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**C^∞ LOCAL SOLUTIONS
OF ELLIPTICAL 2-HESSIAN EQUATION IN \mathbb{R}^3**

G. TIAN, Q. WANG AND C.-J. XU

In remembrance of the late professor Rou-Huai Wang
on the occasion of his 90th Birthday

ABSTRACT. In this work, we study the existence of C^∞ local solutions to 2-Hessian equation in \mathbb{R}^3 . We consider the case that the right hand side function f possibly vanishes, changes the sign, is positively or negatively defined. We also give the convexities of solutions which are related with the annulation or the sign of right-hand side function f . The associated linearized operator are uniformly elliptic.

1. INTRODUCTION

We are interested by the following k -Hessian equation

$$(1.1) \quad S_k[u] = f(y, u, Du)$$

on an open domain $\Omega \subset \mathbb{R}^n$, $1 \leq k \leq n$, $f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Denote $Du = (\partial_1 u, \dots, \partial_n u)$ and D^2u is the Hessian matrix $(\partial_i \partial_j u)_{1 \leq i, j \leq n}$. the Hessian operators $S_k[u]$ is defined as follows:

$$(1.2) \quad S_k[u] = \sigma_k(\lambda(D^2u)), \quad k = 1, \dots, n,$$

where $\lambda(D^2u) = (\lambda_1, \lambda_2, \dots, \lambda_n)$, λ_j is the eigenvalue of the Hessian matrix (D^2u) , and

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k -th elementary symmetric polynomial. Denoting, for $k, j \in \{1, \dots, n\}$,

$$\sigma_{k,j} = \frac{\partial \sigma_{k+1}(\lambda)}{\partial \lambda_j} = \sigma_k|_{\lambda_j=0}.$$

We also introduce the Gårding cone Γ_k which is the open symmetric convex cone in \mathbb{R}^n , with vertex at the origin, given by

$$\Gamma_k = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \forall j = 1, \dots, k\}.$$

When $k = 1$, (1.1) is a semi-linear Poisson equation, and it is Monge-Ampère equation for $k = n$.

We say that a function $u \in C^2$ is k -convex, if

$$\lambda(D^2u) \in \bar{\Gamma}_k,$$

the n -convex function is simply called convex function.

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We say that a function u is a local solution of (1.1) near $y_0 \in \Omega$, if there exists a neighborhood of y_0 , $V_{y_0} \subset \Omega$ such that $u \in C^2(V_{y_0})$ satisfies the equation (1.1) on V_{y_0} .

In this work, we study the existence of C^∞ -local solution of the following 2-Hessian equation in \mathbb{R}^3 ,

$$(1.3) \quad S_2[u] = f(y, u, Du), \quad \text{on } \Omega \subset \mathbb{R}^3,$$

where we also have

$$S_2[u] = u_{11}u_{22} - u_{12}^2 + u_{22}u_{33} - u_{23}^2 + u_{11}u_{33} - u_{13}^2.$$

We have proved the following results.

Theorem 1.1. *Assume that $f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^3)$, then for any $Z_0 = (y_0, z_0, p_0) \in \Omega \times \mathbb{R} \times \mathbb{R}^3$, we have that*

- (1) *if $f(Z_0) = 0$, then (1.3) admits a 1-convex C^∞ local solution which is not convex;*
- (2) *if $f \geq 0$ near Z_0 , then (1.3) admits a 2-convex C^∞ local solution which is not convex. If $f(Z_0) > 0$, (1.3) admits a convex C^∞ local solution.*
- (3) *if $f(Z_0) < 0$, (1.3) admits a 1-convex C^∞ local solution which is not 2-convex.*

Moreover, the equation (1.3) is uniformly elliptic with respect to the above local solutions.

For the local solution, Hong and Zuily [5] obtained the existence of C^∞ local solutions to arbitrary dimensional Monge-Ampère equation, in which f is not only nonnegative but also satisfies a variant of Hörmander rank condition. Lin [8] proved the existence of a local H^s solution in \mathbb{R}^2 with $f \geq 0$. We will follow the ideas of [5] and [8, 9], the existence of the local solution can be obtained by a perturbation of polynomial-typed solution for $S_2[u] = a$ where a is a constant, so that our solution is in the form

$$u(y) = \frac{1}{2} \sum_{j=1}^3 \tau_j y_j^2 + \varepsilon^5 w(\varepsilon^{-2}y), \quad \tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3.$$

The significance of theorem 1.1 is our results break away from the framework of Gårding cone. The sign of f is permitted to change in case (1). For the case (2), we say that it is a degenerate 2-Hessian equation if $f(Z_0) = 0$ (see [10]). The non-convex solution in (1) and (2) never occurs for Monge-Ampère equation. There is also many works about the convexity of solution to Hessian equation, see [11] and reference therein. Besides, these results seems to be strange. However, that is because the relationship between the sign of f and the ellipticity of the nonlinear k -Hessian equation may not be close.

The rest of this paper is arranged as follow: in Section 2, we will give definitions and some known results. Section 3 is devoted to the proof of Theorem 1.1.

2. PRELIMINARIES

In this section, we collect some definitions and known results of k -Hessian equations. Firstly some algebraic properties of Gårding cone.

Proposition 2.1 (See [12]). *Using the notations introduced in Section 1,*

- (1) $\sigma_k(\lambda) = 0$ for $\lambda \in \partial\Gamma_k$ and

$$\Gamma_n \subset \dots \subset \Gamma_k \subset \dots \subset \Gamma_1.$$

(2) *Maclaurin's inequalities, for any $\lambda \in \Gamma_k, 1 \leq l \leq k$,*

$$\left[\frac{\sigma_k(\lambda)}{\binom{n}{k}} \right]^{1/k} \leq \left[\frac{\sigma_l(\lambda)}{\binom{n}{l}} \right]^{1/l}.$$

(3) *we also have*

$$\begin{cases} \sigma_k(\lambda) = \lambda_i \sigma_{k-1;i}(\lambda) + \sigma_{k;i}(\lambda), & \forall \lambda \in \mathbb{R}^n, \\ \sum_{i=1}^n \sigma_{k;i}(\lambda) = (n-k)\sigma_k(\lambda), & \forall \lambda \in \mathbb{R}^n. \end{cases}$$

(4) *Assume that $\lambda \in \Gamma_k$ is in descending order,*

$$\lambda_1 \geq \cdots \lambda_{p-1} \geq \lambda_p > 0 \geq \lambda_{p+1} \geq \cdots \lambda_n,$$

then $p \geq k$ and

$$(2.1) \quad \sigma_{k-1;n}(\lambda) \geq \cdots \geq \sigma_{k-1;1}(\lambda) > 0.$$

When $n = 3$, we see that $\sigma_3(\lambda) > 0$ cannot occur for $\lambda \in \partial\Gamma_2(\lambda)$, therefore we can express $\partial\Gamma_2$ as two parts

$$\partial\Gamma_2(\lambda) = \mathbf{P}_1 \cup \mathbf{P}_2,$$

$$\mathbf{P}_1 = \{\lambda \in \mathbb{R}^3; \sigma_1(\lambda) \geq 0, \sigma_2(\lambda) = \sigma_3(\lambda) = 0\},$$

$$\mathbf{P}_2 = \{\lambda \in \mathbb{R}^3; \sigma_1(\lambda) > 0, \sigma_2(\lambda) = 0, \sigma_3(\lambda) < 0\}.$$

Next, we will recall that what condition can lead to the ellipticity.

As for the framework of ellipticity, we follow the ideas of [6] and [7]. Denote $\text{Sym}(n)$ as the set of symmetric real $n \times n$ matrix. Through the matrix language, we recall the direct condition which leads to the elliptic k -Hessian operator. The ellipticity set of the k -Hessian operator, $k = 1, 2, \dots, n$, is

$$E_k = \{S \in \text{Sym}(n) : S_k(S + t\xi \times \xi) > S_k(S) > 0, |\xi| = 1, t \in \mathbb{R}^+\}$$

and the Gårding cones

$$\Gamma_k = \{S \in \text{Sym}(n) : S_k(S + tId) > S_k(S) > 0, t \in \mathbb{R}^+\},$$

where the definition of $S_k(S)$ is given in (1.2). It is easy to show that $E_k = \Gamma_k$ only for $k = 1, n$ and the example in [7] assures that $\Gamma_k \subset E_k$ and $\text{mess}(E_k \setminus \Gamma_k) > 0$ when $1 < k < n$. Ivochkina, Prokofeva and Yakunina [7] point out that the ellipticity of (1.1) is independent of the sign of f .

We now present an algebraic property of

$$\frac{\partial}{\partial \tau_i} \sigma_2(\tau) = \sigma_{1,i}(\tau), \quad i = 1, 2, 3,$$

for $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbf{P}_2$.

Lemma 2.2. *Assume that $\tau \in \mathbf{P}_2, \tau_1 \geq \tau_2 \geq \tau_3$. Then we have*

$$0 < \sigma_{1,1}(\tau) \leq \sigma_{1,2}(\tau) \leq \sigma_{1,3}(\tau),$$

and

$$(2.2) \quad \tau_3 < 0 < \tau_2 \leq \tau_1.$$

The above result means that for any

$$\psi = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2, \quad \tau \in \mathbf{P}_2$$

it is a solution of 2-Hessian equation $S_2(\psi) = 0$, and the linearized operators of $S_2[u]$ at ψ

$$\mathcal{L} = \sum_{i=1}^3 \sigma_{1,i}(\tau) \partial_i^2$$

is uniformly elliptic,

Proof. Recall that, for any $\tau \in \mathbb{R}^3$,

$$\sigma_2(\tau) = \tau_1 \tau_2 + \tau_2 \tau_3 + \tau_1 \tau_3,$$

and

$$\sigma_{1,1}(\tau) = \tau_2 + \tau_3, \quad \sigma_{1,2}(\tau) = \tau_1 + \tau_3, \quad \sigma_{1,3}(\tau) = \tau_1 + \tau_2.$$

Denote $\lambda + \varepsilon = (\lambda_1 + \varepsilon, \lambda_2 + \varepsilon, \lambda_3 + \varepsilon)$ with $\lambda \in \mathbb{R}^3$ and $\varepsilon \in \mathbb{R}$, then we have the formula

$$\sigma_2(\lambda + \varepsilon) = \sum_{j=0}^2 C(j) \varepsilon^j \sigma_{2-j}(\lambda), \quad C(j) = \frac{\binom{3}{2} \binom{2}{j}}{\binom{3}{2-j}}.$$

For $\tau \in \mathbf{P}_2$, we have

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = 0,$$

then

$$\tau + \varepsilon \in \Gamma_2, \quad \forall \varepsilon > 0.$$

Applying (2.1) to $\tau + \varepsilon$ and letting $\varepsilon \rightarrow 0^+$, we get

$$0 \leq \sigma_{1,1}(\tau) \leq \sigma_{1,2}(\tau) \leq \sigma_{1,3}(\tau).$$

Since $\tau \in \mathbf{P}_2$, we have

$$\sigma_2(\tau) = \tau_1 \sigma_{1,1}(\tau) + \sigma_{2,1}(\tau) = 0.$$

if $\sigma_{1,1}(\tau) = \tau_2 + \tau_3 = 0$, then,

$$\sigma_{2,1}(\tau) = \tau_2 \tau_3 = 0,$$

thus $\sigma_3(\tau) = \tau_1 \tau_1 \tau_3 = 0$, which contradicts with the assumption $\sigma_3(\tau) < 0$. Then, We have proven that, for any $\tau \in \mathbf{P}_2$,

$$0 < \sigma_{1,1}(\tau) \leq \sigma_{1,2}(\tau) \leq \sigma_{1,3}(\tau).$$

We prove now (2.2). Since $\sigma_1(\tau) > 0$, by (4) we have $\tau_1 > 0$. We now claim that $\tau_1 = \tau_2 = \tau_3$ is impossible. Indeed, if that holds, then $\sigma_1(\tau) = 3\tau_1 > 0$ and $\sigma_2(\tau) = 3\tau_1^2 > 0$, which contradicts with the assumption $\sigma_2(\tau) = 0$.

Besides, $\sigma_3(\tau) < 0$ imply that $\tau_i \neq 0$ and τ_i can not be positive at the same time. Then property (4) of Proposition 2.1 implies

$$\tau_3 < 0 < \tau_2 \leq \tau_1.$$

□

We also have the following elliptic results for $\tau \in \Gamma_1 \setminus \bar{\Gamma}_2$.

Lemma 2.3. *For the Gårding cone, we have*

(1) For any given $a < 0$, there exists $\tau \in \Gamma_1 \setminus \bar{\Gamma}_2$, such that

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = a.$$

(2) For any given $b > 0$, there exists $\tau \in \Gamma_2 \setminus \bar{\Gamma}_3$, such that

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = b, \quad \sigma_3(\tau) < 0.$$

(3) For any given $c > 0$, there exists $\tau \in \Gamma_3$, such that

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = c, \quad \sigma_3(\tau) > 0.$$

Moreover, for all above case, we have

$$\sigma_{1,3}(\tau) > \sigma_{1,2}(\tau) > \sigma_{1,1}(\tau) > 0.$$

Proof. We only need to prove the case (1), and to find a $\tau \in \mathbb{R}^3$. We can choose $\alpha > 0$ and $\beta > 0$ such that

$$(1 + \beta)\alpha - 1 < 0.$$

Then take $\Theta > 0$ satisfying

$$\Theta^2(1 + \alpha)[(1 + \alpha)\beta - 1] = a.$$

We claim that τ can be in the following form

$$\tau = (\tau_1, \tau_2, \tau_3) = ((1 + \alpha)(1 + \beta)\Theta, (1 + \alpha)\Theta, -\Theta).$$

Indeed, from $1 + \beta > 1$ and $(1 + \alpha)\Theta > 0$, we have

$$\tau_1 > \tau_2 > \tau_3,$$

$\sigma_1(\tau) > 0$ and $\sigma_2(\tau) = a$. Moreover,

$$\sigma_{1,3}(\tau) = (1 + \alpha)(2 + \beta)\Theta > \sigma_{1,2} = (\alpha\beta + \alpha + \beta)\Theta > \sigma_{1,1}(\tau) = \alpha\Theta > 0.$$

Proof is done. \square

For the linearized operators of k -Hessian equation, we have the following results, the general version of which can be found in section 2, [2].

Lemma 2.4. *The matrix $S_2^{ij}(r(w))$ and $(r_{ij}(w))$ can be diagonalized simultaneously, that is, for any smooth function w , we can find an orthogonal matrix $T(x, \varepsilon)$ satisfying*

$$\begin{cases} T(x, \varepsilon)(S_2^{ij})^t T(x, \varepsilon) = \text{diag} \left[\frac{\partial \sigma_2(\lambda)}{\partial \lambda_1}, \frac{\partial \sigma_2(\lambda)}{\partial \lambda_2}, \frac{\partial \sigma_2(\lambda)}{\partial \lambda_3} \right] \\ T(x, \varepsilon)(r_{ij})^t T(x, \varepsilon) = \text{diag} [\lambda_1(x, \varepsilon), \lambda_2(x, \varepsilon), \lambda_3(x, \varepsilon)], \end{cases}$$

where ${}^t T(x, \varepsilon)$ is the transpose of $T(x, \varepsilon)$ and $S_2^{ij}(r(w)) = \partial S_2 / \partial r_{ij}(r(w))$. Furthermore,

$$T(x, \varepsilon) |_{\varepsilon=0} = Id,$$

where Id is the identity matrix.

Proof. For $T = (T_{ij})$, we have

$$(2.3) \quad \sum_{i=1}^3 T_{si} T_{ti} = \delta_s^t.$$

Now we set (r_{ij}) can be diagonalized by T ,

$$(T_{ij})(r_{ij})^t(T_{ij}) = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \left(\sum_{i,j=1}^3 T_{si}T_{tj}r_{ij} \right)_{st}.$$

Thus, we have, when $s \neq t$

$$(2.4) \quad \sum_{i,j=1}^3 T_{si}T_{tj}r_{ij} \\ = T_{s1}T_{t1}r_{11} + T_{s2}T_{t2}r_{22} + T_{s3}T_{t3}r_{33} + 2T_{s3}T_{t1}r_{31} + 2T_{s1}T_{t2}r_{12} + 2T_{s3}T_{t2}r_{32} = 0$$

Now for

$$(S_2^{ij}(r_{ij})) = \begin{pmatrix} r_{22} + r_{33} & -r_{21} & -r_{31} \\ -r_{12} & r_{11} + r_{33} & -r_{31} \\ -r_{13} & -r_{23} & r_{11} + r_{22} \end{pmatrix},$$

we have

$$(T_{ij})(r_{ij})^t(T_{ij}) = \left(\sum_{i,j=1}^3 T_{si}T_{tj}S_2^{ij} \right)_{st}.$$

If we could prove that $\sum_{i,j=1}^3 (T_{si}T_{tj}S_2^{ij})_{st}$ is a diagonal matrix, our proof was done.

Indeed, when $s \neq t$, we have

$$(2.5) \quad \sum_{i,j=1}^3 T_{si}T_{tj}S_2^{ij} \\ = T_{s1}T_{t1}(r_{22} + r_{33}) + T_{s2}T_{t2}(r_{11} + r_{33}) + T_{s3}T_{t3}(r_{11} + r_{22}) \\ - 2T_{s1}T_{t2}r_{12} - 2T_{s3}T_{t1}r_{31} - 2T_{s3}T_{t2}r_{32}.$$

By (2.4) and (2.3), (2.5) can be

$$\sum_{i,j=1}^3 T_{si}T_{tj}S_2^{ij} = \sum_{i,j=1}^3 T_{si}T_{tj}(r_{11} + r_{22} + r_{33}) = 0.$$

When $\varepsilon = 0$, $S_2^{ij}[r(w)]$ and $(r_{ij}(w))$ are diagonal, thus, T can be the identity matrix Id . \square

From the view above, when $k = 2$ and $f < 0$, the corresponding Hessian operator is possible to be uniformly elliptic. In this paper, we will study some uniformly elliptic 2-Hessian equations which have non-positive right-hand functions f .

3. EXISTENCE OF C^∞ LOCAL SOLUTIONS FOR UNIFORMLY ELLIPTIC CASE

From now on, we fixed $n = 3, k = 2$, by a translation $y \rightarrow y - y_0$ and replacing u by $u - u(0) - y \cdot Du(0)$, we can assume $Z_0 = (0, 0, 0)$ in Theorem 1.1. We prove now the following results,

Theorem 3.1. *Let $f \in C^\infty$ and $f(Z_0) = 0$ for $Z_0 = (0, 0, 0) \in \Omega \times \mathbb{R} \times \mathbb{R}^3$. Then (1.3) admits a 1-convex local solution $u \in C^\infty$ which is not 3-convex and is of the following form*

$$(3.1) \quad u(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y), \quad \forall (\tau_1, \tau_2, \tau_3) \in \mathbf{P}_2$$

in the neighborhood of $y_0 = 0$, $\|w\|_{C^{4,\alpha}} \leq 1$ and $\varepsilon > 0$ very small.

If f is nonnegative near Z_0 , then (1.3) admits a 2-convex local solution $u \in C^\infty$ which is not 3-convex. If $f(Z_0) > 0$, then (1.3) admits a 3-convex local solution $u \in C^\infty$.

Moreover, the equation (1.3) is uniformly elliptic with respect to the solution (3.1).

Remark that, in Theorem 3.1 the function f is permitted to change sign. It is well known that, for Monge-Ampere operator, the type of equation is determined by the sign of $f(y, u, Du)$, it is elliptic if $f > 0$, hyperbolic if $f < 0$ and degenerate elliptic or hyperbolic if f vanishes; it is of mixed type if f changes sign [4]. So that Theorem 3.1 never occurs in Monge-Ampère case.

Theorem 3.1 is exactly the part (1) and (2) of Theorem 1.1.

Let $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbf{P}_2$, then $\psi(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2$ is a polynomial-type solution of

$$S_2[\psi] = 0,$$

we follow Lin [8] to introduce the following function

$$u(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y) = \psi(y) + \varepsilon^5 w(\varepsilon^{-2}y), \quad \tau \in \mathbf{P}_2, \quad \varepsilon > 0,$$

as a candidate of solution for equation (1.1). Noting $y = \varepsilon^2 x$, we have

$$(D_{y_j} u)(x) = \tau_j \varepsilon^2 x_j + \varepsilon^3 w_j(x), \quad j = 1, \dots, 3,$$

and

$$(D_{y_j y_k} u)(x) = \delta_k^j \tau_j + \varepsilon w_{jk}(x), \quad j, k = 1, \dots, 3,$$

where δ_k^j is the Kronecker symbol, $w_j(x) = (D_{y_j} w)(x)$ and $w_{jk}(x) = (D_{y_j y_k}^2 w)(x)$. Then (1.3) transfers to

$$\tilde{S}_2(w) = \tilde{f}_\varepsilon(x, w(x), Dw(x)), \quad x \in B_1(0) = \{x \in \mathbb{R}^3; |x| < 1\}$$

where

$$\tilde{S}_2(w) = S_2(\delta_i^j \tau_i + \varepsilon w_{ij}(x)) = S_2(r(w)),$$

with symmetric matrix $r(w) = (\delta_i^j \tau_i + \varepsilon w_{ij}(x))$, and

$$\tilde{f}_\varepsilon(x, w(x), Dw(x)) = f(\varepsilon^2 x, \varepsilon^4 \psi(x) + \varepsilon^5 w(x), \tau_1 \varepsilon^2 x_1 + \varepsilon^3 w_1(x), \dots, \tau_3 \varepsilon^2 x_3 + \varepsilon^3 w_3(x)).$$

Similar to [8] we consider the nonlinear operators

$$(3.2) \quad G(w) = \frac{1}{\varepsilon} [S_2(r(w)) - \tilde{f}_\varepsilon(x, w, Dw)], \quad \text{on } B_1(0).$$

The linearized operator of G at w is

$$(3.3) \quad L_G(w) = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}} \partial_{ij}^2 + \sum_{i=1}^3 a_i \partial_i + a,$$

where

$$a_i = -\frac{1}{\varepsilon} \frac{\partial \tilde{f}_\varepsilon(x, z, p_i)}{\partial p_i}(x, w, Dw) = -\varepsilon^2 \frac{\partial f}{\partial p_i}$$

$$a = -\frac{1}{\varepsilon} \frac{\partial \tilde{f}_\varepsilon(x, z, p_i)}{\partial z}(x, w, Dw) = -\varepsilon^4 \frac{\partial f}{\partial z}.$$

Hereafter, we denote $S_2^{ij}(r(w)) = \frac{\partial S_2(r(w))}{\partial r_{ij}}$. Since $S_2(r(w)) = \sigma_2(\lambda(r(w)))$ is invariant under orthogonal transformation, by using Lemma 2.4, the matrix $(S_2^{ij}(r(w)))$ and $(r(w))$ can be

diagonalized simultaneously, that is, for any smooth function w , we can find an orthogonal matrix $T(x, \varepsilon)$ satisfying

$$\begin{cases} T(x, \varepsilon) \left(S_2^{ij}(r(w)) \right) {}^t T(x, \varepsilon) = \text{diag} \left[\frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_1}, \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_2}, \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_3} \right] \\ T(x, \varepsilon) \left(r_{ij}(r(w)) \right) {}^t T(x, \varepsilon) = \text{diag} [\lambda_1(r(w)), \lambda_2(r(w)), \lambda_3(r(w))], \end{cases}$$

where ${}^t T(x, \varepsilon)$ is the transpose of $T(x, \varepsilon)$. Since T is not unique, we set $T(x, \varepsilon)|_{\varepsilon=0} = Id$. After this transformation, in order to prove the uniform ellipticity of $L_G(w)$

$$\sum_{i,j=1}^n S_2^{ij}(r(w)) \xi_i \xi_j \geq c |\xi|^2, \quad \forall (x, \xi) \in B_1(0) \times \mathbb{R}^3$$

instead we can prove that, by setting $\xi = {}^t T(x, \varepsilon) \tilde{\xi}$,

$$\sum_{j=1}^3 \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_j} |\tilde{\xi}_j|^2 \geq c |\tilde{\xi}|^2,$$

for some $c > 0$, where

$$\begin{aligned} \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_1} &= \sigma_{1,1}(\lambda(r(w))) = \lambda_2(r(w)) + \lambda_3(r(w)), \\ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_2} &= \sigma_{1,2}(\lambda(r(w))) = \lambda_1(r(w)) + \lambda_3(r(w)), \\ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_3} &= \sigma_{1,3}(\lambda(r(w))) = \lambda_1(r(w)) + \lambda_2(r(w)). \end{aligned}$$

Lemma 3.2. *Assume that $\tau \in \mathbf{P}_2$ and $\|w\|_{C^2(B_1(0))} \leq 1$, then the operator $L_G(w)$ is a uniformly elliptic operator if ε is small enough.*

Proof. To prove the operator $L_G(w)$ is a uniformly elliptic operator, it suffices to prove

$$(3.4) \quad \lambda_i(r(w)) + \lambda_j(r(w)) = \tau_i + \tau_j + O(\varepsilon), \quad i, j = 1, 2, 3, \quad i \neq j.$$

Indeed, for $\tau \in \mathbf{P}_2$ and Lemma 2.2 give $\tau_i + \tau_j > 0$. Thus, for ε small enough, (3.4) imply,

$$\lambda_i + \lambda_j \geq \frac{\tau_i + \tau_j}{2} > 0, \quad i \neq j$$

$L_G(w)$ is then a uniformly elliptic operator.

Next, we prove (3.4). By our choice of $r_{ij}(w)$,

$$r(w) = (r_{ij}(w)) = \begin{pmatrix} \tau_1 + \varepsilon w_{11} & \varepsilon w_{12} & \varepsilon w_{13} \\ \varepsilon w_{21} & \tau_2 + \varepsilon w_{22} & \varepsilon w_{23} \\ \varepsilon w_{31} & \varepsilon w_{32} & \tau_3 + \varepsilon w_{33} \end{pmatrix},$$

we write its characteristic polynomial as

$$g(\lambda) = \det(r(w) - \lambda \mathbf{I}) = \prod_{i=1}^3 (\tau_i - \lambda_i) + R(w, \varepsilon)$$

where

$$R(w, \varepsilon) = \sum_{j=1}^3 \varepsilon R_j(w, \varepsilon) + \sum_{j,k} \varepsilon^2 R_{jk}(w, \varepsilon).$$

For any $\|w\|_{C^2(B_1(0))} \leq 1$ and $0 < \varepsilon \leq 1$

$$|R_j(w, \varepsilon)| \leq C, \quad |R_{jk}(w, \varepsilon)| \leq C$$

with C being independent of x and ε . We have also

$$(3.5) \quad S_1(r(w)) = \sigma_1(\tau) + \varepsilon S_1(w), \quad S_2(r(w)) = \sigma_2(\tau) + \varepsilon \tilde{R}_1(w, \varepsilon),$$

and

$$\det(r(w)) = \sigma_3(\tau) + \varepsilon \tilde{R}_2(w, \varepsilon),$$

where for any $\|w\|_{C^2(B_1(0))} \leq 1$ and $0 < \varepsilon \leq 1$

$$|\tilde{R}_j(w, \varepsilon)| \leq C, \quad |S_1(w)| \leq C.$$

By using Lemma 2.2, we have $\tau_3 < 0 < \tau_2 \leq \tau_1$, then for $0 < \varepsilon \ll |\tau_3|$, we have

$$g\left(\frac{3}{4}\tau_3\right) = \left(\tau_1 - \frac{3}{4}\tau_3\right)\left(\tau_2 - \frac{3}{4}\tau_3\right)\left(\frac{\tau_3}{4}\right) + R(w, \varepsilon) < 0,$$

$$g\left(\frac{5}{4}\tau_3\right) = \left(\tau_1 - \frac{5}{4}\tau_3\right)\left(\tau_2 - \frac{5}{4}\tau_3\right)\left(-\frac{\tau_3}{4}\right) + R(w, \varepsilon) > 0,$$

and we see that, by the virtue of Intermediate Value Theorem, there exists an eigenvalue, denoted by λ_3 , such that

$$\frac{3}{4}\tau_3 > \lambda_3 > \frac{5}{4}\tau_3, \quad g(\lambda_3) = 0.$$

From $0 = g(\lambda_3) = (\tau_1 - \lambda_3)(\tau_2 - \lambda_3)(\tau_3 - \lambda_3) + R(w, \varepsilon)$ and

$$\left(\tau_1 - \frac{5}{4}\tau_3\right)\left(\tau_2 - \frac{5}{4}\tau_3\right) > (\tau_1 - \lambda_3)(\tau_2 - \lambda_3) > \left(\tau_1 - \frac{3}{4}\tau_3\right)\left(\tau_2 - \frac{3}{4}\tau_3\right),$$

it follows that

$$\lambda_3 = \tau_3 + \mathcal{O}_1(w, \varepsilon).$$

Since the trace of a matrix is invariant under the orthogonal transformation, then

$$\lambda_1(w) + \lambda_2(w) + \lambda_3(w) = \sigma_1(\tau) + \varepsilon(w_{11} + w_{22} + w_{33}),$$

from which we see that

$$\lambda_1(w) + \lambda_2(w) = \tau_1 + \tau_2 + \mathcal{O}_2(w, \varepsilon).$$

Using

$$\sigma_2(\tau) + \varepsilon \tilde{R}_1(w, \varepsilon) = S_2(r(w)) = \sigma_2(\lambda(r(w))) = \lambda_3(w)(\lambda_1(w) + \lambda_2(w)) + \lambda_1(w)\lambda_2(w),$$

we obtain

$$\lambda_1\lambda_2 = \tau_1\tau_2 + \mathcal{O}_3(w, \varepsilon),$$

which yields either

$$\lambda_1 = \tau_1 + \mathcal{O}_4(w, \varepsilon), \quad \lambda_2 = \tau_2 + \mathcal{O}_5(w, \varepsilon)$$

or

$$\lambda_1 = \tau_2 + \mathcal{O}_5(w, \varepsilon), \quad \lambda_2 = \tau_1 + \mathcal{O}_4(w, \varepsilon)$$

and then (3.4) is proven. Proof is done. \square

We follow now the idea of Hong and Zuily [5] to prove the existence and a priori estimates of solution for linearized operator. In our case, although $L_G(w)$ is uniformly elliptic, the existence and a priori Schauder estimates of classical solutions are not directly obtainable, because we do not know whether the coefficient a of au in (3.3) is non-positive. If we can prove the existence (Lemma 3.3), we can employ Nash-Moser procedure to prove the existence of local solution for (1.3) in Hölder space rather than Sobolev space. One goal is to see how the procedure depends on the condition $\|w_k\|_{C^{4,\alpha}} \leq A$. We shall use the following schema:

$$(3.6) \quad \begin{cases} w_0 = 0, & w_m = w_{m-1} + \rho_{m-1}, \quad m \geq 1, \\ L_G(w_m)\rho_m = g_m, & \text{in } B_1(0), \\ \rho_m = 0 & \text{on } \partial B_1(0), \\ g_m = -G(w_m), \end{cases}$$

where

$$g_0(x) = \frac{1}{\varepsilon} \left(\sigma_2(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) \right).$$

It is pointed out on page 107, [3] that, if the operator L_G does not satisfy the condition $a \leq 0$, as is well known from simple examples, the Dirichlet problem for $L_G(w)\rho = g$ no longer has a solution in general. Notice a in (3.9) has the factor ε^4 , we will take advantage of smallness of a to obtain the uniqueness and existence of solution for Dirichlet problem (3.9) and then uniformly Schauder estimates of its solution follows.

Lemma 3.3. *Assume that $\|w\|_{C^{4,\alpha}(B_1(0))} \leq A$. Then there exists a unique solution $\rho \in C^{2,\alpha}(\overline{B_1(0)})$ to the following Dirichlet problem*

$$(3.7) \quad \begin{cases} L_G(w)\rho = g, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

for all $g \in C^\alpha(\overline{B_1(0)})$. Moreover,

$$(3.8) \quad \|\rho\|_{C^{4,\alpha}(\overline{B_1(0)})} \leq C \|g\|_{C^{2,\alpha}(\overline{B_1(0)})}, \quad \forall g \in C^{2,\alpha}(\overline{B_1(0)}),$$

where the constant C depends on A, τ and $\|f\|_{C^{4,\alpha}}$. Moreover, C is uniform for $0 < \varepsilon \leq \varepsilon_0$ for some $\varepsilon_0 > 0$.

By virtue of (3.3), we write (3.7) as

$$(3.9) \quad \begin{cases} L_G(w)\rho = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}} \partial_i \partial_j \rho + \sum_{i=1}^3 a_i \partial_i \rho + a\rho = g, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

where

$$a_i = -\varepsilon^2 \frac{\partial f}{\partial p_i}, \quad a = -\varepsilon^4 \frac{\partial f}{\partial z}.$$

Notice that for $\frac{\partial S_2(r(w))}{\partial r_{ij}}$, $a_i = a_i(x, w(x), Dw(x))$, $a = a(x, w(x), Dw(x))$ and $g_m = -G(w_m) = g_m(x, w(x), Dw(x), D^2 w(x))$ by (3.6), we regard them as the functions with variable x . In a word, we regard that all of the coefficients and non-homogeneous term in (3.9) are functions of variable x . For example,

$$\tilde{f}_\varepsilon(x, w(x), Dw(x)) = f(\varepsilon^2 x, \varepsilon^4 \psi(x) + \varepsilon^5 w(x), \tau_1 \varepsilon^2 x_1 + \varepsilon^3 w_1(x), \dots, \tau_3 \varepsilon^2 x_3 + \varepsilon^3 w_3(x)),$$

and

$$\|\tilde{f}_\varepsilon\|_{C^3} = \sup \left\{ |D_x^\beta[\tilde{f}_\varepsilon]|, |0 \leq \beta \leq 3, x \in B_1(0) \right\}$$

$$\|\tilde{f}_\varepsilon\|_{C^{3,\alpha}} = \|\tilde{f}_\varepsilon\|_{C^3} + \sup \left\{ \frac{|D_x^\beta[\tilde{f}_\varepsilon](x) - D_x^\beta[\tilde{f}_\varepsilon](z)|}{|x-z|^\alpha}, |\beta| = 3, x \neq z \in B_1(0) \right\}$$

When we regard \tilde{f}_ε as a function of variable x , usually $\|f\|_{C^{3,\alpha}}$ is denoted as $\|f\|_{C^{3,\alpha}(B_1(0))}$, but it maybe cause confusion because it must be involved in $D^\alpha w$, $0 \leq |\alpha| \leq 3$ as above. Therefore, here and after, we denote the norm as $\|\tilde{f}_\varepsilon\|_{C^3}$, $\|\tilde{f}_\varepsilon\|_{C^{3,\alpha}}$ as above, by dropping $B_1(0)$.

Proof. Let the constant $\mu(\tau) = \inf \left\{ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_i} : \|w\|_{C^{4,\alpha}(B_1(0))} \leq A, i = 1, 2, 3 \right\}$. By Lemma 3.2, $\mu(\tau) > 0$. Applying Theorem 3.7 [3] to the solution $u \in C^0(\overline{B_1(0)}) \cap C^2(B_1(0))$ of

$$\begin{cases} L_G(w)u = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}} \partial_i \partial_j u + \sum_{i=1}^3 a_i \partial_i u = g, & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0) \end{cases}$$

we have

$$(3.10) \quad \sup |u| \leq \frac{C}{\mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})},$$

where $C = \exp^{2(\beta+1)} - 1$ and $\beta = \sup \left\{ \frac{|a_i|}{\mu(\tau)} : i = 1, 2, 3 \right\}$

Let $C_1 = 1 - C \sup \frac{|a_i|}{\mu(\tau)}$ with C being the constant in (3.10). If we choose $\varepsilon_0 > 0$ small (the smallness of a), then $C_1 > \frac{1}{2}$ uniformly for $0 < \varepsilon < \varepsilon_0$. Applying Corollary 3.8 [3] to the solution ρ to Dirichlet problem (3.9), we have

$$(3.11) \quad \sup |\rho| \leq \frac{1}{C_1} \left[\sup_{\partial B_1(0)} |\rho| + \frac{C}{\mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})} \right] = \frac{C}{C_1 \mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})},$$

from which we see that the homogeneous problem

$$\begin{cases} L_G(w)\rho = \sum_{i,j=1}^3 \frac{\partial S_2(r(w))}{\partial r_{ij}} \partial_i \partial_j \rho + \sum_{i=1}^3 a_i \partial_i \rho + a\rho = 0, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

has only the trivial solution. Then we can apply a Fredholm alternative, Theorem 6.15 [3], to the inhomogeneous problem (3.9) for which we can assert that it has a unique $C^{2,\alpha}(\overline{B_1(0)})$ solution for all $g \in C^\alpha(\overline{B_1(0)})$.

With the existence and uniqueness at hand, we can apply Theorem 6.19 [3] to obtain higher regularity up to boundary for solution to (3.9). Besides this, we have the Schauder estimates (see Problem 6.2, [3])

$$(3.12) \quad \|\rho\|_{C^{4,\alpha}} \leq C(A, \tau, \|f\|_{C^{3+\alpha}}) \left[\|\rho_k\|_{C^0(\overline{B_1(0)})} + \|g_k\|_{C^{2,\alpha}(\overline{B_1(0)})} \right],$$

where C depends on $C^{2,\alpha}$ -norm of all of the coefficients; the uniform ellipticity; boundary value and boundary itself. we explain the dependence of $C(A, \tau, \|f\|_{C^{3+\alpha}})$. Firstly, Since the first two derivatives of w have come into the principal coefficients $\frac{\partial S_2(r(w))}{\partial r_{ij}}$, then their $C^{2+\alpha}$ -norms must be involved in $\|w\|_{C^{4,\alpha}}$, and at last $\|w\|_{C^{4,\alpha}} \leq A$ arise into C . Similarly, by virtue of the coefficients a_i and a , $\|f\|_{C^{3,\alpha}}$ and $\|w\|_{C^{3,\alpha}} \leq A$ must arise into C . Secondly, it depends on uniform ellipticity, that is, on

$$\inf \left\{ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_i} : \|w\|_{C^{4,\alpha}(B_1(0))} \leq A, i = 1, 2, 3 \right\}$$

and

$$\sup \left\{ \frac{\partial \sigma_2(\lambda(r(w)))}{\partial \lambda_i} : \|w\|_{C^{4,\alpha}(B_1(0))} \leq A, i = 1, 2, 3, \right\},$$

so $(\tau = (\tau_1, \tau_2, \tau_3))$ and A arise into C .

Thirdly, Since boundary value is $=0$ and boundary $\partial B_1(0)$ is C^∞ , so the two ingredients do not occur into C . Substituting (3.11) into (3.12), we obtain (3.8). \square

It follows from standard elliptic theory (see Theorem 6.17, [3] and Remark 2, [1]) and an iteration argument that we obtain.

Corollary 3.4. *Assume that $u \in C^{2,\alpha}(\Omega)$ is a solution of (1.3), and the linearized operators with respect to u ,*

$$\mathcal{L}_u = \sum_{i,j=1}^3 \frac{\partial S_2(u_{ij})}{\partial r_{ij}} \partial_{ij}^2 - \sum_{i=1}^3 \frac{\partial f}{\partial p_i}(y, u(y), Du(y)) \partial_i - \frac{\partial f}{\partial z}(y, u(y), Du(y))$$

is uniformly elliptic, then $u \in C^\infty(\Omega)$.

Proof. Let v be a function on Ω and denote by e_l , $l = 1, 2, 3$ the unit coordinate vector in the y_l direction. We define the difference quotient of v at y in the direction e_l by

$$\Delta^h v(y) = \Delta_l^h v(y) = \frac{v(y + he_l) - v(y)}{h}.$$

Since

$$\begin{aligned} & S_2(u_{ij}(y + he_l)) - S_2(u_{ij}(y)) \\ &= \int_0^1 \frac{d}{dt} [S_2(tu_{ij}(y + he_l) + (1-t)u_{ij}(y))] dt \\ &= \sum_{i,j=1}^3 \int_0^1 \frac{\partial}{\partial r_{ij}} [S_2(tu_{ij}(y + he_l) + (1-t)u_{ij}(y))] dt [u_{ij}(y + he_l) - u_{ij}(y)] \\ &\equiv \sum_{i,j=1}^3 a_{ij}(y) [u_{ij}(y + he_l) - u_{ij}(y)] \end{aligned}$$

and Taylor expansion give

$$\begin{aligned} & f(y + he_l, u(y + he_l), Du(y + he_l)) - f(y, u(y), Du(y)) \\ &= \sum_{i=1}^3 b_i(y) [u_i(y + he_l) - u_i(y)] + c(y) [u(y + he_l) - u(y)] + g(y)h \end{aligned}$$

with

$$\begin{aligned} b_i(y) &= \int_0^1 \frac{\partial f}{\partial p_i}(t(y + he_l) + (1-t)y, tu(y + he_l) + (1-t)u(y), tDu(y + he_l) + (1-t)D(y)) dt \\ c(y) &= \int_0^1 \frac{\partial f}{\partial z}(t(y + he_l) + (1-t)y, tu(y + he_l) + (1-t)u(y), tDu(y + he_l) + (1-t)D(y)) dt \\ g(y) &= \int_0^1 \frac{\partial f}{\partial y_l}(t(y + he_l) + (1-t)y, tu(y + he_l) + (1-t)u(y), tDu(y + he_l) + (1-t)D(y)) dt. \end{aligned}$$

Taking the difference quotients of both sides of the equation

$$S_2(u_{ij}(y)) = f(y, u, Du),$$

we have

$$\sum_{i,j=1}^3 a_{ij}(y) \partial_i \partial_j \Delta^h u(y) - \sum_{i=1}^3 b_i(y) \partial_i \Delta^h u(y) - c(y) \Delta^h u(y) = g(y).$$

Since $u \in C^{2,\alpha}(\Omega)$, then all the coefficients a_{ij}, b_i, c and inhomogeneous term g are in $C^\alpha(\Omega)$, from the interior estimates of Corollary 6.3 in [3], we can infer

$$\Delta^h u \in C^{2,\alpha}(\Omega).$$

Letting $h \rightarrow 0$, we see $\partial_l u \in C^{2,\alpha}(\Omega), l = 1, 2, 3$ and

$$\sum_{i,j=1}^3 \frac{\partial S_2(D^2 u)}{\partial r_{ij}} \partial_i \partial_j (\partial_l u) - \sum_{i=1}^3 \frac{\partial f}{\partial p_i} \partial_i (\partial_l u) - \frac{\partial f}{\partial z} (\partial_l u) = \frac{\partial f}{\partial y_l}.$$

Repeating the above proof, we obtain $u \in C^\infty(\Omega)$. \square

Using above Lemma 3.3, we can use the procedure (3.6) to construct the sequence $\{w_m\}_{m \in \mathbb{N}}$. Now we study the convergence of $\{w_m\}_{m \in \mathbb{N}}$ and that of $\{g_m\}_{m \in \mathbb{N}}$.

Proposition 3.5. *Let $\{w_m\}_{m \in \mathbb{N}}$ and $\{g_m\}_{m \in \mathbb{N}}$ the sequence in (3.6). Suppose that $\|w_j\|_{C^{4,\alpha}} \leq A$ for $j = 1, 2, \dots, k$. Then we have*

$$(3.13) \quad \|g_{k+1}\|_{C^{2,\alpha}} \leq C[\|g_k\|_{C^{2,\alpha}}^2 + \|g_k\|_{C^{2,\alpha}}^3],$$

where C is some positive constant depends only on τ, A and $\|f\|_{C^{4,\alpha}}$. In particular, C is independent of k .

Proof. Applying Taylor's expansion with integral-typed remainder to (3.2), we have

$$\begin{aligned} -g_{k+1} &= G(w_k + \rho_k) = G(w_k) + L_G(w_k)\rho_k + Q(w_k, \rho_k) \\ &= -g_k + L_G(w_k)\rho_k + Q(w_k, \rho_k) = Q(w_k, \rho_k), \end{aligned}$$

where Q_k is the quadratic error of G which consists of S_2 and f .

$$\begin{aligned} Q(w_k, \rho_k) &= \sum_{i,j,st} \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 S_2(w_k + \mu\rho_k)}{\partial w_{ij} \partial w_{st}} d\mu(\rho_k)_{ij}(\rho_k)_{st} \\ &\quad + \sum_{i,j} \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w_i \partial w_j} d\mu(\rho_k)_i(\rho_k)_j \\ &\quad + \frac{1}{\varepsilon} \sum_i \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w \partial w_i} d\mu(\rho_k)_i(\rho_k) \\ &\quad + \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w^2} d\mu \cdot \rho_k^2 \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

Since $S_2((r(w)))$ is a second-order homogeneous polynomial with variable $r_{ij}(r(w))$ and $\tilde{f}_\varepsilon(x, w, Dw)$ is independent of r_{ij} , we see that

$$\begin{aligned} \left| \frac{\partial^2 S_2(w_k + \mu\rho_k)}{\partial w_{ij} \partial w_{st}} \right| &= \frac{\partial^2 S_2}{\partial w_{ij} \partial w_{st}} (\delta_i^j \tau_i + \varepsilon(w_k + \mu\rho_k)_{ij}) = \varepsilon^2 \quad \text{or} \quad 0, \\ \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w_i \partial w_j} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon^5(w_k + \mu\rho_k), \varepsilon^3 Dw + \varepsilon^3 D(w_k + \mu\rho_k))]}{\partial w_i \partial w_j} \right| \leq \varepsilon^6 \cdot \|f\|_{C^2}, \end{aligned}$$

$$\begin{aligned} \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w \partial w_i} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon^5(w_k + \mu\rho_k), \varepsilon^3 D\psi + \varepsilon^3 D(w_k + \mu\rho_k))]}{\partial w \partial w_i} \right| \leq \varepsilon^8 \|f\|_{C^2}, \\ \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_k + \mu\rho_k)}{\partial w^2} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon^5(w_k + \mu\rho_k), \varepsilon^3 D\psi + \varepsilon^3 D(w_k + \mu\rho_k))]}{\partial w^2} \right| = \varepsilon^{10} \|f\|_{C^2}. \end{aligned}$$

Thus, $I_i (1 \leq i \leq 4)$ in Q_k are under control by $O(\varepsilon)$, $O(\varepsilon^5)$, $O(\varepsilon^7)$ and $O(\varepsilon^9)$, respectively. Therefore

$$\|I_1\|_{C^{2,\alpha}} \leq C \|\rho_k\|_{C^2} \|\rho_k\|_{C^{4,\alpha}}$$

and

$$\begin{aligned} \|I_2\|_{C^{2,\alpha}} &\leq C \|f\|_{C^{4,\alpha}} (\|w_k\|_{C^{3,\alpha}} + \|\rho_k\|_{C^{3,\alpha}}) \|\rho_k\|_{C^1}^2 + C \|f\|_{C^2} \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1} \\ &\leq C \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1}^2 + C \|\rho_k\|_{C^1}^2 + C \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1} \end{aligned}$$

where C depends on A and $\|f\|_{C^{4,\alpha}}$. And $\|I_3\|_{C^{2,\alpha}}$ and $\|I_4\|_{C^{2,\alpha}}$ can be estimated similarly. Accordingly,

$$\begin{aligned} \|g_{k+1}\|_{C^{2,\alpha}} &= \|Q(w_k, \rho_k)\|_{C^{2,\alpha}} \leq \sum_{i=1}^4 \|I_i\|_{C^{2,\alpha}} \\ &\leq C \|\rho_k\|_{C^2} \|\rho_k\|_{C^{4,\alpha}} + C \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1}^2 + \|\rho_k\|_{C^1}^2 + C \|\rho_k\|_{C^{3,\alpha}} \|\rho_k\|_{C^1} \end{aligned}$$

where C is independent of k but dependent of A and $\|f\|_{C^{4,\alpha}}$. Thus, by the interpolation inequalities, we have

$$\|g_{k+1}\|_{C^{2,\alpha}} \leq C \|\rho_k\|_{C^{4,\alpha}}^2 + C \|\rho_k\|_{C^{4,\alpha}}^3,$$

where C is independent of k . By Schauder estimates of Lemma 3.3, we have

$$\|\rho_k\|_{C^{4,\alpha}} \leq C \|g_k\|_{C^{2,\alpha}}.$$

Combining the estimates above, we obtain (3.13). Proof is done. \square

Since C is independent of k , more exactly, A , τ and $\|f\|_{C^{4,\alpha}}$ are independent of k . So here and after, we can assume $A = 1$.

Proof of Theorem 3.1. Set

$$(3.14) \quad d_{k+1} = C \|g_{k+1}\|_{C^{2,\alpha}}.$$

By (3.13) with letting $C \geq 1$ we have

$$d_{k+1} \leq d_k^2 + d_k^3.$$

Take $\tau \in \mathbb{R}^3$ as in Lemmas 2.2 and 2.3 such that $\sigma_2(\tau) = f(0, 0, 0)$, we have

$$\begin{aligned} g_0(x) &= \frac{1}{\varepsilon} (\sigma_2(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3))) \\ &= \frac{1}{\varepsilon} [\sigma_2(\tau) - f(0, 0, 0)] \\ &\quad + \varepsilon \int_0^1 x \cdot (\partial_y f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt \\ &\quad + \varepsilon^3 \int_0^1 \psi(x) (\partial_z f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt \\ &\quad + \varepsilon \int_0^1 (\tau_1 x_1, \tau_2 x_2, \tau_3 x_3) \cdot (\partial_p f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt, \end{aligned}$$

then

$$\|g_0\|_{C^{2,\alpha}(B_1(0))} \leq \varepsilon C_1 \|f\|_{C^{3,\alpha}}.$$

We can choose $0 < \varepsilon \leq \varepsilon_0$ so small such that

$$C\|g_0\|_{C^{2,\alpha}(B_1(0))} \leq 1/4, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Notice ε_0 is independent of k . Since $d_0 = C\|g_0\|_{C^{2,\alpha}}$, we have $d_1 \leq 2d_0^2$ and, by induction,

$$d_{k+1} \leq 2^{2^{k+1}} d_0^{2^{k+1}} \leq (2C)^{2^{k+1}} \|g_0\|_{C^{2,\alpha}}^{2^{k+1}},$$

Thus, by (3.14)

$$\|g_{k+1}\|_{C^{2,\alpha}} \leq (2C)^{2^{k+1}-1} \|g_0\|_{C^{2,\alpha}}^{2^{k+1}}.$$

Firstly, we claim that there exists $\varepsilon > 0$, depending on τ and $\|f\|_{C^{3,\alpha}}$ such that

$$\|w_k\|_{C^{4,\alpha}(B_1(0))} \leq 1, \quad \forall k \geq 1.$$

Indeed, set $w_0 = 0$, we have by (3.13)

$$\begin{aligned} \|w_{k+1}\|_{C^{4,\alpha}(B_1(0))} &= \left\| \sum_{i=0}^k \rho_i \right\|_{C^{4,\alpha}(B_1(0))} \leq \sum_{i=0}^k \|\rho_i\|_{C^{4,\alpha}(B_1(0))} \\ &\leq \sum_{i=0}^k C \|g_i\|_{C^{2,\alpha}(B_1(0))} \leq \sum_{i=0}^k \left(2C \|g_0\|_{C^{2,\alpha}(B_1(0))}\right)^{2^i} \end{aligned}$$

where C is defined in Lemma 3.5. Thus, for any k ,

$$\|w_{k+1}\|_{C^{4,\alpha}(B_1(0))} \leq \sum_{i=0}^{\infty} \left(C \|g_0\|_{C^{2,\alpha}(B_1(0))}\right)^{2^i} \leq \sum_{i=0}^{\infty} 2^{-2^i} \leq 1.$$

Then, by Azelà-Ascoli Theorem, we have

$$w_k \rightarrow w \quad \text{in } C^4(B_1(0)).$$

From (3.13), we see that

$$\|g_{k+1}\|_{C^{2,\alpha}(B_1(0))} \leq \left(\frac{1}{2}\right)^{2^k} \rightarrow 0,$$

and then $g_m = -G(w_m)$ yields

$$G(w) = \frac{1}{\varepsilon} [S_2(r(w)) - \tilde{f}(x, w, Dw)] = 0, \quad \text{on } B_1(0)$$

which yields that the function

$$u(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y) \in C^4(B_{\varepsilon^2}(0)),$$

is a solution of

$$S_2[u] = f(y, u, Du), \quad \text{on } B_{\varepsilon^2}(0).$$

Now if $f(0, 0, 0) = 0$, we take $\tau \in \mathbf{P}_2$, then $\sigma_1(\tau) > 0, \sigma_2(\tau) = 0, \sigma_3(\tau) < 0$, and (3.5) imply,

$$S_j[u] = \sigma_j(\lambda) = \sigma_j(\tau) + O(\varepsilon), \quad j = 1, 2, 3$$

it follows that $S_1[u] > 0, S_3[u] < 0$ on $B_{\varepsilon^2}(0)$ for small $\varepsilon > 0$, that is, u is 1-convex but not convex. Moreover if $S_2[u] = f \geq 0$ near Z_0 and $f(Z_0) = 0$, we see that u is 2-convex by definition, but not 3-convex.

If $S_2[u] = f > 0$ near Z_0 , we take $\tau \in \mathbb{R}^3$ given in (2) and (3) of Lemmas 2.3, then we can get the 3-convex or non convex local solutions.

The C^∞ regularity of solution is given by Corollary 3.4. We have then proved Theorem 3.1. \square

We also have the following elliptic results for negative f

Theorem 3.6. *Let $f \in C^\infty, f(0, 0, 0) < 0$. Then (1.3) admits a 1-convex local solution $u \in C^\infty$ in a neighborhood of $y_0 = 0$ which is not 2-convex, it is of the following form*

$$u(y) = \frac{1}{2} \sum_{i=1}^3 \tau_i y_i^2 + \varepsilon^5 w(\varepsilon^{-2}y),$$

and the equation (1.3) is uniformly elliptic with respect to this solution.

Proof. For $a = f(0, 0, 0) < 0$, take $\tau \in \mathbb{R}^3$ as in (1) of Lemma 2.3 such that

$$\sigma_1(\tau) > 0, \quad \sigma_2(\tau) = f(0, 0, 0) < 0,$$

and

$$\sigma_{1,3}(\tau) > \sigma_{1,2}(\tau) > \sigma_{1,1}(\tau) > 0.$$

Now the proof is exactly same as that of Theorem 3.1 except the estimate of term g_0 , we use Taylor expansion,

$$\begin{aligned} g_0(x) &= -G(0) = \frac{1}{\varepsilon} [S_2(r(0)) - \tilde{f}(x, 0, 0)] \\ &= \frac{1}{\varepsilon} [\sigma_2(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3))] \\ &= \frac{1}{\varepsilon} [\sigma_2(\tau) - f(0, 0, 0)] \\ &\quad + \varepsilon \int_0^1 x \cdot (\partial_y f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt \\ &\quad + \varepsilon^3 \int_0^1 \psi(x) (\partial_z f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt \\ &\quad + \varepsilon \int_0^1 (\tau_1 x_1, \tau_2 x_2, \tau_3 x_3) \cdot (\partial_p f)(t\varepsilon^2 x, t\varepsilon^4 \psi(x), t\varepsilon^2(\tau_1 x_1, \tau_2 x_2, \tau_3 x_3)) dt, \end{aligned}$$

then we can end the proof of Theorem 3.6 exactly as that of Theorem 3.1. \square

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