k,\xi}(t))^2}{\varepsilon_k}} & \text{for } t \in]0;\eta_{k,\xi}[, \\ \chi_{k,\xi}(t) & = & 1 & \text{for } t \in [\eta_{k,\xi};\eta_0[. \end{array}\right. \tag{4.3}$$

We denote $\eta_k = \sup\{\eta_{k,\xi} : \xi \in S_p\}$ and we define p_k as

$$\begin{array}{rl} \forall x \in [p=0], & p_k(x) = 0, \\ \forall(\xi,t) \in S_p \times]0;\eta_0[, & p_k(\xi + t\boldsymbol{\nu}_p(\xi)) = \chi_{k,\xi}(t), \\ \forall x \in [p=1] \setminus V_{\eta_0}, & p_k(x) = 1. \end{array}$$

As $\eta_k \to 0$, then $p_k \to p$ almost everywhere.

In the sequel we take $n = 3$ but we can generalize it for any $n \geq 2$. Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two $\mathcal{C}^1$-function defined on S_p (see figure 4.3) such that for all $\xi \in S_p$ we have

- $(\mathbf{t}_1(\xi), \mathbf{t}_2(\xi), \nu_p(\xi))$ is an orthonormal basis of $\mathbb{R}^3$,
- $(\mathbf{t}_1(\xi), \mathbf{t}_2(\xi))$ is a basis of the tangent plane of S_p at ξ.

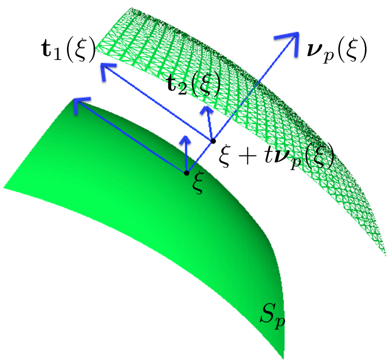

Figure 4.3: Moving basis $(\mathbf{t}_1(\xi), \mathbf{t}_2(\xi), \boldsymbol{\nu}_p(\xi))$ at $\xi + t\nu_p(\xi)$.

Let be $\xi \in S_p$. We can decompose

$$\left\{ \begin{array}{rcl} |\nabla p_k|^2 &=& (\nabla p_k \cdot \boldsymbol{\nu}_p)^2 + (\nabla p_k \cdot \mathbf{t}_1)^2 + (\nabla p_k \cdot \mathbf{t}_2)^2, \\ \mathbf{c} \cdot \nabla p_k &=& (\mathbf{c} \cdot \boldsymbol{\nu}_p)(\nabla p_k \cdot \boldsymbol{\nu}_p) + (\mathbf{c} \cdot \mathbf{t}_1)(\nabla p_k \cdot \mathbf{t}_1) + (\mathbf{c} \cdot \mathbf{t}_2)(\nabla p_k \cdot \mathbf{t}_2), \end{array} \right.$$

We have the following inequality

$$\forall (a,b,c,\varepsilon) \in \mathbb{R}^3 \times \mathbb{R}^+, \qquad (a+b+c)^2 \leq (1+2\varepsilon)a^2 + \left(2 + \frac{1}{\varepsilon}\right) b^2 + \left(2 + \frac{1}{\varepsilon}\right) c^2. \tag{4.4}$$

Indeed,

$$\begin{array}{rcl} (a+b+c)^2 &=& a^2 + b^2 + c^2 + 2ab + 2ac + 2bc, \\ &\leq& a^2 + b^2 + c^2 + \left(\varepsilon a^2 + \dfrac{b^2}{\varepsilon}\right) + \left(\varepsilon a^2 + \dfrac{c^2}{\varepsilon}\right) + \left(b^2 + c^2\right), \\ &\leq& (1+2\varepsilon)a^2 + \left(2 + \dfrac{1}{\varepsilon}\right) b^2 + \left(2 + \dfrac{1}{\varepsilon}\right) c^2. \end{array}$$

If we apply (4.4) with $a = (\mathbf{c} \cdot \boldsymbol{\nu}_p)(\nabla p_k \cdot \boldsymbol{\nu}_p)$, $b = (\mathbf{c} \cdot \mathbf{t}_1)(\nabla p_k \cdot \mathbf{t}_1)$ and $c = (\mathbf{c} \cdot \mathbf{t}_2)(\nabla p_k \cdot \mathbf{t}_2)$ in (4.2.2) we get

$$F_{\varepsilon_k}(p_k, \mathbf{c}) \leq (1 + 2\varepsilon_k)(\star)_{k,0} + \left(2 + \frac{1}{\varepsilon_k}\right)(\star)_{k,1} + \left(2 + \frac{1}{\varepsilon_k}\right)(\star)_{k,2}, \tag{4.5}$$

where we have set

$$(\star)_{k,0} = \iiint_{V_{\eta_k}} 9\varepsilon_k (\nabla p_k \cdot \boldsymbol{\nu}_p)^2 \left[1 + (\mathbf{c} \cdot \boldsymbol{\nu}_p)^2\right] + \frac{p_k^2(1-p_k)^2}{\varepsilon_k}, \tag{4.6}$$

$$(\star)_{k,1} = \iiint_{V_{\eta_k}} 9\varepsilon_k (\nabla p_k \cdot \mathbf{t}_1)^2 \left[1 + (\mathbf{c} \cdot \mathbf{t}_1)^2\right], \tag{4.7}$$

$$(\star)_{k,2} = \iiint_{V_{\eta_k}} 9\varepsilon_k (\nabla p_k \cdot \mathbf{t}_2)^2 \left[1 + (\mathbf{c} \cdot \mathbf{t}_2)^2\right]. \tag{4.8}$$

It is sufficient to prove the two following assertions

1. $$\limsup_{k\to\infty}(\star)_{k,0} \le \iint_{S_p} \sqrt{1+(\mathbf{c}\cdot\boldsymbol{\nu}_p)^2},$$

2. $$\lim_{k\to\infty}\frac{(\star)_{k,1}}{\varepsilon_k}=0, \qquad \lim_{k\to\infty}\frac{(\star)_{k,2}}{\varepsilon_k}=0.$$

Indeed, passing through the limit in (4.5) gives

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{c}) \le \iint_{S_p} \sqrt{1+(\mathbf{c}\cdot\boldsymbol{\nu}_p)^2}.$$

Claim 1: We have the following inequality

$$\limsup_{k\to\infty}(\star)_{k,0} \le \iint_{S_p} \sqrt{1+(\mathbf{c}\cdot\boldsymbol{\nu}_p)^2}.$$

Since

$$\begin{aligned} \frac{\partial p_k}{\partial \boldsymbol{\nu}_p(\xi)}(\xi+t\boldsymbol{\nu}_p(\xi)) &= \lim_{s\to 0}\frac{p_k(\xi+(s+t)\boldsymbol{\nu}_p(\xi))-p_k(\xi+t\boldsymbol{\nu}_p(\xi))}{s}, \\ &= \lim_{s\to 0}\frac{\chi_{k,\xi}(s+t)-\chi_{k,\xi}(t)}{s}, \end{aligned}$$

then, for any $(\xi,t)\in S_p\times]0;\eta_k[$, we have

$$\frac{\partial p_k}{\partial \boldsymbol{\nu}_p(\xi)}(\xi+t\boldsymbol{\nu}_p(\xi)) = \chi'_{k,\xi}(t).$$

This yields $\nabla p_k\cdot\boldsymbol{\nu}_p=\chi'_{k,\xi}$. According to the assumptions of regularity of S_p, as in [9], we may introduce the following change of variable:

$$\iint_{S_p}\int_0^{\eta_0}\frac{\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi)}{\prod_{i=1}^2(1-\kappa_i(\xi)t)} = \iiint_{V_{\eta_0}} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3, \tag{4.9}$$

where $\kappa_1(\xi),\kappa_2(\xi)$ are the principal curvatures of S_p at ξ. As S_p is a $\mathcal{C}^2$ surface, then κ_1 and κ_2 are continuous on S_p. We denote

$$\Pi(\xi,t)=\frac{1}{\prod_{i=1}^2(1-\kappa_i(\xi)t)}.$$

This yields

$$\begin{aligned} (\star)_{k,0} &= \iint_{S_p}\int_0^{\eta_k}\left(9\varepsilon_k(\nabla p_k\cdot\boldsymbol{\nu}_p)^2\left[1+(\mathbf{c}\cdot\boldsymbol{\nu}_p)^2\right]+\frac{p_k^2(1-p_k)^2}{\varepsilon_k}\right)\Pi\,\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi), \\ &= \iint_{S_p}\int_0^{\eta_k}\left(9\varepsilon_k(\chi'_{k,\xi})^2K^2+\frac{p_k^2(1-p_k)^2}{\varepsilon_k}\right)\Pi\,\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi). \end{aligned}$$

In these integrals we remove the dependance variables for the sake of simplicity:

$$\boldsymbol{\nu}_p=\boldsymbol{\nu}_p(\xi),\quad p_k=p_k(\xi+t\boldsymbol{\nu}_p(\xi)),\quad \mathbf{c}=\mathbf{c}(\xi+t\boldsymbol{\nu}_p(\xi)),\quad \Pi=\Pi(\xi,t),$$

$$\chi_{k,\xi} = \chi_{k,\xi}(t), \quad K = K(\xi, t)$$

and we set

$$A = 3\sqrt{\varepsilon_k} K \chi'_{k,\xi}, \qquad B = \frac{\chi_{k,\xi}(1-\chi_{k,\xi})}{\sqrt{\varepsilon_k}}.$$

With the construction of $\chi_{k,\xi}$ in (4.3) we get $A^2 = \frac{1}{\varepsilon_k|\ln(\epsilon_k)|} + B^2$ so that $0 \le B \le A$ on $[0;\eta_k]$ and

$$A^2 + B^2 \le 2AB + \frac{1}{\varepsilon_k|\ln(\epsilon_k)|}.$$

This yields

$$\begin{aligned}(\star)_{k,0} \quad \le \quad & \underbrace{\iint_{S_p}\int_0^{\eta_k} 6K\chi'_{k,\xi}\chi_{k,\xi}(1-\chi_{k,\xi})\Pi \,\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi)}_{(\star)_{k,3}} \\ & + \underbrace{\iint_{S_p}\int_0^{\eta_k} \frac{1}{\varepsilon_k|\ln(\epsilon_k)|}\Pi \,\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi)}_{(\star)_{k,4}}.\end{aligned}$$

The functions K and Π are uniformly bounded with respect to k in $S_p \times]0;\eta_0[$. We denote M their upper bound. We have the following inequalities

$$\begin{aligned}(\star)_{k,4} \quad & \le \quad M\iint_S\int_0^{\eta_k}\frac{1}{\varepsilon_k|\ln(\epsilon_k)|}\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi), \\ & \le \quad M\mathcal{H}^2(S_p)\frac{\eta_k}{\varepsilon_k|\ln(\epsilon_k)|}.\end{aligned}$$

According to (4.2), we have $\eta_k \le 3\sqrt{2}\varepsilon_k\sqrt{|\ln(\varepsilon_k)|}$. This yields

$$(\star)_{k,4} \quad \le \frac{3\sqrt{2}M\mathcal{H}^2(S_p)}{\sqrt{|\ln(\epsilon_k)|}}.$$

and we have $\lim\limits_{k\to\infty}(\star)_{k,4} = 0$. We denote

$$L(\xi,t) = K(\xi,t)\Pi(\xi,t).$$

We have the following decomposition

$$\begin{aligned}(\star)_{k,3} \quad = \quad & \underbrace{\iint_{S_p}\int_0^{\eta_k} 6(L(\xi,t)-L(\xi,0))\chi'_{k,\xi}(t)\chi_{k,\xi}(t)(1-\chi_{k,\xi}(t))\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi)}_{(\star)_{k,5}} \\ & + \underbrace{\iint_{S_p}\int_0^{\eta_k} 6L(\xi,0)\chi'_{k,\xi}(t)\chi_{k,\xi}(t)(1-\chi_{k,\xi}(t))\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi)}_{(\star)_{k,6}},\end{aligned}$$

and the following bound

$$(\star)_{k,5} \quad \le \quad \left(\sup_{(\xi,t)\in S_p\times]0;\eta_k[}(L(\xi,t)-L(\xi,0))\right)\iint_{S_p}\int_0^{\eta_k} 6\chi'_{k,\xi}(t)\chi_{k,\xi}(t)(1-\chi_{k,\xi}(t))\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi).$$

Since $\chi_{k,\xi} \in W^{1,2}(]0;\eta_k[)$, we may use the change of variable $s = \chi_{k,\xi}(t)$ to obtain

$$\begin{aligned}(\star)_{k,5} &\leq 6\left(\sup_{(\xi,t)\in S_p\times]0;\eta_k[}(L(\xi,t)-L(\xi,0))\right)\iint_{S_p}\int_{\chi_{k,\xi}(0)}^{\chi_{k,\xi}(\eta_k)} s(1-s)\mathrm{d}s\ \mathrm{d}\mathcal{H}^2(\xi),\\ &\leq \left(\sup_{(\xi,t)\in S_p\times]0;\eta_k[}(L(\xi,t)-L(\xi,0))\right)\mathcal{H}^2(S_p).\end{aligned}$$

The surface S_p is compact and smooth and the function L is continuous. Then, the family $(L(\cdot,t))_{t>0}$ uniformly converges to $L(\cdot,0)$ when $t\to 0^+$. We can deduce that $\lim\limits_{k\to\infty}(\star)_{k,5}=0$. Using the same change of variable $s=\chi_{k,\xi}(t)$ in $(\star)_{k,6}$ gives

$$\begin{aligned}(\star)_{k,6} &= 6\iint_{S_p} L(\xi,0)\int_{\chi_{k,\xi}(0)}^{\chi_{k,\xi}(\eta_k)} s(1-s)\mathrm{d}s\ \mathrm{d}\mathcal{H}^2(\xi),\\ &= \iint_{S_p}\sqrt{1+(\mathbf{c}(\xi)\cdot\boldsymbol{\nu}_p(\xi))^2}\mathrm{d}\mathcal{H}^2(\xi).\end{aligned}$$

To summarize, we have

$$(\star)_{k,0} = (\star)_{k,4} + (\star)_{k,5} + (\star)_{k,6},$$

and the proof of *Claim 1* is achieved.

$$\limsup_{k\to\infty}(\star)_{k,0} \leq \iint_{S_p}\sqrt{1+(\mathbf{c}(\xi)\cdot\boldsymbol{\nu}_p(\xi))^2}\mathrm{d}\mathcal{H}^2(\xi).$$

Claim 2: We have the following limits

$$\lim_{k\to\infty}\frac{(\star)_{k,1}}{\varepsilon_k}=0,\qquad \lim_{k\to\infty}\frac{(\star)_{k,2}}{\varepsilon_k}=0.$$

We prove the result for $(\star)_{k,1}$, the method for $(\star)_{k,2}$ is the same. As S_p is a $\mathcal{C}^2$ surface, the intersection of the affine plane $P_1 = \xi + \mathrm{Vect}(\mathbf{t}_1(\xi), \nu_p(\xi))$ and S_p at the neighborhood of $\xi \in S_p$ is a $\mathcal{C}^2$-planar curve. Let I be a neighborhood of 0 in $\mathbb{R}$ and $\gamma : I \to S_p$ be a local curvilinear parametrization of this curve such that

$$\begin{cases} \gamma(0)=\xi,\\ \gamma'(0)=\mathbf{t}_1(\xi),\\ \forall t\in I,\quad |\gamma'(t)|=1.\end{cases}$$

As $\nu_p \circ \gamma(s)$ is orthogonal to $\gamma'(s)$ for all $s \in I$ and γ is a planar curve, there exists $\tilde{\kappa}_1 : I \to \mathbb{R}$ such that

$$\frac{\mathrm{d}(\boldsymbol{\nu}_p\circ\gamma)}{\mathrm{d}s}(s) = -\tilde{\kappa}_1(\gamma(s))\mathbf{t}_1(\gamma(s)).$$

As γ is a curve of S_p, $\tilde{\kappa}_1$ is the sectional curvature of S_p in the direction of $\mathbf{t}_1(\gamma(s))$, we have $|\tilde{\kappa}_1| \leq \max(|\kappa_1|, |\kappa_2|)$.

We evaluate

$$\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t) = p_k(\gamma(s)+t\boldsymbol{\nu}_p(\gamma(s))) - p_k(\xi+t\boldsymbol{\nu}_p(\xi))).$$

So, we have the following asymptotic developpement at $s = 0$

$$\gamma(s) + t\boldsymbol{\nu}_p(\gamma(s)) = \xi + t\boldsymbol{\nu}_p(\xi) + s(1 - \tilde{\kappa}_1(\xi)t)\mathbf{t}_1(\xi) + o(s),$$

and we get

$$\lim_{s\to 0} \frac{\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)}{s(1-\tilde{\kappa}_1(\xi)t)} \quad = \quad \nabla p_k(\xi + t\boldsymbol{\nu}_p(\xi)) \cdot \mathbf{t}_1(\xi). \tag{4.10}$$

We calculate the left hand side of (4.10). We recall the equations satisfied by $\chi_{k,\gamma(s)}$ and $\chi_{k,\xi}$ (see figure 4.4)

$$(E_1) : \left\{ \begin{array}{rcll} \chi_{k,\xi}(0) & = & 0, & \\ 3\sqrt{\varepsilon_k}K(\xi,t)\chi'_{k,\xi}(t) & = & \sqrt{\frac{1}{\varepsilon_k|\ln(\varepsilon_k)|} + \frac{(\chi_{k,\xi}(t))^2(1-\chi_{k,\xi}(t))^2}{\varepsilon_k}} & \text{for } t \in]0;\eta_{k,\xi}[, \\ \chi_{k,\xi}(t) & = & 1 & \text{for } t \geq \eta_{k,\xi}. \end{array} \right.$$

$$(E_2) : \left\{ \begin{array}{rcll} \chi_{k,\gamma(s)}(0) & = & 0, & \\ 3\sqrt{\varepsilon_k}K(\gamma(s),t)\chi'_{k,\gamma(s)}(t) & = & \sqrt{\frac{1}{\varepsilon_k|\ln(\varepsilon_k)|} + \frac{\left(\chi_{k,\gamma(s)}(t)\right)^2\left(1-\chi_{k,\gamma(s)}(t)\right)^2}{\varepsilon_k}} & \text{for } t \in]0;\eta_{\gamma(s),k}[, \\ \chi_{k,\gamma(s)}(t) & = & 1 & \text{for } t \geq \eta_{\gamma(s),k}. \end{array} \right.$$

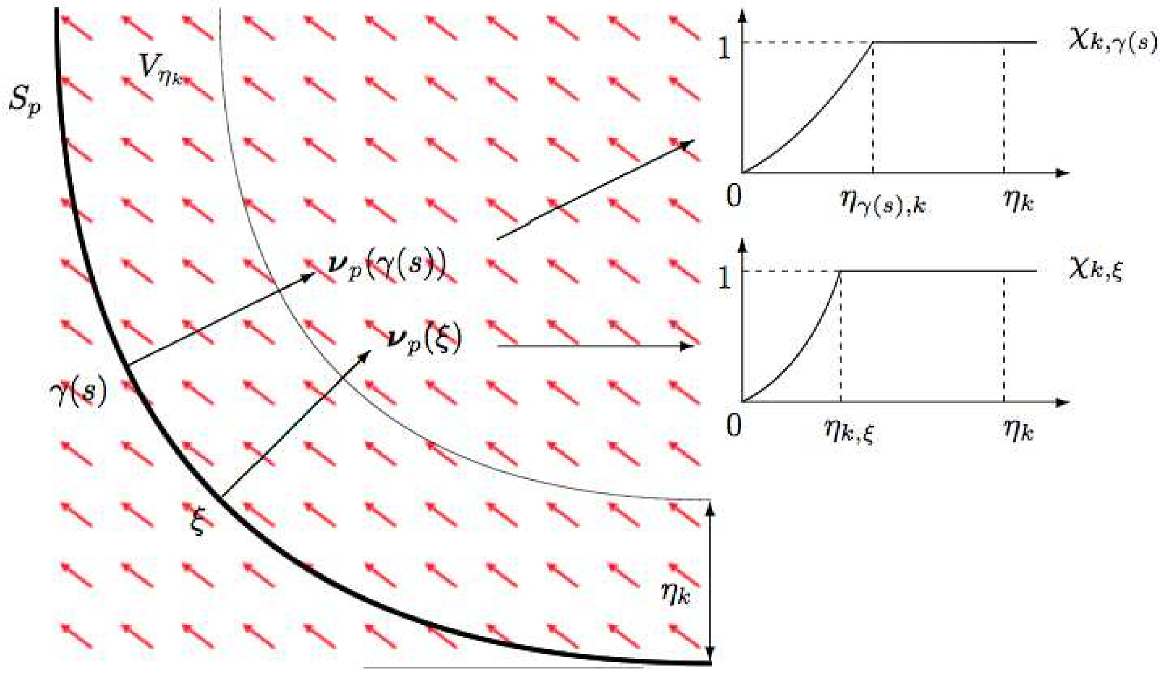

Figure 4.4: Construction of the solution on each slice

We denote $f(x) = \sqrt{\frac{1}{\varepsilon_k|\ln(\varepsilon_k)|} + \frac{x^2(1-x)^2}{\varepsilon_k}}$ and $Y_{k,s}(t) = \chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)$. We calculate $\frac{K(\xi,\cdot)}{K(\gamma(s),\cdot)}(E_2) - (E_1)$ and we denote $\overline{\eta}_{k,s} = \min(\eta_{\gamma(s),k}, \eta_{\xi,k})$. It comes

$$\left\{\begin{array}{rcl} Y_{k,s}(0) & = & 0, \\ 3\sqrt{\varepsilon_k}K(\xi,t)Y'_{k,s}(t) & = & \frac{K(\xi,t)}{K(\gamma(s),t)}f(\chi_{k,\gamma(s)}(t)) - f(\chi_{k,\xi}(t)) \text{ for } t \in]0;\overline{\eta}_{k,s}[. \end{array}\right.$$

As $K(\xi,t) = \sqrt{1+(C(\xi+t\boldsymbol{\nu}_p(\xi))\cdot\boldsymbol{\nu}_p(\xi))^2}$, we get

$$\forall(\xi,t) \in S_p \times]0;\overline{\eta}_{k,s}[, \qquad 1 \leq K(\xi,t) \leq \sqrt{2}. \tag{4.11}$$

As S_p is a $\mathcal{C}^2$-manifold and $C \in \mathcal{C}^\infty \cap W_u^{1,\lambda}(\Omega;\mathbb{R}^n)$, then K is a $\mathcal{C}^1$ function of class $\mathcal{C}^1$ and there exists a constant $\mu > 0$ such that

$$\forall(\xi,\xi',t) \in S_p^2 \times]0;\eta_k[, \qquad |K(\xi,t) - K(\xi',t)| \leq \mu|\xi-\xi'| \tag{4.12}$$

Moreover, the study of f_k gives

$$\forall x \in [0;1], \qquad f_k(x) \leq \sqrt{\frac{1}{\varepsilon_k|\ln(\varepsilon_k)|} + \frac{1}{16\varepsilon_k}} \qquad |f'_k(x)| \leq \frac{1}{\sqrt{\varepsilon_k}}. \tag{4.13}$$

With

$$\begin{array}{rcl} 3\sqrt{\varepsilon_k}K(\xi,t)Y'_{k,s}(t) & = & \dfrac{K(\xi,t)}{K(\gamma(s),t)}f_k(\chi_{k,\gamma(s)}(t)) - f_k(\chi_{k,\xi}(t)), \\ & = & \dfrac{K(\xi,t)}{K(\gamma(s),t)}(f_k(\chi_{k,\gamma(s)}(t)) - f_k(\chi_{k,\xi}(t))) \\ & & +f_k(\chi_{k,\xi}(t))\left(\dfrac{K(\xi,t)-K(\gamma(s),t)}{K(\gamma(s),t)}\right) \end{array}$$

and (4.11), (4.12), (4.13), we get

$$3\sqrt{\varepsilon_k}Y'_{k,s}(t) \leq \frac{\sqrt{2}}{\sqrt{\varepsilon_k}}Y_{k,s}(t) + \mu s\sqrt{\frac{1}{\varepsilon_k|\ln(\varepsilon_k)|} + \frac{1}{16\varepsilon_k}}.$$

Thus, $Y_{k,s}$ is a solution of the following differential inequation

$$\left\{\begin{array}{rcl} Y_{k,s}(0) & = & 0, \\ Y'_{k,s}(t) & \leq & \frac{\sqrt{2}}{3\varepsilon_k}Y_{k,s}(t) + \frac{\mu s}{3\varepsilon_k}\sqrt{\frac{1}{|\ln(\varepsilon_k)|}+\frac{1}{16}} \text{ for } t \in]0;\overline{\eta}_{k,s}[. \end{array}\right. \tag{4.14}$$

So, we have

$$Y_{k,s}(t) \leq \frac{\mu s}{\sqrt{2}}\sqrt{\frac{1}{|\ln(\varepsilon_k)|}+\frac{1}{16}}\left(\exp\left(\frac{\sqrt{2}t}{3\varepsilon_k}\right)-1\right). \tag{4.15}$$

The definition of $Y_{k,s}$ gives

$$\frac{Y_{k,s}(t) - Y_{k,0}(t)}{s} = \frac{\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)}{s},$$

and inequality (4.15) implies

$$\forall t \in]0;\overline{\eta}_{k,s}[, \qquad \frac{\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)}{s} \leq \frac{\mu}{\sqrt{2}}\sqrt{\frac{1}{|\ln(\varepsilon_k)|}+\frac{1}{16}}\left(\exp\left(\frac{\sqrt{2}t}{3\varepsilon_k}\right)-1\right). \tag{4.16}$$

According to the continuous dependance of the solution of the equation (4.14) with respect to the parameter s, then $\overline{\eta}_{k,s}$ converges to $\eta_{k,\xi}$ when s converges to 0. So, the inequality (4.16) remains true in the neighborhood of any point $t\in]0;\eta_{k,\xi}[$. With k, ξ and $t\in]0;\eta_{k,\xi}[$ fixed, we calculate the limit when s converges to 0, and we apply equality (4.10)

$$(1-\tilde{\kappa}_1(\xi)t)\nabla p_k(\xi+t\boldsymbol{\nu}_p(\xi))\cdot\mathbf{t}_1(\xi)\leq\frac{\mu}{\sqrt{2}}\sqrt{\frac{1}{|\ln(\varepsilon_k)|}+\frac{1}{16}}\left(\exp\left(\frac{\sqrt{2}t}{3\varepsilon_k}\right)-1\right).$$

As $\eta_k\to 0$ and $\tilde{\kappa}_1$ is continuous, there exists $r>0$ such that

$$\forall(\xi,t)\in S_p\times]0;\eta_k[,\qquad r<(1-\tilde{\kappa}_1(\xi)t).$$

This gives

$$(\nabla p_k(\xi+t\boldsymbol{\nu}_p(\xi))\cdot\mathbf{t}_1(\xi))^2\leq\frac{\mu^2}{2r^2}\left(\frac{1}{|\ln(\varepsilon_k)|}+\frac{1}{16}\right)\left(\exp\left(\frac{\sqrt{2}t}{3\varepsilon_k}\right)-1\right)^2.\quad(4.17)$$

As $\frac{1}{|\ln(\varepsilon_k)|}\to 0$, there exists $M>0$ such that (4.17) becomes

$$(\nabla p_k(\xi+t\boldsymbol{\nu}_p(\xi))\cdot\mathbf{t}_1(\xi))^2\leq M\exp\left(\frac{2\sqrt{2}t}{3\varepsilon_k}\right)+M.$$

As $\frac{2\sqrt{2}}{3}\leq 1$, we have

$$(\nabla p_k(\xi+t\boldsymbol{\nu}_p(\xi))\cdot\mathbf{t}_1(\xi))^2\leq M\exp\left(\frac{t}{\varepsilon_k}\right)+M.\qquad(4.18)$$

Introducing (4.18) in the definition of $(\star)_{1,k}$ (4.6) gives

$$(\star)_{1,k}\leq\iint_{S_p}\int_0^{\eta_k}9\varepsilon_k\left(M\exp\left(\frac{t}{\varepsilon_k}\right)+M\right)\left[1+(\mathbf{c}\cdot\mathbf{t}_1)^2\right]\Pi(\xi,t)\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi),$$

As $\eta_k\to 0$, the function $\left[1+(\mathbf{c}\cdot\mathbf{t}_1)^2\right]\Pi(\xi,t)$ is bounded and there exists a positive constant, still denoted M, such that

$$(\star)_{1,k}\leq\iint_{S_p}\int_0^{\eta_k}\varepsilon_k\left(M\exp\left(\frac{t}{\varepsilon_k}\right)+M\right)M\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi).$$

Thus, we have

$$(\star)_{1,k}\leq\left(\varepsilon_k^2\left(\exp\left(\frac{\eta_k}{\varepsilon_k}\right)-1\right)+\varepsilon_k\eta_k\right)M^2\mathcal{H}^2(S_p).$$

According to (4.2)

$$\eta_k\leq 3\sqrt{2}\varepsilon_k\sqrt{|\ln(\varepsilon_k)|},$$

so, that

$$\begin{aligned}
(\star)_{1,k}&\leq\left(\varepsilon_k^2\left[\exp\left(3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|}\right)-1\right]+3\sqrt{2}\varepsilon_k^2\sqrt{|\ln(\varepsilon_k)|}\right)M^2\mathcal{H}^2(S_p),\\
\frac{(\star)_{1,k}}{\varepsilon_k}&\leq\left[\varepsilon_k\exp\left(3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|}\right)-\varepsilon_k+3\sqrt{2}\varepsilon_k\sqrt{|\ln(\varepsilon_k)|}\right]M^2\mathcal{H}^2(S_p),\\
\frac{(\star)_{1,k}}{\varepsilon_k}&\leq\left[\exp\left(3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|}+\ln(\varepsilon_k)\right)-\varepsilon_k+3\sqrt{2}\varepsilon_k\sqrt{|\ln(\varepsilon_k)|}\right]M^2\mathcal{H}^2(S_p).
\end{aligned}$$

As $\varepsilon_k \to 0^+$ we have

$$\exp\left(3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|}+\ln(\varepsilon_k)\right) \to 0^+, \qquad \varepsilon_k\sqrt{|\ln(\varepsilon_k)|} \to 0^+.$$

We can conclude that $\frac{(\star)_{1,k}}{\varepsilon_k} \to 0$.

Second step: Assume that $p \in \mathbf{BV}(\Omega)$, p takes its values in $\{0;1\}$ and $\mathbf{c} \in \mathcal{C}^\infty \cap W_u^{1,\lambda}(\Omega;\mathbb{R}^n)$.

In this step, we still we set $\mathbf{c}_k = \mathbf{c}$ for any $k \in \mathbb{N}$. For the same reason than in the previous step, it suffices to construct an appropriate sequence $(p_k)_{k\in\mathbb{N}}$ which converges a.e. to p and such that $\limsup F_{\varepsilon_k}(p_k,\mathbf{c}) \le F(p,\mathbf{c})$.

We denote $A = p^{-1}(\{1\})$. Let us first assume that A and $\Omega \setminus A$ have nonempty interior. We can apply lemma 3.1. So, there exists a sequence $(A_l)_{l\in\mathbb{N}}$ of open bounded subsets of $\mathbb{R}^n$ with smooth boundaries such that

i)
$$\lim_{l\to\infty} \mathcal{L}^n((A_l\cap\Omega)\bigtriangleup A) = 0, \qquad \lim_{l\to\infty} \mathcal{H}^{n-1}(\partial A_l) = \mathcal{H}^{n-1}(\partial A);$$

ii)
$$\mathcal{L}^n(A_l\cap\Omega) = \mathcal{L}^n(A) \qquad \text{for } l \text{ large enough;}$$

iii)
$$\mathcal{H}^{n-1}(\partial A_l\cap\partial\Omega) = 0 \qquad \text{for } l \text{ large enough;}$$

iv)
$$F(p_l,\mathbf{c}) \le F(p,\mathbf{c}) + \frac{1}{l}, \tag{4.19}$$

where $\mathcal{L}^n$ is the Lebesgue measure over Ω and $p_l = \mathbf{1}_{A_l\cap\Omega}$. For (4.19) we use the fact that $\|D\mathbf{1}_{A_l}\|_{\mathcal{M}} \to \|D\mathbf{1}_A\|_{\mathcal{M}}$ and proposition 3.1 ii). With i), ii) and iii), we can say that $(p_l)_{l\in\mathbb{N}}$ is a bounded sequence of $\mathrm{BV}(\Omega)$ which converges to p in $L^1(\Omega)$. According to theorem 3.5, there exists a subsequence, still denoted $(p_l)_{l\in\mathbb{N}}$ which intermediately converges to p in $\mathrm{BV}(\Omega)$. One can apply the result of the first step with $p = p_l$. So, there exists a sequence $(p_{l,k})_{k\in\mathbb{N}}$ which intermediately converges to p_l in $\mathrm{BV}(\Omega)$ such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_{k,l},\mathbf{c}) \le F(p_l,\mathbf{c}). \tag{4.20}$$

With (4.19), (4.20) and a diagonal extraction there exists a sequence $(p_k)_{k\in\mathbb{N}}$ which intermediately converges to p such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{c}) \le F(p,\mathbf{c}).$$

Let us remove the restriction that both A or $\Omega\setminus A$ have non empty interior. First, we notice that if $\mathcal{L}^n(A) = 0$ or $\mathcal{L}^n(A) = \Omega$ the result is obvious by taking for all $l \in \mathbb{N}$, $A_l = \emptyset$ or $A_l = \Omega$. So, we may assume that $0 < \mathcal{L}^n(A) < |\Omega|$. There exists two points x_1, x_2 such that

- $x_1 \in A$ and $\forall r > 0, \quad \mathcal{L}^n(A \cap B(x_1,r)) > 0,$
- $x_2 \in \Omega\setminus A$ and $\forall r > 0, \quad \mathcal{L}^n((\Omega\setminus A) \cap B(x_1,r)) > 0.$

Consider the set $A_{\theta_1,\theta_2} = (A \cup B(x_2,\theta_2)) \setminus B(x_1,\theta_1)$ and the function $\Upsilon(\theta_1,\theta_2) = \mathcal{L}^n(A_{\theta_1,\theta_2})$. As $\Upsilon(0,\theta) > \mathcal{L}^n(A)$ and $\Upsilon(\theta,0) < \mathcal{L}^n(A)$ for any $\theta > 0$, there exists $t \in]0;1[$ depending on θ such that $\Upsilon(t\theta,(1-t)\theta) = \mathcal{L}^n(A)$. By construction, A_θ and $\Omega \setminus A_\theta$ have nonempty interior. The previous result gives the existence of $(p_{\theta,k})_{k\in\mathbb{N}} \subset \mathrm{BV}_p(\Omega)$ which intermediately converges to $p_\theta = \mathbf{1}_{A_\theta}$ in $\mathrm{BV}(\Omega)$ such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_{\theta,k},\mathbf{c}) \leq F(p_\theta,\mathbf{c}). \tag{4.21}$$

Moreover, $\mathcal{L}^n(A \triangle A_\theta)$ tends to 0 as $\theta \to 0^+$, and, using

$$\int_{S_{p_\theta}} \sqrt{1+(\mathbf{c}\cdot\boldsymbol{\nu}_{p_\theta})^2} \leq \int_{S_p} \sqrt{1+(\mathbf{c}\cdot\boldsymbol{\nu}_p)^2} + \mathcal{H}^{n-1}(\partial B(x_1,\theta_1)) \cup \partial B(x_2,\theta_2)),$$

we get

$$\limsup_{\theta\to 0^+} F(p_\theta,\mathbf{c}) \leq F(p,\mathbf{c}).$$

According to (4.21), with a diagonal extraction there exists a sequence $(p_k)_{k\in\mathbb{N}}$ which intermediately converges to p such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{c}) \leq F(p,\mathbf{c}).$$

Last step: Assume that $p \in \mathbf{BV}_p(\Omega)$ and $\mathbf{c} \in W_u^{1,\lambda}(\Omega;\mathbb{R}^n)$.

In this step we do not assume that $(\mathbf{c}_k)_{k\in\mathbb{N}}$ is a constant sequence. As $\mathcal{C}^\infty \cap W^{1,\lambda}(\Omega)$ is dense in $W^{1,\lambda}(\Omega)$, there exists a sequence $(\mathbf{c}_l)_{l\in\mathbb{N}} \subset \mathcal{C}^\infty \cap W^{1,\lambda}(\Omega)$ which converges to $\mathbf{c}$ in $W^{1,\lambda}(\Omega)$. As $\lambda > n$, the inclusion $W^{1,\lambda}(\Omega) \subset \mathcal{C}^0$ is compact so the previous convergence holds for $L^\infty(\Omega)$. We denote $P : \mathbb{R}^n \subset \{0\} \to \mathbb{S}^{n-1}$ the orthogonal projection onto the unit sphere. As $\mathbf{c}$ takes its values in $\mathbb{S}^{n-1}$ and the convergence of $(\mathbf{c}_l)_{l\in\mathbb{N}}$ takes place for L^∞ too, so $(P\circ\mathbf{c}_l)_{l\in\mathbb{N}}$ converges to $\mathbf{c}$ in $W^{1,\lambda}(\Omega)$. We still denote $\mathbf{c}_l$ the function $P\circ\mathbf{c}_l$ and therefore we can assume $(\mathbf{c}_l)_{l\in\mathbb{N}} \subset W_u^{1,\lambda}(\Omega;\mathbb{R}^n)$. As $\mathbf{c}_l \in \mathcal{C}^\infty \cap W_u^{1,\lambda}(\Omega)$, one apply the *Second step* of the proof, it gives

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{c}_l) \leq F(p,\mathbf{c}_l).$$

As in (4.2), we denote $h(x) = \sqrt{1+x^2}$. The function h is 1-Lipschitz, then we have

$$\begin{aligned} |F(p,\mathbf{c}_l) - F(p,\mathbf{c})| &\leq \int_{S_p} |\sqrt{1+(\mathbf{c}_l\cdot\boldsymbol{\nu}_p)^2} - \sqrt{1+(\mathbf{c}\cdot\boldsymbol{\nu}_p)^2}|, \\ &\leq \|\mathbf{c}_l - \mathbf{c}\|_{L^\infty}\mathcal{H}^{n-1}(S_p). \end{aligned}$$

So, we deduce that $(F(p,\mathbf{c}_l))_{l\in\mathbb{N}}$ converges to $F(p,\mathbf{c})$. With a diagonal extraction, we can conclude that there exists $(p_k,\mathbf{c}_k)_{k\in\mathbb{N}} \subset \mathrm{BV}_p(\Omega) \times W_u^{1,\lambda}(\Omega;\mathbb{R}^n)$ which converges for the topology of $\mathcal{X}$ to $(p,\mathbf{c})$ such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{c}_k) \leq F(p,\mathbf{c}).$$

Moreover, as $(p_k)_{k\in\mathbb{N}}$ converges to p almost everywhere, then the dominated convergence implies

$$\lim_{k\to\infty}\int_\Omega (p_k-g)^2 = \int_\Omega (p-g)^2.$$

As $(\mathbf{c}_k)_{k\in\mathbb{N}}$ converges to $\mathbf{c}$, it gives

$$\lim_{k\to\infty}\int_\Omega \|D\mathbf{c}_k\| = \int_\Omega \|D\mathbf{c}\|.$$

We can conclude that

$$\limsup_{k\to\infty} E_{\varepsilon_k}(p_k,\mathbf{c}_k) \le E(p,\mathbf{c}).$$

□

5 Conclusion

We have proved that the approximation process is suitable in the sense of Γ-convergence. We next use the approximated problem for numerical experimentation.The computation of p is done via a classical gradient descent method, while another strategy has to be developed for $\mathbf{c}$: this will be adressed in a future work.

On the other hand, the main hypothesis we did in this paper is the bimodality of histogram: this is quite restrictive for numerics. If this assumption is not ensured the previous model is not valid any longer and has to be modified: we will set a more general formulation that perfoms a similar segmentation without the binary constraint. Roughly speaking, we look for a pair $(f,\mathbf{c})$ where $f:\Omega\to[0;1]$ is a function (not necessarily binary) and $\mathbf{c}:\Omega\to\mathbb{S}^{n-1}$ is still a unitary vector field. The corresponding energy to be minimized is :

$$\int_\Omega (f-g)^2 + \beta\mathcal{H}^{n-1}(S_f) + \mu\int_{S_f} |\mathbf{c}\cdot\boldsymbol{\nu_f}|\,\mathrm{d}\mathcal{H}^{n-1} + \gamma\int_\Omega \|D\mathbf{c}\|^\lambda + \rho\int_{\Omega\setminus S_f} |\nabla f|^2. \tag{5.1}$$

References

[1] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variationnal problems. *Communications on Pure and Applied Mathematics*, XLVII(5):577–685, 1989.

[2] G. Bellettini, E. Paolini, and S. Venturini. Some results on surface measures in calculus of variations. *Annali di Mathematica pura ed applicata*, CLXX:329–359, 1996.

[3] J.M. Morel and S. Solimini. *Variational methods in image segmentation*, volume 14. Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, 1995.

[4] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics, 1992.

[5] H. Attouch, G. Buttazzo, and G. Michaille. *Variational Analysis in Sobolev and BV spaces.* SIAM, 2005.

[6] C. Goffman and J. Serrin. Sublinear functions of measures and variational integrals. *Duke Math. J.*, 1964.

[7] G. Bouchitté and M. Valadier. Multifonctions s.c.i. et régularisée s.c.i. essentielle, fonctions de mesures dans le cas sous-linéaire. *Analyse Non Linéaire (Contributions en l'honneur de J. J. Moreau)*, 1989.

[8] G. Dal Maso. Integral representation on bounded variation spaces of gamma-limits of variational integrals. *Manuscripta Mathematica*, 1980.

[9] L. Modica and S. Mortola. The gradient theory of phase transitions and the minimal interface criterion. *Archive for Rational Mechanics and Analysis*, 98:123–142, 1987.

[10] G. Bouchitté. *Singular Perturbations of Variational Problems Arising from a Two-Phase Transition Model.* Applied Mathematics and Optimization, 1990.