Numerical approximation of doubly reflected BSDEs with jumps and RCLL obstacles

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Abstract

We study a discrete time approximation scheme for the solution of a doubly reflected Backward Stochastic Differential Equation (DBBSDE in short) with jumps, driven by a Brownian motion and an independent compensated Poisson process. Moreover, we suppose that the obstacles are right continuous and left limited (RCLL) processes with predictable and totally inaccessible jumps and satisfy Mokobodzki's condition. Our main contribution consists in the construction of an implementable numerical sheme, based on two random binomial trees and the penalization method, which is shown to converge to the solution of the DBBSDE. Finally, we illustrate the theoretical results with some numerical examples in the case of general jumps.

Key words : Double barrier reflected BSDEs, Backward stochastic differential equations with jumps, Skorohod topology, numerical sheme, penalization method.

MSC 2010 classifications : 60H10,60H35,60J75,34K28.

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1 Introduction

In this paper, we study in the non-markovian setting a discrete time approximation scheme for the solution of a doubly reflected Backward Stochastic Differential Equation (DBBSDE in short) when the noise is given by a Brownian motion and a Poisson random process mutually independent. Moreover, the barriers are supposed to be right-continuous and left-limited (RCLL in short) processes, whose jumps are arbitrary, they can be either predictable or inaccessible. The DBBSDE we solve numerically has the following form:

$$\begin{cases} \text{(i) } Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s, U_s)ds + (A_T - A_t) - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T U_s d\tilde{N}_s, \\ \text{(ii) } \forall t \in [0,T],\ \xi_t \leq Y_t \leq \zeta_t \text{ a.s.,} \\ \text{(iii) } \int_0^T (Y_{t^-} - \xi_{t^-}) dA_t^c = 0 \text{ a.s. and } \int_0^T (\zeta_{t^-} - Y_{t^-}) dK_t^c = 0 \text{ a.s.} \\ \text{(iv) } \forall \tau \text{ predictable stopping time },\ \ \Delta A_\tau^d = \Delta A_\tau^d \mathbf{1}_{Y_{\tau^-} = \xi_{\tau^-}} \text{ and } \Delta K_\tau^d = \Delta K_\tau^d \mathbf{1}_{Y_{\tau^-} = \zeta_{\tau^-}}. \end{cases} \tag{1.1}$$

Here, A^c (resp. K^c) denotes the continuous part of A (resp. K) and A^d (resp. K^d) its discontinuous part, $\{W_t : 0 \leq t \leq T\}$ is a one dimensional standard Brownian motion and $\{\tilde{N}_t := N_t - \lambda t, 0 \leq t \leq T\}$ is a compensated Poisson process. Both processes are independent and they are defined on the probability space $(\Omega, \mathcal{F}_T, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$. The processes A and K have the role to keep the solution between the two obstacles ξ and ζ. Since we consider the general setting when the jumps of the obstacles can be either predictable or totally inaccessible, A and K are also discontinuous.

In the case of a Brownian filtration, non-linear backward stochastic differential equations (BSDEs in short) were introduced by Pardoux and Peng [19]. One barrier reflected BSDEs have been firstly studied by El Karoui et al in [7]. In their setting, one of the components of the solution is forced to stay above a given barrier which is a continuous adapted stochastic process. The main motivation is the pricing of American options especially in constrained markets. The generalization to the case of two reflecting barriers has been carried out by Cvitanic and Karatzas in [5]. It is also well known that doubly reflected BSDEs are related to Dynkin games and in finance to the pricing of Israeli options (or Game options, see [15]). The case of standard BSDEs with jump processes driven by a compensated Poisson random measure was first considered by Tang and Li in [27]. The extension to the case of reflected BSDEs and one reflecting barrier with only inaccessible jumps has been established by Hamadène and Ouknine [11]. Later on, Essaky in [8] and Hamadène and Ouknine in [12] have extended these results to a RCLL obstacle with predictable and inaccessible jumps. Results concerning existence and uniqueness of the solution for doubly reflected BSDEs with jumps can be found in [4],[6], [10], [13] and [9].

Numerical shemes for DBBSDEs driven by the Brownian motion and based on a random tree method have been proposed by Xu in [28] (see also [18] and [21]) and, in the Markovian framework, by Chassagneux in [3]. In the case of a filtration driven also by a Poisson process, some results have been provided only in the non-reflected case. In [1], the authors propose a scheme for Forward-Backward SDEs based on the dynamic programming equation and in [16] the authors propose a fully implementable scheme based on a random binomial tree. This work extends the paper [2], where the authors prove a Donsker type theorem for BSDEs in the Brownian case.

Our aim is to propose an implementable numerical method to approximate the solution of DBBSDEs with jumps and RCLL obstacles (1.1). As for standard BSDEs, the computation of conditional expectations is an important issue. Since we consider reflected BSDEs, we also have to model the constraints. To do this, we consider the following approximations

- we approximate the Brownian motion and the Poisson process by two independent random walks,
- we introduce a sequence of penalized BSDEs to approximate the reflected BSDE.

These approximations enable us to provide a fully implementable scheme, called *explicit penalized discrete scheme* in the following. We prove in Theorem 4.1 that the scheme weakly converges to the solution of (1.1). Moreover, in order to prove the convergence of our sheme, we prove, in the case of jump processes driven by a general Poisson random measure, that the solutions of the penalized equations converge to the solution of the doubly reflected BSDE in the case of a driver depending on the solution, which was not the case in the previous literature (see [9], [10], [13]). This gives another proof for the existence of a solution of DBBSDEs with jumps and RCLL barriers. Our method is based on a combination of penalization, Snell envelope theory, stochastic games, comparison theorem for BSDEs with jumps (see [23], [24]) and a generalized monotonic theorem under the Mokobodzki's condition. It extends [17] to the case when the solution of the DBBSDE also admits totally inaccessible jumps. Finally, we illustrate our theoretical results with some numerical simulations in the case of general jumps. We point out that the practical use of our scheme is restricted to low dimensional cases. Indeed, since we use a random walk to approximate the Brownian motion and the Poisson process, the complexity of the algorithm grows very fast in the number of time steps n (more precisely, in n^d, d being the dimension) and, as we will see in the numerical part, the penalization method requires many time steps to be stable.

The paper is organized as follows: in Section 2 we introduce notation and assumptions. In Section 3, we precise the discrete framework and give the numerical scheme. In Section 4 we provide the convergence by splitting the error : the error due to the approximation by penalization and the error due to the time discretization. Finally, Section 5 presents some numerical examples, where the barriers contain predictable and totally inaccessible jumps. In Appendix, we extend the generalized monotonic theorem and prove some technical results for discrete BSDEs to the case of jumps. For the self-containment of the paper, we also recall some recent results on BSDEs with jumps and reflected BSDEs.

2 Notations and assumptions

Although we propose a numerical scheme for reflected BSDEs driven by a Brownian motion and a Poisson process, one part of the proof of the convergence of our scheme is done in the general setting of jumps driven by a Poisson random measure. Then, we first introduce the general framework, in which we prove the convergence of a sequence of penalized BSDEs to the solution of (1.1).

2.1 General framework

2.1.1 Notation

As said in Introduction, let $(\Omega, \mathbb{F}, P)$ be a probability space, and $\mathcal{P}$ be the predictable σ-algebra on $[0,T]\times\Omega$. W is a one-dimensional Brownian motion and $N(dt, de)$ is a Poisson random measure, independent of W, with compensator $\nu(de)dt$ such that ν is a σ-finite measure on $\mathbb{R}^*$, equipped with its Borel field $\mathcal{B}(\mathbb{R}^*)$. Let $\tilde{N}(dt, du)$ be its compensated process. Let $\mathbb{F} = \{\mathcal{F}_t, 0 \le t \le T\}$ be the natural filtration associated with W and N.

For each $T > 0$, we use the following notations:

- $L^2(\mathcal{F}_T)$ is the set of random variables ξ which are $\mathcal{F}_T$-measurable and square integrable.
- $\mathbb{H}^2$ is the set of real-valued predictable processes ϕ such that $\|\phi\|^2_{\mathbb{H}^2} := \mathbb{E}\left[\int_0^T \phi_t^2 dt\right] < \infty$.
- L^2_ν is the set of Borelian functions $\ell : \mathbb{R}^* \to \mathbb{R}$ such that $\int_{\mathbb{R}^*} |\ell(u)|^2 \nu(du) < +\infty$.
  The set L^2_ν is a Hilbert space equipped with the scalar product $\langle \delta, \ell\rangle_\nu := \int_{\mathbb{R}^*} \delta(u)\ell(u)\nu(du)$ for all $\delta, \ell \in L^2_\nu \times L^2_\nu$, and the norm $\|\ell\|^2_\nu := \int_{\mathbb{R}^*} |\ell(u)|^2\nu(du)$.
- $\mathcal{B}(\mathbb{R}^2)$ (resp $\mathcal{B}(L^2_\nu)$) is the Borelian σ-algebra on $\mathbb{R}^2$ (resp. on L^2_ν).
- $\mathbb{H}^2_\nu$ is the set of processes l which are *predictable*, that is, measurable

  $$l : ([0,T] \times \Omega \times \mathbb{R}^*,\ \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^*)) \to (\mathbb{R}\ , \mathcal{B}(\mathbb{R})); \quad (\omega, t, u) \mapsto l_t(\omega, u)$$

  such that $\|l\|^2_{\mathbb{H}^2_\nu} := \mathbb{E}\left[\int_0^T \|l_t\|^2_\nu\, dt\right] < \infty$.
- $\mathcal{S}^2$ is the set of real-valued RCLL adapted processes ϕ such that $\|\phi\|^2_{\mathcal{S}^2} := \mathbb{E}(\sup_{0\le t\le T} |\phi_t|^2) < \infty$.

- $\mathcal{A}^2$ is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty$.
- $\mathcal{T}_0$ is the set of stopping times τ such that $\tau \in [0,T]$ a.s
- For S in $\mathcal{T}_0$, $\mathcal{T}_S$ is the set of stopping times τ such that $S \le \tau \le T$ a.s.

2.1.2 Definitions and assumptions.

We start this section by recalling the definition of a driver and a Lipschitz driver. We also introduce DBBSDEs and our working assumptions.

Definition 2.1 (Driver, Lipschitz driver). *A function g is said to be a* driver *if*

- $g : \Omega \times [0,T] \times \mathbb{R}^2 \times L^2_\nu \to \mathbb{R}$
  $(\omega, t, y, z, \kappa(\cdot)) \mapsto g(\omega, t, y, z, k(\cdot))$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(L^2_\nu)-$ *measurable,*
- $\|g(.,0,0,0)\|_\infty < \infty$.

A driver g is called a Lipschitz driver *if moreover there exists a constant $C_g \ge 0$ and a bounded, non-decreasing continuous function Λ with $\Lambda(0) = 0$ such that $d\mathbb{P} \otimes dt$-a.s. , for each (s_1, y_1, z_1, k_1), (s_2, y_2, z_2, k_2),*

$$|g(\omega, s_1, y_1, z_1, k_1) - g(\omega, s_2, y_2, z_2, k_2)| \le \Lambda(|s_2 - s_1|) + C_g(|y_1 - y_2| + |z_1 - z_2| + \|k_1 - k_2\|_\nu).$$

In the case of BSDEs with jumps, the coefficient g must satisfy an additional assumption, which allows to apply the comparison theorem for BSDEs with jumps (see Theorem D.1), which extends the result of [25]. More precisely, the driver g satisfies the following assumption:

Assumption 2.2. *A Lipschitz driver g is said to satisfy Assumption 2.2 if the following holds : $dP \otimes dt$ a.s. for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$, we have*

$$g(t, y, z, k_1) - g(t, y, z, k_2) \ge \langle \theta_t^{y,z,k_1,k_2}, k_1 - k_2 \rangle_\nu,$$

with

$$\begin{aligned} \theta :& \Omega \times [0,T] \times \mathbb{R}^2 \times (L^2_\nu)^2 \longmapsto L^2_\nu; \\ & (\omega, t, y, z, k_1, k_2) \longmapsto \theta_t^{y,z,k_1,k_2}(\omega, \cdot) \end{aligned}$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable, bounded, and satisfying $dP \otimes dt \otimes \nu(du)$-a.s., for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,

$$\theta_t^{y,z,k_1,k_2}(u) \ge -1 \text{ and } |\theta_t^{y,z,k_1,k_2}(u)| \le \psi(u),$$

where $\psi \in L^2_\nu$.

We now recall the "Mokobodzki's condition" which is essential in the case of doubly reflected BSDEs, since it ensures the existence of a solution. This condition essentially postulates the existence of a quasimartingale between the barriers.

Definition 2.3 (Mokobodzki's condition). *Let ξ, ζ be in $\mathcal{S}^2$. There exist two nonnegative RCLL supermartingales H and H' in $\mathcal{S}^2$ such that*

$$\forall t \in [0,T], \;\; \xi_t \mathbf{1}_{t<T} \le H_t - H'_t \le \zeta_t \mathbf{1}_{t<T} \text{ a.s.}$$

Assumption 2.4. *ξ and ζ are two adapted RCLL processes with $\xi_T = \zeta_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \le \zeta_t$ for all $t \in [0,T]$, the Mokobodzki's condition holds and g is a Lipschitz driver satisfying Assumption 2.2.*

We introduce the following general reflected BSDE with jumps and two RCLL obstacles

Definition 2.5. *Let $T > 0$ be a fixed terminal time and g be a Lipschitz driver. Let ξ and ζ be two adapted RCLL processes with $\xi_T = \zeta_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \le \zeta_t$ for all $t \in [0,T]$ a.s. A process (Y, Z, U, α) is said to be a solution of the double barrier reflected BSDE (DBBSDE) associated with driver g and barriers ξ, ζ if*

$$\begin{cases} \text{(i) } Y \in \mathcal{S}^2,\ Z \in \mathbb{H}^2,\ U \in \mathbb{H}^2_\nu \text{ and } \alpha \in \mathcal{S}^2, \text{ where } \alpha = A - K \text{ with } A, K \text{ in } \mathcal{A}^2 \\ \text{(ii) } Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s, U_s)ds + (A_T - A_t) - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds, de), \\ \text{(iii) } \forall t \in [0,T],\ \xi_t \le Y_t \le \zeta_t \text{ a.s.,} \\ \text{(iv) } \int_0^T (Y_{t^-} - \xi_{t^-})dA_t = 0 \text{ a.s. and } \int_0^T (\zeta_{t^-} - Y_{t^-})dK_t = 0 \text{ a.s.} \end{cases} \tag{2.1}$$

Remark 2.6. *Condition (iv) is equivalent to the following condition : if $K = K^c + K^d$ and $A = A^c + A^d$, where K^c (resp. K^d) represents the continuous (resp. the discontinous) part of K (the same notation holds for A), then*

$$\int_0^T (Y_t - \xi_t)dA_t^c = 0 \text{ a.s.,} \quad \int_0^T (\zeta_t - Y_t)dK_t^c = 0 \text{ a.s.}$$

and

$$\forall \tau \in \mathcal{T}_0 \text{ predictable,} \quad \Delta A_\tau^d = \Delta A_\tau^d \mathbf{1}_{Y_{\tau^-} = \xi_{\tau^-}} \text{ and } \Delta K_\tau^d = \Delta K_\tau^d \mathbf{1}_{Y_{\tau^-} = \zeta_{\tau^-}}.$$

Theorem 2.7. *([6, Theorem 4.1]) Suppose ξ and ζ are RCLL adapted processes in $\mathcal{S}^2$ such that for all $t \in [0,T]$, $\xi_t \le \zeta_t$ and Mokobodzki's condition holds (see Definition 2.3). Then, DBBSDE (2.1) admits a unique solution (Y, Z, U, α) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{A}^2$.*

Remark 2.8. *As said in [6, Remark 4.3], if for all $t \in]0,T]$ $\xi_{t^-} < \zeta_{t^-}$ a.s., [6, Proposition 4.2] gives the uniqueness of $A, K \in (\mathcal{A}^2)^2$.*

Definition 2.9 (convergence in J_1-Skorokhod topology)**.** *ξ^n is said to converge in probability (resp. in L^2) to ξ for the J_1-Skorokhod topology, if there exists a family $(\psi^n)_{n\in\mathbb{N}}$ of one-to-one random time changes (or stochastic changes of time scale) from $[0,T]$ to $[0,T]$ such that $\sup_{t\in[0,T]} |\psi^n(t) - t| \xrightarrow[n\to\infty]{} 0$ almost surely and $\sup_{t\in[0,T]} |\xi^n_{\psi^n(t)} - \xi_t| \xrightarrow[n\to\infty]{} 0$ in probability (resp. in L^2). Throughout the paper, we denote this convergence $||\xi^n - \xi||_{J_1-\mathbb{P}} \to 0$ (resp. $||\xi^n - \xi||_{J_1-L^2} \to 0$).*

2.2 Framework for our numerical scheme

In order to propose an implementable numerical scheme we consider that the Poisson random measure is simply generated by the jumps of a Poisson process. We consider a Poisson process $\{N_t : 0 \le t \le T\}$ with intensity λ and jumps times $\{\tau_k : k = 0, 1, ...\}$. The random measure is then

$$\tilde{N}(dt, de) = \sum_{k=1}^{N_t} \delta_{\tau_k, 1}(dt, de) - \lambda dt \delta_1(de)$$

where δ_a denotes the Dirac measure at the point a. In the following, $\tilde{N}_t := N_t - \lambda t$. Then, the unknown function $U_s(e)$ does not depend on the magnitude e anymore, and we write $U_s := U_s(1)$.

In this particular case, (2.1) becomes:

$$\begin{cases} \text{(i) } Y \in \mathcal{S}^2,\ Z \in \mathbb{H}^2,\ U \in \mathbb{H}^2 \text{ and } \alpha \in \mathcal{S}^2, \text{ where } \alpha = A - K \text{ with } A, K \text{ in } \mathcal{A}^2 \\ \text{(ii) } Y_t = \xi + \int_t^T g(s, Y_s, Z_s, U_s)ds + (A_T - A_t) - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T U_s d\tilde{N}_s, \\ \text{(iii) } \forall t \in [0,T],\ \xi_t \le Y_t \le \zeta_t \text{ a.s.,} \\ \text{(iv) } \int_0^T (Y_{t^-} - \xi_{t^-})dA_t = 0 \text{ a.s. and } \int_0^T (\zeta_{t^-} - Y_{t^-})dK_t = 0 \text{ a.s.} \end{cases} \tag{2.2}$$

In view of the proof of the convergence of the numerical scheme, we also introduce the penalized version of (2.2):

$$Y_t^p = \xi + \int_t^T g(s, Y_s^p, Z_s^p, U_s^p) ds + A_T^p - A_t^p - (K_T^p - K_t^p) - \int_t^T Z_s^p dW_s - \int_t^T U_s^p d\tilde{N}_s, \tag{2.3}$$

with $A_t^p := p\int_0^t (Y_s^p - \xi_s)^- ds$ and $K_t^p := p\int_0^t (\zeta_s - Y_s^p)^- ds$, and $\alpha_t^p := A_t^p - K_t^p$ for all $t \in [0,T]$.

3 Numerical scheme

The basic idea is to approximate the Brownian motion and the Poisson process by random walks based on the binomial tree model. As explained in Section 3.1.2, these approximations enable to get a martingale representation whose coefficients, involving conditional expectations, can be easily computed. Then, we approximate $(W, \tilde{N})$ in the penalized version of our DBBSDE (i.e. in (2.3)) by using these random walks. Taking conditional expectation and using the martingale representation leads to the *explicit penalized discrete scheme* (3.9). In view of the proof of the convergence of this explicit scheme, we introduce an implicit intermediate scheme (3.5).

3.1 Discrete time Approximation

We adopt the framework of [16], presented below.

3.1.1 Random walk approximation of $(W, \tilde{N})$

For $n \in \mathbb{N}$, we introduce $\delta_n := \frac{T}{n}$ and the regular grid $(t_j)_{j=0,\dots,n}$ with step size δ_n (i.e. $t_j := j\delta_n$) to discretize $[0,T]$. In order to approximate W, we introduce the following random walk

$$\begin{cases} W_0^n = 0 \\ W_t^n = \sqrt{\delta_n} \sum_{i=1}^{[t/\delta_n]} e_i^n \end{cases} \tag{3.1}$$

where $e_1^n, e_2^n, ..., e_n^n$ are independent identically distributed random variables with the following symmetric Bernoulli law:

$$P(e_1^n = 1) = P(e_1^n = -1) = \frac{1}{2}.$$

To approximate $\tilde{N}$, we introduce a second random walk

$$\begin{cases} \tilde{N}_0^n = 0 \\ \tilde{N}_t^n = \sum_{i=1}^{[t/\delta_n]} \eta_i^n \end{cases} \tag{3.2}$$

where $\eta_1^n, \eta_2^n, ..., \eta_n^n$ are independent and identically distributed random variables with law

$$P(\eta_1^n = \kappa_n - 1) = 1 - P(\eta_1^n = k_n) = \kappa_n$$

where $\kappa_n = e^{-\frac{\lambda}{n}}$. We assume that both sequences $e_1^n, ..., e_n^n$ and $\eta_1^n, \eta_2^n, ..., \eta_n^n$ are defined on the original probability space $(\Omega, \mathbb{F}, P)$. The (discrete) filtration in the probability space is $\mathbb{F}^n = \{\mathcal{F}_j^n : j = 0, ..., n\}$ with $\mathcal{F}_0^n = \{\Omega, \emptyset\}$ and $\mathcal{F}_j^n = \sigma\{e_1^n, ..., e_j^n, \eta_1^n, ..., \eta_j^n\}$ for $j = 1, ..., n$.

The following result states the convergence of $(W^n, \tilde{N}^n)$ to $(W, \tilde{N})$ for the J_1-Skorokhod topology, and the convergence of W^n to W in any L^p, $p \geq 1$, for the topology of uniform convergence on $[0,T]$. We refer to [16, Section 3] for more results on the convergence in probability of $\mathcal{F}^n$-martingales.

Lemma 3.1. *([16, Lemma3, (III)], and [2, Proof of Corollary 2.2]) The couple $(W^n, \tilde{N}^n)$ converges in probability to $(W, \tilde{N})$ for the J_1-Skorokhod topology, and*

$$\sup_{0 \leq t \leq T} |W_t^n - W_t| \to 0 \text{ as } n \to \infty$$

in probability and in L^p, for any $1 \leq p < \infty$.

3.1.2 Martingale representation

Let y_{j+1} denote a $\mathcal{F}^n_{j+1}$-measurable random variable. As said in [16], we need a set of three strongly orthogonal martingales to represent the martingale difference $m_{j+1} := y_{j+1} - \mathbb{E}(y_{j+1}|\mathcal{F}^n_j)$. We introduce a third martingale increments sequence $\{\mu^n_j = e^n_j \eta^n_j, j = 0, \cdots, n\}$. In this context there exists a unique triplet (z_j, u_j, v_j) of $\mathcal{F}^n_j$-random variables such that

$$m_{j+1} := y_{j+1} - \mathbb{E}(y_{j+1}|\mathcal{F}^n_j) = \sqrt{\delta_n} z_j e^n_{j+1} + u_j \eta^n_{j+1} + v_j \mu^n_{j+1},$$

and

$$\begin{cases} z_j = \dfrac{1}{\sqrt{\delta_n}} \mathbb{E}(y_{j+1} e^n_{j+1}|\mathcal{F}^n_j), \\ u_j = \dfrac{\mathbb{E}(y_{j+1}\eta^n_{j+1}|\mathcal{F}^n_j)}{\mathbb{E}((\eta^n_{j+1})^2|\mathcal{F}^n_j)} = \dfrac{1}{\kappa_n(1-\kappa_n)} \mathbb{E}(y_{j+1}\eta^n_{j+1}|\mathcal{F}^n_j), \\ v_j = \dfrac{\mathbb{E}(y_{j+1}\mu^n_{j+1}|\mathcal{F}^n_j)}{\mathbb{E}((\mu^n_{j+1})^2|\mathcal{F}^n_j)} = \dfrac{1}{\kappa_n(1-\kappa_n)} \mathbb{E}(y_{j+1}\mu^n_{j+1}|\mathcal{F}^n_j) \end{cases} \tag{3.3}$$

Remark 3.2. *(Computing the conditional expectations) Let Φ denote a function from $\mathbb{R}^{2j+2}$ to $\mathbb{R}$. We use the following formula to compute the conditional expectations*

$$\begin{aligned} \mathbb{E}(\Phi(e^n_1, \cdots, e^n_{j+1}, \eta^n_1, \cdots, \eta^n_{j+1})|\mathcal{F}^n_j) = & \frac{\kappa_n}{2} \Phi(e^n_1, \cdots, e^n_j, 1, \eta^n_1, \cdots, \eta^n_j, \kappa_n - 1) \\ & + \frac{\kappa_n}{2} \Phi(e^n_1, \cdots, e^n_j, -1, \eta^n_1, \cdots, \eta^n_j, \kappa_n - 1) \\ & + \frac{1-\kappa_n}{2} \Phi(e^n_1, \cdots, e^n_j, 1, \eta^n_1, \cdots, \eta^n_j, \kappa_n) \\ & + \frac{1-\kappa_n}{2} \Phi(e^n_1, \cdots, e^n_j, -1, \eta^n_1, \cdots, \eta^n_j, \kappa_n). \end{aligned}$$

3.2 Fully implementable numerical scheme

In this Section we present two numerical schemes to approximate the solution of the penalized equation (2.3): the first one, (3.5), is an implicit intermediate scheme, useful for the proof of convergence. We also introduce the main scheme (3.9), which is explicit. The implicit scheme (3.5) is not easy to solve numerically, since it involves to inverse a function, as we will see below. However, it plays an important role in the proof of the convergence of the explicit scheme, that's why we introduce it.

In both schemes, we approximate the barrier $(\xi_t)_t$ (resp. $(\zeta_t)_t$) by $(\xi^n_j)_{j=0,\cdots,n}$ (resp. $(\zeta^n_j)_{j=0,\cdots,n}$). We also introduce their continuous time versions:

$$\overline{\xi}^n_t := \xi^n_{[t/\delta_n]}, \;\; \overline{\zeta}^n_t := \zeta^n_{[t/\delta_n]}.$$

These approximations satisfy

Assumption 3.3.

(i) For some $r > 2$, $\sup_{n\in\mathbb{N}} \max_{j\le n} \mathbb{E}(|\xi^n_j|^r) + \sup_{n\in\mathbb{N}} \max_{j\le n} \mathbb{E}(|\zeta^n_j|^r) + \sup_{t\le T} \mathbb{E}|\xi_t|^r + \sup_{t\le T} \mathbb{E}|\zeta_t|^r < \infty$

(ii) $\overline{\xi}^n$ (resp $\overline{\zeta}^n$) converges in probability to ξ (resp. ζ) for the J_1-Skorokhod topology.

Remark 3.4. *Assumption 3.3 implies that for all t in $[0,T]$ $\overline{\xi}^n_{\psi^n(t)}$ (resp. $\overline{\zeta}^n_{\psi^n(t)}$) converges to ξ_t (resp. ζ_t) in L^2.*

Remark 3.5. *Let us give different examples of barriers in $\mathcal{S}^2$ satisfying Assumption 3.3. In this Remark, X represents either ξ or ζ.*

1. *X satisfies the following SDE*

$$X_t = X_0 + \int_0^t b_X(X_{s^-})ds + \int_0^t \sigma_X(X_{s^-})dW_s + \int_0^t c_X(X_{s^-})d\tilde{N}_s$$

   where b_X, σ_X and c_X are Lipschitz functions. We approximate it by

$$\overline{X}^n_t = \overline{X}^n_0 + \sum_{j=0}^{[t/\delta_n]-1} b_X(\overline{X}^n_{j\delta_n})\delta_n + \int_0^t \sigma_X(\overline{X}^n_{s^-})dW^n_s + \int_0^t c_X(\overline{X}^n_{s^-})d\tilde{N}^n_s$$

   Since $(W^n, \tilde{N}^n)$ converges in probability to $(W, \tilde{N})$ for the J_1-topology, [26, Corollary 1] gives that $\overline{X}^n$ converges to X in probability for the J_1-topology (for more details on the convergence of sequences of stochastic integrals on the space of RCLL functions endowed with the J_1-Skorokhod topology, we refer to [14]). Then, $\overline{X}^n$ satisfies Assumption 3.3 (ii). We deduce from Doob and Burkhölder-Davis-Gundy inequalities that X and $\overline{X}^n$ satisfy Assumption 3.3 (i) and that X belongs to $\mathcal{S}^2$.

2. *X is defined by $X_t := \Phi(t, W_t, \tilde{N}_t)$, where Φ satisfies the following assumptions*

   (a) *$\Phi(t, x, y)$ is uniformly continuous in (t, y) uniformly in x, i.e. there exist two continuous non decreasing functions $g_0(\cdot)$ and $g_1(\cdot)$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ with linear growth and satisfying $g_0(0) = g_1(0) = 0$ such that*

$$\forall\ (t, t', x, y, y'),\ \ |\Phi(t, x, y) - \Phi(t', x, y')| \le g_0(|t - t'|) + g_1(|y - y'|).$$

   We denote a_0 (resp. a_1) the constant of linear growth for g_0 (resp. g_1) i.e. $\forall\ (t, y) \in (\mathbb{R}_+)^2$, $0 \le g_0(t) + g_1(y) \le a_0(1 + t) + a_1(1 + y)$,

   (b) *$\Phi(t, x, y)$ is "strongly" locally Lispchitz in x uniformly in (t, y), i.e. there exists a constant K_0 and an integer p_0 such that*

$$\forall\ (t, x, x', y),\ \ |\Phi(t, x, y) - \Phi(t, x', y)| \le K_0(1 + |x|^{p_0} + |x'|^{p_0})|x - x'|.$$

   Then, $\forall (t, x, y)$ we have $|\Phi(t, x, y)| \le a_0|t| + a_1|y| + K_0(1 + |x|^{p_0})|x| + |\Phi(0, 0, 0)| + a_0 + a_1$. From this inequality, we prove that X satisfies Assumption 3.3 (i) by standard computations. Since $(\tilde{N}^n)$ converges in probability to $(\tilde{N})$ for the J_1-topology and $\lim_{n\to\infty} \sup_t |W^n_t - W_t| = 0$ in L^p for any p (see Lemma 3.1), we get that $(X^n_t)_t := (\Phi(\delta_n[t/\delta_n], W^n_t, \tilde{N}^n_t))_t$ converges in probability to X for the J_1-topology.

3.2.1 Intermediate penalized implicit discrete scheme

After the discretization of the penalized equation (2.3) on time intervals $[t_j, t_{j+1}]_{0\le j\le n-1}$, we get the following discrete backward equation. For all j in $\{0, \cdots, n-1\}$

$$\begin{cases} y^{p,n}_j = y^{p,n}_{j+1} + g(t_j, y^{p,n}_j, z^{p,n}_j, u^{p,n}_j)\delta_n + a^{p,n}_j - k^{p,n}_j - (z^{p,n}_j\sqrt{\delta_n}e^n_{j+1} + u^{p,n}_j\eta^n_{j+1} + v^{p,n}_j\mu^n_{j+1}) \\ a^{p,n}_j = p\delta_n(y^{p,n}_j - \xi^n_j)^-;\ k^{p,n}_j = p\delta_n(\zeta^n_j - y^{p,n}_j)^-, \\ y^{p,n}_n := \xi^n_n. \end{cases} \tag{3.4}$$

Following (3.3), the triplet $(z^{p,n}_j, u^{p,n}_j, v^{p,n}_j)$ can be computed as follows

$$\begin{cases} z^{p,n}_j = \dfrac{1}{\sqrt{\delta_n}}\mathbb{E}(y^{p,n}_{j+1}e^n_{j+1}|\mathcal{F}^n_j), \\ u^{p,n}_j = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(y^{p,n}_{j+1}\eta^n_{j+1}|\mathcal{F}^n_j), \\ v^{p,n}_j = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(y^{p,n}_{j+1}\mu^n_{j+1}|\mathcal{F}^n_j), \end{cases}$$

where we refer to Remark 3.2 for the computation of conditional expectations. By taking the conditional expectation w.r.t. $\mathcal{F}_j^n$ in (3.4), we get the following scheme, called *implicit penalized discrete scheme*: $y_n^{p,n} := \xi_n^n$ and for $j = n-1, \cdots, 0$

$$\begin{cases} y_j^{p,n} = (\Theta^{p,n})^{-1}(\mathbb{E}(y_{j+1}^{p,n}|\mathcal{F}_j^n)), \\ a_j^{p,n} = p\delta_n(y_j^{p,n} - \xi_j^n)^-;\ k_j^{p,n} = p\delta_n(\zeta_j^n - y_j^{p,n})^-, \\ z_j^{p,n} = \dfrac{1}{\sqrt{\delta_n}}\mathbb{E}(y_{j+1}^{p,n}e_{j+1}^n|\mathcal{F}_j^n), \\ u_j^{p,n} = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(y_{j+1}^{p,n}\eta_{j+1}^n|\mathcal{F}_j^n), \end{cases} \tag{3.5}$$

where $\Theta^{p,n}(y) = y - g(j\delta_n, y, z_j^{p,n}, u_j^{p,n})\delta_n - p\delta_n(y-\xi_j^n)^- + p\delta_n(\zeta_j^n - y)^-$.

We also introduce the continuous time version $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n}, A_t^{p,n}, K_t^{p,n})_{0\le t\le T}$ of the solution to (3.5):

$$Y_t^{p,n} := y_{[t/\delta_n]}^{p,n}, Z_t^{p,n} := z_{[t/\delta_n]}^{p,n}, U_t^{p,n} := u_{[t/\delta_n]}^{p,n}, A_t^{p,n} := \sum_{i=0}^{[t/\delta_n]} a_i^{p,n}, K_t^{p,n} := \sum_{i=0}^{[t/\delta_n]} k_i^{p,n}. \tag{3.6}$$

We also introduce $\alpha_t^{p,n} := A_t^{p,n} - K_t^{p,n}$, for all $t \in [0,T]$.

3.2.2 Main scheme

As said before, the numerical inversion of the operator $\Theta^{p,n}$ is not easy and is time consuming. If we replace $y_j^{p,n}$ by $\mathbb{E}(y_{j+1}^{p,n}|\mathcal{F}_j^n)$ in g, (3.4) becomes

$$\begin{cases} \overline{y}_j^{p,n} = \overline{y}_{j+1}^{p,n} + g(t_j, \mathbb{E}(\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n), \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n + \overline{a}_j^{p,n} - \overline{k}_j^{p,n} - (\overline{z}_j^{p,n}\sqrt{\delta_n}e_{j+1}^n + \overline{u}_j^{p,n}\eta_{j+1}^n + \overline{v}_j^{p,n}\mu_{j+1}^n) \\ \overline{a}_j^{p,n} = p\delta_n(\overline{y}_j^{p,n} - \xi_j^n)^-;\ \overline{k}_j^{p,n} = p\delta_n(\zeta_j^n - \overline{y}_j^{p,n})^-, \\ \overline{y}_n^{p,n} := \xi_n^n. \end{cases} \tag{3.7}$$

Now, by taking the conditional expectation in the above equation, we obtain:

$$\overline{y}_j^{p,n} = \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n + \overline{a}_j^{p,n} - \overline{k}_j^{p,n}. \tag{3.8}$$

Solving this equation, we get the following scheme, called *explicit penalized scheme*: $\overline{y}_n^{p,n} := \xi_n^n$ and for $j = n-1, \cdots, 0$

$$\begin{cases} \overline{y}_j^{p,n} = \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + g(t_j, \mathbb{E}(\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n), \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n + \overline{a}_j^{p,n} - \overline{k}_j^{p,n}, \\ \overline{a}_j^{p,n} = \dfrac{p\delta_n}{1+p\delta_n}\left(\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n}) - \xi_j^n\right)^-, \\ \overline{k}_j^{p,n} = \dfrac{p\delta_n}{1+p\delta_n}\left(\zeta_j^n - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] - \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\right)^- \\ \overline{z}_j^{p,n} = \dfrac{1}{\sqrt{\delta_n}}\mathbb{E}(\overline{y}_{j+1}^{p,n}e_{j+1}^n|\mathcal{F}_j^n), \\ \overline{u}_j^{p,n} = \dfrac{1}{\kappa_n(1-\kappa_n)}\mathbb{E}(\overline{y}_{j+1}^{p,n}\eta_{j+1}^n|\mathcal{F}_j^n). \end{cases} \tag{3.9}$$

Remark 3.6 (Explanations on the derivation of the main scheme)**.** *We give below some explanations concerning the derivation of the values of $\overline{a}_j^{p,n}$ and $\overline{k}_j^{p,n}$. We consider the following cases:*

- *If $\xi_j^n < \overline{y}_j^{p,n} < \zeta_j^n$, then by (3.7) we get $\overline{a}_j^{p,n} = \overline{k}_j^{p,n} = 0$, which corresponds to* $\dfrac{p\delta_n}{1+p\delta_n}\left(\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n}) - \xi_j^n\right)^- = \dfrac{p\delta_n}{1+p\delta_n}\left(\overline{y}_j^{p,n} - \xi_j^n\right)^- = 0$ *and* $\dfrac{p\delta_n}{1+p\delta_n}\left(\zeta_j^n - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] - \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\right)^- = \dfrac{p\delta_n}{1+p\delta_n}(\zeta_j^n - \overline{y}_j^{p,n})^- = 0.$

- *If $\xi_j^n \geq \overline{y}_j^{p,n}$, then by* (3.7) *we have $\overline{a}_j^{p,n} = p\delta_n(\xi_j^n - \overline{y}_j^{p,n})$ and $\overline{k}_j^{p,n} = 0$; we then replace $\overline{a}_j^{p,n}$ and $\overline{k}_j^{p,n}$ in* (3.8) *and we get $\overline{a}_j^{p,n} = \frac{p\delta_n}{1+p\delta_n}\left(\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] + g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n - \xi_j^n\right)^-$. We also have $\frac{p\delta_n}{1+p\delta_n}\left(\zeta_j^n - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] - \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\right)^- = 0$ and hence $\overline{k}_j^{p,n} = \frac{p\delta_n}{1+p\delta_n}\left(\zeta_j^n - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] - \delta_n g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\right)^-$.*
- *The case $\zeta_j^n \leq \overline{y}_j^{p,n}$ is symmetric to the one studied above: $\xi_j^n \geq \overline{y}_j^{p,n}$.*

As for the implicit scheme, we define the continuous time version $(\overline{Y}_t^{p,n}, \overline{Z}_t^{p,n}, \overline{U}_t^{p,n}, \overline{A}_t^{p,n}, \overline{K}_t^{p,n})_{0\leq t\leq T}$ of the solution to (3.9):

$$\overline{Y}_t^{p,n} = \overline{y}_{[t/\delta_n]}^{p,n}, \quad \overline{Z}_t^{p,n} = \overline{z}_{[t/\delta_n]}^{p,n}, \quad \overline{U}_t^{p,n} = \overline{u}_{[t/\delta_n]}^{p,n}, \quad \overline{A}_t^{p,n} = \sum_{j=0}^{[t/\delta_n]} \overline{a}_j^{p,n} \quad \overline{K}_t^{p,n} = \sum_{j=0}^{[t/\delta_n]} \overline{k}_j^{p,n}. \tag{3.10}$$

We also introduce $\overline{\alpha}_t^{p,n} := \overline{A}_t^{p,n} - \overline{K}_t^{p,n}$, for all $t \in [0,T]$.

4 Convergence result

The following result states the convergence of $\overline{\Theta}^{p,n} := (\overline{Y}^{p,n}, \overline{Z}^{p,n}, \overline{U}^{p,n}, \overline{\alpha}^{p,n})$ to $\Theta := (Y, Z, U, \alpha)$, the solution of the DBBSDE (2.2).

Theorem 4.1. *Assume that Assumptions 2.4 and 3.3 hold. The sequence $(\overline{Y}^{p,n}, \overline{Z}^{p,n}, \overline{U}^{p,n})$ defined by* (3.10) *converges to (Y, Z, U), the solution of the DBBSDE* (2.2)*, in the following sense: $\forall r \in [1, 2[$*

$$\lim_{p\to\infty}\lim_{n\to\infty}\left(\mathbb{E}\left[\int_0^T |\overline{Y}_s^{p,n} - Y_s|^2 ds\right] + \mathbb{E}\left[\int_0^T |\overline{Z}_s^{p,n} - Z_s|^r ds\right] + \mathbb{E}\left[\int_0^T |\overline{U}_s^{p,n} - U_s|^r ds\right]\right) = 0. \tag{4.1}$$

Moreover, $\overline{Z}^{p,n}$ (resp. $\overline{U}^{p,n}$) weakly converges in $\mathbb{H}^2$ to Z (resp. to U) and for $0 \leq t \leq T$, $\overline{\alpha}_{\psi^n(t)}^{p,n}$ converges weakly to α_t in $L^2(\mathcal{F}_T)$ as $n \to \infty$ and $p \to \infty$.

In order to prove this result, we split the error in three terms, by introducing $\Theta_t^{p,n} := (Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n}, \alpha_t^{p,n})$, the solution of the implicit penalized discrete scheme (3.6) and $\Theta_t^p := (Y_t^p, Z_t^p, U_t^p, \alpha_t^p)$, the penalized version of (2.2), defined by (2.3). For the error on Y, we get

$$\mathbb{E}[\int_0^T |\overline{Y}_s^{p,n} - Y_s|^2 ds] \leq 3\left(\mathbb{E}[\int_0^T |\overline{Y}_s^{p,n} - Y_s^{p,n}|^2 ds] + \mathbb{E}[\int_0^T |Y_s^{p,n} - Y_s^p|^2 ds] + \mathbb{E}[\int_0^T |Y_s^p - Y_s|^2 ds]\right),$$

and the same splitting holds for $|\overline{Z}^{p,n} - Z|^r$ and $|\overline{U}^{p,n} - U|^r$. For the increasing processes, we have:

$$\mathbb{E}[|\overline{\alpha}_{\psi^n(t)}^{p,n} - \alpha_t|^2] \leq 3\left(\mathbb{E}[|\overline{\alpha}_{\psi^n(t)}^{p,n} - \alpha_{\psi^n(t)}^{p,n}|^2] + \mathbb{E}[|\alpha_{\psi^n(t)}^{p,n} - \alpha_t^p|^2] + \mathbb{E}[|\alpha_t^p - \alpha_t|^2]\right). \tag{4.2}$$

The proof of Theorem 4.1 ensues from Proposition 4.2, Corollary 4.4 and Proposition 4.5. Proposition 4.2 states the convergence of the error between $\overline{\Theta}^{p,n}$, the explicit penalization scheme defined in (3.10), and $\Theta^{p,n}$, the implicit penalization scheme. It generalizes the results of [21]. We refer to Section 4.1. Corollary 4.4 states the convergence (in n) of $\Theta^{p,n}$ to Θ^p. This is based on the convergence of a standard BSDE with jumps in discrete time setting to the associated BSDE with jumps in continuous time setting, which is proved in [16]. We refer to Section 4.2. Finally, Proposition 4.5 proves the convergence (in p) of the penalized BSDE with jumps Θ^p to Θ, the solution of the DBBSDE (2.2). In fact, we prove a more general result in Section 4.3, since we show the convergence of penalized BSDEs to (2.1) in the case of jumps driven by a general Poisson random measure.

The rest of the Section is devoted to the proof of these results.

4.1 Error between explicit and implicit penalization schemes

We prove the convergence of the error between the explicit penalization scheme and the implicit one. The scheme of the proof is inspired from [21, Proposition 5].

Proposition 4.2. *Assume Assumption 3.3 (i) and g is a Lipschitz driver. We have*

$$\lim_{n\to\infty}\sup_{0\le t\le T}\left(\mathbb{E}[|\overline{Y}_t^{p,n}-Y_t^{p,n}|^2]+\mathbb{E}[\int_0^T|\overline{Z}_s^{p,n}-Z_s^{p,n}|^2ds]+\mathbb{E}[\int_0^T|\overline{U}_s^{p,n}-U_s^{p,n}|^2ds]\right)=0.$$

Moreover, $\lim_{n\to\infty}(\overline{\alpha}_t^{p,n}-\alpha_t^{p,n})=0$ *in* $L^2(\mathcal{F}_t)$*, for* $t\in[0,T]$.

Recall that

$$Y_t^{p,n}=y_{[t/\delta_n]}^{p,n},\, Z_t^{p,n}=z_{[t/\delta_n]}^{p,n},\, U_t^{p,n}=u_{[t/\delta_n]}^{p,n},\, A_t^{p,n}=\sum_{i=0}^{[t/\delta_n]}a_i^{p,n},\, K_t^{p,n}=\sum_{i=0}^{[t/\delta_n]}k_i^{p,n}.$$

In a similar way we have defined the continuous time versions of $(\bar{y}^{p,n},\bar{z}^{p,n},\bar{u}^{p,n},\bar{a}^{p,n},\bar{k}^{p,n})$, denoted by $(\bar{Y}^{p,n},\bar{Z}^{p,n},\bar{U}^{p,n},\bar{A}^{p,n},\bar{K}^{p,n})$.

Proof. By using the definitions of the implicit and explicit schemes (3.4) and (3.7), we obtain that:

$$\begin{aligned}y_{j+1}^{p,n}-\overline{y}_{j+1}^{p,n}=&(y_j^{p,n}-\overline{y}_j^{p,n})+(g_p(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{y}_j^{p,n},\overline{z}_j^{p,n},\overline{u}_j^{p,n})-g(t_j,y_j^{p,n},y_j^{p,n},z_j^{p,n},u_j^{p,n}))\delta_n\\&+(z_j^{p,n}-\overline{z}_j^{p,n})e_{j+1}^n\sqrt{\delta}_n+(u_j^{p,n}-\overline{u}_j^{p,n})\eta_{j+1}^n+(v_j^{p,n}-\overline{v}_j^{p,n})\mu_{j+1}^n\end{aligned}$$

where $g_p(t,y_1,y_2,z,u)=g(t,y_1,z,u)+p(y_2-\overline{\xi}_t^n)^--p(\overline{\zeta}_t^n-y_2)^-$. It implies that:

$$\begin{aligned}\mathbb{E}[(y_j^{p,n}-\overline{y}_j^{p,n})^2]=&\mathbb{E}[(y_{j+1}^{p,n}-\overline{y}_{j+1}^{p,n})^2]-\mathbb{E}[(g_p(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{y}_j^{p,n},\overline{z}_j^{p,n},\overline{u}_j^{p,n})-g_p(t_j,y_j^{p,n},y_j^{p,n},z_j^{p,n},u_j^{p,n}))^2]\delta_n^2\\&-\mathbb{E}[(z_j^{p,n}-\overline{z}_j^{p,n})^2]\delta_n-\mathbb{E}[(u_j^{p,n}-\overline{u}_j^{p,n})^2](1-\kappa_n)\kappa_n-\mathbb{E}[(v_j^{p,n}-\overline{v}_j^{p,n})^2](1-\kappa_n)\kappa_n\\&+2\mathbb{E}[(g_p(t_j,y_j^{p,n},y_j^{p,n},z_j^{p,n},u_j^{p,n})-g_p(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{y}_j^{p,n},\overline{z}_j^{p,n},\overline{u}_j^{p,n}))(y_j^{p,n}-\overline{y}_j^{p,n})]\delta_n.\end{aligned}$$

In the above relation, we take the sum over j from i to $n-1$. We have:

$$\begin{aligned}&\mathbb{E}[(y_i^{p,n}-\overline{y}_i^{p,n})^2]+\delta_n\sum_{j=i}^{n-1}\mathbb{E}[(z_j^{p,n}-\overline{z}_j^{p,n})^2]+(1-\kappa_n)\kappa_n\sum_{j=i}^{n-1}\mathbb{E}[(u_j^{p,n}-\overline{u}_j^{p,n})^2]\\&\qquad\le 2\delta_n\sum_{j=i}^{n-1}\mathbb{E}[(g_p(t_j,y_j^{p,n},y_j^{p,n},z_j^{p,n},u_j^{p,n})-g_p(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{y}_j^{p,n},\overline{z}_j^{p,n},\overline{u}_j^{p,n}))(y_j^{p,n}-\overline{y}_j^{p,n})].\end{aligned}$$

Let us introduce $f:y\longmapsto(y-\overline{\xi}_t^n)^--(\overline{\zeta}_t^n-y)^-$. We have $g_p(t,y_1,y_2,z,u)=g(t,y_1,z,u)+pf(y_2)$. The last expectation of the previous inequality can be written

$$\mathbb{E}[(g(t_j,y_j^{p,n},z_j^{p,n},u_j^{p,n})-g(t_j,\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n],\overline{z}_j^{p,n},\overline{u}_j^{p,n}))(y_j^{p,n}-\overline{y}_j^{p,n})+p(f(y_j^{p,n})-f(\overline{y}_j^{p,n}))(y_j^{p,n}-\overline{y}_j^{p,n})]$$

Since f is decreasing and g is Lipschitz, we obtain:

$$\begin{aligned}&\mathbb{E}[(y_i^{p,n}-\overline{y}_i^{p,n})^2]+\delta_n\sum_{j=i}^{n-1}\mathbb{E}[(z_j^{p,n}-\overline{z}_j^{p,n})^2]+(1-\kappa_n)\kappa_n\sum_{j=i}^{n-1}\mathbb{E}[(u_j^{p,n}-\overline{u}_j^{p,n})^2]\\&\qquad\le 2\delta_n\sum_{j=i}^{n-1}\mathbb{E}\left[(C_g|y_j^{p,n}-\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n]|+C_g|z_j^{p,n}-\overline{z}_j^{p,n}|+C_g|u_j^{p,n}-\overline{u}_j^{p,n}|)|y_j^{p,n}-\overline{y}_j^{p,n}|\right].\end{aligned}$$

Consequently, by applying the inequality $2ab \le a^2 + b^2$ for $a = C_g|y_j^{p,n} - \overline{y}_j^{p,n}|\sqrt{2\delta_n}$; $b = \sqrt{\frac{\delta_n}{2}}|z_j^{p,n} - \overline{z}_j^{p,n}|$ and $a = C_g|y_j^{p,n} - \overline{y}_j^{p,n}|\sqrt{2}\frac{\delta_n}{\sqrt{\kappa_n(1-\kappa_n)}}$; $b = \sqrt{\frac{\kappa_n(1-\kappa_n)}{2}}|u_j^{p,n} - \overline{u}_j^{p,n}|$ we get that:

$$
\begin{aligned}
&\mathbb{E}[(y_i^{p,n} - \overline{y}_i^{p,n})^2] + \delta_n \sum_{j=i}^{n-1} \mathbb{E}[(z_j^{p,n} - \overline{z}_j^{p,n})^2] + (1-\kappa_n)\kappa_n \sum_{j=i}^{n-1} \mathbb{E}[(u_j^{p,n} - \overline{u}_j^{p,n})^2] \\
&\qquad \le 2\delta_n C_g^2 \sum_{j=i}^{n-1} \mathbb{E}[(y_j^{p,n} - \overline{y}_j^{p,n})^2] + \frac{\delta_n}{2} \sum_{j=i}^{n-1} \mathbb{E}[(z_j^{p,n} - \overline{z}_j^{p,n})^2] + \frac{2C_g^2\delta_n^2}{\kappa_n(1-\kappa_n)} \sum_{j=i}^{n-1} \mathbb{E}[(y_j^{p,n} - \overline{y}_j^{p,n})^2] \\
&\qquad + \frac{(1-\kappa_n)\kappa_n}{2} \sum_{j=i}^{n-1} \mathbb{E}[(u_j^{p,n} - \overline{u}_j^{p,n})^2] + 2C_g\delta_n \mathbb{E}[\sum_{j=i}^{n-1} |y_j^{p,n} - \overline{y}_j^{p,n}||y_j^{p,n} - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n]|].
\end{aligned}
$$

Now, since $\overline{y}_j^{p,n} - \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n] = g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})\delta_n$, the last term is dominated by

$$
\delta_n \sum_{j=i}^{n-1} (2C_g + 1)\mathbb{E}[(y_j^{p,n} - \overline{y}_j^{p,n})^2] + C_g^2\delta_n^3 \sum_{j=i}^{n-1} \mathbb{E}[g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{y}_j^{p,n}, \overline{z}_j^{p,n}, \overline{u}_j^{p,n})^2].
$$

Using the definition of g_p yields

$$
\begin{aligned}
g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{y}_j^{p,n}, \overline{z}_j^{p,n}, \overline{u}_j^{p,n}) &\le |g(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})| + p(|\overline{y}_j^{p,n}| + |\xi_j^n| + |\zeta_j^n|), \\
&\le |g(t_j, 0, 0, 0)| + C_g(|\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n]| + |\overline{z}_j^{p,n}| + |\overline{u}_j^{p,n}|) + p(|\overline{y}_j^{p,n}| + |\xi_j^n| + |\zeta_j^n|).
\end{aligned}
$$

We get

$$
\begin{aligned}
\delta_n^3 \sum_{j=i}^{n-1} \mathbb{E}[g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{y}_j^{p,n}, \overline{z}_j^{p,n}, \overline{u}_j^{p,n})^2] \le & C_0\delta_n^2(\delta_n \sum_{j=i}^{n-1} |g(t_j, 0, 0, 0)|^2 + \delta_n \sum_{j=i}^{n-1} |\overline{z}_j^{p,n}|^2 + \delta_n \sum_{j=i}^{n-1} |\overline{u}_j^{p,n}|^2) \\
&+ C_0(p\delta_n)^2(\max_j \mathbb{E}(|\xi_j^n|^2) + \max_j \mathbb{E}(|\zeta_j^n|^2)) \\
&+ C_0\delta_n^2(1+p^2)\max_j \mathbb{E}(|\overline{y}_j^{p,n}|^2)
\end{aligned}
$$

where C_0 denotes a generic constant depending on C_g. Since $\frac{\delta_n}{(1-\kappa_n)\kappa_n} = \frac{1}{\lambda}\frac{\lambda\delta_n}{(1-e^{-\lambda\delta_n})e^{-\lambda\delta_n}}$ and $e^x \le \frac{xe^{2x}}{e^x - 1} \le e^{2x}$, we get $\frac{\delta_n}{(1-\kappa_n)\kappa_n} \le \frac{1}{\lambda}e^{2\lambda T}$. Hence, for δ_n small enough such that $(3 + 2p + 2C_g + 2C_g^2(1 + \frac{1}{\lambda}e^{2\lambda T}))\delta_n < 1$, Lemma C.1 enables to write:

$$
\begin{aligned}
\mathbb{E}[(y_i^{p,n} - \overline{y}_i^{p,n})^2] + &\frac{\delta_n}{2}\mathbb{E}[\sum_{j=i}^{n-1}(z_j^{p,n} - \overline{z}_j^{p,n})^2] + \frac{1}{2}(1-\kappa_n)\kappa_n \mathbb{E}[\sum_{j=i}^{n-1}(u_j^{p,n} - \overline{u}_j^{p,n})^2] \\
&\le \left(1 + 2C_g + 2C_g^2 + \frac{2C_g^2\delta_n}{(1-\kappa_n)\kappa_n}\right)\delta_n \mathbb{E}[\sum_{j=i}^{n-1}(y_j^{p,n} - \overline{y}_j^{p,n})^2] + C_1(p)\delta_n^2, \qquad (4.3)
\end{aligned}
$$

where $C_1(p) = C_0(\|g(\cdot, 0, 0, 0)\|_\infty^2 + p^2(\sup_n \max_j \mathbb{E}|\xi_j^n|^2 + \sup_n \max_j \mathbb{E}|\zeta_j^n|^2) + (1+p^2)K_{\text{Lem.}C.1})$, $K_{\text{Lem.}C.1}$ denotes the constant appearing in Lemma C.1. Discrete Gronwall's Lemma (see [21, Lemma 3]) gives

$$
\sup_{i \le n} \mathbb{E}[(y_i^{p,n} - \overline{y}_i^{p,n})^2] \le C_1(p)\delta_n^2 e^{(1+2C_g+2C_g^2(1+\frac{1}{\lambda}e^{2\lambda T}))T}.
$$

Since $\delta_n \le T$, $(1-\kappa_n)\kappa_n \ge \lambda\delta_n e^{-2\lambda T}$, and Equation (4.3) gives

$$
\mathbb{E}[\int_0^T |\overline{Z}_s^{p,n} - Z_s^{p,n}|^2 ds] + \mathbb{E}[\int_0^T |\overline{U}_s^{p,n} - U_s^{p,n}|^2 ds] \le C_1'(p)\delta_n^2,
$$

where $C_1'(p)$ is another constant depending on C_g, λ, T and $C_1(p)$. It remains to prove the convergence for the increasing processes. We have

$$\overline{A}_t^{p,n} - \overline{K}_t^{p,n} = \overline{Y}_0^{p,n} - \overline{Y}_t^{p,n} - \int_0^t g(s, \overline{Y}_s^{p,n}, \overline{Z}_s^{p,n}, \overline{U}_s^{p,n})ds + \int_0^t \overline{Z}_s^{p,n} dW_s^n + \int_0^t \overline{U}_s^{p,n} d\tilde{N}_s^n,$$
$$A_t^{p,n} - K_t^{p,n} = Y_0^{p,n} - Y_t^{p,n} - \int_0^t g(s, Y_s^{p,n}, Z_s^{p,n}, U_s^{p,n})ds + \int_0^t Z_s^{p,n} dW_s^n + \int_0^t U_s^{p,n} d\tilde{N}_s^n.$$

Using the Lispchitz property of g and the convergence of $(\overline{Y}_s^{p,n} - Y_s^{p,n}, \overline{Z}_s^{p,n} - Z_s^{p,n}, \overline{U}_s^{p,n} - U_s^{p,n})$, we get the result. $\square$

4.2 Convergence of the discrete time setting to the continuous time setting

The following Proposition ensues from [16].

Proposition 4.3. *Let g be a Lipschitz driver and assume that Assumption 3.3 (ii) holds. For any $p \in \mathbb{N}^*$, the sequence $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n})$ converges to (Y_t^p, Z_t^p, U_t^p) in the following sense:*

$$\lim_{n\to\infty} \left(||Y^{p,n} - Y^p||^2_{J_1-L^2} + \mathbb{E}[\int_0^T |Z_s^{p,n} - Z_s^p|^2 ds + \int_0^T |U_s^{p,n} - U_s^p|^2 ds] \right) = 0. \tag{4.4}$$

Proof. For a fixed p, we have the following:

$$Y^{p,n} - Y^p = (Y^{p,n} - Y^{p,n,q}) + (Y^{p,n,q} - Y^{p,\infty,q}) + (Y^{p,\infty,q} - Y^p). \tag{4.5}$$

where $(Y^{p,\infty,q}, Z^{p,\infty,q}, U^{p,\infty,q})$ is the Picard approximation of (Y^p, Z^p, U^p) and $(Y^{p,n,q}, Z^{p,n,q}, U^{p,n,q})$ represents the continuous time version of the discrete Picard approximation of $(y_k^{p,n}, z_k^{p,n}, u_k^{p,n})$, denoted by $(y_k^{p,n,q}, z_k^{p,n,q}, u_k^{p,n,q})$. Note that $(y_k^{p,n,q+1}, z_k^{p,n,q+1}, u_k^{p,n,q+1})$ is defined inductively as the solution of the backward recursion given by [16, Eq. (3.16)], for the penalized driver $g_n(\omega, t, y, z, u) := g(\omega, t, y, z, u) + p(y - \overline{\xi}_t^n(\omega))^- - p(\overline{\zeta}_t^n(\omega) - y)^-$. Since $\overline{\xi}^n$ and $\overline{\zeta}^n$ satisfy Assumption 3.3 (ii), $(g_n(\omega, \cdot, \cdot, \cdot, \cdot))_n$ converges uniformly to $g(\omega, \cdot, \cdot, \cdot, \cdot) + p(y - \xi_t(\omega))^- - p(\zeta_t(\omega) - y)^-$ almost surely up to a subsequence (i.e. g_n satisfies [16, Assumption (A')]).
Now, by using (4.5), [16, Proposition 1], [16, Proposition 3] and [16, Eq. (3.17)], one can easily show that (4.4) holds. $\square$

The following Corollary ensues from Proposition 4.3.

Corollary 4.4. *Let g be a Lipschitz driver, ξ and ζ belong to $\mathcal{S}^2$, ψ^n is the random mapping introduced in Proposition 4.3 and assume that Assumption 3.3 holds. For any $p \in \mathbb{N}^*$, the sequence $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n})$ converges to (Y_t^p, Z_t^p, U_t^p) in the following sense:*

$$\lim_{n\to\infty} \mathbb{E}[\int_0^T |Y_s^{p,n} - Y_s^p|^2 ds + \int_0^T |Z_s^{p,n} - Z_s^p|^2 ds + \int_0^T |U_s^{p,n} - U_s^p|^2 ds] = 0,$$

Moreover, $A^{p,n}$ (resp. $K^{p,n}$) converges to A^p (resp. K^p) when n tends to infinity in L^2 for the J_1-Skorokhod topology.

Proof. Note that:

$$\int_0^T |Y_s^{p,n} - Y_s^p|^2 ds \leq 2 \int_0^T |Y_s^{p,n} - Y_{\eta^n(s)}^p|^2 dt + 2 \int_0^T |Y_{\eta^n(s)}^p - Y_s^p|^2 ds,$$

where $\eta^n(s)$ represents the inverse of $\psi^n(s)$.

Proposition 4.3 gives that the first term in the right-hand side converges to 0. Concerning the second term, $s \mapsto Y_s^p$ is continuous except at the times at which the Poisson process jumps. Consequently, $Y_{\eta^n(s)}^p$

converges to Y_s^p for almost every s and as Y^p belongs to $\mathcal{S}^2$, we get that $\mathbb{E}[\int_0^T |Y^p_{\eta^n(s)} - Y_s^p|^2 ds] \to 0$ when $n \to \infty$.

Now, remark that we can rewrite $A_t^{p,n}$ and A_t^p as follows:

$$A_t^{p,n} = p\int_0^t (Y_s^{p,n} - \overline{\xi}_s^n)^- ds \qquad A_t^p = p\int_0^t (Y_s^p - \xi_s)^- ds. \tag{4.6}$$

Then

$$\begin{aligned}\sup_{t\in[0,T]} |A^{p,n}_{\psi^n(t)} - A_t^p| &= \sup_{t\in[0,T]} |A_t^{p,n} - A^p_{\eta^n(t)}| \\ &= \sup_{k\in\{0,\cdots,n\}} |A^{p,n}_{t_k} - A^p_{t_k}| + \sup_{k\in\{0,\cdots,n\}} \sup_{t\in[t_k,t_{k+1}]} |A^p_{t_k} - A^p_{\eta^n(t)}|.\end{aligned}$$

since ξ and Y^p belong to $\mathcal{S}^2$, we get that the second term in the right hand side tends to 0 in L^2 when $n \to \infty$.

$$\sup_{k\in\{0,\cdots,n\}} |A^{p,n}_{t_k} - A^p_{t_k}| \le p\int_0^T |Y_s^{p,n} - Y_s^p| + |\overline{\xi}_s^n - \xi_s| ds.$$

Since $\lim_{n\to\infty} \mathbb{E}[\int_0^T |Y_s^{p,n} - Y_s^p|^2 ds] = 0$, $\lim_{n\to\infty} \mathbb{E}|\overline{\xi}_s^n - \xi_{\eta(s)}|^2 = 0$ (see Remark 3.4) and $\lim_{n\to\infty} \mathbb{E}[\int_0^T |\xi_{\eta^n(s)} - \xi_s|^2 ds] = 0$ (ξ is RCLL, its jumps are countable), we get that $\sup_{k\in\{0,\cdots,n\}} |A^{p,n}_{t_k} - A^p_{t_k}|$ converges to 0 in L^2 in n, which ends the proof.

□

4.3 Convergence of the penalized BSDE to the reflected BSDE

As said in the Introduction, this part of the proof deals with the convergence of the penalized BSDE when the jumps are driven by a general Poisson random measure. We state in Proposition 4.5 that a sequence of penalized BSDEs converges to the solution to (2.1). To do so, we give in Section 4.3.1 an other proof of existence of solutions to reflected BSDEs with jumps and RCLL barriers based on the penalization method. We extend the proof of [17, Section 4] to the case of totally inacessible jumps. We are able to generalize their proof thanks to Mokobodzki's condition (which in particular enables to get Lemma 4.7, generalizing [17, Lemma 4.1]), to the comparison Theorem for BSDEs with jumps (see Theorem D.1 and Theorem D.2) and to the caracterization of the solution of the DBBSDE as the value function of a stochastic game (proved in Proposition D.5).

We introduce the penalization scheme, generalizing (2.3) to the case of Poisson random measure :

$$\begin{aligned} Y_t^p =& \xi_T + \int_t^T g(s, Y_s^p, Z_s^p, U_s^p) ds + p\int_t^T (Y_s^p - \xi_s)^- ds - p\int_t^T (\zeta_s - Y_s^p)^- ds - \int_t^T Z_s^p dW_s \\ &- \int_t^T \int_{\mathbb{R}^*} U_s^p(e) \tilde{N}(ds, de) \end{aligned} \tag{4.7}$$

with $A_t^p = p\int_0^t (Y_s^p - \xi_s)^- ds$ and $K_t^p = p\int_0^t (\zeta_s - Y_s^p)^- ds$.

Proposition 4.5. *Under Hypothesis 2.4, Y^p converges to Y in $\mathbb{H}^2$, Z^p weakly converges in $\mathbb{H}^2$ to Z, U^p weakly converges in $\mathbb{H}^2_\nu$ to U, and $\alpha_t^p := A_t^p - K_t^p$ weakly converges to α_t in $L^2(\mathcal{F}_t)$. Moreover, for all $r \in [1, 2[$, the following strong convergence holds*

$$\lim_{p\to\infty} \mathbb{E}\left[\int_0^T |Y_s^p - Y_s|^2 ds\right] + \mathbb{E}\left[\int_0^T |Z_s^p - Z_s|^r ds + \int_0^T \left(\int_{\mathbb{R}^*} |U_s^p - U_s|^2 \nu(de)\right)^{\frac{r}{2}} ds\right] = 0. \tag{4.8}$$

The proof of Proposition 4.5 is postponed to Section 4.3.2.

4.3.1 Intermediate result

For each p,q in $\mathbb{N}$, since the driver $g(s,y,z,u)+q(y-\xi_s)^- - p(\zeta_s-y)^-$ is Lipschitz in (y,z,u), the following classical BSDE with jumps admits a unique solution $(Y^{p,q},Z^{p,q},U^{p,q})$ (see [27])

$$\begin{aligned} Y_t^{p,q} =& \xi_T + \int_t^T g(s,Y_s^{p,q},Z_s^{p,q},U_s^{p,q})ds + q\int_t^T (Y_s^{p,q}-\xi_s)^- ds - p\int_t^T (\zeta_s - Y_s^{p,q})^- ds - \int_t^T Z_s^{p,q} dW_s \\ & - \int_t^T \int_{\mathbb{R}^*} U_s^{p,q}(e)\tilde{N}(ds,de). \end{aligned} \tag{4.9}$$

We set $A_t^{p,q} = q\int_0^t (Y_s^{p,q}-\xi_s)^- ds$ and $K_t^{p,q} = p\int_0^t (\zeta_s - Y_s^{p,q})^- ds$.

Theorem 4.6. *Let us assume that Assumption 2.4 holds. The quadruple $(Y^{p,q},Z^{p,q},U^{p,q},\alpha^{p,q})$, where $\alpha^{p,q} = A^{p,q} - K^{p,q}$, converges to (Y,Z,U,α), the solution of (2.1), as $p\to\infty$ then $q\to\infty$ (or equivalently as $q\to\infty$ then $p\to\infty$) in the following sense : $Y^{p,q}$ converges to Y in $\mathbb{H}^2$, $Z^{p,q}$ weakly converges to Z in $\mathbb{H}^2$, $U^{p,q}$ weakly converges to U in $\mathbb{H}^2_\nu$, $\alpha_t^{p,q}$ weakly converges to α_t in $L^2(\mathcal{F}_t)$. Moreover, for each $r\in[1,2[$, the following strong convergence holds*

$$\lim_{p\to\infty}\lim_{q\to\infty} \mathbb{E}\left(\int_0^T |Y_s^{p,q}-Y_s|^2 ds\right) + \mathbb{E}\left(\int_0^T |Z_s^{p,q}-Z_s|^r ds + \int_0^T \left(\int_{\mathbb{R}^*} |U_s^{p,q}-U_s|^2\nu(de)\right)^{\frac{r}{2}} ds\right) = 0. \tag{4.10}$$

The proof of Theorem 4.6 is divided in several steps. We prove

1. the quadruple $(Y^{p,q},Z^{p,q},U^{p,q},\alpha^{p,q})$ converges as $q\to\infty$ then $p\to\infty$
2. the quadruple $(Y^{p,q},Z^{p,q},U^{p,q},\alpha^{p,q})$ converges as $p\to\infty$ then $q\to\infty$
3. the two limits are equal (see Lemma 4.11)
4. the limit of the penalized BSDE is the solution of the reflected BSDE (2.1) (see Theorem 4.3.1)
5. Equation (4.10) ensues from (4.27) and (4.29).

Proof of point 1.
Let us first state the following preliminary result.

Lemma 4.7. *Suppose that $H,H'\in\mathcal{S}^2$ are two supermartingales such that Assumption 2.4 holds. Let Y^* be the RCLL adapted process defined by $Y_t^* := (H_t - H_t')\mathbf{1}_{t<T} + \xi_T\mathbf{1}_{t=T}$. There exists $(Z^*,U^*,A^*,K^*)\in \mathbb{H}^2\times\mathbb{H}^2_\nu\times\mathcal{A}^2\times\mathcal{A}^2$ such that (Y^*,Z^*,U^*,A^*,K^*) solves $(i),(ii),(iii)$ of (2.1).*

Proof. By assumption, H and H' are square integrable supermartingales. The process Y^* is thus well defined. By the Doob-Meyer decomposition of supermartingales, there exist two square integrable martingales M and M', two square integrable nondecreasing predictable RCLL processes V and V' with $V_0 = V_0' = 0$ such that:

$$dH_t = dM_t - dV_t \quad ; \quad dH_t' = dM_t' - dV_t'. \tag{4.11}$$

Define

$$\overline{M}_t := M_t - M_t'.$$

By the above relation and (4.11), we derive $dY_t^* = d\overline{M}_t - dV_t + dV_t'$. Now, by the martingale representation theorem, there exist $Z^*\in\mathbb{H}^2, U^*\in\mathbb{H}^2_\nu$ such that:

$$d\overline{M}_t = Z_t^* dW_t + \int_{\mathbb{R}^*} U_t^*(e)\tilde{N}(de,dt). \tag{4.12}$$

Consequently, (4.11) and (4.12) imply that:

$$
\begin{aligned}
Y_t^* =& \xi_T + \int_t^T g(s, Y_s^*, Z_s^*, U_s^*)ds - \left(\int_t^T g(s, Y_s^*, Z_s^*, U_s^*)ds + (V_T - V_t) - (V_T' - V_t') \right) - \int_t^T Z_s^* dW_s \\
& - \int_t^T \int_{\mathbb{R}^*} U_s^*(e) \tilde{N}(ds, de).
\end{aligned}
$$

Now let g^+ (resp. g^-) denote the positive (resp. negative) part of the function g. By setting $A_t^* := V_t + \int_0^t g^+(s, Y_s^*, Z_s^*, U_s^*)ds$ and $K_t^* := V_t' + \int_0^t g^-(s, Y_s^*, Z_s^*, U_s^*)ds$, the result follows. □

Proposition 4.8. *Suppose Assumption 2.4 holds. Then, there exists a constant C, independent of p and q such that we have :*

$$
\mathbb{E}\left[\sup_{0 \le t \le T} (Y_t^{p,q})^2\right] + \mathbb{E}\left[\int_0^T |Z_t^{p,q}|^2 dt\right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^*} |U_t^{p,q}(e)|^2 \nu(de) dt\right] + \mathbb{E}[(A_T^{p,q})^2] + \mathbb{E}[(K_T^{p,q})^2] \le C. \tag{4.13}
$$

Proof. This proof generalizes the proof of [17, Proposition 4.1] to the case of jumps. Since p and q play symmetric roles, the calculations over p and q are uniform throughout this proof. From Lemma 4.7, we know that there exists $(Y^*, Z^*, U^*, A^*, K^*)$ in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2 \times \mathcal{A}^2 \times \mathcal{A}^2$ such that

$$
Y_t^* = \xi_T + \int_t^T g(s, \theta_s^*)ds + (A_T^* - A_t^*) - (K_T^* - K_t^*) - \int_t^T Z_s^* dW_s - \int_t^T \int_{\mathbb{R}^*} U_s^*(e) \tilde{N}(ds, de)
$$

and $\xi_t \le Y_t^* \le \zeta_t$ $dP \otimes dt$ a.s. (θ_s^* denotes (Y_s^*, Z_s^*, U_s^*)). Then, for $p, q \in \mathbb{N}$, we also have

$$
\begin{aligned}
Y_t^* =& \xi_T + \int_t^T g(s, \theta_s^*)ds + (A_T^* - A_t^*) - (K_T^* - K_t^*) + q \int_t^T (\xi_s - Y_s^*)^+ ds - p \int_t^T (Y_s^* - \zeta_s)^+ ds \\
& - \int_t^T Z_s^* dW_s - \int_t^T \int_{\mathbb{R}^*} U_s^*(e) \tilde{N}(ds, de).
\end{aligned}
$$

Let $\overline{\theta}^{p,q} := (\overline{Y}^{p,q}, \overline{Z}^{p,q}, \overline{U}^{p,q})$ and $\tilde{\theta}^{p,q} = (\tilde{Y}^{p,q}, \tilde{Z}^{p,q}, \tilde{U}^{p,q})$ be the solutions of the following equations

$$
\overline{Y}_t^{p,q} = \xi_T + \int_t^T g(s, \overline{\theta}_s^{p,q})ds + (A_T^* - A_t^*) + q \int_t^T (\xi_s - \overline{Y}_s^{p,q})^+ ds - p \int_t^T (\overline{Y}_s^{p,q} - \zeta_s)^+ ds \tag{4.14}
$$

$$
- \int_t^T \overline{Z}_s^{p,q} dW_s - \int_t^T \int_{\mathbb{R}^*} \overline{U}_s^{p,q}(e) \tilde{N}(ds, de). \tag{4.15}
$$

$$
\tilde{Y}_t^{p,q} = \xi_T + \int_t^T g(s, \tilde{\theta}_s^{p,q})ds - (K_T^* - K_t^*) + q \int_t^T (\xi_s - \tilde{Y}_s^{p,q})^+ ds - p \int_t^T (\tilde{Y}_s^{p,q} - \zeta_s)^+ ds \tag{4.16}
$$

$$
- \int_t^T \tilde{Z}_s^{p,q} dW_s - \int_t^T \int_{\mathbb{R}^*} \tilde{U}_s^{p,q}(e) \tilde{N}(ds, de). \tag{4.17}
$$

By the comparison theorem for BSDEs with jumps (see Theorem D.1), we get that for all p, q in $\mathbb{N}$, $\tilde{Y}_t^{p,q} \le Y_t^{p,q} \le \overline{Y}_t^{p,q}$, $\xi_t \le Y_t^* \le \overline{Y}_t^{p,q}$ and $\tilde{Y}_t^{p,q} \le Y_t^* \le \zeta_t$. Applying this result to (4.14) gives that $(\overline{Y}^{p,q}, \overline{Z}^{p,q}, \overline{U}^{p,q})$ is also solution to

$$
\overline{Y}_t^{p,q} = \xi_T + \int_t^T g(s, \overline{\theta}_s^{p,q})ds + (A_T^* - A_t^*) - p \int_t^T (\overline{Y}_s^{p,q} - \zeta_s)^+ ds - \int_t^T \overline{Z}_s^{p,q} dW_s - \int_t^T \int_{\mathbb{R}^*} \overline{U}_s^{p,q}(e) \tilde{N}(ds, de). \tag{4.18}
$$

Doing the same with (4.16) gives that $(\tilde{Y}^{p,q}, \tilde{Z}^{p,q}, \tilde{U}^{p,q})$ is also solution to

$$\tilde{Y}_t^{p,q} = \xi_T + \int_t^T g(s, \tilde{\theta}_s^{p,q})ds - (K_T^* - K_t^*) + q\int_t^T (\xi_s - \tilde{Y}_s^{p,q})^+ ds - \int_t^T \tilde{Z}_s^{p,q} dW_s - \int_t^T \int_{\mathbb{R}^*} \tilde{U}_s^{p,q}(e)\tilde{N}(ds, de). \tag{4.19}$$

Let us consider the following BSDEs

$$Y_t^+ = \xi_T + \int_t^T g(s, \theta_s^+)ds + (A_T^* - A_t^*) - \int_t^T Z_s^+ dW_s - \int_t^T \int_{\mathbb{R}^*} \tilde{U}_s^+(e)\tilde{N}(ds, de), \tag{4.20}$$

$$Y_t^- = \xi_T + \int_t^T g(s, \theta_s^-)ds - (K_T^* - K_t^*) - \int_t^T Z_s^- dW_s - \int_t^T \int_{\mathbb{R}^*} \tilde{U}_s^-(e)\tilde{N}(ds, de), \tag{4.21}$$

where $\theta_s^+ := (Y_s^+, Z_s^+, U_s^+)$ and $\theta_s^- := (Y_s^-, Z_s^-, U_s^-)$. Since $\overline{K}_t^{p,q} := p\int_0^t (\overline{Y}_s^{p,q} - \zeta_s)^+ ds$ and $\tilde{A}_t^{p,q} := q\int_0^t (\xi_s - \tilde{Y}_s^{p,q})^+ ds$ are increasing processes, Theorem D.1 applied to (4.18) and (4.20) (resp. to (4.19) and (4.21)) gives $\overline{Y}_t^{p,q} \le Y_t^+$ (resp. $Y_t^- \le \tilde{Y}_t^{p,q}$). Combining theses results with the inequality $\tilde{Y}_t^{p,q} \le Y_t^{p,q} \le \overline{Y}_t^{p,q}$ leads to

$$\forall (p,q) \in \mathbb{N}^2,\ \forall t \in [0,T],\ \ Y_t^- \le \tilde{Y}_t^{p,q} \le Y_t^{p,q} \le \overline{Y}_t^{p,q} \le Y_t^+. \tag{4.22}$$

Then we have

$$\mathbb{E}[\sup_{0\le t\le T} (Y_t^{p,q})^2] \le \max\{\mathbb{E}[\sup_{0\le t\le T} (Y_t^+)^2], \mathbb{E}[\sup_{0\le t\le T} (Y_t^-)^2]\}. \tag{4.23}$$

Since A^* and K^* belong to $\mathcal{A}^2$, Itô's formula, BDG inequality and Gronwall's Lemma give $\mathbb{E}[\sup_{0\le t\le T}(Y_t^+)^2] \le C$ and $\mathbb{E}[\sup_{0\le t\le T}(Y_t^-)^2] \le C$. Then we get

$$\mathbb{E}[\sup_{0\le t\le T} (Y_t^{p,q})^2] \le C. \tag{4.24}$$

Let us now prove that $\mathbb{E}[(A_T^{p,q})^2] + \mathbb{E}[(K_T^{p,q})^2] \le C$. Since for all p, q in $\mathbb{N}$, $\tilde{Y}_t^{p,q} \le Y_t^{p,q} \le \overline{Y}_t^{p,q}$, then $\tilde{A}_t^{p,q} \ge A_t^{p,q} \ge 0$ and $\overline{K}_t^{p,q} \ge K_t^{p,q} \ge 0$. It boils down to prove $\mathbb{E}[(\tilde{A}_T^{p,q})^2] + \mathbb{E}[(\overline{K}_T^{p,q})^2] \le C$. Let us first prove that $\mathbb{E}[(\tilde{A}_T^{p,q})^2] \le C$. To do so, we apply [8, Equation (17)] to (4.19) (as a sequence in q). In the same way, we apply [8, Equation (17)] to (4.18) (as a sequence in p). We get $\mathbb{E}[(\overline{K}_T^{p,q})^2] \le C$.

It remains to prove $\mathbb{E}\left[\int_0^T |Z_t^{p,q}|^2 dt\right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^*} |U_t^{p,q}(e)|^2 \nu(de)dt\right] \le C$. By applying Itô's formula to $|Y_t^{p,q}|^2$, we get

$$\begin{aligned}
&\mathbb{E}\left[|Y_t^{p,q}|^2\right] + \mathbb{E}\left[\int_t^T |Z_s^{p,q}|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^*} |U_s^{p,q}(e)|^2 \nu(de)ds\right] \\
=&\mathbb{E}[\xi_T^2] + 2\mathbb{E}\left[\int_t^T Y_s^{p,q} g(s, Y_s^{p,q}, Z_s^{p,q}, U_s^{p,q})ds\right] + 2\mathbb{E}\left[\int_t^T Y_s^{p,q} q(Y_s^{p,q} - \xi_s)^- ds\right] - 2\mathbb{E}\left[\int_t^T Y_s^{p,q} p(\zeta_s - Y_s^{p,q})^- ds\right].
\end{aligned}$$

The third term of the right hand side is zero if $Y_s^{p,q} \ge \xi_s$. Then we can bound it by $2\mathbb{E}\left[\sup_{0\le t\le T} |\xi_t|(A_T^{p,q} - A_t^{p,q})\right]$. The last term of the right hand side is bounded in the same way. We bound it by $2\mathbb{E}\left[\sup_{0\le t\le T} |\zeta_t|(K_T^{p,q} - K_t^{p,q})\right]$. By using that g is Lipschitz, we bound the second term of the right hand side

$$2\mathbb{E}\left[\int_t^T Y_s^{p,q} g(s, Y_s^{p,q}, Z_s^{p,q}, U_s^{p,q})ds\right] \le 2\mathbb{E}\left[\int_t^T |Y_s^{p,q}|(\|g(\cdot, 0,0,0)\|_\infty + C_g(|Y_s^{p,q}| + |Z_s^{p,q}| + |U_s^{p,q}|))ds\right].$$

By applying Young's inequality, we get

$$
\begin{aligned}
&\mathbb{E}\left[|Y_t^{p,q}|^2\right] + \mathbb{E}\left[\int_t^T |Z_s^{p,q}|^2 ds\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^*} |U_s^{p,q}(e)|^2 \nu(de) ds\right] \qquad (4.25)\\
\leq &\|g(\cdot,0,0,0)\|_\infty^2 + (1+2C_g+4C_g^2)\mathbb{E}\left[\int_t^T |Y_s^{p,q}|^2 ds\right] + \frac{1}{2}\mathbb{E}\left[\int_t^T |Z_s^{p,q}|^2 ds\right] + \frac{1}{2}\mathbb{E}\left[\int_t^T \int_{\mathbb{R}^*} |U_s^{p,q}(e)|^2 \nu(de) ds\right]\\
&+ \mathbb{E}[\sup_{0\leq t\leq T} \xi_t^2] + \mathbb{E}[\sup_{0\leq t\leq T} \zeta_t^2] + \mathbb{E}[(A_T^{p,q})^2] + \mathbb{E}[(K_T^{p,q})^2].
\end{aligned}
$$

By combining the assumptions on ξ, ζ, (4.24) and the previous result bounding $\mathbb{E}[(A_T^{p,q})^2] + E[(K_T^{p,q})^2]$, we get $\mathbb{E}[\int_t^T |Z_s^{p,q}|^2 ds] + \mathbb{E}[\int_t^T \int_{\mathbb{R}^*} |U_s^{p,q}(e)|^2 \nu(de) ds] \leq C$. $\square$

In (4.9), for fixed p we set $g_p(s,y,z,u) = g(s,y,z,u) - p(\zeta_s - y)^-$. g_p is Lipschitz and

$$
\mathbb{E}\left(\int_0^T (g_p(s,0,0,0))^2 ds\right) \leq 2\mathbb{E}\left(\int_0^T (g(s,0,0,0))^2 ds\right) + 2p^2 T\mathbb{E}(\sup_{0\leq t\leq T} (\zeta_t)^2) < \infty.
$$

By Theorem D.1, we know that $(Y^{p,q})$ is increasing in q for all p. Thanks to Theorem D.4, we know that $(Y^{p,q}, Z^{p,q}, U^{p,q})_{q\in\mathbb{N}}$ has a limit $(Y^{p,\infty}, Z^{p,\infty}, U^{p,\infty}) := \theta^{p,\infty}$ such that $(Y^{p,q})_q$ converges increasingly to $Y^{p,\infty} \in \mathcal{S}^2$, and thanks to Theorem D.3, we know that there exists $Z^{p,\infty} \in \mathbb{H}^2$, $U^{p,\infty} \in \mathbb{H}^2_\nu$ and $A^{p,\infty} \in \mathcal{A}^2$ such that $(Y^{p,\infty}, Z^{p,\infty}, U^{p,\infty}, A^{p,\infty})$ satisfies the following equation

$$
\begin{aligned}
Y_t^{p,\infty} =&\xi_T + \int_t^T g(s,\theta_s^{p,\infty}) ds + (A_T^{p,\infty} - A_t^{p,\infty}) - p\int_t^T (\zeta_s - Y_s^{p,\infty})^- ds - \int_t^T Z_s^{p,\infty} dW_s\\
&- \int_t^T \int_{\mathbb{R}^*} U_s^{p,\infty}(e) \tilde{N}(ds,de) \qquad (4.26)
\end{aligned}
$$

$Z^{p,\infty}$ is the weak limit of $(Z^{p,q})_q$ in $\mathbb{H}^2$, $U^{p,\infty}$ is the weak limit of $(U^{p,q})_q$ in $\mathbb{H}^2_\nu$ and $A_t^{p,\infty}$ is the weak limit of $(A_t^{p,q})_q$ in $L^2(\mathcal{F}_t)$. Moreover, for each $r \in [1,2[$, the following strong convergence holds

$$
\lim_{q\to\infty} \mathbb{E}\left(\int_0^T |Y_s^{p,q} - Y_s^{p,\infty}|^2 ds\right) + \mathbb{E}\left(\int_0^T |Z_s^{p,q} - Z_s^{p,\infty}|^r ds + \int_0^T \left(\int_{\mathbb{R}^*} |U_s^{p,q} - U_s^{p,\infty}|^2 \nu(de)\right)^{\frac{r}{2}} ds\right) = 0. \qquad (4.27)
$$

From [8, Theorem 5.1], we also get that $\forall t \in [0,T]$, $Y_t^{p,\infty} \geq \xi_t$ and $\int_0^T (Y_{t^-}^{p,\infty} - \xi_{t^-}) dA_t^{p,\infty} = 0$ a.s. Set $K_t^{p,\infty} = p\int_0^t (\zeta_s - Y_s^{p,\infty})^- ds$. Since $Y^{p,q} \nearrow Y^{p,\infty}$ when $q \to \infty$, $K^{p,q} \nearrow K^{p,\infty}$ when $q \to \infty$. By the monotone convergence theorem and (4.13), we get that $\mathbb{E}((K_T^{p,\infty})^2) \leq C$. Then we get the following Lemma.

Lemma 4.9. *There exists a constant C independent of p such that*

$$
\mathbb{E}\left[\sup_{0\leq t\leq T} (Y_t^{p,\infty})^2\right] + \mathbb{E}\left[\int_0^T |Z_t^{p,\infty}|^2 dt\right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^*} |U_t^{p,\infty}(e)|^2 \nu(de) dt\right] + \mathbb{E}[(A_T^{p,\infty})^2] + \mathbb{E}[(K_T^{p,\infty})^2] \leq C.
$$

From Theorem D.2, we have $Y_t^{p,\infty} \geq Y_t^{p+1,\infty}$, then there exists a process Y such that $Y^{p,\infty} \searrow Y$. By using Fatou's lemma, we get

$$
\mathbb{E}\left(\sup_{0\leq t\leq T} (Y_t)^2\right) \leq C,
$$

and the dominated convergence theorem gives us that $\lim_{p\to\infty} Y^{p,\infty} = Y$ in $\mathbb{H}^2$. Since $(Y^{p,q})_p$ is a decreasing sequence, $(A^{p,q})_p$ is an increasing sequence, and by passing to the limit ($(A_t^{p,q})_q$ weakly converges to $A_t^{p,\infty}$), we get $A_t^{p,\infty} \leq A_t^{p+1,\infty}$. Then, we deduce from Lemma 4.9 that there exists a process A such that $A^{p,\infty} \nearrow A$

and $\mathbb{E}(A_T^2) < \infty$. Since $A_t^{p,q} - A_s^{p,q} = \int_s^t q(\xi_r - Y_r^{p,q})^+ dr \leq \int_s^t q(\xi_r - Y_r^{p+1,q})^+ dr = A_t^{p+1,q} - A_s^{p+1,q}$, we get that

$$A_t^{p,\infty} - A_s^{p,\infty} \leq A_t^{p+1,\infty} - A_s^{p+1,\infty} \;\; \forall\, 0 \leq s \leq t \leq T.$$

Thanks to Lemma 4.9, we can apply the "generalized monotonic Theorem" A.1: there exist $Z \in \mathbb{H}^2$, $U \in \mathbb{H}^2_\nu$ and $K \in \mathcal{A}^2$ such that

$$Y_t = \xi_T + \int_t^T g(s, Y_s, Z_s, U_s)ds + A_T - A_t - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds, de), \quad (4.28)$$

K_t is the weak limit of $K_t^{p,\infty}$ in $L^2(\mathcal{F}_t)$, Z is the weak limit of $Z^{p,\infty}$ in $\mathbb{H}^2$ and U is the weak limit of $U^{p,\infty}$ in $\mathbb{H}^2_\nu$. Moreover, $A_t^{p,\infty}$ strongly converges to A_t in $L^2(\mathcal{F}_t)$ and $A \in \mathcal{A}^2$, and we have for each $r \in [1, 2[$,

$$\lim_{p\to\infty} \mathbb{E}\left(\int_0^T |Y_s^{p,\infty} - Y_s|^2 ds\right) + \mathbb{E}\left(\int_0^T |Z_s^{p,\infty} - Z_s|^r ds + \int_0^T \left(\int_{\mathbb{R}^*} |U_s^{p,\infty} - U_s|^2 \nu(de)\right)^{\frac{r}{2}} ds\right) = 0. \quad (4.29)$$

Proof of point 2.

Similarly, $(Y^{p,q})_p$ is decreasing for any fixed q. The same arguments as before give that $(Y^{p,q}, Z^{p,q}, U^{p,q})_{p\in\mathbb{N}}$ has a limit $(Y^{\infty,q}, Z^{\infty,q}, U^{\infty,q}) := \theta^{\infty,q}$ such that $(Y^{p,q})_p$ converges decreasingly to $Y^{\infty,q} \in \mathcal{S}^2$, and thanks to Theorem D.3, we know that there exists $Z^{\infty,q} \in \mathbb{H}^2$, $U^{\infty,q} \in \mathbb{H}^2_\nu$ and $K^{\infty,q} \in \mathcal{A}^2$ such that $(Y^{\infty,q}, Z^{\infty,q}, U^{\infty,q}, K^{\infty,q})$ satisfies the following equation

$$\begin{aligned} Y_t^{\infty,q} =& \xi_T + \int_t^T g(s, \theta_s^{\infty,q})ds + q\int_t^T (Y_s^{\infty,q} - \xi_s)^- ds - (K_T^{\infty,q} - K_t^{\infty,q}) - \int_t^T Z_s^{\infty,q} dW_s \\ & - \int_t^T \int_{\mathbb{R}^*} U_s^{\infty,q}(e)\tilde{N}(ds, de) \end{aligned} \quad (4.30)$$

$Z^{\infty,q}$ is the weak limit of $(Z^{p,q})_p$ in $\mathbb{H}^2$, $U^{\infty,q}$ is the weak limit of $(U^{p,q})_p$ in $\mathbb{H}^2_\nu$ and $K_t^{\infty,q}$ is the weak limit of $(K_t^{p,q})_p$ in $L^2(\mathcal{F}_t)$. From [8, Theorem 5.1], we also get that $\forall t \in [0, T]$, $Y_t^{\infty,q} \leq \zeta_t$ and $\int_0^T (Y_{t^-}^{\infty,q} - \zeta_{t^-})dK_t^{\infty,q} = 0$ a.s. Set $A_t^{\infty,q} = q\int_0^t (Y_s^{\infty,q} - \xi_s)^- ds$. Since $Y^{p,q} \searrow Y^{\infty,q}$ when $p \to \infty$, $A^{p,q} \nearrow A^{\infty,q}$ when $p \to \infty$. By the monotone convergence theorem and (4.13), we get that $\mathbb{E}((A_T^{\infty,q})^2) \leq C$. We get the following result, equivalent to Lemma 4.9

Lemma 4.10. *There exists a constant C independent of q such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} (Y_t^{\infty,q})^2\right] + \mathbb{E}\left[\int_0^T |Z_t^{\infty,q}|^2 dt\right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^*} |U_t^{\infty,q}(e)|^2 \nu(de)dt\right] + \mathbb{E}[(A_T^{\infty,q})^2] + \mathbb{E}[(K_T^{\infty,q})^2] \leq C.$$

From Theorem D.2, we have $Y_t^{\infty,q} \leq Y_t^{\infty,q+1}$, then there exists a process Y' such that $Y^{\infty,q} \nearrow Y'$. By using Fatou's lemma, we get that Y' belongs to $\mathcal{S}^2$, and the convergence also holds in $\mathbb{H}^2$. By using the same proof as before, we can apply Theorem A.1: there exist $Z' \in \mathbb{H}^2$, $U' \in \mathbb{H}^2_\nu$ and $A' \in \mathcal{A}^2$ such that

$$Y'_t = \xi_T + \int_t^T g(s, Y'_s, Z'_s, U'_s)ds + A'_T - A'_t - (K'_T - K'_t) - \int_t^T Z'_s dW_s - \int_t^T \int_{\mathbb{R}^*} U'_s(e)\tilde{N}(ds, de),$$

A'_t is the weak limit of $A_t^{\infty,q}$ in $L^2(\mathcal{F}_t)$, Z' is the weak limit of $Z^{\infty,q}$ in $\mathbb{H}^2$ and U' is the weak limit of $U^{\infty,q}$ in $\mathbb{H}^2_\nu$. Moreover, $K_t^{\infty,q}$ strongly converges to K'_t in $L^2(\mathcal{F}_t)$ and $K' \in \mathcal{A}^2$. We will now prove that the two limits are equal.

Proof of point 3.

Lemma 4.11. *The two limits Y and Y' are equal. Moreover $Z = Z'$, $U = U'$ and $A - K = A' - K'$.*

Proof. Since $Y^{p,q} \nearrow Y^{p,\infty}$ and $Y^{p,q} \searrow Y^{\infty,q}$, we get that for all $p, q \in \mathbb{N}$, $Y^{\infty,q} \leq Y^{p,q} \leq Y^{p,\infty}$. Then, since $Y^{p,\infty} \searrow Y$ and $Y^{\infty,q} \nearrow Y'$, we get $Y' \leq Y$. On the other hand, since $Y^{\infty,q} \leq Y^{p,q}$, we get that for all $0 \leq s \leq t \leq T$

$$A_t^{p,q} - A_s^{p,q} \leq A_t^{\infty,q} - A_s^{\infty,q}.$$

Since $(A_t^{p,q})_q$ weakly converges to $A_t^{p,\infty}$ in $L^2(\mathcal{F}_t)$, $(A_t^{\infty,q})_q$ weakly converges to A'_t in $L^2(\mathcal{F}_t)$, and $(A_t^{p,\infty})_p$ strongly converges to A_t in $L^2(\mathcal{F}_t)$, taking limit in q and then limit in p gives

$$A_t - A_s \leq A'_t - A'_s. \tag{4.31}$$

Since $Y^{p,q} \leq Y^{p,\infty}$, we get that for all $0 \leq s \leq t \leq T$

$$K_t^{p,q} - K_s^{p,q} \leq K_t^{p,\infty} - K_s^{p,\infty}.$$

Letting $p \to \infty$ and $q \to \infty$ leads to

$$K'_t - K'_s \leq K_t - K_s. \tag{4.32}$$

Combining (4.31) and (4.32) gives that for all $0 \leq s \leq t \leq T$

$$A_t - A_s - (K_t - K_s) \leq A'_t - A'_s - (K'_t - K'_s).$$

Thanks to Theorem D.1, we get that $Y' \geq Y$. Then $Y' = Y$, and we get $Z' = Z$, $U' = U$, and $A' - K' = A - K$. □

Proof of point 4.

It remains to prove that the limit $(Y, Z, U, A - K)$ of the penalized BSDE is the solution of the reflected BSDE with two RCLL barriers ξ and ζ. To do so, we use the links between Dynkin games and DBBSDEs (see Proposition D.5) and Snell envelope theory (see Appendix B).

Theorem 4.12. *Let $\alpha := A - K$. The quartuple (Y, Z, U, α) solving* (4.28) *is the unique solution to* (2.1).

Proof. We know from Theorem 2.7 that (2.1) has a unique solution. We already know that (Y, Z, U, A, K) belongs to $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{A}^2 \times \mathcal{A}^2$ and satisfies (ii). It remains to check (iii) and (iv). We first check (iii). From (4.26), we know that $(Y^{p,\infty}, Z^{p,\infty}, U^{p,\infty}, A^{p,\infty})$ is the solution of a reflected BSDE (RBSDE in the following) with one lower barrier ξ. Let $\alpha^{p,\infty} := A^{p,\infty} - K^{p,\infty}$. Then, $(Y^{p,\infty}, Z^{p,\infty}, U^{p,\infty}, \alpha^{p,\infty})$ can be considered as the solution of a RBSDE with two barriers ξ and $\zeta + (\zeta - Y^{p,\infty})^-$, since we have

$$\xi \leq Y^{p,\infty} \leq \zeta + (\zeta - Y^{p,\infty})^-, \quad \int_0^T (Y_t^{p,\infty} - \xi_t) dA_t^{p,\infty} = 0$$

and

$$\int_0^T (Y_t^{p,\infty} - \zeta_t - (\zeta - Y^{p,\infty})_t^-) dK_t^{p,\infty} = -p \int_0^T (Y_t^{p,\infty} - \zeta_t)^- (\zeta_t - Y_t^{p,\infty})^- dt = 0.$$

From Proposition D.5 we know that

$$\begin{aligned}
Y_t^{p,\infty} &= \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma \wedge \tau} g(s, \theta_s^{p,\infty}) ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} + (\zeta_\sigma - Y_\sigma^{p,\infty})^- \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) \\
&\geq \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma \wedge \tau} g(s, \theta_s^{p,\infty}) ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) \\
&\geq \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma \wedge \tau} g(s, \theta_s) ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) \\
&\quad - C_g \mathbb{E}\left(\int_0^T |Y_s^{p,\infty} - Y_s| + |Z_s^{p,\infty} - Z_s| + \|U_s^{p,\infty} - U_s\|_\nu ds | \mathcal{F}_t \right).
\end{aligned}$$

Since $Y^{p,\infty} \to Y$ in $\mathbb{H}^2$, $Z^{p,\infty} \to Z$ in $\mathbb{H}^r$ for $r < 2$, and $U^{p,\infty} \to U$ in $\mathbb{H}^r_\nu$ for $r < 2$, there exists a subsequence p_j such that the last conditional expectation converges to 0 a.s. Taking the limit in p in the last inequality gives

$$Y_t \geq \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s,\theta_s)ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right). \tag{4.33}$$

In the same way, we know that $(Y^{\infty,q}, Z^{\infty,q}, U^{\infty,q}, K^{\infty,q})$ is the solution of a RBSDE with one upper barrier ζ. Let $\alpha^{\infty,q} := A^{\infty,q} - K^{\infty,q}$. Then $(Y^{\infty,q}, Z^{\infty,q}, U^{\infty,q}, \alpha^{\infty,q})$ is the solution of a RBSDE with two barriers $\xi - (Y^{\infty,q} - \xi)^-$ and ζ. By Proposition D.5 we know that

$$\begin{aligned} Y_t^{\infty,q} \leq & \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s,\theta_s)ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) \\ & + C_g \mathbb{E}\left(\int_0^T |Y_s^{\infty,q} - Y_s| + |Z_s^{\infty,q} - Z_s| + \|U_s^{\infty,q} - U_s\|_\nu ds | \mathcal{F}_t \right). \end{aligned}$$

Since $Y^{\infty,q} \to Y$ in $\mathbb{H}^2$, $Z^{\infty,q} \to Z$ in $\mathbb{H}^r$ for $r < 2$, and $U^{\infty,q} \to U$ in $\mathbb{H}^r_\nu$ for $r < 2$, there exists a subsequence q_j such that the last conditional expectation converges to 0 a.s. Taking the limit in q in the last inequality gives

$$Y_t \leq \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s,\theta_s)ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right). \tag{4.34}$$

Comparing (4.33) and (4.34) and since esssup essinf $\leq$ essinf esssup, we deduce

$$\begin{aligned} Y_t &= \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s,\theta_s)ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) \\ &= \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\int_t^{\sigma\wedge\tau} g(s,\theta_s)ds + \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right). \end{aligned}$$

Let $M_t := \mathbb{E}(\xi_T + \int_0^T g(s,\theta_s)ds|\mathcal{F}_t) - \int_0^t g(s,\theta_s)ds$, $\tilde{\xi}_t = \xi_t - M_t$ and $\tilde{\zeta}_t = \zeta_t - M_t$. We can rewrite Y in the following form

$$\begin{aligned} Y_t &= \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \mathbb{E}\left(\tilde{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) + M_t \\ &= \operatorname*{essinf}_{\sigma \in \mathcal{T}_t} \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left(\tilde{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\sigma < \tau} \big| \mathcal{F}_t \right) + M_t \end{aligned}$$

Then $Y_t - M_t$ is the value of a stochastic game problem with payoff $I_t(\tau,\sigma) = \tilde{\xi}_\tau \mathbf{1}_{\tau \leq \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\sigma < \tau}$. Let us check that $\tilde{\xi}$ and $\tilde{\zeta}$ are in $\mathcal{S}^2$. Since ξ and ζ are in $\mathcal{S}^2$, we only have to check that $M \in \mathcal{S}^2$. Using Doob's inequality

$$\begin{aligned} \mathbb{E}(\sup_{0 \leq t \leq T} (M_t)^2) &\leq 2\mathbb{E}\left(\sup_{0 \leq t \leq T} \left(E(\xi + \int_0^T g(s,\theta_s)ds|\mathcal{F}_t) \right)^2 + \left(\int_0^T |g(s,\theta_s)|ds \right)^2 \right), \\ &\leq C(1 + \mathbb{E}\int_0^T |Y_s|^2 + |Z_s|^2 + \|U_s\|_\nu^2 ds) < \infty. \end{aligned}$$

Since $\tilde{\xi}_T = \tilde{\zeta}_T = 0$ and ξ and ζ satisfy Mokobodzki'condition, we can apply [17, Theorem 5.1]: there exists a pair of non-negative RCLL supermatingales (X^+, X^-) in $\mathcal{S}^2$ such that

$$\begin{aligned} X_t^+ &= \mathcal{R}_t(X^- + \tilde{\xi}), \\ X_t^- &= \mathcal{R}_t(X^+ - \tilde{\zeta}) \end{aligned} \tag{4.35}$$

where $\mathcal{R}_t(\phi)$ denotes the Snell enveloppe of ϕ (see Appendix B). Thanks to [17, Theorem 5.2], we know that $Y_t - M_t = X_t^+ - X_t^-$. Moreover, by the Doob-Meyer decomposition theorem, we get

$$X_t^+ = \mathbb{E}(A_T^1|\mathcal{F}_t) - A_t^1, \; X_t^- = \mathbb{E}(K_T^1|\mathcal{F}_t) - K_t^1$$

where A^1, K^1 are predictable increasing processes belonging to $\mathcal{A}^2$. With the representation theorem for the martingale part we know that there exists $Z^1 \in \mathbb{H}^2$ and $U^1 \in \mathbb{H}^2_\nu$ such that

$$\begin{aligned} Y_t &= M_t + X_t^+ - X_t^- \\ &= \mathbb{E}(\xi + \int_0^T g(s,\theta_s)ds + A_T^1 - K_T^1|\mathcal{F}_t) - \int_0^t g(s,\theta_s)ds - A_t^1 + K_t^1, \\ &= Y_0 + \int_0^t Z_s^1 dW_s + \int_0^t \int_{\mathbb{R}^*} U_s^1(e)\tilde{N}(ds,de) - \int_0^t g(s,\theta_s)ds - A_t^1 + K_t^1. \end{aligned}$$

Then, we compare the forward form of (4.28) and the previous equality, we get

$$(A_t - K_t) - (A_t^1 - K_t^1) = \int_0^t (Z_s - Z_s^1)dW_s + \int_0^t \int_{\mathbb{R}^*} (U_s(e) - U_s^1(e))\tilde{N}(ds,de)$$

and then $Z_t = Z_t^1$, $U_t = U_t^1$ and $K_t - A_t = K_t^1 - A_t^1$. By using the properties of the Snell envelope in (4.35) (see Proposition B.3), we get the $X^+ \geq X^- + \tilde{\xi}$ and $X^- \geq X^+ - \tilde{\zeta}$, which leads to

$$\xi = M + \tilde{\xi} \leq Y = M + X^+ - X^- \leq M + \tilde{\zeta} = \zeta$$

and (iii) follows.
It remains to check (iv). By Proposition B.4, we get that

$$0 = \int_0^T (X_{t^-}^+ - (\tilde{\xi}_{t^-} + X_{t^-}^-))dA_t^1 = \int_0^T (X_{t^-}^+ - X_{t^-}^- - \xi_{t^-} + M_{t^-})dA_t^1 = \int_0^T (Y_{t^-} - \xi_{t^-})dA_t^1,$$

and

$$0 = \int_0^T (X_{t^-}^- - (X_{t^-}^+ - \tilde{\zeta}_{t^-}))dK_t^1 = \int_0^T (X_{t^-}^- - X_{t^-}^+ + \zeta_{t^-} - M_{t^-})dK_t^1 = \int_0^T (\zeta_{t^-} - Y_{t^-})dK_t^1,$$

which ends the proof. □

4.3.2 Proof of Proposition 4.5

In order to prove the convergence of $(Y^p, Z^p, U^p, \alpha^p)$, we rewrite (4.26), the solution of the reflected BSDE with one lower obstacle ξ

$$\begin{aligned} Y_t^{p,\infty} =& \xi + \int_t^T g(s,\theta_s^{p,\infty})ds + (A_T^{p,\infty} - A_t^{p,\infty}) - p\int_t^T (\zeta_s - Y_s^{p,\infty})^- ds - \int_t^T Z_s^{p,\infty}dW_s \\ &- \int_t^T \int_{\mathbb{R}^*} U_s^{p,\infty}(e)\tilde{N}(ds,de), \end{aligned}$$

and (4.30), the solution of the reflected BSDE with one upper obstacle ζ

$$\begin{aligned} Y_t^{\infty,p} =& \xi + \int_t^T g(s,\theta_s^{\infty,p})ds + p\int_t^T (Y_s^{\infty,p} - \xi_s)^- ds - (K_T^{\infty,p} - K_t^{\infty,p}) - \int_t^T Z_s^{\infty,p}dW_s \\ &- \int_t^T \int_{\mathbb{R}^*} U_s^{\infty,p}(e)\tilde{N}(ds,de). \end{aligned}$$

Since $Y_t^{p,\infty} \geq \xi_t$ and $Y^{\infty,p} \leq \zeta_t$, we can substract $p\int_t^T (Y_s^{p,\infty} - \xi_s)^- ds$ to the first BSDE and we can add $p\int_t^T (\zeta_s - Y_s^{\infty,p})^- ds$ to the second BSDE. By the comparison theorem we get $Y_t^{\infty,p} \leq Y_t^p \leq Y_t^{p,\infty}$. Since

$Y^{p,\infty} \searrow Y$ and $Y^{\infty,p} \nearrow Y$ when $p \to \infty$, we get that $Y_t^p \to Y_t$ almost surely, for all $t \in [0,T]$. From (4.29) and the corresponding result for $Y^{\infty,p}$, we get that $\lim_{p\to\infty} \mathbb{E}(\int_0^T |Y_s^p - Y_s|^2 ds) = 0$.
Applying Itô's formula to $\mathbb{E}(|Y_t^p - Y_t|^2)$ between $[\sigma,\tau]$, a pair of stopping times such that $t \le \sigma \le \tau \le T$, we get

$$
\begin{aligned}
&\mathbb{E}\left(|Y_\sigma^p - Y_\sigma|^2 + \int_\sigma^\tau |Z_s^p - Z_s|^2 ds + \int_\sigma^\tau \int_{\mathbb{R}^*} |U_s^p(e) - U_s(e)|^2 \nu(de) ds\right) = \mathbb{E}(|Y_\tau^p - Y_\tau|^2) \\
&+ 2\mathbb{E}(\int_\sigma^\tau (Y_s^p - Y_s)(g(s,\theta_s^p) - g(s,\theta_s))ds) + \sum_{\sigma\le s\le\tau} (\Delta_s A)^2 + \sum_{\sigma\le s\le\tau} (\Delta_s K)^2 + 2\sum_{\sigma\le s\le\tau} \Delta_s A \Delta_s K \\
&+ 2\int_\sigma^\tau (Y_s^p - Y_s) d(A^p - A)_s - 2\int_\sigma^\tau (Y_s^p - Y_s) d(K^p - K)_s.
\end{aligned}
$$

By using the Cauchy-Schwarz inequality, the convergence of Y^p to Y in $\mathbb{H}^2$, and the fact that $g(s,\theta_s^p)$ and $g(s,\theta_s)$ are bounded in $L^2(\Omega \times [0,T])$, we get that the second term of the r.h.s. tends to zero when p tends to ∞. From the dominated convergence theorem the last two terms of the r.h.s. also tend to zero. Since $2\sum_{\sigma\le s\le\tau} \Delta_s A \Delta_s K \le \sum_{\sigma\le s\le\tau}(\Delta_s A)^2 + \sum_{\sigma\le s\le\tau}(\Delta_s K)^2$, we are back to Theorem D.3, which ends the proof of (4.8).

It remains to prove that Z^p weakly converges to Z in $\mathbb{H}^2$, U^p weakly converges to U in $\mathbb{H}^2_\nu$ and α_t^p weakly converges to α in $L^2(\mathcal{F}_t)$. Since $Y_t^{\infty,p} \le Y_t^p \le Y_t^{p,\infty}$, we get $A_t^p \le A_t^{\infty,p}$ and $K_t^p \le K_t^{p,\infty}$. Then, by using Lemmas 4.9 and 4.10, we obtain $\mathbb{E}((A_T^p)^2) + \mathbb{E}((K_T^p)^2) \le C$, where C does not depend on p. By applying Itô's formula to $|Y_t^p|^2$ and by using Young's inequality as in (4.25) we get $\mathbb{E}(\int_0^T |Z_t^p|^2 dt + \int_0^T(\int_{\mathbb{R}^*} |U_s^p(e)|^2 \nu(de) ds)) \le C$, where C does not depend on p. The sequences $(Z^p)_{p\ge 0}$, $(U^p)_{p\ge0}$, $(A_t^p)_{p\ge0}$ and $(K_t^p)_{p\ge0}$ are bounded in the respective spaces $\mathbb{H}^2$, $\mathbb{H}^2_\nu$, $L^2(\mathcal{F}_t)$ and $L^2(\mathcal{F}_t)$. Then, we can extract subsequences which weakly converge in the related spaces. Let us denote Z', U', A' and K' the respective limits. Since (Z^p, U^p) strongly converge to (Z,U) for any $q < 2$ (see (4.8)), we get that $Z = Z'$ and $U = U'$.

Let us prove that $A' - K' = A - K$. We have

$$
\begin{aligned}
A_t^p - K_t^p &= Y_0^p - Y_t^p - \int_0^t g(s,\theta_s^p) ds + \int_0^t Z_s^p dW_s + \int_0^t \int_{\mathbb{R}^*} U_s^p(e) \tilde{N}(ds,de), \\
A_t - K_t &= Y_0 - Y_t - \int_0^t g(s,\theta_s) ds + \int_0^t Z_s dW_s + \int_0^t \int_{\mathbb{R}^*} U_s(e) \tilde{N}(ds,de).
\end{aligned}
$$

Taking the limit in p in the first equation, we get $A_t' - K_t' = A_t - K_t$.

5 Numerical simulations

In this section, we illustrate the convergence of our scheme with two examples. The difficulty in the choice of examples is given by the hypothesis we assume, in particular the Mokobodzi's condition which is difficult to check in practice.

Example 1 : inaccessible jumps

We consider the simulation of the solution of a DBBSDE with obstacles having only totally inaccessible jumps. More precisely, we take the barriers and driver of the following form: $\xi_t := (W_t)^2 + \tilde{N}_t + (T-t)$, $\zeta_t := (W_t)^2 + \tilde{N}_t + 3(T-t)$, $g(t,\omega,y,z,u) := -5|y+z| + 6u - 1$.

Our example satisfies the assumptions assumed in the theoretical part, in particular Hypotheses 2.4 and 3.3 (see Remark 3.5, point 2.). Assumption (2.4), which represents the Mokobodzki's condition, is fulfilled, since $H_t := (W_t)^2 + \tilde{N}_t + 2(T-t)$ satisfies $\xi_t \leq H_t \leq \zeta_t$ and $H_t = M_t + A_t$, where $M_t := (W_t)^2 + \tilde{N}_t + T - t$ is a martingale and $A_t := T - t$ is a decreasing finite variation process.

Table 1 gives the values of Y_0 with respect to parameters n and p of our explicit sheme. We notice that the algorithm converges quite fast in p and n. However, when n is too small ($n = 20$ and $n = 50$), the result for $p = 20000$ is quite far from the "reference" result ($n = 600$ and $p = 20000$). Concerning the computational time, we notice that it is low, even for big values of p and n.

Table 1: The solution $\overline{y}^{p,n}$ at time $t = 0$

$Y_0^{p,n}$	n=20	n=50	n=100	n=200	n=400	n=500	n=600
p=20	1.1736	1.2051	1.2181	1.2245	1.2277	1.2283	1.2288
p=50	1.2077	1.2482	1.2648	1.2728	1.2767	1.2775	1.2780
p=100	1.2214	1.2634	1.2808	1.2894	1.2936	1.2945	1.2950
p=500	1.2350	1.2753	1.2939	1.3033	1.3079	1.3088	1.3094
p=1000	1.2365	1.2767	1.2957	1.3051	1.3098	1.3107	1.3113
p=5000	1.2376	1.2778	1.2971	1.3066	1.3113	1.3122	1.3129
p=20000	1.2377	1.2780	1.2974	1.3069	1.3116	1.3125	1.3132
CPU time for p=20000	0.00071	0.0084	0.0644	0.6622	6.3560	12.5970	20.0062

Figure 1 represents one path of $(\overline{y}_t^{p,n}, \overline{\xi}_t^n, \zeta_t^n)_{t\geq 0}$. We notice that for all t, $\overline{y}_t^{p,n}$ stays between the two obstacles.

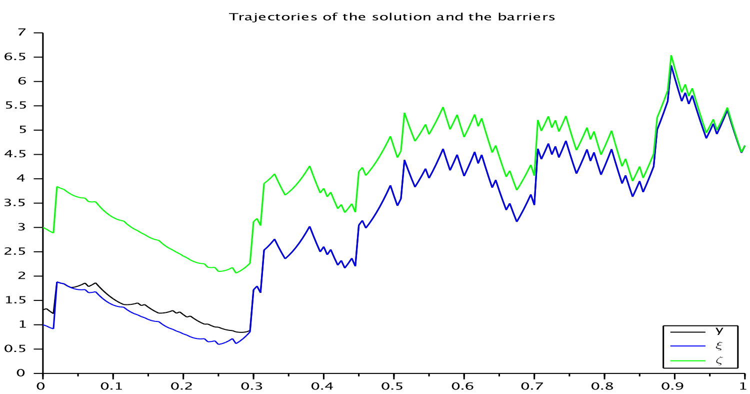

Figure 1: Trajectories of the solution $\overline{y}^{p,n}$ and the barriers $\overline{\xi}^n$ and $\overline{\zeta}^n$ for $\lambda = 5$, $N = 200$, $p = 20000$.

Example 2 : predictable and totally inaccessible jumps

We consider now the simulation of the DBBSDE with obstacles having general jumps (totally inaccessible and predictable). More precisely, we take the barriers and driver of the following form: $\xi_t := (W_t)^2 + \tilde{N}_t + (T-t)(1 - \mathbf{1}_{W_t \geq a}), \zeta_t := (W_t)^2 + \tilde{N}_t + (T-t)(2 + \mathbf{1}_{W_t \geq a})$, $g(t, \omega, y, z, u) := -5|y + z| + 6u - 1$.
We first give the numerical results for two different values of a, in order to show the influence of the predictable jumps given by $\mathbf{1}_{W_t \geq a}$ on the solution Y and also the convergence in n and p of the numerical explicit scheme (see Tables 2 and 3).
Then, Figures 2, 3 and 4 allow to distinguish the predictable jumps of totally inaccesible ones and their influence on the barriers (for e.g. the first jump of the barriers is totally inaccessible, the second and third ones are predictable). Moreover, we remark, as in the previous example, that the solution Y stays between the two obstacles ξ and ζ.

Table 2: The solution Y at time $t = 0$ for a=-1

$Y_0^{p,n}$	n=100	n=200	n=400	n=500	n=600
p=20	1.0745	1.0698	1.0782	1.0748	1.0759
p=50	1.1138	1.1103	1.1191	1.1159	1.1170
p=100	1.1266	1.1238	1.1328	1.1297	1.1308
p=500	1.1373	1.1353	1.1448	1.1419	1.1431
p=1000	1.1387	1.1369	1.1465	1.1437	1.1449
p=5000	1.1399	1.1382	1.1481	1.1453	1.1466
p=20000	1.1401	1.1385	1.1484	1.1456	1.1469

Table 3: The solution Y at time $t = 0$ for a=1

$Y_0^{p,n}$	n=100	n=200	n=400	n=500	n=600
p=20	1.2125	1.2177	1.2203	1.2208	1.2212
p=50	1.2582	1.2647	1.2680	1.2686	1.2690
p=100	1.2738	1.2808	1.2843	1.2850	1.2855
p=500	1.2866	1.2944	1.2982	1.2990	1.2995
p=1000	1.2884	1.2962	1.3001	1.3008	1.3013
p=5000	1.2898	1.2976	1.3016	1.3023	1.3029
p=20000	1.2900	1.2979	1.3018	1.3026	1.3032

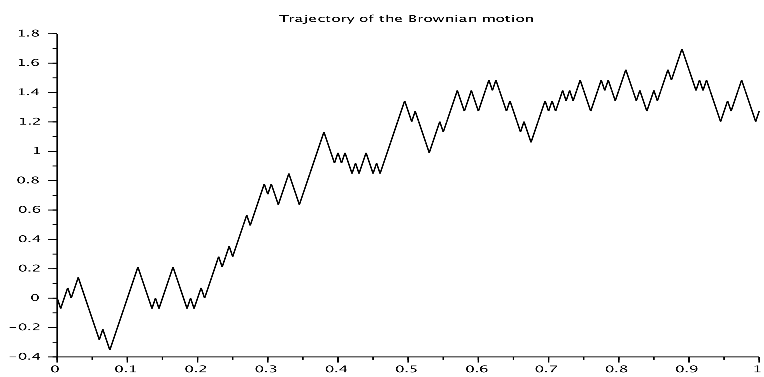

Figure 2: Trajectories of the Brownian motion for $a = -0.2$, $N = 200$.

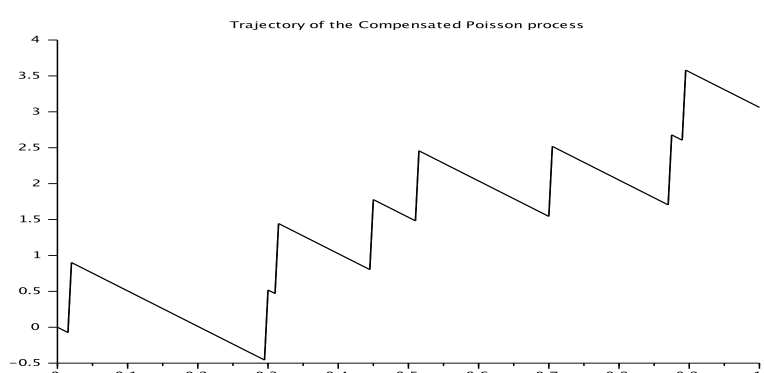

Figure 3: Trajectories of the Compensated Poisson process for $\lambda = 5$, $N = 200$.

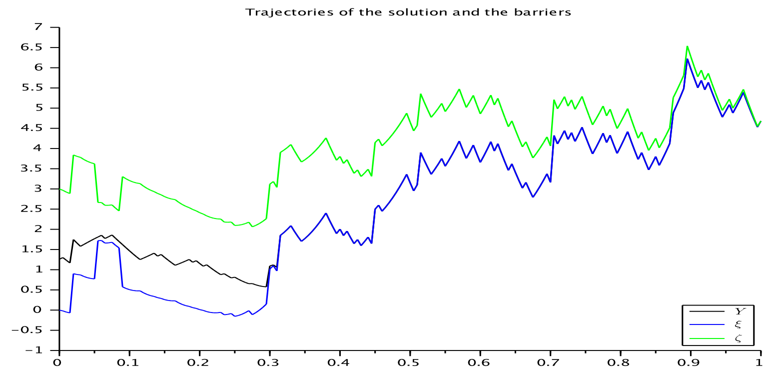

Figure 4: Trajectories of the solution Y and the barriers ξ and ζ for a=-0.2, $\lambda = 5$, $N = 200$.

A Generalized monotonic limit theorem

The following Theorem generalizes [20, Theorem 3.1] and Theorem D.3 to the case of doubly reflected BSDEs with jumps.

Theorem A.1 (Monotonic limit theorem)**.** *Assume that g satisfies Assumption 2.2, and ξ belongs to $L^2(\mathcal{F}_T)$. We consider the following sequence (in n) of BSDEs :*

$$Y_t^n = \xi + \int_t^T g(s, Y_s^n, Z_s^n, U_s^n)ds + (A_T^n - A_t^n) - (K_T^n - K_t^n) - \int_t^T Z_s^n dW_s - \int_t^T \int_{\mathbb{R}^*} U_s^n(e)\tilde{N}(ds, de)$$

such that $Y^n \in \mathcal{S}^2$, A^n and K^n are in $\mathcal{A}^2$, and $\sup_n \mathbb{E}(\int_0^T |Z_s^n|^2 ds) + \sup_n \mathbb{E}(\int_0^T \int_{\mathbb{R}^*} |U_s^n(e)|^2 \nu(de)ds) < \infty$. We also assume that for each $n \in \mathbb{N}$

1. *$(A^n)_n$ is continuous and increasing and such that $A_0^n = 0$ and $\sup_n \mathbb{E}((A_T^n)^2) < \infty$*
2. *$K_t^j - K_s^j \geq K_t^i - K_s^i$, for all $0 \leq s \leq t \leq T$ and for all $i \leq j$*
3. *for all $t \in [0, T]$, $(K_t^n)_n \nearrow K_t$ and $E(K_T^2) < \infty$*
4. *$(Y_t^n)_n$ increasingly converges to Y_t with $\mathbb{E}(\sup_{0 \leq t \leq T} |Y_t|^2) < \infty$.*

Then $K \in \mathcal{A}^2$ and there exist $Z \in \mathbb{H}^2$, $A \in \mathcal{A}^2$ and $U \in \mathbb{H}_\nu^2$ such that

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s, U_s)ds + A_T - A_t - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds, de).$$

Z is the weak limit of $(Z^n)_n$ in $\mathbb{H}^2$, K_t is the strong limit of $(K_t^n)_n$ in $L^2(\mathcal{F}_t)$, A_t is the weak limit of $(A_t^n)_n$ in $L^2(\mathcal{F}_t)$ and U is the weak limit of $(U^n)_n$ in $\mathbb{H}_\nu^2$. Moreover, for all $r \in [1, 2[$, the following strong convergence holds

$$\lim_{n \to \infty} \mathbb{E}\left(\int_0^T |Y_s^n - Y_s|^2 ds + \int_0^T |Z_s^n - Z_s|^r ds + \int_0^T \left(\int_{\mathbb{R}^*} |U_s^n(e) - U_s(e)|^2 \nu(de)\right)^{\frac{r}{2}} ds\right) = 0.$$

Proof of Theorem A.1. This proof follows the proofs of Theorem D.3 and [20, Theorem 3.1]. From the hypotheses, the sequences $(Z^n)_n$, $(U^n)_n$ and $(g(\cdot, Y^n, Z^n, U^n))_n$ are bounded in $\mathbb{H}^2$, $\mathbb{H}_\nu^2$ and $L^2([0, T] \times \Omega)$, then we can extract subsequences which weakly converge in the related spaces. Let Z, U and g_0 denote the respective weak limits. Thus, for each stopping time $\tau \leq T$, the following weak convergence holds in $L^2(\mathcal{F}_\tau)$

$$\int_0^\tau g(s, Y_s^n, Z_s^n, U_s^n)ds \underset{n\to\infty}{\rightharpoonup} \int_0^\tau g_0(s)ds, \quad \int_0^\tau Z_s^n dW_s \underset{n\to\infty}{\rightharpoonup} \int_0^\tau Z_s dW_s$$

and

$$\int_0^\tau \int_{\mathbb{R}^*} U_s^n(e)\tilde{N}(ds,de) \underset{n\to\infty}{\rightharpoonup} \int_0^\tau \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds,de), \quad K_\tau^n \underset{n\to\infty}{\rightharpoonup} K_\tau$$

since $(K_t^n)_n \nearrow K_t$ in $L^2(\mathcal{F}_t)$.

$$A_\tau^n = Y_0^n - Y_\tau^n - \int_0^\tau g(s,Y_s^n,Z_s^n,U_s^n)ds + K_\tau^n + \int_0^\tau Z_s^n dW_s + \int_0^\tau \int_{\mathbb{R}^*} U_s^n(e)\tilde{N}(ds,de)$$

we also have the following weak convergence in $L^2(\mathcal{F}_\tau)$

$$A_\tau^n \rightharpoonup A_\tau := Y_0 - Y_\tau - \int_0^\tau g_0(s)ds + K_\tau + \int_0^\tau Z_s dW_s + \int_0^\tau \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds,de).$$

Then $\mathbb{E}(A_T^2) < \infty$. Since the process $(A_t^n)_t$ is increasing, predictable and such that $A_0^n = 0$, the limit process A remains an increasing predictable process with $A_0 = 0$. We deduce from [20, Lemma 3.2] that K is a RCLL process, and from [20, Lemma 3.1] that A and Y are RCLL processes. Then Y has the form

$$Y_t = \xi + \int_t^T g_0(s)ds + A_T - A_t - (K_T - K_t) - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds,de).$$

It remains to prove that for all $r \in [1,2[$

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T |Z_s^n - Z_s|^r ds + \int_0^T \left(\int_{\mathbb{R}^*} |U_s^n(e) - U_s(e)|^2 \nu(de)\right)^{\frac{r}{2}} ds\right) = 0$$

and for all $t \in [0,T]$

$$\int_0^t g_0(s)ds = \int_0^t g(s,Y_s,Z_s,U_s)ds.$$

Let $N_t = \int_0^t \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds,de)$ and $N_t^n = \int_0^t \int_{\mathbb{R}^*} U_s^n(e)\tilde{N}(ds,de)$. We have $\Delta_s(Y^n - Y) = \Delta_s(N^n - N + K^n - K + A)$. We appply Itô's formula to $(Y_t^n - Y_t)^2$ on each subinterval $]\sigma,\tau]$, where σ and τ are two predictable stopping times such that $0 \le \sigma \le \tau \le T$. Let θ_s^n denotes (Y_s^n, Z_s^n, U_s^n)

$$\begin{aligned}
&(Y_\sigma^n - Y_\sigma)^2 + \int_\sigma^\tau |Z_s^n - Z_s|^2 ds + \sum_{\sigma\le s\le\tau} \Delta_s(Y^n - Y)^2 \\
&= (Y_\tau^n - Y_\tau)^2 + 2\int_\sigma^\tau (Y_s^n - Y_s)(g(s,\theta_s^n) - g_0(s))ds + 2\int_\sigma^\tau (Y_s^n - Y_s)dA_s^n - 2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})dA_s \\
&- 2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})d(K_s^n - K_s) - 2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})(Z_s^n - Z_s)dW_s - 2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})(U_s^n(e) - U_s(e))\tilde{N}(ds,de).
\end{aligned}$$

Since $\int_\sigma^\tau (Y_s^n - Y_s)dA_s^n \le 0$, $-2\int_\sigma^\tau (Y_{s^-}^n - Y_{s^-})d(K_s^n - K_s) \le 0$ and

$$\sum_{\sigma\le s\le\tau} \Delta_s(Y^n - Y)^2 = \sum_{\sigma\le s\le\tau} \Delta_s(N^n - N)^2 + \sum_{\sigma\le s\le\tau} \Delta_s(K^n - K)^2 + \sum_{\sigma\le s\le\tau} (\Delta_s A)^2 + 2\sum_{\sigma\le s\le\tau} \Delta_s A \Delta_s(K^n - K).$$

By taking expectation and using $Y_{s^-}^n - Y_{s^-} = (Y_s^n - Y_s) - \Delta_s(Y^n - Y)$, we get

$$\begin{aligned}
&\mathbb{E}(Y_\sigma^n - Y_\sigma)^2 + \mathbb{E}\int_\sigma^\tau |Z_s^n - Z_s|^2 ds + \mathbb{E}\int_\sigma^\tau \int_{\mathbb{R}^*} |U_s^n(e) - U_s(e)|^2 \nu(de)ds + \mathbb{E}\sum_{\sigma\le s\le\tau} \Delta_s(K^n - K)^2 \\
&\le \mathbb{E}(Y_\tau^n - Y_\tau)^2 + 2\mathbb{E}\int_\sigma^\tau (Y_s^n - Y_s)(g(s,\theta_s^n) - g_0(s))ds - 2\mathbb{E}\int_\sigma^\tau (Y_s^n - Y_s)dA_s + \mathbb{E}\sum_{\sigma\le s\le\tau} (\Delta_s A)^2.
\end{aligned}$$

It comes down to [8, Equation (10)], we refer to this paper for the end of the proof. □

B Snell envelope theory

Definition B.1. *Any $\mathcal{F}_t$-adapted RCLL process $\eta = (\eta_t)_{0\leq t\leq T}$ is of class $\mathcal{D}[0,T]$ if the family $\{\eta(\tau)\}_{\tau\in\mathcal{T}_0}$ is uniformly integrable.*

Definition B.2. *Let $\eta = (\eta_t)_{t\leq T}$ be a $\mathcal{F}_t$-adapted RCLL process of class $\mathcal{D}[0,T]$. Its Snell envelope $\mathcal{R}_t(\eta)$ is defined as*

$$\mathcal{R}_t(\eta) = \operatorname*{esssup}_{\nu\in\mathcal{T}_t} \mathbb{E}(\eta_\nu|\mathcal{F}_t).$$

Proposition B.3. *$\mathcal{R}_t(\eta)$ is the lowest RCLL $\mathcal{F}_t$-supermartingale of class $\mathcal{D}[0,T]$ which dominates η, i.e. $\mathbb{P}$-a.s., for all $t\in[0,T]$, $\mathcal{R}(\eta)_t \geq \eta_t$.*

Proposition B.4. *(Doob-Meyer decomposition of Snell envelopes) Let $\eta := (\eta_t)_{t\leq T}$ be of class $\mathcal{D}([0,T])$. There exists a unique decomposition of the Snell envelope*

$$\mathcal{R}_t(\eta) = M_t - K_t^c - K_t^d,$$

where M_t is a RCLL $\mathcal{F}_t$-martingale, K^c is a continuous integrable increasing process with $K_0^c = 0$, and K^d is a pure jump integrable increasing predictable RCLL process with $K_0^d = 0$. Moreover, we have

$$\int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-})dK_t = 0,$$

where $K := K^c + K^d$.

Proof. The first part of the proposition corresponds to the Doob-Meyer decomposition of supermartingales of class $\mathcal{D}[0,T]$. To prove the second part of the proof, we write

$$\int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-})dK_t = \int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-})dK_t^d + \int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-})dK_t^c.$$

The first term of the right hand side is null, since $\{\Delta K^d > 0\} \subset \{\mathcal{R}(\eta)_- = \eta_-\}$ (see [12, Property A.2, (ii)]). Let us prove that the second term of the r.h.s. is also null. We know that $(\mathcal{R}_t(\eta) + K_t^d)_t = (M_t - K_t^c)_t$ is a supermartingale satisfying $\mathcal{R}_t(\eta) + K_t^d \geq \eta_t + K_t^d$, then $\mathcal{R}_t(\eta) + K_t^d \geq \mathcal{R}(\eta_t + K_t^d)$. On the other hand, for every supermartingale N_t such that $N_t \geq \eta_t + K_t^d$, we have $N_t - K_t^d \geq \eta_t$, and then $N_t - K_t^d \geq \mathcal{R}(\eta)_t$ (since $(N_t - K_t^d)_t$ is a supermartingale), then $N_t \geq \mathcal{R}(\eta)_t + K_t^d$. By choosing $N_t := \mathcal{R}(\eta + K^d)_t$, we get $\mathcal{R}_t(\eta) + K_t^d = \mathcal{R}(\eta_t + K_t^d)$. Since K^c is continuous, $(\mathcal{R}_t(\eta) + K_t^d)_t$ is regular (see [22, Exercise 27]). Then, from [12, Property A3], we get that $\tau_t := \inf\{s \geq t : K_s^c - K_t^c > 0\}$ is optimal after t. This yields $\int_t^{\tau_t} (\mathcal{R}(\eta)_s + K_s^d - (\eta_s + K_s^d))dK_s^c = 0$ for all $t \leq T$. Then, we get $\int_0^T (\mathcal{R}_{t^-}(\eta) - \eta_{t^-})dK_t^c = 0$. $\square$

C Technical result for standard BSDEs with jumps

Lemma C.1. *We assume that δ_n is small enough such that $(3 + 2p + 2C_g + 2C_g^2(1 + \frac{1}{\lambda}e^{2\lambda T}))\delta_n < 1$. Then we have:*

$$\sup_{j\leq n} \mathbb{E}[|\overline{y}_j^{p,n}|^2] + \delta_n \sum_{j=0}^{n-1} \mathbb{E}[|\overline{z}_j^{p,n}|^2] + (1-\kappa_n)\kappa_n \sum_{j=0}^{n-1} \mathbb{E}[|\overline{u}_j^{p,n}|^2] \leq K_{Lem.C.1}.$$

where $K_{Lem.C.1} = (\|g(\cdot,0,0,0)\|_\infty^2 + (p^2 + C_gT)(\sup_n \max_j \mathbb{E}[|\xi_j^n|^2] + \sup_n \max_j \mathbb{E}[|\zeta_j^n|^2]))e^{(3+2p+2C_g+2C_g^2(2+\frac{1}{\lambda}e^{2\lambda T}))}$.

Proof. From the explicit scheme, we derive that:

$$\begin{aligned}\mathbb{E}[|\overline{y}_j^{p,n}|^2] - \mathbb{E}[|\overline{y}_{j+1}^{p,n}|^2] =& -\delta_n \mathbb{E}[|\overline{z}_j^{p,n}|^2] - (1-\kappa_n)\kappa_n \mathbb{E}[|\overline{u}_j^{p,n}|^2] - (1-\kappa_n)\kappa_n \mathbb{E}[|\overline{v}_j^{p,n}|^2] \\ &- \delta_n^2 \mathbb{E}[g_p^2(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})] + 2\delta_n \mathbb{E}[\overline{y}_j^{p,n} g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})].\end{aligned}$$

Taking the sum for $j = i, ..., n-1$ yields

$$\begin{aligned}\mathbb{E}[|\overline{y}_i^{p,n}|^2] &\le \mathbb{E}[|\xi^n|^2] - \delta_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{z}_j^{p,n}|^2] - (1-\kappa_n)\kappa_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{u}_j^{p,n}|^2] + 2\delta_n \sum_{j=i}^{n-1} \mathbb{E}[\overline{y}_j^{p,n} g_p(t_j, \mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n], \overline{z}_j^{p,n}, \overline{u}_j^{p,n})]\\
&\le \mathbb{E}[|\xi^n|^2] - \delta_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{z}_j^{p,n}|^2] - (1-\kappa_n)\kappa_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{u}_j^{p,n}|^2]\\
&+ 2\delta_n \sum_{j=i}^{n-1} \mathbb{E}[|\overline{y}_j^{p,n}|(|g(t_j,0,0,0)| + C_g|\mathbb{E}[\overline{y}_{j+1}^{p,n}|\mathcal{F}_j^n]| + C_g|\overline{z}_j^{p,n}| + C_g|\overline{u}_j^{p,n}| + p(|\overline{y}_j^{p,n}| + |\xi_j^n| + |\zeta_j^n|))]\end{aligned}$$

Hence, we get that:

$$\begin{aligned}\mathbb{E}[|\overline{y}_i^{p,n}|^2] + \frac{\delta_n}{2} \sum_{j=i}^{n-1} \mathbb{E}[|\overline{z}_j^{p,n}|^2] + \frac{(1-\kappa_n)\kappa_n}{2} \sum_{j=i}^{n-1} \mathbb{E}[|\overline{u}_j^{p,n}|^2] &\le \delta_n \sum_{j=i}^{n-1} \mathbb{E}[|g(t_j,0,0,0)|^2]\\
&+ (p^2 + C_g\delta_n)(\max_j \mathbb{E}[|\xi_j^n|^2] + \max_j \mathbb{E}[|\zeta_j^n|^2]) + \delta_n \left(3 + 2p + 2C_g + 2C_g^2 + \frac{2C_g^2\delta_n}{(1-\kappa_n)\kappa_n}\right) \sum_{j=i}^{n-1} \mathbb{E}[|\overline{y}_j^{p,n}|^2].\end{aligned}$$

Since $\dfrac{\delta_n}{\kappa_n(1-\kappa_n)} \le \dfrac{1}{\lambda} e^{2\lambda T}$, the assumption on δ_n enables to apply Gronwall's Lemma, and the result follows.

□

D Some recent results on BSDEs and reflected BSDEs with jumps

For the self-containment of the paper, we recall in this Section some recent results used several times in the paper.

D.1 Comparison Theorem for BSDEs and reflected BSDEs with jumps

Theorem D.1 (Comparison Theorem for BSDEs with jumps ([23], Theorem 4.2))**.** *Let ξ_1 and ξ_2 be in $L^2(\mathcal{F}_T)$. Let f_1 be a Lipschitz driver and f_2 be a driver. For $i = 1, 2$ let (X_t^i, π_t^i, l_t^i) be a solution in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$ of the BSDE*

$$-dX_t^i = f_i(t, X_t^i, \pi_t^i, l_t^i)dt - \pi_t^i dW_t - \int_{\mathbb{R}^*} l_t^i(u)\tilde{N}(dt, du);\ X_T^i = \xi_i. \tag{D.1}$$

Assume that there exists a bounded predictable process (γ_t) such that $dt \otimes dP \otimes \nu(du)$-a.s.

$$\gamma_t(u) \ge -1 \quad \text{and } |\gamma_t(u)| \le \psi(u),$$

where $\psi \in L_\nu^2$ and such that

$$f_1(t, X_t^2, \pi_t^2, l_t^1) - f_1(t, X_t^2, \pi_t^2, l_t^2) \ge \langle \gamma_t, l_t^1 - l_t^2 \rangle_\nu,\ t \in [0,T], dt \otimes dP \ a.s. \tag{D.2}$$

Assume that

$$\xi_1 \ge \xi_2 \ a.s. \text{ and } f_1(t, X_t^2, \pi_t^2, l_t^2) \ge f_2(t, X_t^2, \pi_t^2, l_t^2)\ t \in [0,T], dt \otimes dP \ a.s. \tag{D.3}$$

Then we have

$$X_t^1 \ge X_t^2 \ a.s. \text{ for all } t \in [0,T]. \tag{D.4}$$

Moreover, if inequality (D.3) *is satisfied for (X_t^1, π_t^1, l_t^1) instead of (X_t^2, π_t^2, l_t^2) and if f_2 (instead of f_1) is Lipschitz and satisfies* (D.2)*, then* (D.4) *still holds.*

Theorem D.2 (Comparison Theorem for reflected BSDEs with jumps ([24], Theorem 5.1))**.** *Let ξ^1, ξ^2 be two RCLL obstacle processes in $\mathcal{S}^2$. Let f_1 and f_2 be Lipschitz drivers satisfying Assumption 2.2. Suppose that*

$$\xi_t^2 \leq \xi_t^1,\ 0 \leq t \leq T \ a.s.$$
$$f_2(t,y,z,k) \leq f_1(t,y,z,k),\ \text{for all } (y,z,k) \in \mathbb{R}^2 \times L_\nu^2,\ dP \otimes dt \ a.s.$$

Let (Y^i, Z^i, k^i, A^i) be a solution in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2 \times \mathcal{S}^2$ of the reflected BSDE

$$-dY_t^i = f_i(t, Y_t^i, Z_t^i, k_t^i(\cdot))dt + dA_t^i - Z_t^i dW_t - \int_{\mathbb{R}^*} k_t^i(u)\tilde{N}(dt,du);\ Y_T^i = \xi_T^i, \tag{D.5}$$
$$Y_t^i \geq \xi_t^i,\ 0 \leq t \leq T \ a.s. \tag{D.6}$$
$$\tag{D.7}$$

and A^i is a non decreasing RCLL predictable process with $A_0^i = 0$ and such that

$$\int_0^T (Y_t^i - \xi_t^i) dA_t^{i,c} = 0 \ a.s. \text{ and } \Delta A_t^{i,d} = -\Delta Y_t^i \mathbf{1}_{Y_{t^-}^i = \xi_{t^-}^i} \ a.s.$$

Then $Y_t^2 \leq Y_t^1$ for all t in $[0,T]$ a.s.

D.2 Convergence results on reflected BSDEs with jumps

Theorem D.3 (Monotonic limit theorem for reflected BSDEs with jumps ([8], Theorem 3.1))**.** *Assume that f satisfies [8, Assumption A.2], $\xi \in L^2$ and K^n is a continuous and increasing process such that $\sup_{n\in\mathbb{N}} \mathbb{E}(K_T^n)^2 < \infty$ and $K_0^n = 0$ for any $n \in \mathbb{N}$. Let (Y^n, Z^n, V^n) be the solution of the following BSDE*

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n, V_s^n)ds + K_T^n - K_t^n - \int_t^T Z_s^n dW_s - \int_t^T \int_U V_s^n(u)\tilde{N}(ds,du),\ t \leq T,$$

where $\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T |Z_s^n|^2 ds < \infty$ and $\sup_{n\in\mathbb{N}} \mathbb{E}\int_0^T \int_U |V_s^n(u)|^2 \nu(du)ds < \infty$. If Y^n converges increasingly to Y with $\mathbb{E}(\sup_{0\leq t\leq T} Y_t^2) < \infty$, then there exists $Z \in \mathbb{H}^2$, $K \in \mathcal{A}^2$ and $V \in \mathbb{H}_\nu^2$ such that the triple (Z,K,V) satisfies the following equation

$$Y_t = \xi + \int_0^T f(s, Y_s, Z_s, V_s)ds + K_T - K_t - \int_t^T Z_s dW_s - \int_t^T \int_U V_s(u)\tilde{N}(ds,du),\ t \leq T.$$

Here Z is the weak limit of $(Z^n)_n$ in $\mathbb{H}^2$, K_t is the weak limit of $(K_t^n)_n$ in $L^2(\mathcal{F}_t)$ and V is the weak limit of $(V^n)_n$ in $\mathbb{H}_\nu^2$. Moreover, for every $p \in [1,2[$, the following strong convergence holds

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |Y_s^n - Y_s|^2 ds\right] + \mathbb{E}\left[\int_0^T |Z_s^n - Z_s|^p ds + \int_0^T \left(\int_U |V_s^n(u) - V_s(u)|^2 \nu(du)\right)^{\frac{p}{2}} ds\right] = 0.$$

Now we introduce the following penalized equation

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n, V_s^n)ds + K_T^n - K_t^n - \int_t^T Z_s^n dW_s - \int_t^T \int_U V_s^n(u)\tilde{N}(ds,du),\ t \leq T,$$

where $K_t^n = n\int_0^t (Y_s^n - S_s)^- ds$. We have

Theorem D.4 ([8], Theorem 4.2)**.** *The sequence $(Y^n, Z^n, V^n)_n$ has a limit (Y,Z,V) such that Y^n converges to Y in $\mathcal{S}^2$ and Z is the weak limit in $\mathbb{H}^2$, K_t is the weak limit of $(K_t^n)_n$ in $L^2(\mathcal{F}_t)$ and V is the weak limit in $\mathbb{H}_\nu^2$.*

D.3 Dynkin games and DBBSDEs

In this section, we briefly recall the definition of a Dynkin game, as well as its connection with doubly reflected BSDEs, established for the first time in [5] in the case of a Brownian filtration and regular obstacles. This link has also been investigated in the case of jumps and irregular obstacles (see e.g. [17]).

The setting of a Dynkin game is very simple. Two players observe two processes ξ and ζ. Player 1 chooses a stopping time $\sigma \in \mathcal{T}$, and Player 2 chooses a stopping time $\tau \in \mathcal{T}$. Player 2 pays Player 1 the amount $I(\tau,\sigma) := \xi_{\tau \leq \sigma} + \zeta_{\sigma < \tau}$ at the stopping time $\tau \wedge \sigma$. Player 1 wishes to maximize $\mathbb{E}[I(\tau,\sigma)]$ while Player 2 wishes to miminize it. It is then natural to define the lower and upper values of the game:

$$\overline{V} := \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathbb{E}[I(\tau,\sigma)]; \ \underline{V} := \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}[I(\tau,\sigma)].$$

The game is said to admit a value if $\overline{V} = \underline{V}$.

Let us now give the characterization of the solution of the DBBSDE as the value function of a Dynkin game.

Proposition D.5. *Let $(Y,Z,U,\alpha) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{A}^2$ be a solution of the DBBSDE (2.1). For any $S \in \mathcal{T}_0$ and any stopping times $\tau, \sigma \in \mathcal{T}_S$, consider the payoff:*

$$I_S(\tau,\sigma) = \int_S^{\tau\wedge\sigma} g(s, Y_s, Z_s, U_s(\cdot))ds + \xi_{\tau \leq \sigma} + \zeta_{\sigma<\tau}. \tag{D.8}$$

The upper and lower value functions at time S associated to the Dynkin game are defined respectively by

$$\overline{V}(S) := \operatorname*{essinf}_{\sigma \in \mathcal{T}_S} \operatorname*{esssup}_{\tau \in \mathcal{T}_S} \mathbb{E}[I_S(\tau,\sigma)|\mathcal{F}_S]. \tag{D.9}$$

$$\underline{V}(S) := \operatorname*{esssup}_{\tau \in \mathcal{T}_S} \operatorname*{essinf}_{\sigma \in \mathcal{T}_S} \mathbb{E}[I_S(\tau,\sigma)|\mathcal{F}_S] \tag{D.10}$$

This game has a value V, given by the state-process Y solution of DBBSDE, i.e.

$$Y_S = \overline{V}(S) = \underline{V}(S). \tag{D.11}$$

Note that in the definition (D.8), $(g(s, Y_s, Z_s, U_s(\cdot))_{s \leq \tau\wedge\sigma}$ represents the instantaneous reward, while $\xi_{\tau\leq\sigma} + \zeta_{\sigma<\tau}$ the terminal one.

Proof. For each $S \in \mathcal{T}_0$ and for each $\varepsilon > 0$, let

$$\tau_S^\varepsilon := \inf\{t \geq S,\ Y_t \leq \xi_t + \varepsilon\} \quad \sigma_S^\varepsilon := \inf\{t \geq S,\ Y_t \geq \zeta_t - \varepsilon\}. \tag{D.12}$$

Remark that σ_S^ε and $\tau_S^\varepsilon \in \mathcal{T}_S$. Fix $\varepsilon > 0$. We have that almost surely, if $t \in [S, \tau_S^\varepsilon[$, then $Y_t > \xi_t + \varepsilon$ and hence $Y_t > \xi_t$. It follows that the function $t \mapsto A_t^c$ is constant a.s. on $[S, \tau_S^\varepsilon]$ and $t \mapsto A_t^d$ is constant a.s. on $[S, \tau_S^\varepsilon[$. Also, $Y_{(\tau_S^\varepsilon)^-} \geq \xi_{(\tau_S^\varepsilon)^-} + \varepsilon$ a.s. Since $\varepsilon > 0$, it follows that $Y_{(\tau_S^\varepsilon)^-} > \xi_{(\tau_S^\varepsilon)^-}$ a.s. , which implies that $\Delta A^d_{\tau_S^\varepsilon} = 0$ a.s. (see Remark 2.6). Hence, the process A is constant on $[S, \tau_S^\varepsilon]$. Furthermore, by the right-continuity of (ξ_t) and (Y_t), we clearly have $Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon$ a.s. Similarly, one can show that the process K is constant on $[S, \sigma_S^\varepsilon]$ and that $Y_{\sigma_S^\varepsilon} \geq \zeta_{\sigma_S^\varepsilon} - \varepsilon$ a.s.

Let us now consider two cases. First, on the set $\{\sigma_S^\varepsilon < \tau\}$, by using the definition of the stopping times and the fact that K is constant on $[S, \sigma_S^\varepsilon]$, we have:

$$I_S(\tau, \sigma_S^\varepsilon) \leq \int_S^{\sigma_S^\varepsilon} g(s, Y_s, Z_s, U_s(\cdot))ds + Y_{\sigma_S^\varepsilon} + \varepsilon - (K_{\sigma_S^\varepsilon} - K_S) + (A_{\sigma_S^\varepsilon} - A_S) \tag{D.13}$$

$$\leq Y_S + \int_S^{\sigma_S^\varepsilon} Z_s dW_s + \int_S^{\sigma_S^\varepsilon} \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds,de) + \varepsilon.$$

On the set $\{\tau \leq \sigma_S^\varepsilon\}$, we obtain:

$$I_S(\tau, \sigma_S^\varepsilon) \leq \int_S^\tau g(s, Y_s, Z_s, U_s(\cdot))ds + Y_\tau - (K_\tau - K_S) + (A_\tau - A_S)$$

$$\leq Y_S + \int_S^\tau Z_s dW_s + \int_S^\tau \int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds, de).$$

The two above inequalities imply:

$$\mathbb{E}[I_S(\tau, \sigma_S^\varepsilon)|\mathcal{F}_S] \leq Y_S + \varepsilon.$$

Similarly, one can show that:

$$\mathbb{E}[I_S(\tau_S^\varepsilon, \sigma)|\mathcal{F}_S] \geq Y_S - \varepsilon.$$

Consequently, we get that for each $\varepsilon > 0$

$$\operatorname*{esssup}_{\tau \in \mathcal{T}_s} E[I_S(\tau, \sigma_S^\varepsilon)|\mathcal{F}_S] - \varepsilon \ \leq \ Y_S \ \leq \ \operatorname*{essinf}_{\sigma \in \mathcal{T}_S} E[I_S(\tau_S^\varepsilon, \sigma)|\mathcal{F}_S] + \varepsilon \ \text{ a.s.},$$

that is $\overline{V}(S) - \varepsilon \ \leq \ Y_S \ \leq \ \underline{V}(S) + \varepsilon$ a.s. Since $\underline{V}(S) \leq \overline{V}(S)$ a.s., the result follows.

□

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