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C^∞ LOCAL SOLUTIONS OF ELLIPTICAL 2−HESSIAN EQUATION IN $\mathbb{R}^3$

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In remembrance of the late professor Rou-Huai Wang
on the occasion of his 90th Birthday

Abstract. In this work, we study the existence of C^∞ local solutions to 2-Hessian equation in $\mathbb{R}^3$. We consider the case that the right hand side function f possibly vanishes, changes the sign, is positively or negatively defined. We also give the convexities of solutions which are related with the annulation or the sign of right-hand side function f. The associated linearized operator are uniformly elliptic.

1. Introduction

We are interested by the following k-Hessian equation

$$S_k[u] = f(y, u, Du) \tag{1.1}$$

on an open domain $\Omega \subset \mathbb{R}^n, 1 \le k \le n, f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Denote $Du = (\partial_1 u, \dots, \partial_n u)$ and D^2u is the Hessian matrix $(\partial_i \partial_j u)_{1\le i,j\le n}$. the Hessian operators $S_k[u]$ is defined as follows:

$$S_k[u] = \sigma_k(\lambda(D^2u)), \quad k = 1, \dots, n, \tag{1.2}$$

where $\lambda(D^2u) = (\lambda_1, \lambda_2, \dots, \lambda_n)$, λ_j is the eigenvalue of the Hessian matrix (D^2u), and

$$\sigma_k(\lambda) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k-th elementary symmetric polynomial. Denoting, for $k, j \in \{1, \cdots, n\}$,

$$\sigma_{k,j} = \frac{\partial \sigma_{k+1}(\lambda)}{\partial \lambda_j} = \sigma_k|_{\lambda_j=0}.$$

We also introduce the Gårding cone Γ_k which is the open symmetric convex cone in $\mathbb{R}^n$, with vertex at the origin, given by

$$\Gamma_k = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \ \sigma_j(\lambda) > 0, \forall\, j = 1, \dots, k\}.$$

When $k = 1$, (1.1) is a semi-linear Poisson equation, and it is Monge-Ampère equation for $k = n$.

We say that a function $u \in C^2$ is k-convex, if

$$\lambda(D^2u) \in \bar{\Gamma}_k,$$

the n-convex function is simply called convex function.

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We say that a function u is a local solution of (1.1) near $y_0 \in \Omega$, if there exists a neighborhood of y_0, $V_{y_0} \subset \Omega$ such that $u \in C^2(V_{y_0})$ satisfies the equation (1.1) on V_{y_0}.

In this work, we study the existence of C^∞-local solution of the following 2-Hessian equation in $\mathbb{R}^3$,

$$S_2[u] = f(y, u, Du), \quad \text{on } \Omega \subset \mathbb{R}^3, \tag{1.3}$$

where we also have

$$S_2[u] = u_{11}u_{22} - u_{12}^2 + u_{22}u_{33} - u_{23}^2 + u_{11}u_{33} - u_{13}^2.$$

We have proved the following results.

Theorem 1.1. *Assume that $f \in C^\infty(\Omega\times\mathbb{R}\times\mathbb{R}^3)$, then for any $Z_0 = (y_0, z_0, p_0) \in \Omega\times\mathbb{R}\times\mathbb{R}^3$, we have that*

(1) *if $f(Z_0) = 0$, then* (1.3) *admits a 1-convex C^∞ local solution which is not convex;*
(2) *if $f \geq 0$ near Z_0, then* (1.3) *admits a 2-convex C^∞ local solution which is not convex. If $f(Z_0) > 0$,* (1.3) *admits a convex C^∞ local solution.*
(3) *if $f(Z_0) < 0$,* (1.3) *admits a 1-convex C^∞ local solution which is not 2-convex.*

Moreover, the equation (1.3) *is uniformly elliptic with respect to the above local solutions.*

For the local solution, Hong and Zuily [5] obtained the existence of C^∞ local solutions to arbitrary dimensional Monge-Ampére equation, in which f is not only nonnegative but also satisfies a variant of Hörmander rank condition. Lin [8] proved the existence of a local H^s solution in $\mathbb{R}^2$ with $f \geq 0$. We will follow the ideas of [5] and [8, 9], the existence of the local solution can be obtained by a perturbation of polynomial-typed solution for $S_2[u] = a$ where a is a constant, so that our solution is in the form

$$u(y) = \frac{1}{2}\sum_{j=1}^{3} \tau_j y_j^2 + \varepsilon^5 w(\varepsilon^{-2}y), \quad \tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3.$$

The significance of theorem 1.1 is our results break away from the framework of Gårding cone. The sign of f is permitted to change in case (1). For the case (2), we say that it is a degenerate 2-Hessian equation if $f(Z_0) = 0$(see [10]). The non-convex solution in (1) and (2) never occurs for Monge-Ampére equation. There is also many works about the convexity of solution to Hessian equation, see [11] and reference therein. Besides, these results seems to be strange. However, that is because the relationship between the sign of f and the ellipticity of the nonlinear k-Hessian equation may not be close.

The rest of this paper is arranged as follow: in Section 2, we will give definitions and some known results. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

In this section, we collect some definitions and known results of k-Hessian equations. Firstly some algebraic properties of Gårding cone.

Proposition 2.1 (See [12])**.** *Using the notations introduced in Section 1,*

(1) *$\sigma_k(\lambda) = 0$ for $\lambda \in \partial\Gamma_k$ and*

$$\Gamma_n \subset \ldots \subset \Gamma_k \subset \ldots \subset \Gamma_1 .$$

(2) *Maclaurin's inequalities, for any* $\lambda \in \Gamma_k, 1 \le l \le k$,

$$\left[\frac{\sigma_k(\lambda)}{\binom{n}{k}}\right]^{1/k} \le \left[\frac{\sigma_l(\lambda)}{\binom{n}{l}}\right]^{1/l}.$$

(3) *we also have*

$$\begin{cases} \sigma_k(\lambda) = \lambda_i \sigma_{k-1;i}(\lambda) + \sigma_{k;i}(\lambda), & \forall \lambda \in \mathbb{R}^n, \\ \sum_{i=1}^n \sigma_{k,i}(\lambda) = (n-k)\sigma_k(\lambda), & \forall \lambda \in \mathbb{R}^n. \end{cases}$$

(4) *Assume that* $\lambda \in \Gamma_k$ *is in descending order,*

$$\lambda_1 \ge \cdots \lambda_{p-1} \ge \lambda_p > 0 \ge \lambda_{p+1} \ge \cdots \lambda_n,$$

then $p \ge k$ *and*

$$\sigma_{k-1;n}(\lambda) \ge \cdots \ge \sigma_{k-1;1}(\lambda) > 0. \tag{2.1}$$

When $n = 3$, we see that $\sigma_3(\lambda) > 0$ cannot occur for $\lambda \in \partial\Gamma_2(\lambda)$, therefore we can express $\partial\Gamma_2$ as two parts

$$\begin{aligned} &\partial\Gamma_2(\lambda) = \mathbf{P}_1 \cup \mathbf{P}_2, \\ &\mathbf{P}_1 = \{\lambda \in \mathbb{R}^3; \sigma_1(\lambda) \ge 0,\ \sigma_2(\lambda) = \sigma_3(\lambda) = 0\}, \\ &\mathbf{P}_2 = \{\lambda \in \mathbb{R}^3; \sigma_1(\lambda) > 0,\ \sigma_2(\lambda) = 0,\ \sigma_3(\lambda) < 0\}. \end{aligned}$$

Next, we will recall that what condition can lead to the ellipticity.

As for the framework of ellipticity, we follow the ideas of [6] and [7]. Denote $\mathrm{Sym}(n)$ as the set of symmetric real $n \times n$ matrix. Through the matrix language, we recall the direct condition which leads to the elliptic k-Hessian operator. The ellipticity set of the k-Hessian operator, $k = 1, 2, \ldots, n$, is

$$E_k = \{S \in \mathrm{Sym}(n) : S_k(S + t\xi \times \xi) > S_k(S) > 0, |\xi| = 1, t \in \mathbb{R}^+\}$$

and the Gårding cones

$$\Gamma_k = \{S \in \mathrm{Sym}(n) : S_k(S + tId) > S_k(S) > 0, t \in \mathbb{R}^+\},$$

where the definition of $S_k(S)$ is given in (1.2). It is easy to show that $E_k = \Gamma_k$ only for $k = 1, n$ and the example in [7] assures that $\Gamma_k \subset E_k$ and $mess(E_k \setminus \Gamma_k) > 0$ when $1 < k < n$. Ivochkina, Prokofeva and Yakunina [7] point out that the ellipticity of (1.1) is independent of the sign of f.

We now present an algebraic property of

$$\frac{\partial}{\partial \tau_i}\sigma_2(\tau) = \sigma_{1,i}(\tau), \quad i = 1, 2, 3,$$

for $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbf{P}_2$.

Lemma 2.2. *Assume that* $\tau \in \mathbf{P}_2$, $\tau_1 \ge \tau_2 \ge \tau_3$. *Then we have*

$$0 < \sigma_{1,1}(\tau) \le \sigma_{1,2}(\tau) \le \sigma_{1,3}(\tau),$$

and

$$\tau_3 < 0 < \tau_2 \le \tau_1. \tag{2.2}$$

The above result means that for any

$$\psi = \frac{1}{2}\sum_{i=1}^{3} \tau_i y_i^2, \qquad \tau \in \mathbf{P}_2$$

it is a solution of 2-Hessian equation $S_2(\psi) = 0$, and the linearized operators of $S_2[u]$ at ψ

$$\mathcal{L} = \sum_{i=1}^{3} \sigma_{1,i}(\tau)\partial_i^2$$

is uniformly elliptic,

Proof. Recall that, for any $\tau \in \mathbb{R}^3$,

$$\sigma_2(\tau) = \tau_1\tau_2 + \tau_2\tau_3 + \tau_1\tau_3,$$

and

$$\sigma_{1;1}(\tau) = \tau_2 + \tau_3, \;\; \sigma_{1;2}(\tau) = \tau_1 + \tau_3, \;\; \sigma_{1;3}(\tau) = \tau_1 + \tau_2.$$

Denote $\lambda + \varepsilon = (\lambda_1 + \varepsilon, \lambda_2 + \varepsilon, \lambda_3 + \varepsilon)$ with $\lambda \in \mathbb{R}^3$ and $\varepsilon \in \mathbb{R}$, then we have the formula

$$\sigma_2(\lambda + \varepsilon) = \sum_{j=0}^{2} C(j)\varepsilon^j \sigma_{2-j}(\lambda), \qquad C(j) = \frac{\binom{3}{2}\binom{2}{j}}{\binom{3}{2-j}}.$$

For $\tau \in \mathbf{P}_2$, we have

$$\sigma_1(\tau) > 0, \qquad \sigma_2(\tau) = 0,$$

then

$$\tau + \varepsilon \in \Gamma_2, \qquad \forall \varepsilon > 0.$$

Applying (2.1) to $\tau + \varepsilon$ and letting $\varepsilon \to 0^+$, we get

$$0 \le \sigma_{1,1}(\tau) \le \sigma_{1,2}(\tau) \le \sigma_{1,3}(\tau).$$

Since $\tau \in \mathbf{P}_2$, we have

$$\sigma_2(\tau) = \tau_1\sigma_{1,1}(\tau) + \sigma_{2,1}(\tau) = 0.$$

if $\sigma_{1,1}(\tau) = \tau_2 + \tau_3 = 0$, then,

$$\sigma_{2,1}(\tau) = \tau_2\tau_3 = 0,$$

thus $\sigma_3(\tau) = \tau_1\tau_1\tau_3 = 0$, which contradicts with the assumption $\sigma_3(\tau) < 0$. Then, We have proven that, for any $\tau \in \mathbf{P}_2$,

$$0 < \sigma_{1,1}(\tau) \le \sigma_{1,2}(\tau) \le \sigma_{1,2}(\tau).$$

We prove now (2.2). Since $\sigma_1(\tau) > 0$, by (4) we have $\tau_1 > 0$. We now claim that $\tau_1 = \tau_2 = \tau_3$ is impossible. Indeed, if that holds, then $\sigma_1(\tau) = 3\tau_1 > 0$ and $\sigma_2(\tau) = 3\tau_1^2 > 0$, which contradicts with the assumption $\sigma_2(\tau) = 0$.

Besides, $\sigma_3(\tau) < 0$ imply that $\tau_i \neq 0$ and τ_i can not be positive at the same time. Then property (4) of Proposition 2.1 implies

$$\tau_3 < 0 < \tau_2 \le \tau_1.$$

□

We also have the following elliptic results for $\tau \in \Gamma_1 \setminus \bar{\Gamma}_2$.

Lemma 2.3. *For the Gårding cone, we have*

(1) *For any given $a < 0$, there exists $\tau \in \Gamma_1 \setminus \bar{\Gamma}_2$, such that*

$$\sigma_1(\tau) > 0, \qquad \sigma_2(\tau) = a.$$

(2) *For any given $b > 0$, there exists $\tau \in \Gamma_2 \setminus \bar{\Gamma}_3$, such that*

$$\sigma_1(\tau) > 0, \qquad \sigma_2(\tau) = b, \qquad \sigma_3(\tau) < 0.$$

(3) *For any given $c > 0$, there exists $\tau \in \Gamma_3$, such that*

$$\sigma_1(\tau) > 0, \qquad \sigma_2(\tau) = c, \qquad \sigma_3(\tau) > 0.$$

Moreover, for all above case, we have

$$\sigma_{1,3}(\tau) > \sigma_{1,2}(\tau) > \sigma_{1,1}(\tau) > 0.$$

Proof. We only need to prove the case (1), and to find a $\tau \in \mathbb{R}^3$. We can choose $\alpha > 0$ and $\beta > 0$ such that

$$(1+\beta)\alpha - 1 < 0.$$

Then take $\Theta > 0$ satisfying

$$\Theta^2(1+\alpha)[(1+\alpha)\beta - 1] = a.$$

We claim that τ can be in the following form

$$\tau = (\tau_1, \tau_2, \tau_3) = ((1+\alpha)(1+\beta)\Theta, (1+\alpha)\Theta, -\Theta).$$

Indeed, from $1+\beta > 1$ and $(1+\alpha)\Theta > 0$, we have

$$\tau_1 > \tau_2 > \tau_3,$$

$\sigma_1(\tau) > 0$ and $\sigma_2(\tau) = a$. Moreover,

$$\sigma_{1,3}(\tau) = (1+\alpha)(2+\beta)\Theta > \sigma_{1,2} = (\alpha\beta + \alpha + \beta)\Theta > \sigma_{1,1}(\tau) = \alpha\Theta > 0.$$

Proof is done. □

For the linearized operators of k-Hessian equation, we have the following results, the general version of which can be found in section 2, [2].

Lemma 2.4. *The matrix $S_2^{ij}(r(w))$ and $(r_{ij}(w))$ can be diagonalized simultaneously, that is, for any smooth function w, we can find an orthogonal matrix $T(x,\varepsilon)$ satisfying*

$$\begin{cases} T(x,\varepsilon)(S_2^{ij})\ {}^tT(x,\varepsilon) = \operatorname{diag}\left[\frac{\partial\sigma_2(\lambda)}{\partial\lambda_1}, \frac{\partial\sigma_2(\lambda)}{\partial\lambda_2}, \frac{\partial\sigma_2(\lambda)}{\partial\lambda_3}\right] \\ T(x,\varepsilon)(r_{ij})\ {}^tT(x,\varepsilon) = \operatorname{diag}\left[\lambda_1(x,\varepsilon), \lambda_2(x,\varepsilon), \lambda_3(x,\varepsilon)\right], \end{cases}$$

where ${}^tT(x,\varepsilon)$ is the transpose of $T(x,\varepsilon)$ and $S_2^{ij}(r(w)) = \partial S_2/\partial r_{ij}(r(w))$. Furthermore,

$$T(x,\varepsilon)\mid_{\varepsilon=0} = Id,$$

where Id is the identity matrix.

Proof. For $T = (T_{ij})$, we have

$$\sum_{i=1}^{3} T_{si}T_{ti} = \delta_s^t. \tag{2.3}$$

Now we set (r_{ij}) can be diagonalized by T,

$$(T_{ij})(r_{ij})^t(T_{ij}) = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \left(\sum_{i,j=1}^{3} T_{si}T_{tj}r_{ij} \right)_{st}.$$

Thus, we have, when $s \neq t$

$$\begin{aligned} &\sum_{i,j=1}^{3} T_{si}T_{tj}r_{ij} \\ =&T_{s1}T_{t1}r_{11} + T_{s2}T_{t2}r_{22} + T_{s3}T_{t3}r_{33} + 2T_{s3}T_{t1}r_{31} + 2T_{s1}T_{t2}r_{12} + 2T_{s3}T_{t2}r_{32} = 0 \end{aligned} \tag{2.4}$$

Now for

$$\left(S_2^{ij}(r_{ij})\right) = \begin{pmatrix} r_{22}+r_{33} & -r_{21} & -r_{31} \\ -r_{12} & r_{11}+r_{33} & -r_{31} \\ -r_{13} & -r_{23} & r_{11}+r_{22} \end{pmatrix},$$

we have

$$(T_{ij})(r_{ij})^t(T_{ij}) = \left(\sum_{i,j=1}^{3} T_{si}T_{tj}S_2^{ij} \right)_{st}.$$

If we could prove that $\sum_{i,j=1}^{3} \left(T_{si}T_{tj}S_2^{ij}\right)_{st}$ is a diagonal matrix, our proof was done.

Indeed, when $s \neq t$, we have

$$\begin{aligned} &\sum_{i,j=1}^{3} T_{si}T_{tj}S_2^{ij} \\ =&T_{s1}T_{t1}(r_{22}+r_{33}) + T_{s2}T_{t2}(r_{11}+r_{33}) + T_{s3}T_{t3}(r_{11}+r_{22}) \\ &- 2T_{s1}T_{t2}r_{12} - 2T_{s3}T_{t1}r_{31} - 2T_{s3}T_{t2}r_{32}. \end{aligned} \tag{2.5}$$

By (2.4) and (2.3), (2.5) can be

$$\sum_{i,j=1}^{3} T_{si}T_{tj}S_2^{ij} = \sum_{i,j=1}^{3} T_{si}T_{tj}(r_{11}+r_{22}+r_{33}) = 0.$$

When $\varepsilon = 0$, $S_2^{ij}[r(w)]$ and $(r_{ij}(w))$ are diagonal, thus, T can be the identity matrix Id. □

From the view above, when $k = 2$ and $f < 0$, the corresponding Hessian operator is possible to be uniformly elliptic. In this paper, we will study some uniformly elliptic 2-Hessian equations which have non-positive right-hand functions f.

3. Existence of C^∞ local Solutions for uniformly elliptic case

From now on, we fixed $n = 3, k = 2$, by a translation $y \longrightarrow y - y_0$ and replacing u by $u - u(0) - y \cdot Du(0)$, we can assume $Z_0 = (0,0,0)$ in Theorem 1.1. We prove now the following results,

Theorem 3.1. *Let $f \in C^\infty$ and $f(Z_0) = 0$ for $Z_0 = (0,0,0) \in \Omega \times \mathbb{R} \times \mathbb{R}^3$. Then (1.3) admits a 1-convex local solution $u \in C^\infty$ which is not 3-convex and is of the following form*

$$u(y) = \frac{1}{2}\sum_{i=1}^{3} \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y), \qquad \forall (\tau_1,\tau_2,\tau_3) \in \mathbf{P}_2 \tag{3.1}$$

in the neighborhood of $y_0 = 0$, $\|w\|_{C^{4,\alpha}} \leq 1$ and $\varepsilon > 0$ very small.

If f is nonnegative near Z_0, then (1.3) *admits a 2-convex local solution $u \in C^\infty$ which is not 3-convex. If $f(Z_0) > 0$, then* (1.3) *admits a 3-convex local solution $u \in C^\infty$.*

Moreover, the equation (1.3) *is uniformly elliptic with respect to the solution* (3.1)*.*

Remark that, in Theorem 3.1 the function f is permitted to change sign. It is well known that, for Monge-Ampere operator, the type of equation is determined by the sign of $f(y, u, Du)$, it is elliptic if $f > 0$, hyperbolic if $f < 0$ and degenerate elliptic or hyperbolic if f vanishes; it is of mixed type if f changes sign [4]. So that Theorem 3.1 never occurs in Monge-Ampére case.

Theorem 3.1 is exactly the part (1) and (2) of Theorem 1.1.

Let $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbf{P}_2$, then $\psi(y) = \frac{1}{2}\sum_{i=1}^3 \tau_i y_i^2$ is a polynomial-type solution of

$$S_2[\psi] = 0,$$

we follow Lin [8] to introduce the following function

$$u(y) = \frac{1}{2}\sum_{i=1}^3 \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y) = \psi(y) + \varepsilon^5 w(\varepsilon^{-2}y), \qquad \tau \in \mathbf{P}_2,\ \varepsilon > 0,$$

as a candidate of solution for equation (1.1). Noting $y = \varepsilon^2 x$, we have

$$(D_{y_j} u)(x) = \tau_j \varepsilon^2 x_j + \varepsilon^3 w_j(x), \qquad j = 1, \cdots, 3,$$

and

$$(D_{y_j y_k} u)(x) = \delta_k^j \tau_j + \varepsilon w_{jk}(x), \qquad j, k = 1, \cdots, 3,$$

where δ_k^j is the Kronecker symbol, $w_j(x) = (D_{y_j} w)(x)$ and $w_{jk}(x) = (D^2_{y_{jk}} w)(x)$. Then (1.3) transfers to

$$\tilde{S}_2(w) = \tilde{f}_\varepsilon(x, w(x), Dw(x)), \qquad x \in B_1(0) = \{x \in \mathbb{R}^3; |x| < 1\}$$

where

$$\tilde{S}_2(w) = S_2(\delta_i^j \tau_i + \varepsilon w_{ij}(x)) = S_2(r(w)),$$

with symmetric matrix $r(w) = (\delta_i^j \tau_i + \varepsilon w_{ij}(x))$, and

$$\tilde{f}_\varepsilon(x, w(x), Dw(x)) = f(\varepsilon^2 x, \varepsilon^4 \psi(x) + \varepsilon^5 w(x), \tau_1 \varepsilon^2 x_1 + \varepsilon^3 w_1(x), \cdots, \tau_3 \varepsilon^2 x_3 + \varepsilon^3 w_3(x)).$$

Similar to [8] we consider the nonlinear operators

$$G(w) = \frac{1}{\varepsilon}[S_2(r(w)) - \tilde{f}_\varepsilon(x, w, Dw)], \qquad \text{on } B_1(0). \tag{3.2}$$

The linearized operator of G at w is

$$L_G(w) = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}} \partial_{ij}^2 + \sum_{i=1}^3 a_i \partial_i + a, \tag{3.3}$$

where

$$a_i = -\frac{1}{\varepsilon}\frac{\partial \widetilde{f}_\varepsilon(x, z, p_i)}{\partial p_i}(x, w, Dw) = -\varepsilon^2 \frac{\partial f}{\partial p_i}$$

$$a = -\frac{1}{\varepsilon}\frac{\partial \widetilde{f}_\varepsilon(x, z, p_i)}{\partial z}(x, w, Dw) = -\varepsilon^4 \frac{\partial f}{\partial z}.$$

Hereafter, we denote $S_2^{ij}(r(w)) = \frac{\partial S_2(r(w))}{\partial r_{ij}}$. Since $S_2(r(w)) = \sigma_2(\lambda(r(w)))$ is invariant under orthogonal transformation, by using Lemma 2.4, the matrix $\left(S_2^{ij}(r(w))\right)$ and $(r(w))$ can be

diagonalized simultaneously, that is, for any smooth function w, we can find an orthogonal matrix $T(x,\varepsilon)$ satisfying

$$\begin{cases} T(x,\varepsilon)\left(S_2^{ij}(r(w))\right)\ {}^tT(x,\varepsilon) = \mathrm{diag}\left[\frac{\partial\sigma_2(\lambda(r(w)))}{\partial\lambda_1}, \frac{\partial\sigma_2(\lambda(r(w))}{\partial\lambda_2}, \frac{\partial\sigma_2(\lambda(r(w)))}{\partial\lambda_3}\right] \\ T(x,\varepsilon)\left(r_{ij}(r(w))\right)\ {}^tT(x,\varepsilon) = \mathrm{diag}\left[\lambda_1(r(w)), \lambda_2(r(w)), \lambda_3(r(w))\right], \end{cases}$$

where ${}^tT(x,\varepsilon)$ is the transpose of $T(x,\varepsilon)$. Since T is not unique, we set $T(x,\varepsilon)\mid_{\varepsilon=0}= Id$. After this transformation, in order to prove the uniform ellipticity of $L_G(w)$

$$\sum_{i,j=1}^{n} S_2^{ij}(r(w)\xi_i\xi_j \geq c|\xi|^2, \qquad \forall (x,\xi) \in B_1(0)\times R^3$$

instead we can prove that , by setting $\xi = {}^tT(x,\varepsilon)\tilde{\xi}$,

$$\sum_{j=1}^{3} \frac{\partial\sigma_2(\lambda(r(w)))}{\partial\lambda_j}|\tilde{\xi}_j|^2 \geq c|\tilde{\xi}|^2,$$

for some $c > 0$, where

$$\begin{aligned} \frac{\partial\sigma_2(\lambda(r(w)))}{\partial\lambda_1} &= \sigma_{1,1}(\lambda(r(w))) = \lambda_2(r(w)) + \lambda_3(r(w)), \\ \frac{\partial\sigma_2(\lambda(r(w)))}{\partial\lambda_2} &= \sigma_{1,2}(\lambda(r(w))) = \lambda_1(r(w)) + \lambda_3(r(w)), \\ \frac{\partial\sigma_2(\lambda(r(w)))}{\partial\lambda_3} &= \sigma_{1,3}(\lambda(r(w))) = \lambda_1(r(w)) + \lambda_2(r(w)). \end{aligned}$$

Lemma 3.2. *Assume that $\tau \in \mathbf{P}_2$ and $\|w\|_{C^2(B_1(0))} \leq 1$, then the operator $L_G(w)$ is a uniformly elliptic operator if ε is small enough.*

Proof. To prove the operator $L_G(w)$ is a uniformly elliptic operator, it suffices to prove

$$\lambda_i(r(w)) + \lambda_j(r(w)) = \tau_i + \tau_j + O(\varepsilon), \qquad i,j = 1,2,3, \qquad i \neq j. \tag{3.4}$$

Indeed, for $\tau \in \mathbf{P}_2$ and Lemma 2.2 give $\tau_i + \tau_j > 0$. Thus, for ε small enough, (3.4) imply,

$$\lambda_i + \lambda_j \geq \frac{\tau_i + \tau_j}{2} > 0, \qquad i \neq j$$

$L_G(w)$ is then a uniformly elliptic operator.

Next, we prove (3.4). By our choice of $r_{ij}(w)$,

$$r(w) = (r_{ij}(w)) = \begin{pmatrix} \tau_1 + \varepsilon w_{11} & \varepsilon w_{12} & \varepsilon w_{13} \\ \varepsilon w_{21} & \tau_2 + \varepsilon w_{22} & \varepsilon w_{23} \\ \varepsilon w_{31} & \varepsilon w_{32} & \tau_3 + \varepsilon w_{33} \end{pmatrix},$$

we write its characteristic polynomial as

$$g(\lambda) = \det(r(w) - \lambda\,\mathbf{I}) = \prod_{i=1}^{3}(\tau_i - \lambda_i) + R(w,\varepsilon)$$

where

$$R(w,\varepsilon) = \sum_{j=1}^{3} \varepsilon R_j(w,\varepsilon) + \sum_{j,k} \varepsilon^2 R_{jk}(w,\varepsilon).$$

For any $\|w\|_{C^2(B_1(0))} \leq 1$ and $0 < \varepsilon \leq 1$

$$|R_j(w,\varepsilon)| \leq C, \qquad |R_{jk}(w,\varepsilon)| \leq C$$

with C being independent of x and ε. We have also

$$S_1(r(w)) = \sigma_1(\tau) + \varepsilon S_1(w), \qquad S_2(r(w)) = \sigma_2(\tau) + \varepsilon\tilde{R}_1(w,\varepsilon), \tag{3.5}$$

and

$$\det(r(w)) = \sigma_3(\tau) + \varepsilon\tilde{R}_2(w,\varepsilon),$$

where for any $\|w\|_{C^2(B_1(0))} \le 1$ and $0 < \varepsilon \le 1$

$$|\tilde{R}_j(w,\varepsilon)| \le C, \quad |S_1(w)| \le C.$$

By using Lemma 2.2, we have $\tau_3 < 0 < \tau_2 \le \tau_1$, then for $0 < \varepsilon \ll |\tau_3|$, we have

$$g(\frac{3}{4}\tau_3) = (\tau_1 - \frac{3}{4}\tau_3)(\tau_2 - \frac{3}{4}\tau_3)(\frac{\tau_3}{4}) + R(w,\varepsilon) < 0,$$

$$g(\frac{5}{4}\tau_3) = (\tau_1 - \frac{5\tau_3}{4})(\tau_2 - \frac{5\tau_3}{4})(-\frac{\tau_3}{4}) + R(w,\varepsilon) > 0,$$

and we see that, by the virtue of Intermediate Value Theorem, there exists an eigenvalue, denoted by λ_3, such that

$$\frac{3}{4}\tau_3 > \lambda_3 > \frac{5}{4}\tau_3, \ \ g(\lambda_3) = 0.$$

From $0 = g(\lambda_3) = (\tau_1 - \lambda_3)(\tau_2 - \lambda_3)(\tau_3 - \lambda_3) + R(w,\varepsilon)$ and

$$(\tau_1 - \frac{5\tau_3}{4})(\tau_2 - \frac{5\tau_3}{4}) > (\tau_1 - \lambda_3)(\tau_2 - \lambda_3) > (\tau_1 - \frac{3\tau_3}{4})(\tau_2 - \frac{3\tau_3}{4}),$$

it follows that

$$\lambda_3 = \tau_3 + O_1(w,\varepsilon).$$

Since the trace of a matrix is invariant under the orthogonal transformation, then

$$\lambda_1(w) + \lambda_2(w) + \lambda_3(w) = \sigma_1(\tau) + \varepsilon(w_{11} + w_{22} + w_{33}),$$

from which we see that

$$\lambda_1(w) + \lambda_2(w) = \tau_1 + \tau_2 + O_2(w,\varepsilon).$$

Using

$$\sigma_2(\tau) + \varepsilon\tilde{R}_1(w,\varepsilon) = S_2(r(w)) = \sigma_2(\lambda(r(w))) = \lambda_3(w)(\lambda_1(w) + \lambda_2(w)) + \lambda_1(w)\lambda_2(w),$$

we obtain

$$\lambda_1\lambda_2 = \tau_1\tau_2 + O_3(w,\varepsilon),$$

which yields either

$$\lambda_1 = \tau_1 + O_4(w,\varepsilon), \quad \lambda_2 = \tau_2 + O_5(w,\varepsilon)$$

or

$$\lambda_1 = \tau_2 + O_5(w,\varepsilon), \quad \lambda_2 = \tau_1 + O_4(w,\varepsilon)$$

and then (3.4) is proven. Proof is done. □

We follows now the idea of Hong and Zuily [5] to prove the existence and a priori estimates of solution for linearized operator. In our case, although $L_G(w)$ is uniformly elliptic, the existence and a priori Schauder estimates of classical solutions are not directly obtainable, because we do not know whether the coefficient a of au in (3.3) is non-positive. If we can prove the existence (Lemma 3.3), we can employ Nash-Moser procedure to prove the existence of local solution for (1.3) in Hölder space rather than Sobolev space. One goal is to see how the procedure depends on the condition $\|w_k\|_{C^{4,\alpha}} \leq A$. We shall use the following schema:

$$\begin{cases} w_0 = 0, \quad w_m = w_{m-1} + \rho_{m-1}, \ m \geq 1, \\ L_G(w_m)\rho_m = g_m, \text{in } B_1(0), \\ \rho_m = 0 \quad \text{on } \partial B_1(0), \\ g_m = -G(w_m)\,, \end{cases} \tag{3.6}$$

where

$$g_0(x) = \frac{1}{\varepsilon}\Big(\sigma_2(\tau) - f(\varepsilon^2 x, \varepsilon^4\psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3))\Big).$$

It is pointed out on page 107, [3] that, if the operator L_G does not satisfy the condition $a \leq 0$, as is well known from simple examples, the Dirichlet problem for $L_G(w)\rho = g$ no longer has a solution in general. Notice a in (3.9) has the factor ε^4, we will take advantage of smallness of a to obtain the uniqueness and existence of solution for Dirichlet problem (3.9) and then uniformly Schauder estimates of its solution follows.

Lemma 3.3. *Assume that $\|w\|_{C^{4,\alpha}(B_1(0))} \leq A$. Then there exists a unique solution $\rho \in C^{2,\alpha}(\overline{B_1(0)})$ to the following Dirichlet problem*

$$\begin{cases} L_G(w)\rho = g, \quad \textit{in } B_1(0), \\ \rho = 0 \quad \textit{on } \partial B_1(0) \end{cases} \tag{3.7}$$

for all $g \in C^{\alpha}(\overline{B_1(0)})$. Moreover,

$$\|\rho\|_{C^{4,\alpha}(\overline{B_1(0)})} \leq C\|g\|_{C^{2,\alpha}(\overline{B_1(0)})}, \quad \forall g \in C^{2,\alpha}(\overline{B_1(0)}), \tag{3.8}$$

where the constant C depends on A, τ and $\|f\|_{C^{4,\alpha}}$. Moreover, C is unform for $0 < \varepsilon \leq \varepsilon_0$ for some $\varepsilon_0 > 0$.

By virtue of (3.3), we write (3.7) as

$$\begin{cases} L_G(w)\rho = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}}\partial_i\partial_j\rho + \sum_{i=1}^3 a_i\partial_i\rho + a\rho = g, \quad \text{in } B_1(0), \\ \rho = 0 \quad \text{on } \partial B_1(0) \end{cases} \tag{3.9}$$

where

$$a_i = -\varepsilon^2\frac{\partial f}{\partial p_i}, \quad a = -\varepsilon^4\frac{\partial f}{\partial z}.$$

Notice that for $\frac{\partial S_2(r(w))}{\partial r_{ij}}$, $a_i = a_i(x, w(x), Dw(x))$, $a = a(x, w(x), Dw(x))$ and $g_m = -G(w_m) = g_m(x, w(x), Dw(x), D^2w(x))$ by (3.6), we regard them as the functions with variable x. In a word, we regard that all of the coefficients and non-homogeneous term in (3.9) are functions of variable x. For example,

$$\tilde{f_\varepsilon}(x, w(x), Dw(x)) = f(\varepsilon^2 x, \varepsilon^4\psi(x) + \varepsilon^5 w(x), \tau_1\varepsilon^2 x_1 + \varepsilon^3 w_1(x), \cdots, \tau_3\varepsilon^2 x_3 + \varepsilon^3 w_3(x)),$$

and

$$\|\tilde{f_\varepsilon}\|_{C^3} = \sup\left\{|D_x^\beta[\tilde{f_\varepsilon}]|, |0 \le \beta \le 3, x \in B_1(0)\right\}$$

$$\|\tilde{f_\varepsilon}\|_{C^{3,\alpha}} = \|\tilde{f_\varepsilon}\|_{C^3} + \sup\left\{\frac{|D_x^\beta[\tilde{f_\varepsilon}](x) - D_x^\beta[\tilde{f_\varepsilon}](z)|}{|x-z|^\alpha}, |\beta| = 3, x \ne z \in B_1(0)\right\}$$

When we regard $\tilde{f_\varepsilon}$ as a function of variable x, usually $\|f\|_{C^{3,\alpha}}$ is denoted as $\|f\|_{C^{3,\alpha}(B_1(0))}$, but it maybe cause confusion because it must be involved in $D^\alpha w, 0 \le |\alpha| \le 3$ as above. Therefore, here and after, we denote the norm as $\|\tilde{f_\varepsilon}\|_{C^3}$, $\|\tilde{f_\varepsilon}\|_{C^{3,\alpha}}$ as above, by dropping $B_1(0)$.

Proof. Let the constant $\mu(\tau) = \inf\left\{\frac{\partial\sigma_2(\lambda(r(w)))}{\partial\lambda_i} : \|w\|_{C^{4,\alpha}(B_1(0))} \le A, i = 1, 2, 3, \right\}$. By Lemma 3.2, $\mu(\tau) > 0$. Applying Theorem 3.7 [3] to the solution $u \in C^0(\overline{B_1(0)}) \cap C^2(B_1(0))$ of

$$\begin{cases} L_G(w)u = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}}\partial_i\partial_j u + \sum_{i=1}^3 a_i\partial_i u = g, & \text{in } B_1(0), \\ u = 0 \quad \text{on } \partial B_1(0) \end{cases}$$

we have

$$\sup|u| \le \frac{C}{\mu(\tau)}\|g\|_{C^0(\overline{B_1(0)})}, \tag{3.10}$$

where $C = \exp^{2(\beta+1)} - 1$ and $\beta = \sup\left\{\frac{|a_i|}{\mu(\tau)} : i = 1, 2, 3.\right\}$

Let $C_1 = 1 - C\sup\frac{|a|}{\mu(\tau)}$ with C being the constant in (3.10). If we choose $\varepsilon_0 > 0$ small (the smallness of a), then $C_1 > \frac{1}{2}$ uniformly for $0 < \varepsilon < \varepsilon_0$. Applying Corollary 3.8 [3] to the solution ρ to Dirichlet problem (3.9), we have

$$\sup|\rho| \le \frac{1}{C_1}\left[\sup_{\partial B_1(0)}|\rho| + \frac{C}{\mu(\tau)}\|g\|_{C^0(\overline{B_1(0)})}\right] = \frac{C}{C_1\mu(\tau)}\|g\|_{C^0(\overline{B_1(0)})}, \tag{3.11}$$

from which we see that the homogeneous problem

$$\begin{cases} L_G(w)\rho = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}}\partial_i\partial_j\rho + \sum_{i=1}^3 a_i\partial_i\rho + a\rho = 0, & \text{in } B_1(0), \\ \rho = 0 \quad \text{on } \partial B_1(0) \end{cases}$$

has only the trivial solution. Then we can apply a Fredholm alternative, Theorem 6.15 [3], to the inhomogeneous problem (3.9) for which we can assert that it has a unique $C^{2,\alpha}(\overline{B_1(0)})$ solution for all $g \in C^\alpha(\overline{B_1(0)})$.

With the existence and uniqueness at hand, we can apply Theorem 6.19 [3] to obtain higher regularity up to boundary for solution to (3.9). Besides this, we have the Schauder estimates (see Problem 6.2 , [3])

$$\|\rho\|_{C^{4,\alpha}} \le C(A, \tau, \|f\|_{C^{3+\alpha}})\left[\|\rho_k\|_{C^0(\overline{B_1(0)})} + \|g_k\|_{C^{2,\alpha}(\overline{B_1(0)})}\right], \tag{3.12}$$

where C depends on $C^{2,\alpha}-$norm of all of the coefficients; the uniform ellipticity; boundary value and boundary itself . we explain the dependence of $C(A, \tau, \|f\|_{C^{3+\alpha}})$. Firstly, Since the first two derivatives of w have come into the principal coefficients $\frac{\partial S_2(r(w))}{\partial r_{ij}}$, then their $C^{2+\alpha}$-norms must be involved in $\|w\|_{C^{4,\alpha}}$, and at last $\|w\|_{C^{4,\alpha}} \le A$ arise into C. Similarly, by virtue of the coefficients a_i and a, $\|f\|_{C^{3,\alpha}}$ and $\|w\|_{C^{3,\alpha}} \le A$ must arise into C. Secondly, it depends on uniform ellipticity, that is, on

$$\inf\left\{\frac{\partial\sigma_2(\lambda(r(w)))}{\partial\lambda_i} : \|w\|_{C^{4,\alpha}(B_1(0))} \le A, i = 1, 2, 3, \right\}$$

and

$$\sup\left\{\frac{\partial\sigma_2(\lambda(r(w)))}{\partial\lambda_i} : \|w\|_{C^{4,\alpha}(B_1(0))} \le A, i = 1, 2, 3,\right\},$$

so ($\tau = (\tau_1, \tau_2, \tau_3)$) and A arise into C.

Thirdly, Since boundary value is =0 and boundary $\partial B_1(0)$ is C^∞, so the two ingredients do not occur into C. Substituting (3.11) into (3.12), we obtain (3.8). □

It follows from standard elliptic theory (see Theorem 6.17, [3] and Remark 2, [1]) and an iteration argument that we obtain.

Corollary 3.4. *Assume that $u \in C^{2,\alpha}(\Omega)$ is a solution of* (1.3)*, and the linearized operators with respect to u,*

$$\mathcal{L}_u = \sum_{i,j=1}^{3} \frac{\partial S_2(u_{ij})}{\partial r_{ij}}\partial^2_{ij} - \sum_{i=1}^{3} \frac{\partial f}{\partial p_i}(y, u(y), Du(y))\partial_i - \frac{\partial f}{\partial z}(y, u(y), Du(y))$$

is uniformly elliptic, then $u \in C^\infty(\Omega)$.

Proof. Let v be a function on Ω and denote by e_l, $l = 1, 2, 3$ the unit coordinate vector in the y_l direction. We define the difference quotient of v at y in the direction e_l by

$$\triangle^h v(y) = \triangle^h_l v(y) = \frac{v(y + he_l) - v(y)}{h}.$$

Since

$$\begin{aligned}
&S_2(u_{ij}(y + he_l)) - S_2(u_{ij}(y))\\
&= \int_0^1 \frac{d}{dt}[S_2(tu_{ij}(y + he_l) + (1 - t)u_{ij}(y))]dt\\
&= \sum_{i,j=1}^{3} \int_0^1 \frac{\partial}{\partial r_{ij}}[S_2(tu_{ij}(y + he_l) + (1 - t)u_{ij}(y))]dt[u_{ij}(y + he_l) - u_{ij}(y)]\\
&\equiv \sum_{i,j=1}^{3} a_{ij}(y)[u_{ij}(y + he_l) - u_{ij}(y)]
\end{aligned}$$

and Taylor expansion give

$$\begin{aligned}
&f(y + he_l, u(y + he_l), Du(y + he_l)) - f(y, u(y), Du(y))\\
&= \sum_{i=1}^{3} b_i(y)[u_i(y + he_l) - u_i(y)] + c(y)[u(y + he_l) - u(y)] + g(y)h
\end{aligned}$$

with

$$\begin{aligned}
b_i(y) &= \int_0^1 \frac{\partial f}{\partial p_i}(t(y + he_l) + (1 - t)y, tu(y + he_l) + (1 - t)u(y), tDu(y + he_l) + (1 - t)D(y))dt\\
c(y) &= \int_0^1 \frac{\partial f}{\partial z}(t(y + he_l) + (1 - t)y, tu(y + he_l) + (1 - t)u(y), tDu(y + he_l) + (1 - t)D(y))dt\\
g(y) &= \int_0^1 \frac{\partial f}{\partial y_l}(t(y + he_l) + (1 - t)y, tu(y + he_l) + (1 - t)u(y), tDu(y + he_l) + (1 - t)D(y))dt.
\end{aligned}$$

Taking the difference quotients of both sides of the equation

$$S_2(u_{ij}(y)) = f(y, u, Du),$$

we have

$$\sum_{i,j=1}^{3} a_{ij}(y)\partial_i\partial_j\triangle^h u(y) - \sum_{i=1}^{3} b_i(y)\partial_i\triangle^h u(y) - c(y)\triangle^h u(y) = g(y).$$

Since $u \in C^{2,\alpha}(\Omega)$, then all the coefficients a_{ij}, b_i, c and inhomogeneous term g are in $C^{\alpha}(\Omega)$, from the interior estimates of Corollary 6.3 in [3], we can infer

$$\triangle^h u \in C^{2,\alpha}(\Omega).$$

Letting $h \to 0$, we see $\partial_l u \in C^{2,\alpha}(\Omega), l = 1, 2, 3$ and

$$\sum_{i,j=1}^{3} \frac{\partial S_2(D^2u)}{\partial r_{ij}}\partial_i\partial_j(\partial_l u) - \sum_{i=1}^{3} \frac{\partial f}{\partial p_i}\partial_i(\partial_l u) - \frac{\partial f}{\partial z}(\partial_l u) = \frac{\partial f}{\partial y_l}.$$

Repeating the above proof, we obtain $u \in C^{\infty}(\Omega)$. □

Using above Lemma 3.3, we can use the procedure (3.6) to construct the sequence $\{w_m\}_{m\in\mathbb{N}}$. Now we study the convergence of $\{w_m\}_{m\in\mathbb{N}}$ and that of $\{g_m\}_{m\in\mathbb{N}}$.

Proposition 3.5. *Let $\{w_m\}_{m\in\mathbb{N}}$ and $\{g_m\}_{m\in\mathbb{N}}$ the sequence in* (3.6)*. Suppose that $\|w_j\|_{C^{4,\alpha}} \le A$ for $j = 1, 2, \ldots, k$. Then we have*

$$\|g_{k+1}\|_{C^{2,\alpha}} \le C[\|g_k\|^2_{C^{2,\alpha}} + \|g_k\|^3_{C^{2,\alpha}}], \tag{3.13}$$

where C is some positive constant depends only on τ ,A and $\|f\|_{C^{4,\alpha}}$. In particular, C is independent of k.

Proof. Applying Taylor's expansion with integral-typed remainder to (3.2), we have

$$\begin{aligned} -g_{k+1} &= G(w_k + \rho_k) = G(w_k) + L_G(w_k)\rho_k + Q(w_k, \rho_k) \\ &= -g_k + L_G(w_k)\rho_k + Q(w_k, \rho_k) = Q(w_k, \rho_k), \end{aligned}$$

where Q_k is the quadratic error of G which consists of S_2 and f.

$$\begin{aligned} Q(w_k, \rho_k) = &\sum_{ij,st} \frac{1}{\varepsilon}\int (1-\mu)\frac{\partial^2 S_2(w_k + \mu\rho_k)}{\partial w_{ij}\partial w_{st}} d\mu(\rho_k)_{ij}(\rho_k)_{st} \\ &+ \sum_{i,j} \frac{1}{\varepsilon}\int (1-\mu)\frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w_i\partial w_j} d\mu(\rho_k)_i(\rho_k)_j \\ &+ \frac{1}{\varepsilon}\sum_{i} \int (1-\mu)\frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w\partial w_i} d\mu(\rho_k)_i(\rho_k) \\ &+ \frac{1}{\varepsilon}\int (1-\mu)\frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w^2} d\mu \cdot \rho_k^2 \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

Since $S_2((r(w)))$ is a second-order homogeneous polynomial with variable $r_{ij}(r(w))$ and $\tilde{f}_\varepsilon(x, w, Dw)$ is independent of r_{ij}, we see that

$$\left|\frac{\partial^2 S_2(w_k + \mu\rho_k)}{\partial w_{ij} w_{st}}\right| = \frac{\partial^2 S_2}{\partial w_{ij}\partial w_{st}}(\delta_i^j\tau_i + \varepsilon(w_k + \mu\rho_k)_{ij}) = \varepsilon^2 \quad \text{or} \quad 0,$$

$$\left|\frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w_i\partial w_j}\right| = \left|\frac{\partial^2[f(\varepsilon x, \varepsilon^4\psi + \varepsilon^5(w_k + \mu\rho_k), \varepsilon^3 D\psi + \varepsilon^3 D(w_k + \mu\rho_k))]}{\partial w_i\partial w_j}\right| \le \varepsilon^6 \cdot \|f\|_{C^2},$$

$$\left|\frac{\partial^2 \tilde{f}_\varepsilon(w_k+\mu\rho_k)}{\partial w\partial w_i}\right| = \left|\frac{\partial^2[f(\varepsilon x, \varepsilon^4\psi+\varepsilon^5(w_k+\mu\rho_k), \varepsilon^3 D\psi+\varepsilon^3 D(w_k+\mu\rho_k))]}{\partial w\partial w_i}\right| \le \varepsilon^8\|f\|_{C^2},$$
$$\left|\frac{\partial^2 \tilde{f}_\varepsilon(w_k+\mu\rho_k)}{\partial w^2}\right| = \left|\frac{\partial^2[f(\varepsilon x, \varepsilon^4\psi+\varepsilon^5(w_k+\mu\rho_k), \varepsilon^3 D\psi+\varepsilon^3 D(w_k+\mu\rho_k))]}{\partial w^2}\right| = \varepsilon^{10}\|f\|_{C^2}.$$

Thus, $I_i (1 \le i \le 4)$ in Q_k are under control by $O(\varepsilon)$, $O(\varepsilon^5)$, $O(\varepsilon^7)$ and $O(\varepsilon^9)$, repectively. Therefore

$$\|I_1\|_{C^{2,\alpha}} \le C\|\rho_k\|_{C^2}\|\rho_k\|_{C^{4,\alpha}}$$

and

$$\begin{aligned}\|I_2\|_{C^{2,\alpha}} \le& C\|f\|_{C^{4,\alpha}}(\|w_k\|_{C^{3,\alpha}}+\|\rho_k\|_{C^{3,\alpha}})\|\rho_k\|^2_{C^1} + C\|f\|_{C^2}\|\rho_k\|_{C^{3,\alpha}}\|\rho_k\|_{C^1}\\ \le& C\|\rho_k\|_{C^{3,\alpha}}\|\rho_k\|^2_{C^1} + C\|\rho_k\|^2_{C^1} + C\|\rho_k\|_{C^{3,\alpha}}\|\rho_k\|_{C^1}\end{aligned}$$

where C depends on A and $\|f\|_{C^{4,\alpha}}$. And $\|I_3\|_{C^{2,\alpha}}$ and $\|I_4\|_{C^{2,\alpha}}$ can be estimated similarly. Accordingly,

$$\begin{aligned}\|g_{k+1}\|_{C^{2,\alpha}} =& \|Q(w_k,\rho_k)\|_{C^{2,\alpha}} \le \sum_{i=1}^4 \|I_i\|_{C^{2,\alpha}}\\ \le& C\|\rho_k\|_{C^2}\|\rho_k\|_{C^{4,\alpha}} + C\|\rho_k\|_{C^{3,\alpha}}\|\rho_k\|^2_{C^1} + \|\rho_k\|^2_{C^1} + C\|\rho_k\|_{C^{3,\alpha}}\|\rho_k\|_{C^1}\end{aligned}$$

where C is independent of k but dependent of A and $\|f\|_{C^{4,\alpha}}$. Thus, by the interpolation inequalities, we have

$$\|g_{k+1}\|_{C^{2,\alpha}} \le C\|\rho_k\|^2_{C^{4,\alpha}} + C\|\rho_k\|^3_{C^{4,\alpha}},$$

where C is independent of k. By Schauder estimates of Lemma 3.3, we have

$$\|\rho_k\|_{C^{4,\alpha}} \le C\|g_k\|_{C^{2,\alpha}}.$$

Combining the estimates above, we obtain (3.13). Proof is done. □

Since C is independent of k, more exactly, A, τ and $\|f\|_{C^{4,\alpha}}$ are independent of k. So here and after, we can assume $A = 1$.

Proof of Theorem 3.1. Set

$$d_{k+1} = C\|g_{k+1}\|_{C^{2,\alpha}}. \tag{3.14}$$

By (3.13) with letting $C \ge 1$ we have

$$d_{k+1} \le d_k^2 + d_k^3.$$

Take $\tau \in \mathbb{R}^3$ as in Lemmas 2.2 and 2.3 such that $\sigma_2(\tau) = f(0,0,0)$, we have

$$\begin{aligned}g_0(x) =& \frac{1}{\varepsilon}\Big(\sigma_2(\tau) - f\big(\varepsilon^2 x, \varepsilon^4\psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)\big)\Big)\\ =& \frac{1}{\varepsilon}[\sigma_2(\tau) - f(0,0,0)]\\ &+ \varepsilon\int_0^1 x\cdot(\partial_y f)\Big(t\varepsilon^2 x, t\varepsilon^4\psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)\Big)dt\\ &+ \varepsilon^3\int_0^1 \psi(x)(\partial_z f)\Big(t\varepsilon^2 x, t\varepsilon^4\psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)\Big)dt\\ &+ \varepsilon\int_0^1 (\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)\cdot(\partial_p f)\Big(t\varepsilon^2 x, t\varepsilon^4\psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)\Big)dt,\end{aligned}$$

then

$$\|g_0\|_{C^{2,\alpha}(B_1(0))} \le \varepsilon C_1 \|f\|_{C^{3,\alpha}}.$$

We can choose $0 < \varepsilon \le \varepsilon_0$ so small such that

$$C\|g_0\|_{C^{2,\alpha}(B_1(0))} \le 1/4, \quad 0 < \varepsilon \le \varepsilon_0.$$

Notice ε_0 is independent of k. Since $d_0 = C\|g_0\|_{C^{2,\alpha}}$, we have $d_1 \le 2d_0^2$ and, by induction,

$$d_{k+1} \le 2^{2^{k+1}} d_0^{2^{k+1}} \le (2C)^{2^{k+1}} \|g_0\|_{C^{2,\alpha}}^{2^{k+1}},$$

Thus, by (3.14)

$$\|g_{k+1}\|_{C^{2,\alpha}} \le (2C)^{2^{k+1}-1} \|g_0\|_{C^{2,\alpha}}^{2^{k+1}}.$$

Firstly, we claim that there exists $\varepsilon > 0$, depending on τ and $\|f\|_{C^{3,\alpha}}$ such that

$$\|w_k\|_{C^{4,\alpha}(B_1(0))} \le 1, \ \ \forall k \ge 1.$$

Indeed, set $w_0 = 0$, we have by (3.13)

$$\begin{aligned}\|w_{k+1}\|_{C^{4,\alpha}(B_1(0))} &= \|\sum_{i=0}^{k} \rho_i\|_{C^{4,\alpha}(B_1(0))} \le \sum_{i=0}^{k} \|\rho_i\|_{C^{4,\alpha}(B_1(0))} \\ &\le \sum_{i=0}^{k} C\|g_i\|_{C^{2,\alpha}(B_1(0))} \le \sum_{i=0}^{k} \left(2C\|g_0\|_{C^{2,\alpha}(B_1(0))}\right)^{2^i}\end{aligned}$$

where C is defined in Lemma 3.5. Thus, for any k,

$$\|w_{k+1}\|_{C^{4,\alpha}(B_1(0))} \le \sum_{i=0}^{\infty} \left(C\|g_0\|_{C^{2,\alpha}(B_1(0))}\right)^{2^i} \le \sum_{i=0}^{\infty} 2^{-2^i} \le 1.$$

Then, by Azelà-Ascoli Theorem, we have

$$w_k \to w \quad \text{in } C^4((B_1(0))).$$

From (3.13), we see that

$$\|g_{k+1}\|_{C^{2,\alpha}(B_1(0))} \le \left(\frac{1}{2}\right)^{2^k} \ \to \ 0,$$

and then $g_m = -G(w_m)$ yields

$$G(w) = \frac{1}{\varepsilon}[S_2(r(w)) - \tilde{f}(x, w, Dw)] = 0, \qquad \text{on } \ B_1(0)$$

which yields that the function

$$u(y) = \frac{1}{2}\sum_{i=1}^{3} \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2} y) \in C^4(B_{\varepsilon^2}(0)),$$

is a solution of

$$S_2[u] = f(y, u, Du), \qquad \text{on } \ B_{\varepsilon^2}(0).$$

Now if $f(0,0,0) = 0$, we take $\tau \in \mathbf{P}_2$, then $\sigma_1(\tau) > 0, \sigma_2(\tau) = 0, \sigma_3(\tau) < 0$, and (3.5) imply,

$$S_j[u] = \sigma_j(\lambda) = \sigma_j(\tau) + O(\varepsilon), \qquad j = 1, 2, 3$$

it follows that $S_1[u] > 0, S_3[u] < 0$ on $B_{\varepsilon^2}(0)$ for small $\varepsilon > 0$, that is, u is 1−convex but not convex. Moreover if $S_2[u] = f \geq 0$ near Z_0 and $f(Z_0) = 0$, we see that u is 2-convex by definition, but not 3-convex.

If $S_2[u] = f > 0$ near Z_0, we take $\tau \in \mathbb{R}^3$ given in (2) and (3) of Lemmas 2.3, then we can get the 3-convex or non convex local solutions.

The C^∞ regularity of solution is given by Corollary 3.4. We have then proved Theorem 3.1. □

We also have the following elliptic results for negative f

Theorem 3.6. *Let $f \in C^\infty, f(0,0,0) < 0$. Then* (1.3) *admits a* 1−*convex local solution $u \in C^\infty$ in a neighborhood of $y_0 = 0$ which is not* 2−*convex, it is of the following form*

$$u(y) = \frac{1}{2}\sum_{i=1}^{3} \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2} y)\,,$$

and the equation (1.3) *is uniformly elliptic with respect to this solution.*

Proof. For $a = f(0,0,0) < 0$, take $\tau \in \mathbb{R}^3$ as in (1) of Lemma 2.3 such that

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = f(0,0,0) < 0,$$

and

$$\sigma_{1,3}(\tau) > \sigma_{1,2}(\tau) > \sigma_{1,1}(\tau) > 0.$$

Now the proof is exactly same as that of Theorem 3.1 except the estimate of term g_0, we use Taylor expansion,

$$\begin{aligned}
g_0(x) = -G(0) &= \frac{1}{\varepsilon}[S_2(r(0)) - \tilde{f}(x,0,0)] \\
&= \frac{1}{\varepsilon}\Big[\sigma_2(\tau) - f(\varepsilon^2 x, \varepsilon^4\psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3))\Big] \\
&= \frac{1}{\varepsilon}[\sigma_2(\tau) - f(0,0,0)] \\
&\quad + \varepsilon \int_0^1 x \cdot (\partial_y f)\Big(t\varepsilon^2 x, t\varepsilon^4\psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)\Big)dt \\
&\quad + \varepsilon^3 \int_0^1 \psi(x)(\partial_z f)\Big(t\varepsilon^2 x, t\varepsilon^4\psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)\Big)dt \\
&\quad + \varepsilon \int_0^1 (\tau_1 x_1, \tau_2 x_2, \tau_3 x_3) \cdot (\partial_p f)\Big(t\varepsilon^2 x, t\varepsilon^4\psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)\Big)dt,
\end{aligned}$$

then we can end the proof of Theorem 3.6 exactly as that of Theorem 3.1. □

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