



**HAL**  
open science

## $C^\infty$ local solutions of elliptical 2–Hessian equation in $R^3$

Guiji Tian, Qi Wang, Chao-Jiang Xu

► **To cite this version:**

Guiji Tian, Qi Wang, Chao-Jiang Xu.  $C^\infty$  local solutions of elliptical 2–Hessian equation in  $R^3$ . 2014.  
hal-01006141

**HAL Id: hal-01006141**

**<https://hal.science/hal-01006141v1>**

Preprint submitted on 13 Jun 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**$C^\infty$  LOCAL SOLUTIONS  
OF ELLIPTICAL 2-HESSIAN EQUATION IN  $\mathbb{R}^3$**

G. TIAN, Q. WANG AND C.-J. XU

In remembrance of the late professor Rou-Huai Wang  
on the occasion of his 90th Birthday

**ABSTRACT.** In this work, we study the existence of  $C^\infty$  local solutions to 2-Hessian equation in  $\mathbb{R}^3$ . We consider the case that the right hand side function  $f$  possibly vanishes, changes the sign, is positively or negatively defined. We also give the convexities of solutions which are related with the annulation or the sign of right-hand side function  $f$ . The associated linearized operator are uniformly elliptic.

1. INTRODUCTION

We are interested by the following  $k$ -Hessian equation

$$(1.1) \quad S_k[u] = f(y, u, Du)$$

on an open domain  $\Omega \subset \mathbb{R}^n$ ,  $1 \leq k \leq n$ ,  $f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ . Denote  $Du = (\partial_1 u, \dots, \partial_n u)$  and  $D^2u$  is the Hessian matrix  $(\partial_i \partial_j u)_{1 \leq i, j \leq n}$ . the Hessian operators  $S_k[u]$  is defined as follows:

$$(1.2) \quad S_k[u] = \sigma_k(\lambda(D^2u)), \quad k = 1, \dots, n,$$

where  $\lambda(D^2u) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ ,  $\lambda_j$  is the eigenvalue of the Hessian matrix  $(D^2u)$ , and

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the  $k$ -th elementary symmetric polynomial. Denoting, for  $k, j \in \{1, \dots, n\}$ ,

$$\sigma_{k,j} = \frac{\partial \sigma_{k+1}(\lambda)}{\partial \lambda_j} = \sigma_k|_{\lambda_j=0}.$$

We also introduce the Gårding cone  $\Gamma_k$  which is the open symmetric convex cone in  $\mathbb{R}^n$ , with vertex at the origin, given by

$$\Gamma_k = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \forall j = 1, \dots, k\}.$$

When  $k = 1$ , (1.1) is a semi-linear Poisson equation, and it is Monge-Ampère equation for  $k = n$ .

We say that a function  $u \in C^2$  is  $k$ -convex, if

$$\lambda(D^2u) \in \bar{\Gamma}_k,$$

the  $n$ -convex function is simply called convex function.

---

2000 *Mathematics Subject Classification.* 35J60; 35J70.

*Key words and phrases.* Elliptical  $k$ -Hessian equation, existence of local solutions, convexities.

We say that a function  $u$  is a local solution of (1.1) near  $y_0 \in \Omega$ , if there exists a neighborhood of  $y_0$ ,  $V_{y_0} \subset \Omega$  such that  $u \in C^2(V_{y_0})$  satisfies the equation (1.1) on  $V_{y_0}$ .

In this work, we study the existence of  $C^\infty$ -local solution of the following 2-Hessian equation in  $\mathbb{R}^3$ ,

$$(1.3) \quad S_2[u] = f(y, u, Du), \quad \text{on } \Omega \subset \mathbb{R}^3,$$

where we also have

$$S_2[u] = u_{11}u_{22} - u_{12}^2 + u_{22}u_{33} - u_{23}^2 + u_{11}u_{33} - u_{13}^2.$$

We have proved the following results.

**Theorem 1.1.** *Assume that  $f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^3)$ , then for any  $Z_0 = (y_0, z_0, p_0) \in \Omega \times \mathbb{R} \times \mathbb{R}^3$ , we have that*

- (1) *if  $f(Z_0) = 0$ , then (1.3) admits a 1-convex  $C^\infty$  local solution which is not convex;*
- (2) *if  $f \geq 0$  near  $Z_0$ , then (1.3) admits a 2-convex  $C^\infty$  local solution which is not convex. If  $f(Z_0) > 0$ , (1.3) admits a convex  $C^\infty$  local solution.*
- (3) *if  $f(Z_0) < 0$ , (1.3) admits a 1-convex  $C^\infty$  local solution which is not 2-convex.*

*Moreover, the equation (1.3) is uniformly elliptic with respect to the above local solutions.*

For the local solution, Hong and Zuily [5] obtained the existence of  $C^\infty$  local solutions to arbitrary dimensional Monge-Ampère equation, in which  $f$  is not only nonnegative but also satisfies a variant of Hörmander rank condition. Lin [8] proved the existence of a local  $H^s$  solution in  $\mathbb{R}^2$  with  $f \geq 0$ . We will follow the ideas of [5] and [8, 9], the existence of the local solution can be obtained by a perturbation of polynomial-typed solution for  $S_2[u] = a$  where  $a$  is a constant, so that our solution is in the form

$$u(y) = \frac{1}{2} \sum_{j=1}^3 \tau_j y_j^2 + \varepsilon^5 w(\varepsilon^{-2}y), \quad \tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3.$$

The significance of theorem 1.1 is our results break away from the framework of Gårding cone. The sign of  $f$  is permitted to change in case (1). For the case (2), we say that it is a degenerate 2-Hessian equation if  $f(Z_0) = 0$  (see [10]). The non-convex solution in (1) and (2) never occurs for Monge-Ampère equation. There is also many works about the convexity of solution to Hessian equation, see [11] and reference therein. Besides, these results seems to be strange. However, that is because the relationship between the sign of  $f$  and the ellipticity of the nonlinear  $k$ -Hessian equation may not be close.

The rest of this paper is arranged as follow: in Section 2, we will give definitions and some known results. Section 3 is devoted to the proof of Theorem 1.1.

## 2. PRELIMINARIES

In this section, we collect some definitions and known results of  $k$ -Hessian equations. Firstly some algebraic properties of Gårding cone.

**Proposition 2.1** (See [12]). *Using the notations introduced in Section 1,*

- (1)  $\sigma_k(\lambda) = 0$  for  $\lambda \in \partial\Gamma_k$  and

$$\Gamma_n \subset \dots \subset \Gamma_k \subset \dots \subset \Gamma_1.$$

(2) *Maclaurin's inequalities, for any  $\lambda \in \Gamma_k, 1 \leq l \leq k$ ,*

$$\left[ \frac{\sigma_k(\lambda)}{\binom{n}{k}} \right]^{1/k} \leq \left[ \frac{\sigma_l(\lambda)}{\binom{n}{l}} \right]^{1/l}.$$

(3) *we also have*

$$\begin{cases} \sigma_k(\lambda) = \lambda_i \sigma_{k-1;i}(\lambda) + \sigma_{k;i}(\lambda), & \forall \lambda \in \mathbb{R}^n, \\ \sum_{i=1}^n \sigma_{k;i}(\lambda) = (n-k)\sigma_k(\lambda), & \forall \lambda \in \mathbb{R}^n. \end{cases}$$

(4) *Assume that  $\lambda \in \Gamma_k$  is in descending order,*

$$\lambda_1 \geq \cdots \geq \lambda_{p-1} \geq \lambda_p > 0 \geq \lambda_{p+1} \geq \cdots \geq \lambda_n,$$

*then  $p \geq k$  and*

$$(2.1) \quad \sigma_{k-1;n}(\lambda) \geq \cdots \geq \sigma_{k-1;1}(\lambda) > 0.$$

When  $n = 3$ , we see that  $\sigma_3(\lambda) > 0$  cannot occur for  $\lambda \in \partial\Gamma_2(\lambda)$ , therefore we can express  $\partial\Gamma_2$  as two parts

$$\partial\Gamma_2(\lambda) = \mathbf{P}_1 \cup \mathbf{P}_2,$$

$$\mathbf{P}_1 = \{\lambda \in \mathbb{R}^3; \sigma_1(\lambda) \geq 0, \sigma_2(\lambda) = \sigma_3(\lambda) = 0\},$$

$$\mathbf{P}_2 = \{\lambda \in \mathbb{R}^3; \sigma_1(\lambda) > 0, \sigma_2(\lambda) = 0, \sigma_3(\lambda) < 0\}.$$

Next, we will recall that what condition can lead to the ellipticity.

As for the framework of ellipticity, we follow the ideas of [6] and [7]. Denote  $\text{Sym}(n)$  as the set of symmetric real  $n \times n$  matrix. Through the matrix language, we recall the direct condition which leads to the elliptic  $k$ -Hessian operator. The ellipticity set of the  $k$ -Hessian operator,  $k = 1, 2, \dots, n$ , is

$$E_k = \{S \in \text{Sym}(n) : S_k(S + t\xi \times \xi) > S_k(S) > 0, |\xi| = 1, t \in \mathbb{R}^+\}$$

and the Gårding cones

$$\Gamma_k = \{S \in \text{Sym}(n) : S_k(S + tId) > S_k(S) > 0, t \in \mathbb{R}^+\},$$

where the definition of  $S_k(S)$  is given in (1.2). It is easy to show that  $E_k = \Gamma_k$  only for  $k = 1, n$  and the example in [7] assures that  $\Gamma_k \subset E_k$  and  $\text{mess}(E_k \setminus \Gamma_k) > 0$  when  $1 < k < n$ . Ivochkina, Prokofeva and Yakunina [7] point out that the ellipticity of (1.1) is independent of the sign of  $f$ .

We now present an algebraic property of

$$\frac{\partial}{\partial \tau_i} \sigma_2(\tau) = \sigma_{1,i}(\tau), \quad i = 1, 2, 3,$$

for  $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbf{P}_2$ .

**Lemma 2.2.** *Assume that  $\tau \in \mathbf{P}_2, \tau_1 \geq \tau_2 \geq \tau_3$ . Then we have*

$$0 < \sigma_{1,1}(\tau) \leq \sigma_{1,2}(\tau) \leq \sigma_{1,3}(\tau),$$

*and*

$$(2.2) \quad \tau_3 < 0 < \tau_2 \leq \tau_1.$$

The above result means that for any

$$\psi = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2, \quad \tau \in \mathbf{P}_2$$

it is a solution of 2-Hessian equation  $S_2(\psi) = 0$ , and the linearized operators of  $S_2[u]$  at  $\psi$

$$\mathcal{L} = \sum_{i=1}^3 \sigma_{1,i}(\tau) \partial_i^2$$

is uniformly elliptic,

*Proof.* Recall that, for any  $\tau \in \mathbb{R}^3$ ,

$$\sigma_2(\tau) = \tau_1 \tau_2 + \tau_2 \tau_3 + \tau_1 \tau_3,$$

and

$$\sigma_{1,1}(\tau) = \tau_2 + \tau_3, \quad \sigma_{1,2}(\tau) = \tau_1 + \tau_3, \quad \sigma_{1,3}(\tau) = \tau_1 + \tau_2.$$

Denote  $\lambda + \varepsilon = (\lambda_1 + \varepsilon, \lambda_2 + \varepsilon, \lambda_3 + \varepsilon)$  with  $\lambda \in \mathbb{R}^3$  and  $\varepsilon \in \mathbb{R}$ , then we have the formula

$$\sigma_2(\lambda + \varepsilon) = \sum_{j=0}^2 C(j) \varepsilon^j \sigma_{2-j}(\lambda), \quad C(j) = \frac{\binom{3}{2} \binom{2}{j}}{\binom{3}{2-j}}.$$

For  $\tau \in \mathbf{P}_2$ , we have

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = 0,$$

then

$$\tau + \varepsilon \in \Gamma_2, \quad \forall \varepsilon > 0.$$

Applying (2.1) to  $\tau + \varepsilon$  and letting  $\varepsilon \rightarrow 0^+$ , we get

$$0 \leq \sigma_{1,1}(\tau) \leq \sigma_{1,2}(\tau) \leq \sigma_{1,3}(\tau).$$

Since  $\tau \in \mathbf{P}_2$ , we have

$$\sigma_2(\tau) = \tau_1 \sigma_{1,1}(\tau) + \sigma_{2,1}(\tau) = 0.$$

if  $\sigma_{1,1}(\tau) = \tau_2 + \tau_3 = 0$ , then,

$$\sigma_{2,1}(\tau) = \tau_2 \tau_3 = 0,$$

thus  $\sigma_3(\tau) = \tau_1 \tau_1 \tau_3 = 0$ , which contradicts with the assumption  $\sigma_3(\tau) < 0$ . Then, We have proven that, for any  $\tau \in \mathbf{P}_2$ ,

$$0 < \sigma_{1,1}(\tau) \leq \sigma_{1,2}(\tau) \leq \sigma_{1,3}(\tau).$$

We prove now (2.2). Since  $\sigma_1(\tau) > 0$ , by (4) we have  $\tau_1 > 0$ . We now claim that  $\tau_1 = \tau_2 = \tau_3$  is impossible. Indeed, if that holds, then  $\sigma_1(\tau) = 3\tau_1 > 0$  and  $\sigma_2(\tau) = 3\tau_1^2 > 0$ , which contradicts with the assumption  $\sigma_2(\tau) = 0$ .

Besides,  $\sigma_3(\tau) < 0$  imply that  $\tau_i \neq 0$  and  $\tau_i$  can not be positive at the same time. Then property (4) of Proposition 2.1 implies

$$\tau_3 < 0 < \tau_2 \leq \tau_1.$$

□

We also have the following elliptic results for  $\tau \in \Gamma_1 \setminus \bar{\Gamma}_2$ .

**Lemma 2.3.** *For the Gårding cone, we have*

(1) For any given  $a < 0$ , there exists  $\tau \in \Gamma_1 \setminus \bar{\Gamma}_2$ , such that

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = a.$$

(2) For any given  $b > 0$ , there exists  $\tau \in \Gamma_2 \setminus \bar{\Gamma}_3$ , such that

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = b, \quad \sigma_3(\tau) < 0.$$

(3) For any given  $c > 0$ , there exists  $\tau \in \Gamma_3$ , such that

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = c, \quad \sigma_3(\tau) > 0.$$

Moreover, for all above case, we have

$$\sigma_{1,3}(\tau) > \sigma_{1,2}(\tau) > \sigma_{1,1}(\tau) > 0.$$

*Proof.* We only need to prove the case (1), and to find a  $\tau \in \mathbb{R}^3$ . We can choose  $\alpha > 0$  and  $\beta > 0$  such that

$$(1 + \beta)\alpha - 1 < 0.$$

Then take  $\Theta > 0$  satisfying

$$\Theta^2(1 + \alpha)[(1 + \alpha)\beta - 1] = a.$$

We claim that  $\tau$  can be in the following form

$$\tau = (\tau_1, \tau_2, \tau_3) = ((1 + \alpha)(1 + \beta)\Theta, (1 + \alpha)\Theta, -\Theta).$$

Indeed, from  $1 + \beta > 1$  and  $(1 + \alpha)\Theta > 0$ , we have

$$\tau_1 > \tau_2 > \tau_3,$$

$\sigma_1(\tau) > 0$  and  $\sigma_2(\tau) = a$ . Moreover,

$$\sigma_{1,3}(\tau) = (1 + \alpha)(2 + \beta)\Theta > \sigma_{1,2} = (\alpha\beta + \alpha + \beta)\Theta > \sigma_{1,1}(\tau) = \alpha\Theta > 0.$$

Proof is done.  $\square$

For the linearized operators of  $k$ -Hessian equation, we have the following results, the general version of which can be found in section 2, [2].

**Lemma 2.4.** *The matrix  $S_2^{ij}(r(w))$  and  $(r_{ij}(w))$  can be diagonalized simultaneously, that is, for any smooth function  $w$ , we can find an orthogonal matrix  $T(x, \varepsilon)$  satisfying*

$$\begin{cases} T(x, \varepsilon)(S_2^{ij})^t T(x, \varepsilon) = \text{diag} \left[ \frac{\partial \sigma_2(\lambda)}{\partial \lambda_1}, \frac{\partial \sigma_2(\lambda)}{\partial \lambda_2}, \frac{\partial \sigma_2(\lambda)}{\partial \lambda_3} \right] \\ T(x, \varepsilon)(r_{ij})^t T(x, \varepsilon) = \text{diag} [\lambda_1(x, \varepsilon), \lambda_2(x, \varepsilon), \lambda_3(x, \varepsilon)], \end{cases}$$

where  ${}^t T(x, \varepsilon)$  is the transpose of  $T(x, \varepsilon)$  and  $S_2^{ij}(r(w)) = \partial S_2 / \partial r_{ij}(r(w))$ . Furthermore,

$$T(x, \varepsilon) |_{\varepsilon=0} = Id,$$

where  $Id$  is the identity matrix.

*Proof.* For  $T = (T_{ij})$ , we have

$$(2.3) \quad \sum_{i=1}^3 T_{si} T_{ti} = \delta_s^t.$$

Now we set  $(r_{ij})$  can be diagonalized by  $T$ ,

$$(T_{ij})(r_{ij})^t(T_{ij}) = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \left( \sum_{i,j=1}^3 T_{si}T_{tj}r_{ij} \right)_{st}.$$

Thus, we have, when  $s \neq t$

$$(2.4) \quad \sum_{i,j=1}^3 T_{si}T_{tj}r_{ij} = T_{s1}T_{t1}r_{11} + T_{s2}T_{t2}r_{22} + T_{s3}T_{t3}r_{33} + 2T_{s3}T_{t1}r_{31} + 2T_{s1}T_{t2}r_{12} + 2T_{s3}T_{t2}r_{32} = 0$$

Now for

$$(S_2^{ij}(r_{ij})) = \begin{pmatrix} r_{22} + r_{33} & -r_{21} & -r_{31} \\ -r_{12} & r_{11} + r_{33} & -r_{31} \\ -r_{13} & -r_{23} & r_{11} + r_{22} \end{pmatrix},$$

we have

$$(T_{ij})(r_{ij})^t(T_{ij}) = \left( \sum_{i,j=1}^3 T_{si}T_{tj}S_2^{ij} \right)_{st}.$$

If we could prove that  $\sum_{i,j=1}^3 (T_{si}T_{tj}S_2^{ij})_{st}$  is a diagonal matrix, our proof was done.

Indeed, when  $s \neq t$ , we have

$$(2.5) \quad \sum_{i,j=1}^3 T_{si}T_{tj}S_2^{ij} = T_{s1}T_{t1}(r_{22} + r_{33}) + T_{s2}T_{t2}(r_{11} + r_{33}) + T_{s3}T_{t3}(r_{11} + r_{22}) - 2T_{s1}T_{t2}r_{12} - 2T_{s3}T_{t1}r_{31} - 2T_{s3}T_{t2}r_{32}.$$

By (2.4) and (2.3), (2.5) can be

$$\sum_{i,j=1}^3 T_{si}T_{tj}S_2^{ij} = \sum_{i,j=1}^3 T_{si}T_{tj}(r_{11} + r_{22} + r_{33}) = 0.$$

When  $\varepsilon = 0$ ,  $S_2^{ij}[r(w)]$  and  $(r_{ij}(w))$  are diagonal, thus,  $T$  can be the identity matrix  $Id$ .  $\square$

From the view above, when  $k = 2$  and  $f < 0$ , the corresponding Hessian operator is possible to be uniformly elliptic. In this paper, we will study some uniformly elliptic 2-Hessian equations which have non-positive right-hand functions  $f$ .

### 3. EXISTENCE OF $C^\infty$ LOCAL SOLUTIONS FOR UNIFORMLY ELLIPTIC CASE

From now on, we fixed  $n = 3, k = 2$ , by a translation  $y \rightarrow y - y_0$  and replacing  $u$  by  $u - u(0) - y \cdot Du(0)$ , we can assume  $Z_0 = (0, 0, 0)$  in Theorem 1.1. We prove now the following results,

**Theorem 3.1.** *Let  $f \in C^\infty$  and  $f(Z_0) = 0$  for  $Z_0 = (0, 0, 0) \in \Omega \times \mathbb{R} \times \mathbb{R}^3$ . Then (1.3) admits a 1-convex local solution  $u \in C^\infty$  which is not 3-convex and is of the following form*

$$(3.1) \quad u(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y), \quad \forall (\tau_1, \tau_2, \tau_3) \in \mathbf{P}_2$$

in the neighborhood of  $y_0 = 0$ ,  $\|w\|_{C^{4,\alpha}} \leq 1$  and  $\varepsilon > 0$  very small.

If  $f$  is nonnegative near  $Z_0$ , then (1.3) admits a 2-convex local solution  $u \in C^\infty$  which is not 3-convex. If  $f(Z_0) > 0$ , then (1.3) admits a 3-convex local solution  $u \in C^\infty$ .

Moreover, the equation (1.3) is uniformly elliptic with respect to the solution (3.1).

Remark that, in Theorem 3.1 the function  $f$  is permitted to change sign. It is well known that, for Monge-Ampere operator, the type of equation is determined by the sign of  $f(y, u, Du)$ , it is elliptic if  $f > 0$ , hyperbolic if  $f < 0$  and degenerate elliptic or hyperbolic if  $f$  vanishes; it is of mixed type if  $f$  changes sign [4]. So that Theorem 3.1 never occurs in Monge-Ampère case.

Theorem 3.1 is exactly the part (1) and (2) of Theorem 1.1.

Let  $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbf{P}_2$ , then  $\psi(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2$  is a polynomial-type solution of

$$S_2[\psi] = 0,$$

we follow Lin [8] to introduce the following function

$$u(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y) = \psi(y) + \varepsilon^5 w(\varepsilon^{-2}y), \quad \tau \in \mathbf{P}_2, \quad \varepsilon > 0,$$

as a candidate of solution for equation (1.1). Noting  $y = \varepsilon^2 x$ , we have

$$(D_{y_j} u)(x) = \tau_j \varepsilon^2 x_j + \varepsilon^3 w_j(x), \quad j = 1, \dots, 3,$$

and

$$(D_{y_j y_k} u)(x) = \delta_k^j \tau_j + \varepsilon w_{jk}(x), \quad j, k = 1, \dots, 3,$$

where  $\delta_k^j$  is the Kronecker symbol,  $w_j(x) = (D_{y_j} w)(x)$  and  $w_{jk}(x) = (D_{y_j y_k} w)(x)$ . Then (1.3) transfers to

$$\tilde{S}_2(w) = \tilde{f}_\varepsilon(x, w(x), Dw(x)), \quad x \in B_1(0) = \{x \in \mathbb{R}^3; |x| < 1\}$$

where

$$\tilde{S}_2(w) = S_2(\delta_i^j \tau_j + \varepsilon w_{ij}(x)) = S_2(r(w)),$$

with symmetric matrix  $r(w) = (\delta_i^j \tau_j + \varepsilon w_{ij}(x))$ , and

$$\tilde{f}_\varepsilon(x, w(x), Dw(x)) = f(\varepsilon^2 x, \varepsilon^4 \psi(x) + \varepsilon^5 w(x), \tau_1 \varepsilon^2 x_1 + \varepsilon^3 w_1(x), \dots, \tau_3 \varepsilon^2 x_3 + \varepsilon^3 w_3(x)).$$

Similar to [8] we consider the nonlinear operators

$$(3.2) \quad G(w) = \frac{1}{\varepsilon} [S_2(r(w)) - \tilde{f}_\varepsilon(x, w, Dw)], \quad \text{on } B_1(0).$$

The linearized operator of  $G$  at  $w$  is

$$(3.3) \quad L_G(w) = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}} \partial_{ij}^2 + \sum_{i=1}^3 a_i \partial_i + a,$$

where

$$a_i = -\frac{1}{\varepsilon} \frac{\partial \tilde{f}_\varepsilon(x, z, p_i)}{\partial p_i}(x, w, Dw) = -\varepsilon^2 \frac{\partial f}{\partial p_i}$$

$$a = -\frac{1}{\varepsilon} \frac{\partial \tilde{f}_\varepsilon(x, z, p_i)}{\partial z}(x, w, Dw) = -\varepsilon^4 \frac{\partial f}{\partial z}.$$

Hereafter, we denote  $S_2^{ij}(r(w)) = \frac{\partial S_2(r(w))}{\partial r_{ij}}$ . Since  $S_2(r(w)) = \sigma_2(\lambda(r(w)))$  is invariant under orthogonal transformation, by using Lemma 2.4, the matrix  $(S_2^{ij}(r(w)))$  and  $(r(w))$  can be



diagonalized simultaneously, that is, for any smooth function  $w$ , we can find an orthogonal matrix  $T(x, \varepsilon)$  satisfying

$$\begin{cases} T(x, \varepsilon) \left( S_2^{ij}(r(w)) \right) {}^t T(x, \varepsilon) = \text{diag} \left[ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_1}, \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_2}, \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_3} \right] \\ T(x, \varepsilon) \left( r_{ij}(r(w)) \right) {}^t T(x, \varepsilon) = \text{diag} [\lambda_1(r(w)), \lambda_2(r(w)), \lambda_3(r(w))], \end{cases}$$

where  ${}^t T(x, \varepsilon)$  is the transpose of  $T(x, \varepsilon)$ . Since  $T$  is not unique, we set  $T(x, \varepsilon)|_{\varepsilon=0} = Id$ . After this transformation, in order to prove the uniform ellipticity of  $L_G(w)$

$$\sum_{i,j=1}^n S_2^{ij}(r(w)) \xi_i \xi_j \geq c |\xi|^2, \quad \forall (x, \xi) \in B_1(0) \times \mathbb{R}^3$$

instead we can prove that, by setting  $\xi = {}^t T(x, \varepsilon) \tilde{\xi}$ ,

$$\sum_{j=1}^3 \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_j} |\tilde{\xi}_j|^2 \geq c |\tilde{\xi}|^2,$$

for some  $c > 0$ , where

$$\begin{aligned} \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_1} &= \sigma_{1,1}(\lambda(r(w))) = \lambda_2(r(w)) + \lambda_3(r(w)), \\ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_2} &= \sigma_{1,2}(\lambda(r(w))) = \lambda_1(r(w)) + \lambda_3(r(w)), \\ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_3} &= \sigma_{1,3}(\lambda(r(w))) = \lambda_1(r(w)) + \lambda_2(r(w)). \end{aligned}$$

**Lemma 3.2.** *Assume that  $\tau \in \mathbf{P}_2$  and  $\|w\|_{C^2(B_1(0))} \leq 1$ , then the operator  $L_G(w)$  is a uniformly elliptic operator if  $\varepsilon$  is small enough.*

*Proof.* To prove the operator  $L_G(w)$  is a uniformly elliptic operator, it suffices to prove

$$(3.4) \quad \lambda_i(r(w)) + \lambda_j(r(w)) = \tau_i + \tau_j + O(\varepsilon), \quad i, j = 1, 2, 3, \quad i \neq j.$$

Indeed, for  $\tau \in \mathbf{P}_2$  and Lemma 2.2 give  $\tau_i + \tau_j > 0$ . Thus, for  $\varepsilon$  small enough, (3.4) imply,

$$\lambda_i + \lambda_j \geq \frac{\tau_i + \tau_j}{2} > 0, \quad i \neq j$$

$L_G(w)$  is then a uniformly elliptic operator.

Next, we prove (3.4). By our choice of  $r_{ij}(w)$ ,

$$r(w) = (r_{ij}(w)) = \begin{pmatrix} \tau_1 + \varepsilon w_{11} & \varepsilon w_{12} & \varepsilon w_{13} \\ \varepsilon w_{21} & \tau_2 + \varepsilon w_{22} & \varepsilon w_{23} \\ \varepsilon w_{31} & \varepsilon w_{32} & \tau_3 + \varepsilon w_{33} \end{pmatrix},$$

we write its characteristic polynomial as

$$g(\lambda) = \det(r(w) - \lambda \mathbf{I}) = \prod_{i=1}^3 (\tau_i - \lambda_i) + R(w, \varepsilon)$$

where

$$R(w, \varepsilon) = \sum_{j=1}^3 \varepsilon R_j(w, \varepsilon) + \sum_{j,k} \varepsilon^2 R_{jk}(w, \varepsilon).$$

For any  $\|w\|_{C^2(B_1(0))} \leq 1$  and  $0 < \varepsilon \leq 1$

$$|R_j(w, \varepsilon)| \leq C, \quad |R_{jk}(w, \varepsilon)| \leq C$$

with  $C$  being independent of  $x$  and  $\varepsilon$ . We have also

$$(3.5) \quad S_1(r(w)) = \sigma_1(\tau) + \varepsilon S_1(w), \quad S_2(r(w)) = \sigma_2(\tau) + \varepsilon \tilde{R}_1(w, \varepsilon),$$

and

$$\det(r(w)) = \sigma_3(\tau) + \varepsilon \tilde{R}_2(w, \varepsilon),$$

where for any  $\|w\|_{C^2(B_1(0))} \leq 1$  and  $0 < \varepsilon \leq 1$

$$|\tilde{R}_j(w, \varepsilon)| \leq C, \quad |S_1(w)| \leq C.$$

By using Lemma 2.2, we have  $\tau_3 < 0 < \tau_2 \leq \tau_1$ , then for  $0 < \varepsilon \ll |\tau_3|$ , we have

$$g\left(\frac{3}{4}\tau_3\right) = \left(\tau_1 - \frac{3}{4}\tau_3\right)\left(\tau_2 - \frac{3}{4}\tau_3\right)\left(\frac{\tau_3}{4}\right) + R(w, \varepsilon) < 0,$$

$$g\left(\frac{5}{4}\tau_3\right) = \left(\tau_1 - \frac{5}{4}\tau_3\right)\left(\tau_2 - \frac{5}{4}\tau_3\right)\left(-\frac{\tau_3}{4}\right) + R(w, \varepsilon) > 0,$$

and we see that, by the virtue of Intermediate Value Theorem, there exists an eigenvalue, denoted by  $\lambda_3$ , such that

$$\frac{3}{4}\tau_3 > \lambda_3 > \frac{5}{4}\tau_3, \quad g(\lambda_3) = 0.$$

From  $0 = g(\lambda_3) = (\tau_1 - \lambda_3)(\tau_2 - \lambda_3)(\tau_3 - \lambda_3) + R(w, \varepsilon)$  and

$$\left(\tau_1 - \frac{5}{4}\tau_3\right)\left(\tau_2 - \frac{5}{4}\tau_3\right) > (\tau_1 - \lambda_3)(\tau_2 - \lambda_3) > \left(\tau_1 - \frac{3}{4}\tau_3\right)\left(\tau_2 - \frac{3}{4}\tau_3\right),$$

it follows that

$$\lambda_3 = \tau_3 + \mathcal{O}_1(w, \varepsilon).$$

Since the trace of a matrix is invariant under the orthogonal transformation, then

$$\lambda_1(w) + \lambda_2(w) + \lambda_3(w) = \sigma_1(\tau) + \varepsilon(w_{11} + w_{22} + w_{33}),$$

from which we see that

$$\lambda_1(w) + \lambda_2(w) = \tau_1 + \tau_2 + \mathcal{O}_2(w, \varepsilon).$$

Using

$$\sigma_2(\tau) + \varepsilon \tilde{R}_1(w, \varepsilon) = S_2(r(w)) = \sigma_2(\lambda(r(w))) = \lambda_3(w)(\lambda_1(w) + \lambda_2(w)) + \lambda_1(w)\lambda_2(w),$$

we obtain

$$\lambda_1\lambda_2 = \tau_1\tau_2 + \mathcal{O}_3(w, \varepsilon),$$

which yields either

$$\lambda_1 = \tau_1 + \mathcal{O}_4(w, \varepsilon), \quad \lambda_2 = \tau_2 + \mathcal{O}_5(w, \varepsilon)$$

or

$$\lambda_1 = \tau_2 + \mathcal{O}_5(w, \varepsilon), \quad \lambda_2 = \tau_1 + \mathcal{O}_4(w, \varepsilon)$$

and then (3.4) is proven. Proof is done.  $\square$

We follow now the idea of Hong and Zuily [5] to prove the existence and a priori estimates of solution for linearized operator. In our case, although  $L_G(w)$  is uniformly elliptic, the existence and a priori Schauder estimates of classical solutions are not directly obtainable, because we do not know whether the coefficient  $a$  of  $au$  in (3.3) is non-positive. If we can prove the existence (Lemma 3.3), we can employ Nash-Moser procedure to prove the existence of local solution for (1.3) in Hölder space rather than Sobolev space. One goal is to see how the procedure depends on the condition  $\|w_k\|_{C^{4,\alpha}} \leq A$ . We shall use the following schema:

$$(3.6) \quad \begin{cases} w_0 = 0, & w_m = w_{m-1} + \rho_{m-1}, \quad m \geq 1, \\ L_G(w_m)\rho_m = g_m, & \text{in } B_1(0), \\ \rho_m = 0 & \text{on } \partial B_1(0), \\ g_m = -G(w_m), \end{cases}$$

where

$$g_0(x) = \frac{1}{\varepsilon} \left( \sigma_2(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) \right).$$

It is pointed out on page 107, [3] that, if the operator  $L_G$  does not satisfy the condition  $a \leq 0$ , as is well known from simple examples, the Dirichlet problem for  $L_G(w)\rho = g$  no longer has a solution in general. Notice  $a$  in (3.9) has the factor  $\varepsilon^4$ , we will take advantage of smallness of  $a$  to obtain the uniqueness and existence of solution for Dirichlet problem (3.9) and then uniformly Schauder estimates of its solution follows.

**Lemma 3.3.** *Assume that  $\|w\|_{C^{4,\alpha}(B_1(0))} \leq A$ . Then there exists a unique solution  $\rho \in C^{2,\alpha}(\overline{B_1(0)})$  to the following Dirichlet problem*

$$(3.7) \quad \begin{cases} L_G(w)\rho = g, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

for all  $g \in C^\alpha(\overline{B_1(0)})$ . Moreover,

$$(3.8) \quad \|\rho\|_{C^{4,\alpha}(\overline{B_1(0)})} \leq C \|g\|_{C^{2,\alpha}(\overline{B_1(0)})}, \quad \forall g \in C^{2,\alpha}(\overline{B_1(0)}),$$

where the constant  $C$  depends on  $A, \tau$  and  $\|f\|_{C^{4,\alpha}}$ . Moreover,  $C$  is uniform for  $0 < \varepsilon \leq \varepsilon_0$  for some  $\varepsilon_0 > 0$ .

By virtue of (3.3), we write (3.7) as

$$(3.9) \quad \begin{cases} L_G(w)\rho = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}} \partial_i \partial_j \rho + \sum_{i=1}^3 a_i \partial_i \rho + a \rho = g, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

where

$$a_i = -\varepsilon^2 \frac{\partial f}{\partial p_i}, \quad a = -\varepsilon^4 \frac{\partial f}{\partial z}.$$

Notice that for  $\frac{\partial S_2(r(w))}{\partial r_{ij}}$ ,  $a_i = a_i(x, w(x), Dw(x))$ ,  $a = a(x, w(x), Dw(x))$  and  $g_m = -G(w_m) = g_m(x, w(x), Dw(x), D^2 w(x))$  by (3.6), we regard them as the functions with variable  $x$ . In a word, we regard that all of the coefficients and non-homogeneous term in (3.9) are functions of variable  $x$ . For example,

$$\tilde{f}_\varepsilon(x, w(x), Dw(x)) = f(\varepsilon^2 x, \varepsilon^4 \psi(x) + \varepsilon^5 w(x), \tau_1 \varepsilon^2 x_1 + \varepsilon^3 w_1(x), \dots, \tau_3 \varepsilon^2 x_3 + \varepsilon^3 w_3(x)),$$

and

$$\|\tilde{f}_\varepsilon\|_{C^3} = \sup \left\{ |D_x^\beta[\tilde{f}_\varepsilon]|, |0 \leq \beta \leq 3, x \in B_1(0) \right\}$$

$$\|\tilde{f}_\varepsilon\|_{C^{3,\alpha}} = \|\tilde{f}_\varepsilon\|_{C^3} + \sup \left\{ \frac{|D_x^\beta[\tilde{f}_\varepsilon](x) - D_x^\beta[\tilde{f}_\varepsilon](z)|}{|x-z|^\alpha}, |\beta| = 3, x \neq z \in B_1(0) \right\}$$

When we regard  $\tilde{f}_\varepsilon$  as a function of variable  $x$ , usually  $\|f\|_{C^{3,\alpha}}$  is denoted as  $\|f\|_{C^{3,\alpha}(B_1(0))}$ , but it maybe cause confusion because it must be involved in  $D^\alpha w, 0 \leq |\alpha| \leq 3$  as above. Therefore, here and after, we denote the norm as  $\|\tilde{f}_\varepsilon\|_{C^3}, \|\tilde{f}_\varepsilon\|_{C^{3,\alpha}}$  as above, by dropping  $B_1(0)$ .

*Proof.* Let the constant  $\mu(\tau) = \inf \left\{ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_i} : \|w\|_{C^{4,\alpha}(B_1(0))} \leq A, i = 1, 2, 3 \right\}$ . By Lemma 3.2,  $\mu(\tau) > 0$ . Applying Theorem 3.7 [3] to the solution  $u \in C^0(\overline{B_1(0)}) \cap C^2(B_1(0))$  of

$$\begin{cases} L_G(w)u = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}} \partial_i \partial_j u + \sum_{i=1}^3 a_i \partial_i u = g, & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0) \end{cases}$$

we have

$$(3.10) \quad \sup |u| \leq \frac{C}{\mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})},$$

where  $C = \exp^{2(\beta+1)} - 1$  and  $\beta = \sup \left\{ \frac{|a_i|}{\mu(\tau)} : i = 1, 2, 3 \right\}$

Let  $C_1 = 1 - C \sup \frac{|a_i|}{\mu(\tau)}$  with  $C$  being the constant in (3.10). If we choose  $\varepsilon_0 > 0$  small (the smallness of  $a$ ), then  $C_1 > \frac{1}{2}$  uniformly for  $0 < \varepsilon < \varepsilon_0$ . Applying Corollary 3.8 [3] to the solution  $\rho$  to Dirichlet problem (3.9), we have

$$(3.11) \quad \sup |\rho| \leq \frac{1}{C_1} \left[ \sup_{\partial B_1(0)} |\rho| + \frac{C}{\mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})} \right] = \frac{C}{C_1 \mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})},$$

from which we see that the homogeneous problem

$$\begin{cases} L_G(w)\rho = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}} \partial_i \partial_j \rho + \sum_{i=1}^3 a_i \partial_i \rho + a\rho = 0, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

has only the trivial solution. Then we can apply a Fredholm alternative, Theorem 6.15 [3], to the inhomogeneous problem (3.9) for which we can assert that it has a unique  $C^{2,\alpha}(\overline{B_1(0)})$  solution for all  $g \in C^\alpha(\overline{B_1(0)})$ .

With the existence and uniqueness at hand, we can apply Theorem 6.19 [3] to obtain higher regularity up to boundary for solution to (3.9). Besides this, we have the Schauder estimates (see Problem 6.2, [3])

$$(3.12) \quad \|\rho\|_{C^{4,\alpha}} \leq C(A, \tau, \|f\|_{C^{3+\alpha}}) \left[ \|\rho_k\|_{C^0(\overline{B_1(0)})} + \|g_k\|_{C^{2,\alpha}(\overline{B_1(0)})} \right],$$

where  $C$  depends on  $C^{2,\alpha}$ -norm of all of the coefficients; the uniform ellipticity; boundary value and boundary itself. we explain the dependence of  $C(A, \tau, \|f\|_{C^{3+\alpha}})$ . Firstly, Since the first two derivatives of  $w$  have come into the principal coefficients  $\frac{\partial S_2(r(w))}{\partial r_{ij}}$ , then their  $C^{2+\alpha}$ -norms must be involved in  $\|w\|_{C^{4,\alpha}}$ , and at last  $\|w\|_{C^{4,\alpha}} \leq A$  arise into  $C$ . Similarly, by virtue of the coefficients  $a_i$  and  $a$ ,  $\|f\|_{C^{3,\alpha}}$  and  $\|w\|_{C^{3,\alpha}} \leq A$  must arise into  $C$ . Secondly, it depends on uniform ellipticity, that is, on

$$\inf \left\{ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_i} : \|w\|_{C^{4,\alpha}(B_1(0))} \leq A, i = 1, 2, 3 \right\}$$

and

$$\sup \left\{ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_i} : \|w\|_{C^{4,\alpha}(B_1(0))} \leq A, i = 1, 2, 3, \right\},$$

so  $(\tau = (\tau_1, \tau_2, \tau_3))$  and  $A$  arise into  $C$ .

Thirdly, Since boundary value is  $=0$  and boundary  $\partial B_1(0)$  is  $C^\infty$ , so the two ingredients do not occur into  $C$ . Substituting (3.11) into (3.12), we obtain (3.8).  $\square$

It follows from standard elliptic theory (see Theorem 6.17, [3] and Remark 2, [1]) and an iteration argument that we obtain.

**Corollary 3.4.** *Assume that  $u \in C^{2,\alpha}(\Omega)$  is a solution of (1.3), and the linearized operators with respect to  $u$ ,*

$$\mathcal{L}_u = \sum_{i,j=1}^3 \frac{\partial S_2(u_{ij})}{\partial r_{ij}} \partial_{ij}^2 - \sum_{i=1}^3 \frac{\partial f}{\partial p_i}(y, u(y), Du(y)) \partial_i - \frac{\partial f}{\partial z}(y, u(y), Du(y))$$

is uniformly elliptic, then  $u \in C^\infty(\Omega)$ .

*Proof.* Let  $v$  be a function on  $\Omega$  and denote by  $e_l$ ,  $l = 1, 2, 3$  the unit coordinate vector in the  $y_l$  direction. We define the difference quotient of  $v$  at  $y$  in the direction  $e_l$  by

$$\Delta^h v(y) = \Delta_l^h v(y) = \frac{v(y + he_l) - v(y)}{h}.$$

Since

$$\begin{aligned} & S_2(u_{ij}(y + he_l)) - S_2(u_{ij}(y)) \\ &= \int_0^1 \frac{d}{dt} [S_2(tu_{ij}(y + he_l) + (1-t)u_{ij}(y))] dt \\ &= \sum_{i,j=1}^3 \int_0^1 \frac{\partial}{\partial r_{ij}} [S_2(tu_{ij}(y + he_l) + (1-t)u_{ij}(y))] dt [u_{ij}(y + he_l) - u_{ij}(y)] \\ &\equiv \sum_{i,j=1}^3 a_{ij}(y) [u_{ij}(y + he_l) - u_{ij}(y)] \end{aligned}$$

and Taylor expansion give

$$\begin{aligned} & f(y + he_l, u(y + he_l), Du(y + he_l)) - f(y, u(y), Du(y)) \\ &= \sum_{i=1}^3 b_i(y) [u_i(y + he_l) - u_i(y)] + c(y) [u(y + he_l) - u(y)] + g(y)h \end{aligned}$$

with

$$b_i(y) = \int_0^1 \frac{\partial f}{\partial p_i}(t(y + he_l) + (1-t)y, tu(y + he_l) + (1-t)u(y), tDu(y + he_l) + (1-t)D(y)) dt$$

$$c(y) = \int_0^1 \frac{\partial f}{\partial z}(t(y + he_l) + (1-t)y, tu(y + he_l) + (1-t)u(y), tDu(y + he_l) + (1-t)D(y)) dt$$

$$g(y) = \int_0^1 \frac{\partial f}{\partial y_l}(t(y + he_l) + (1-t)y, tu(y + he_l) + (1-t)u(y), tDu(y + he_l) + (1-t)D(y)) dt.$$

Taking the difference quotients of both sides of the equation

$$S_2(u_{ij}(y)) = f(y, u, Du),$$

we have

$$\sum_{i,j=1}^3 a_{ij}(y) \partial_i \partial_j \Delta^h u(y) - \sum_{i=1}^3 b_i(y) \partial_i \Delta^h u(y) - c(y) \Delta^h u(y) = g(y).$$

Since  $u \in C^{2,\alpha}(\Omega)$ , then all the coefficients  $a_{ij}, b_i, c$  and inhomogeneous term  $g$  are in  $C^\alpha(\Omega)$ , from the interior estimates of Corollary 6.3 in [3], we can infer

$$\Delta^h u \in C^{2,\alpha}(\Omega).$$

Letting  $h \rightarrow 0$ , we see  $\partial_l u \in C^{2,\alpha}(\Omega), l = 1, 2, 3$  and

$$\sum_{i,j=1}^3 \frac{\partial S_2(D^2 u)}{\partial r_{ij}} \partial_i \partial_j (\partial_l u) - \sum_{i=1}^3 \frac{\partial f}{\partial p_i} \partial_i (\partial_l u) - \frac{\partial f}{\partial z} (\partial_l u) = \frac{\partial f}{\partial y_l}.$$

Repeating the above proof, we obtain  $u \in C^\infty(\Omega)$ .  $\square$

Using above Lemma 3.3, we can use the procedure (3.6) to construct the sequence  $\{w_m\}_{m \in \mathbb{N}}$ . Now we study the convergence of  $\{w_m\}_{m \in \mathbb{N}}$  and that of  $\{g_m\}_{m \in \mathbb{N}}$ .

**Proposition 3.5.** *Let  $\{w_m\}_{m \in \mathbb{N}}$  and  $\{g_m\}_{m \in \mathbb{N}}$  the sequence in (3.6). Suppose that  $\|w_j\|_{C^{4,\alpha}} \leq A$  for  $j = 1, 2, \dots, k$ . Then we have*

$$(3.13) \quad \|g_{k+1}\|_{C^{2,\alpha}} \leq C[\|g_k\|_{C^{2,\alpha}}^2 + \|g_k\|_{C^{2,\alpha}}^3],$$

where  $C$  is some positive constant depends only on  $\tau, A$  and  $\|f\|_{C^{4,\alpha}}$ . In particular,  $C$  is independent of  $k$ .

*Proof.* Applying Taylor's expansion with integral-typed remainder to (3.2), we have

$$\begin{aligned} -g_{k+1} &= G(w_k + \rho_k) = G(w_k) + L_G(w_k)\rho_k + Q(w_k, \rho_k) \\ &= -g_k + L_G(w_k)\rho_k + Q(w_k, \rho_k) = Q(w_k, \rho_k), \end{aligned}$$

where  $Q_k$  is the quadratic error of  $G$  which consists of  $S_2$  and  $f$ .

$$\begin{aligned} Q(w_k, \rho_k) &= \sum_{i,j,st} \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 S_2(w_k + \mu\rho_k)}{\partial w_{ij} \partial w_{st}} d\mu(\rho_k)_{ij}(\rho_k)_{st} \\ &\quad + \sum_{i,j} \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w_i \partial w_j} d\mu(\rho_k)_i(\rho_k)_j \\ &\quad + \frac{1}{\varepsilon} \sum_i \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w \partial w_i} d\mu(\rho_k)_i(\rho_k) \\ &\quad + \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w^2} d\mu \cdot \rho_k^2 \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

Since  $S_2((r(w)))$  is a second-order homogeneous polynomial with variable  $r_{ij}(r(w))$  and  $\tilde{f}_\varepsilon(x, w, Dw)$  is independent of  $r_{ij}$ , we see that

$$\begin{aligned} \left| \frac{\partial^2 S_2(w_k + \mu\rho_k)}{\partial w_{ij} \partial w_{st}} \right| &= \frac{\partial^2 S_2}{\partial w_{ij} \partial w_{st}} (\delta_i^j \tau_i + \varepsilon(w_k + \mu\rho_k)_{ij}) = \varepsilon^2 \quad \text{or} \quad 0, \\ \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w_i \partial w_j} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon^5(w_k + \mu\rho_k), \varepsilon^3 Dw + \varepsilon^3 D(w_k + \mu\rho_k))]}{\partial w_i \partial w_j} \right| \leq \varepsilon^6 \cdot \|f\|_{C^2}, \end{aligned}$$

$$\begin{aligned} \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w \partial w_i} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon^5(w_k + \mu\rho_k), \varepsilon^3 D\psi + \varepsilon^3 D(w_k + \mu\rho_k))]}{\partial w \partial w_i} \right| \leq \varepsilon^8 \|f\|_{C^2}, \\ \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w^2} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon^5(w_k + \mu\rho_k), \varepsilon^3 D\psi + \varepsilon^3 D(w_k + \mu\rho_k))]}{\partial w^2} \right| = \varepsilon^{10} \|f\|_{C^2}. \end{aligned}$$

Thus,  $I_i$  ( $1 \leq i \leq 4$ ) in  $Q_k$  are under control by  $O(\varepsilon)$ ,  $O(\varepsilon^5)$ ,  $O(\varepsilon^7)$  and  $O(\varepsilon^9)$ , respectively. Therefore

$$\|I_1\|_{C^{2,\alpha}} \leq C \|\rho_k\|_{C^2} \|\rho_k\|_{C^{4,\alpha}}$$

and

$$\begin{aligned} \|I_2\|_{C^{2,\alpha}} &\leq C \|f\|_{C^{4,\alpha}} (\|w_k\|_{C^{3,\alpha}} + \|\rho_k\|_{C^{3,\alpha}}) \|\rho_k\|_{C^1}^2 + C \|f\|_{C^2} \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1} \\ &\leq C \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1}^2 + C \|\rho_k\|_{C^1}^2 + C \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1} \end{aligned}$$

where  $C$  depends on  $A$  and  $\|f\|_{C^{4,\alpha}}$ . And  $\|I_3\|_{C^{2,\alpha}}$  and  $\|I_4\|_{C^{2,\alpha}}$  can be estimated similarly. Accordingly,

$$\begin{aligned} \|g_{k+1}\|_{C^{2,\alpha}} &= \|Q(w_k, \rho_k)\|_{C^{2,\alpha}} \leq \sum_{i=1}^4 \|I_i\|_{C^{2,\alpha}} \\ &\leq C \|\rho_k\|_{C^2} \|\rho_k\|_{C^{4,\alpha}} + C \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1}^2 + \|\rho_k\|_{C^1}^2 + C \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1} \end{aligned}$$

where  $C$  is independent of  $k$  but dependent of  $A$  and  $\|f\|_{C^{4,\alpha}}$ . Thus, by the interpolation inequalities, we have

$$\|g_{k+1}\|_{C^{2,\alpha}} \leq C \|\rho_k\|_{C^{4,\alpha}}^2 + C \|\rho_k\|_{C^{4,\alpha}}^3,$$

where  $C$  is independent of  $k$ . By Schauder estimates of Lemma 3.3, we have

$$\|\rho_k\|_{C^{4,\alpha}} \leq C \|g_k\|_{C^{2,\alpha}}.$$

Combining the estimates above, we obtain (3.13). Proof is done.  $\square$

Since  $C$  is independent of  $k$ , more exactly,  $A$ ,  $\tau$  and  $\|f\|_{C^{4,\alpha}}$  are independent of  $k$ . So here and after, we can assume  $A = 1$ .

**Proof of Theorem 3.1.** Set

$$(3.14) \quad d_{k+1} = C \|g_{k+1}\|_{C^{2,\alpha}}.$$

By (3.13) with letting  $C \geq 1$  we have

$$d_{k+1} \leq d_k^2 + d_k^3.$$

Take  $\tau \in \mathbb{R}^3$  as in Lemmas 2.2 and 2.3 such that  $\sigma_2(\tau) = f(0, 0, 0)$ , we have

$$\begin{aligned} g_0(x) &= \frac{1}{\varepsilon} (\sigma_2(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3))) \\ &= \frac{1}{\varepsilon} [\sigma_2(\tau) - f(0, 0, 0)] \\ &\quad + \varepsilon \int_0^1 x \cdot (\partial_y f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt \\ &\quad + \varepsilon^3 \int_0^1 \psi(x) (\partial_z f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt \\ &\quad + \varepsilon \int_0^1 (\tau_1 x_1, \tau_2 x_2, \tau_3 x_3) \cdot (\partial_p f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt, \end{aligned}$$

then

$$\|g_0\|_{C^{2,\alpha}(B_1(0))} \leq \varepsilon C_1 \|f\|_{C^{3,\alpha}}.$$

We can choose  $0 < \varepsilon \leq \varepsilon_0$  so small such that

$$C\|g_0\|_{C^{2,\alpha}(B_1(0))} \leq 1/4, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Notice  $\varepsilon_0$  is independent of  $k$ . Since  $d_0 = C\|g_0\|_{C^{2,\alpha}}$ , we have  $d_1 \leq 2d_0^2$  and, by induction,

$$d_{k+1} \leq 2^{2^{k+1}} d_0^{2^{k+1}} \leq (2C)^{2^{k+1}} \|g_0\|_{C^{2,\alpha}}^{2^{k+1}},$$

Thus, by (3.14)

$$\|g_{k+1}\|_{C^{2,\alpha}} \leq (2C)^{2^{k+1}-1} \|g_0\|_{C^{2,\alpha}}^{2^{k+1}}.$$

Firstly, we claim that there exists  $\varepsilon > 0$ , depending on  $\tau$  and  $\|f\|_{C^{3,\alpha}}$  such that

$$\|w_k\|_{C^{4,\alpha}(B_1(0))} \leq 1, \quad \forall k \geq 1.$$

Indeed, set  $w_0 = 0$ , we have by (3.13)

$$\begin{aligned} \|w_{k+1}\|_{C^{4,\alpha}(B_1(0))} &= \left\| \sum_{i=0}^k \rho_i \right\|_{C^{4,\alpha}(B_1(0))} \leq \sum_{i=0}^k \|\rho_i\|_{C^{4,\alpha}(B_1(0))} \\ &\leq \sum_{i=0}^k C \|g_i\|_{C^{2,\alpha}(B_1(0))} \leq \sum_{i=0}^k \left(2C \|g_0\|_{C^{2,\alpha}(B_1(0))}\right)^{2^i} \end{aligned}$$

where  $C$  is defined in Lemma 3.5. Thus, for any  $k$ ,

$$\|w_{k+1}\|_{C^{4,\alpha}(B_1(0))} \leq \sum_{i=0}^{\infty} \left(C \|g_0\|_{C^{2,\alpha}(B_1(0))}\right)^{2^i} \leq \sum_{i=0}^{\infty} 2^{-2^i} \leq 1.$$

Then, by Azelà-Ascoli Theorem, we have

$$w_k \rightarrow w \quad \text{in } C^4(B_1(0)).$$

From (3.13), we see that

$$\|g_{k+1}\|_{C^{2,\alpha}(B_1(0))} \leq \left(\frac{1}{2}\right)^{2^k} \rightarrow 0,$$

and then  $g_m = -G(w_m)$  yields

$$G(w) = \frac{1}{\varepsilon} [S_2(r(w)) - \tilde{f}(x, w, Dw)] = 0, \quad \text{on } B_1(0)$$

which yields that the function

$$u(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y) \in C^4(B_{\varepsilon^2}(0)),$$

is a solution of

$$S_2[u] = f(y, u, Du), \quad \text{on } B_{\varepsilon^2}(0).$$

Now if  $f(0, 0, 0) = 0$ , we take  $\tau \in \mathbf{P}_2$ , then  $\sigma_1(\tau) > 0$ ,  $\sigma_2(\tau) = 0$ ,  $\sigma_3(\tau) < 0$ , and (3.5) imply,

$$S_j[u] = \sigma_j(\lambda) = \sigma_j(\tau) + O(\varepsilon), \quad j = 1, 2, 3$$



it follows that  $S_1[u] > 0, S_3[u] < 0$  on  $B_{\varepsilon^2}(0)$  for small  $\varepsilon > 0$ , that is,  $u$  is 1-convex but not convex. Moreover if  $S_2[u] = f \geq 0$  near  $Z_0$  and  $f(Z_0) = 0$ , we see that  $u$  is 2-convex by definition, but not 3-convex.

If  $S_2[u] = f > 0$  near  $Z_0$ , we take  $\tau \in \mathbb{R}^3$  given in (2) and (3) of Lemmas 2.3, then we can get the 3-convex or non convex local solutions.

The  $C^\infty$  regularity of solution is given by Corollary 3.4. We have then proved Theorem 3.1.  $\square$

We also have the following elliptic results for negative  $f$

**Theorem 3.6.** *Let  $f \in C^\infty, f(0, 0, 0) < 0$ . Then (1.3) admits a 1-convex local solution  $u \in C^\infty$  in a neighborhood of  $y_0 = 0$  which is not 2-convex, it is of the following form*

$$u(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y),$$

and the equation (1.3) is uniformly elliptic with respect to this solution.

*Proof.* For  $a = f(0, 0, 0) < 0$ , take  $\tau \in \mathbb{R}^3$  as in (1) of Lemma 2.3 such that

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = f(0, 0, 0) < 0,$$

and

$$\sigma_{1,3}(\tau) > \sigma_{1,2}(\tau) > \sigma_{1,1}(\tau) > 0.$$

Now the proof is exactly same as that of Theorem 3.1 except the estimate of term  $g_0$ , we use Taylor expansion,

$$\begin{aligned} g_0(x) &= -G(0) = \frac{1}{\varepsilon} [S_2(r(0)) - \tilde{f}(x, 0, 0)] \\ &= \frac{1}{\varepsilon} [\sigma_2(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3))] \\ &= \frac{1}{\varepsilon} [\sigma_2(\tau) - f(0, 0, 0)] \\ &\quad + \varepsilon \int_0^1 x \cdot (\partial_y f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt \\ &\quad + \varepsilon^3 \int_0^1 \psi(x) (\partial_z f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt \\ &\quad + \varepsilon \int_0^1 (\tau_1 x_1, \tau_2 x_2, \tau_3 x_3) \cdot (\partial_p f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt, \end{aligned}$$

then we can end the proof of Theorem 3.6 exactly as that of Theorem 3.1.  $\square$

**Acknowledgements.** The research of first author is supported by the National Science Foundation of China No.11171339 and Partially supported by National Center for Mathematics and Interdisciplinary Sciences. The research of the second author and the last author is supported partially by ‘‘The Fundamental Research Funds for Central Universities’’ and the National Science Foundation of China No. 11171261.

## REFERENCES

- [1] L. Caffarelli, L. Nirenberg, J. Spruck, Dirichlet problem for nonlinear second order elliptic equations I, Monge-Ampere equations, Comm. Pure Appl. Math., 37(1984), 369-402.
- [2] B. Guan, J. Spruck, Locally convex hypersurfaces of constant curvature with boundary, Comm. Pure Appl. Math. 57 (2004) 1311-1331.
- [3] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer, 1983.
- [4] Q. Han, Local solutions to a class of Monge-Ampere equations of mixed type, Duke Math. J., Volume 136, Number 3 (2007), 401-618
- [5] J. Hong, C. Zuily, Existence of  $C^\infty$  local solutions for the Monge-Ampère equation, Invent.Math.,89(1987), 645-661.
- [6] N.M. Ivochkina , A description of the stability cones generated by differential operators of Monge-Ampere type, English transl.: Math. USSR Sbornik, 50(1)(1985), : 259-268.
- [7] N. M. Ivochkina, S. I. Prokofeva, G. V. Yakunina , The Gårding cones in the modern theory of fully nonlinear second order differential equations, Journal of Mathematical Sciences , Volume 184(2012), 295-315.
- [8] C.S. Lin, The local isometric embedding in  $\mathbb{R}^3$  of 2-dimensional Riemannian manifolds with non negative curvature. J. of Diff. Equations 21(1985): 213-230.
- [9] C.S. Lin, The local isometric embedding in  $\mathbb{R}^3$  of two dimensional Riemannian manifolds with Gaussian curvature changing sign clearly, Comm. Pure Appl. Math.,39(1986),867-887.
- [10] Q. Wang, C.-J. Xu ,  $C^{1,1}$  solution of the Dirichlet problem for degenerate k-Hessian equations, Nonlinear Analysis 104 (2014) 133-146.
- [11] X.-N. Ma, L. Xu, The convexity of solution of a class Hessian equation in bounded convex domain in  $\mathbb{R}^3$ , J. Funct. Anal.,255(2008),1713-1723.
- [12] X.-J. Wang, The  $k$ -Hessian equation, Lecture Notes in Math., Springer, Dordrecht, 2009.

GUJI TIAN, WUHAN INSTITUTE OF PHYSICS AND MATHEMATICS, CHINESE ACADEMY OF SCIENCES, WUHAN, P.R. CHINA  
*E-mail address:* tianguji@wipm.ac.cn

QI WANG, SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY 430072, WUHAN, P. R. CHINA  
*E-mail address:* qiwan88@whu.edu.cn

CHAO-JIANG XU, SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY 430072, WUHAN, P. R. CHINA,  
AND UNIVERSITÉ DE ROUEN, CNRS UMR 6085-LABORATOIRE DE MATHÉMATIQUES, 76801 SAINT-ÉTIENNE DU ROUVRAY,  
FRANCE  
*E-mail address:* chao-jiang.xu@univ-rouen.fr