Concise formulae for the cumulant matrices of a random vector

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Abstract

Concise formulae are given for the cumulant matrices of a random vector up to order 6. In addition to usual matrix operations, they involve only the Kronecker product, the vec operator, and the commutation matrix. Orders 5 and 6 are provided here for the first time; the same method as provided in the paper can be applied to compute higher orders. An immediate consequence of these formulae is to return 1) upper bounds on the rank of the cumulant matrices and 2) the expression of the sixth-order moment matrix of a Gaussian vector. Due to their conciseness, the proposed formulae also have a computational advantage as compared to the repeated use of Leonov and Shiryaev formula.

Keywords: Moments, cumulants, cumulant matrix, commutation matrix, independent component analysis, rank of cumulant matrices, moment matrix of a Gaussian vector

1. Introduction

Cumulants are useful quantities for characterizing some important statistical properties, such as independence, gaussianity, or possible (a)symmetry of probability distributions. They were first introduced in 1899 by T.N. Thiele under the name halfinvariants (see Hald [1] where original Thiele's paper is translated into English). In particular, higher order cumulants make it possible to solve problems that are otherwise intractable on the second order. This is typically the case in Independent Component Analysis (ICA, as introduced in Comon [2]), where fourth-order cumulants are used to blindly separate (over-)determined (as many as or less sources than sensors) mixtures of independent sources (e.g. Cardoso and Soulomiac [3], Albera et al. [4]. Recently, several researchers have investigated the use of sixth-order cumulants (Albera et al. [5]) and, more generally, cumulants of order $2k$, $k > 2$, (Albera et al. [6]) for the separation of under-determined mixtures of sources (more sources than sensors).

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This paper is concerned with the matrix expression of the cumulants of a random vector. Loosely speaking, the cumulant matrix of a random vector is a collection of the various cross-cumulants between its elements. It is often convenient to arrange such a collection into a matrix (a square symmetric matrix in the case of even order cumulants) for both notational and practical reasons. This makes it easier to to handle cumulants for algorithmic perspectives and to exploit some of their statistical properties, such as "matrix redundancy" (see e.g. Albera et al. [5, 4, 6]). An important example is ICA where the blind separation of statistically independent sources is solved by zeroing all cross-cumulants between independent sources: if properly set up, the issue then boils down to the diagonalization of a cumulant matrix (see Cardoso and Soulomiac [3], Kollo [7]).

It is a well-known fact that scalar cumulants can be evaluated from lower-order moments by use of the classical Leonov and Shiryaev formula (see Leonov and Shiryaev [8] or Speed [9], Mac Cullagh [10]). This trivially applies to all (cross-)cumulants of a vector which can eventually be rearranged into a matrix. The main drawback of that approach is to involve lengthy formulae and, when calculation is of concern, a computational burden, which both become rapidly prohibitive as the dimension of the random vector increases. It is therefore advisable to search for direct and more concise expressions of cumulant matrices. One solution is actually returned by Neudecker matrix derivative (Neudecker [11]) (see e.g. Kollo and von Rosen [12, pages 187-188]). Unfortunately, it faces two difficulties which limit its practical use:

(i) the resulting cumulant matrix is not square except on the second order;

(ii) matrix derivatives of the logarithm of the characteristic function are hardly tractable for orders higher than four and, according to the authors knowledge, no formulae have ever been derived for those cases.

Limitation (*i*) can be overcome on the fourth order by using an algebraic technique such as proposed by Jinadasa and Tracy [13] and Tracy and Sultan [14] to obtain the matrix expression of the fourth-order moment of a Gaussian vector; the same result is found in Kollo [7], Loperfido [15]. However, point (*ii*) has not been solved till now. In this paper we give the expressions of fifth and sixth-order cumulant matrices of a real-valued random vector. The expression of the sixth-order cumulant matrix is symmetric.

The paper is organized as follows. Section 2 resumes the basic definitions of moments and cumulants of a random vector and deduces Leonov and Shiryaev formula as a consequence of de Faà di Bruno formula (following the lines in Lukacs [16] and Hardy [17]). The main result of the paper – a theorem providing the formulae of cumulant matrices up to order six – is given in section 3. At the same time a general strategy is described as how to obtain similar results to any order. In section 4, the formulae are used to get, without pain, i) upper bounds on the rank of cumulant matrices and ii) the sixth-order moment matrix of a Gaussian vector. Since the latter is a known formula (see for instance Magnus and Neudecker [18], Tracy and Sultan [14], and more recently Schott [19]), it somehow validates the expression of the cumulant matrix from which it is derived.

Finally, section 5 concludes on the advantages implied by the proposed matrix expressions as compared to the classical use of Leonov and Shiryaev formula.

2. Generalities on moment and cumulant matrices

This section first recalls the link between the scalar moments and cumulants of a random vector, in order to pave the way for our main theorem given in section 3.

2.1. Scalar moments and cumulants

Let $\mathbf{x} = (x_1, \ldots, x_p)'$ be a real-valued random vector of dimension $p \times 1$. Its characteristic function ϕ is defined on $\mathbb{R}^p$ as

$$\phi(\mathbf{u}) = \mathbb{E}\left[\exp(i(\mathbf{u}, \mathbf{x}))\right] = \mathbb{E}\left[\prod_{j=1}^{p} \exp(iu_j x_j)\right]. \tag{1}$$

Let $n \in \mathbb{N}$; vector $\mathbf{x}$ is said to admit moments of order n if

$$\mathbb{E}\left[|\mathbf{x}|^n\right] < \infty, \tag{2}$$

where $|\mathbf{x}| = \left(\sum_{j=1}^{p} x_j^2\right)^{1/2}$ stands for the euclidian norm of $\mathbb{R}^p$. In that case, (i) ϕ is n times differentiable and (ii) all the scalar (cross-)moments of order $k \leq n$ of $\mathbf{x}$ are well defined as

$$(\forall (j_1, \ldots, j_k) \in [\![1, p]\!]^k) \qquad mom^{(k)}(x_{j_1}, \ldots, x_{j_k}) \overset{\text{def}}{=} \mathbb{E}\left[x_{j_1} \cdots x_{j_k}\right] = (-i)^k \frac{\partial^k \phi}{\partial u_{j_1} \cdots \partial u_{j_k}}(\mathbf{0}). \tag{3}$$

Moreover, since ϕ is a continuous function on $\mathbb{R}^p$ and $\phi(\mathbf{0}) = 1$, it is necessarily non-zero in a neighborhood of $\mathbf{0} \in \mathbb{R}^p$. The second characteristic function (also referred to as the "cumulant function" in the literature),

$$\psi(\mathbf{u}) = \log\left(\phi(\mathbf{u})\right), \tag{4}$$

is thus considered. By composition under condition (2), it is n times differentiable in a neighborhood of $\mathbf{0}$; this makes possible the definition of scalar (cross-)cumulants of order $k \leq n$ of $\mathbf{x}$ as

$$(\forall (j_1, \ldots, j_k) \in [\![1, p]\!]^k) \qquad cum^{(k)}(x_{j_1}, \ldots, x_{j_k}) \overset{\text{déf}}{=} (-i)^k \frac{\partial^k \psi}{\partial u_{j_1} \cdots \partial u_{j_k}}(\mathbf{0}). \tag{5}$$

Since Eq. (4) implicitly connects moments to cumulants, it suggests two methods to compute the later in terms of the former:

1. expand up to order n the following function

$$\log\left(1 + \sum_{k=1}^{n} \frac{(i)^k}{k!} \sum_{(j_1, \ldots, j_k) \in [\![1, p]\!]^k} mom^{(k)}(x_{j_1}, \ldots, x_{j_k}) u_{j_1} \cdots u_{j_k}\right) \tag{6}$$

   where the sum in the brackets is the Taylor series to order n of $\phi(\mathbf{u})$;

2. use de Faà di Bruno formula which returns the partial derivatives of a composition of applications.

The first method is carried out for instance in Mac Cullagh [10]. However, calculation becomes rapidly intractable as the order increases: Mac Cullagh provides formulae only up to order six; a few authors, like Albera and Comon [20], have attempted to pursue the calculations up to order eight. The second method has been used by Lukacs [16] in the case of random variables ($p = 1$); using a generalization of de Faà di Bruno formula to the multivariable case, several authors have proposed general formulae to calculate cumulants of a random vector in terms of its moments, which only differ in the way terms are collected. One of the simplest is provided by Hardy [17]; with the notations of the present paper, it reads

$$\frac{\partial^k}{\partial u_{j_1} \cdots \partial u_{j_k}} f \circ \phi(\mathbf{u}) = \sum_{\pi \in P([\![1,k]\!])} f^{(|\pi|)} \circ \phi(\mathbf{u}) \prod_{B \in \pi} \frac{\partial^{|B|}}{\prod_{l \in B} \partial u_{j_l}} \phi(\mathbf{u}) \tag{7}$$

where f is a function of one variable, $P([\![1,k]\!])$ denotes the set of partitions of $[\![1,k]\!]$, and $|E|$ the cardinality of set E such that, for instance, with $B = \{2; 3; 6\}$,

$$\frac{\partial^{|B|}}{\prod_{l \in B} \partial u_{j_l}} = \frac{\partial^3}{\partial u_{j_2} \partial u_{j_3} \partial u_{j_6}}. \tag{8}$$

The main advantage of Eq. (7) is to be fully developed, that is it no longer contains factorials assigned to the collection of equal partial derivatives when some indices j_l happen to coincide.

In order to compute cumulants in terms of moments, one sets $f = \log$, $\mathbf{u} = \mathbf{0}$ and $\phi(\mathbf{0}) = 1$ in Eq. (7). In particular,

$$f^{(n)}(1) = (-1)^{n-1}(n-1)! \tag{9}$$

and, in accordance with (3),

$$\frac{\partial^{|B|}}{\prod_{l \in B} \partial u_{j_l}} \phi(\mathbf{0}) = i^{|B|} mom^{(|B|)}(x_{j_l} \ : \ l \in B). \tag{10}$$

By noting that for any $\pi \in P([\![1,k]\!])$ it comes $\sum_{B \in \pi} |B| = k$, one eventually arrives at the Leonov and Shiryaev formula:

$$cum^{(k)}(x_{j_1}, \ldots, x_{j_k}) = \sum_{\pi \in P([\![1,k]\!])} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B \in \pi} mom^{(|B|)}(x_{j_l} \ : \ l \in B). \tag{11}$$

Alternatively, in order to compute moments in terms of cumulants, one sets $f = \exp$, ψ in place of Φ, $\mathbf{u} = \mathbf{0}$ and $\psi(\mathbf{0}) = 0$ in Eq. (7), so as to obtain

$$mom^{(k)}(x_{j_1}, \ldots, x_{j_k}) = \sum_{\pi \in P([\![1,k]\!])} \prod_{B \in \pi} cum^{(|B|)}(x_{j_l} \ : \ l \in B). \tag{12}$$

This formula will be used in section 4.2 to find the expression of the sixth-order moment, $m_6(\mathbf{x})$, of a centered Gaussian random vector $\mathbf{x}$.

2.2. Moment and cumulant matrices

There are many different ways to arrange the p^k scalar moments of Eq.(3) in a matrix. It is natural to select those arrangements which are maximally symmetric, as is often practiced in the literature (see e.g. Schott [19], Jinadasa and Tracy [13], Tracy and Sultan [14]).

Definition 1 (The moment matrix). The moment matrix of a real-valued random vector $\mathbf{x} = (x_1, \ldots, x_p)'$, $\mathbb{E}\left[|\mathbf{x}|^n\right] < \infty$, $n \in \mathbb{N}$, of even order $2k \leq n$ is defined as

$$m_{2k}(\mathbf{x}) \;=\; \mathbb{E}\left[\otimes^k \mathbf{x}\mathbf{x}'\right] \qquad \text{(dimension } p^k \times p^k) \tag{13}$$

and that of odd order, $2k + 1 \leq n$, as

$$m_{2k+1}(\mathbf{x}) \;=\; \mathbb{E}\left[\otimes^k \mathbf{x}\mathbf{x}' \otimes \mathbf{x}\right] \qquad \text{(dimension } p^{k+1} \times p^k), \tag{14}$$

where $\otimes$ is the Kronecker product and $\otimes^k \mathbf{x}\mathbf{x}' = \underbrace{\mathbf{x}\mathbf{x}' \otimes \cdots \otimes \mathbf{x}\mathbf{x}'}_{k \text{ matrices } p\times p}$.

Note there are several ways to write down the moment matrix since, for any pair of vectors $\mathbf{a}$ and $\mathbf{b}$ of dimension $p \times 1$,

$$\mathbf{a}\mathbf{b}' = \mathbf{a} \otimes \mathbf{b}' = \mathbf{b}' \otimes \mathbf{a}. \tag{15}$$

It is easy to verify that all scalar (cross-)moments are included in definition 1. To do so, let decompose $\mathbf{x} = \sum_{j \in [\![1,p]\!]} x_j \mathbf{e}_j$ onto the canonical basis of $\mathbb{R}^p$; thus, from the multilinearity property of the Kronecker product and the linearity of the expected value,

$$\begin{aligned}
m_{2k}(\mathbf{x}) &= \mathbb{E}\left[\otimes^k \mathbf{x}\mathbf{x}'\right] = \mathbb{E}\left[\otimes^k \sum_{(j_1,j_2)\in[\![1,p]\!]^2} x_{j_1} x_{j_2} \mathbf{e}_{j_1} \mathbf{e}'_{j_2}\right] \\
&= \mathbb{E}\left[\sum_{(j_1,j_2)\in[\![1,p]\!]^2} x_{j_1} x_{j_2} \mathbf{e}_{j_1} \mathbf{e}'_{j_2} \otimes \cdots \otimes \sum_{(j_{2k-1},j_{2k})\in[\![1,p]\!]^2} x_{j_{2k-1}} x_{j_{2k}} \mathbf{e}_{j_{2k-1}} \mathbf{e}'_{j_{2k}}\right] \\
&= \sum_{(j_1,j_2,\ldots,j_{2k-1},j_{2k})\in[\![1,p]\!]^{2k}} \mathbb{E}\left[x_{j_1} x_{j_2} \cdots x_{j_{2k-1}} x_{j_{2k}}\right] \mathbf{e}_{j_1} \mathbf{e}'_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_{2k-1}} \mathbf{e}'_{j_{2k}} \\
&= \sum_{(j_1,j_2,\ldots,j_{2k-1},j_{2k})\in[\![1,p]\!]^{2k}} mom^{(2k)}(x_{j_1}, \ldots, x_{j_{2k}}) \mathbf{e}_{j_1} \mathbf{e}'_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_{2k-1}} \mathbf{e}'_{j_{2k}}.
\end{aligned} \tag{16}$$

Similarly, for odd orders,

$$m_{2k+1}(\mathbf{x}) = \sum_{(j_1,j_2,\ldots,j_{2k-1},j_{2k},j_{2k+1})\in[\![1,p]\!]^{2k+1}} mom^{(2k+1)}(x_{j_1}, \ldots, x_{j_{2k+1}}) \mathbf{e}_{j_1} \mathbf{e}'_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_{2k-1}} \mathbf{e}'_{j_{2k}} \otimes \mathbf{e}_{j_{2k+1}}. \tag{17}$$

It is then possible to define the cumulant matrix in a similar way as the moment matrix.

Definition 2 (The cumulant matrix). Let $\mathbf{x} = (x_1, \ldots, x_p)'$ be a real-valued random vector of dimension $p \times 1$, such that $\mathbb{E}\left[|\mathbf{x}|^n\right] < \infty$ for a given $n \in \mathbb{N}$. The cumulant matrix of even order, $2k \leq n$, of $\mathbf{x}$ is a $p^k \times p^k$ matrix defined as

$$c_{2k}(\mathbf{x}) = \sum_{(j_1, j_2, \ldots, j_{2k-1}, j_{2k}) \in [\![1,p]\!]^{2k}} cum^{(2k)}(x_{j_1} x_{j_2} \cdots x_{j_{2k-1}} x_{j_{2k}}) \mathbf{e}_{j_1} \mathbf{e}'_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_{2k-1}} \mathbf{e}'_{j_{2k}} \tag{18}$$

whereas the cumulant matrix of odd order, $2k + 1 \leq n$, is a $p^{k+1} \times p^k$ matrix defined as

$$c_{2k+1}(\mathbf{x}) = \sum_{(j_1, j_2, \ldots, j_{2k-1}, j_{2k}, j_{2k+1}) \in [\![1,p]\!]^{2k+1}} cum^{(2k+1)}(x_{j_1} x_{j_2} \cdots x_{j_{2k-1}} x_{j_{2k}} x_{j_{2k+1}}) \mathbf{e}_{j_1} \mathbf{e}'_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_{2k-1}} \mathbf{e}'_{j_{2k}} \otimes \mathbf{e}_{j_{2k+1}}. \tag{19}$$

Except for the special case of a Gaussian random vector, cumulants are much more difficult to compute than moments. Hitherto, the only way to computing the cumulant matrix $c_n(\mathbf{x})$ has consisted in

1. deriving the p^n scalar cumulants from the scalar moments of orders lower than n via Eq. (11),
2. arranging the scalar cumulants in a matrix according definition 2.

However, Eq. (11) tends to return lengthy expressions that are hardly tractable beyond order four or so; for instance, the sixth-order cumulant requires the consideration (and computation) of 41 moments. It is the object of this paper to compute the cumulant matrix $c_n(\mathbf{x})$ directly from the moment matrices of orders lower than n, thus providing formulae that are much more concise and convenient to handle.

3. Cumulant matrices from lower-order moment matrices

3.1. Statement of results for orders 2 *to* 6

In order to state the theorem of the paper, it is first necessary to introduce the commutation matrix and the **vec** operator. The following definitions and properties are borrowed from Magnus and Neudecker [18].

Lemma 1 (The commutation matrix). *Let $p \geq 1$ be an integer. The commutation matrix $\mathbf{K}_{pp}$ of dimension $p^2 \times p^2$ is defined as*

$$\mathbf{K}_{pp} = \sum_{(i,j) \in [\![1,p]\!]^2} \mathbf{e}_i \mathbf{e}'_j \otimes \mathbf{e}_j \mathbf{e}'_i, \tag{20}$$

where $(\mathbf{e}_i)_{i \in [\![1,p]\!]}$ stands for the canonical basis of $\mathbb{R}^p$.

Let us define two other related commutation matrices, of dimension $p^3 \times p^3$:

$$\mathcal{P}_{3,2} = \mathbf{I}_p \otimes \mathbf{K}_{pp} \tag{21}$$

$$\mathcal{P}_{2,1} = \mathbf{K}_{pp} \otimes \mathbf{I}_p. \tag{22}$$

Therefore, if $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are three vectors of dimension* $p \times 1$, *the following equalities hold:*

$$\begin{aligned}
\mathbf{K}_p(\mathbf{a}\otimes\mathbf{b}) &= \mathbf{b}\otimes\mathbf{a} && (23)\\
\mathcal{P}_{3,2}\,(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) &= \mathbf{a}\otimes\mathbf{c}\otimes\mathbf{b} && (24)\\
\mathcal{P}_{2,1}\,(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) &= \mathbf{b}\otimes\mathbf{a}\otimes\mathbf{c}. && (25)
\end{aligned}$$

Since $K'_{pp} = K_{pp}$, *it comes* $\mathcal{P}'_{3,2} = \mathcal{P}_{3,2}$, $\mathcal{P}'_{2,1} = \mathcal{P}_{2,1}$, *and*

$$\begin{aligned}
(\mathbf{a}'\otimes\mathbf{b}')\mathbf{K}_p &= \mathbf{b}'\otimes\mathbf{a}' && (26)\\
(\mathbf{a}'\otimes\mathbf{b}'\otimes\mathbf{c}')\,\mathcal{P}_{3,2} &= \mathbf{a}'\otimes\mathbf{c}'\otimes\mathbf{b}' && (27)\\
(\mathbf{a}'\otimes\mathbf{b}'\otimes\mathbf{c}')\,\mathcal{P}_{2,1} &= \mathbf{b}'\otimes\mathbf{a}'\otimes\mathbf{c}'. && (28)
\end{aligned}$$

Proof.

$$\begin{aligned}
\mathbf{K}_p(\mathbf{a}\otimes\mathbf{b}) &= \sum_{(i,j)\in[\![1,p]\!]^2} \mathbf{e}_i\mathbf{e}'_j\otimes\mathbf{e}_j\mathbf{e}'_i\,(\mathbf{a}\otimes\mathbf{b})\\
&= \sum_{(i,j)\in[\![1,p]\!]^2} \underbrace{(\mathbf{e}_i\mathbf{e}'_j\mathbf{a})}_{a_j\mathbf{e}_i}\otimes\underbrace{(\mathbf{e}_j\mathbf{e}'_i\mathbf{b})}_{b_i\mathbf{e}_j} = \sum_{(i,j)\in[\![1,p]\!]^2} b_i a_j\mathbf{e}_i\otimes\mathbf{e}_j = \mathbf{b}\otimes\mathbf{a} && (29)
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{3,2}\,(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) &= \Big(\mathbf{I}_p\otimes\mathbf{K}_{pp}\Big)(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) = \mathbf{a}\otimes\mathbf{K}_{pp}\,(\mathbf{b}\otimes\mathbf{c}) = \mathbf{a}\otimes\mathbf{c}\otimes\mathbf{b} && (30)\\
\mathcal{P}_{2,1}\,(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) &= \Big(\mathbf{K}_{pp}\otimes\mathbf{I}_p\Big)(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) = \mathbf{K}_{pp}\,(\mathbf{a}\otimes\mathbf{b})\otimes\mathbf{c} = \mathbf{b}\otimes\mathbf{a}\otimes\mathbf{c} && (31)
\end{aligned}$$

□

We are now in a position to state the main result of the paper.

Theorem 2. *Let* $\mathbf{x} = (x_1,\ldots,x_p)'$ *be a real-valued random vector of dimension* $p\times 1$, *with assumed zero-mean for simplicity (*$\mathbb{E}[\mathbf{x}] = 0$*). Let us further assume that* $\mathbb{E}[|\mathbf{x}|^n] < \infty$ *for a given integer* $n \geq 6$ *so as to guarantee the existence of moments and cumulants up to order six.*
Next let us define the four matrices:

$$\begin{aligned}
\mathbf{K} &= \mathbf{I}_{p^2} + \mathbf{K}_{pp} && (32)\\
\mathbf{P} &= \mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1} && (33)\\
\mathbf{R} &= \mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2} && (34)\\
\mathbf{Q} &= \mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1} + \mathcal{P}_{2,1}\mathcal{P}_{3,2} + \mathcal{P}_{3,2}\mathcal{P}_{2,1} + \mathcal{P}_{2,1}\mathcal{P}_{3,2}\mathcal{P}_{2,1} && (35)
\end{aligned}$$

which depend only on dimension p. *Thus, the cumulant matrices of orders* 2 *to* 6 *read*

$$\begin{aligned}
c_2(\mathbf{x}) &= m_2(\mathbf{x}) = \mathbb{E}[\mathbf{x}\mathbf{x}'] && (36)\\
c_3(\mathbf{x}) &= m_3(\mathbf{x}) = \mathbb{E}[\mathbf{x}\mathbf{x}'\otimes\mathbf{x}] && (37)\\
c_4(\mathbf{x}) &= m_4(\mathbf{x}) - \mathbf{K}\Big\{m_2(\mathbf{x})\otimes m_2(\mathbf{x})\Big\} - \mathbf{vec}m_2(\mathbf{x})\mathbf{vec}'m_2(\mathbf{x}) && (38)\\
c_5(\mathbf{x}) &= m_5(\mathbf{x}) - \mathbf{R}\Big\{m_3(\mathbf{x})\otimes m_2(\mathbf{x})\Big\}\mathbf{K} - \mathbf{P}\Big\{m_3(\mathbf{x})'\otimes\mathbf{vec}m_2(\mathbf{x})\Big\} - \mathbf{vec}m_3(\mathbf{x})\mathbf{vec}'m_2(\mathbf{x}) && (39)\\
c_6(\mathbf{x}) &= m_6(\mathbf{x}) - \Gamma_{(4,2)}(\mathbf{x}) - \Gamma_{(3,3)}(\mathbf{x}) + 2\Gamma_{(2,2,2)}(\mathbf{x}), && (40)
\end{aligned}$$

respectively, where

$$\begin{aligned}
\Gamma_{(4,2)}(\mathbf{x}) &= \mathbf{R}\Big\{m_4(\mathbf{x})\otimes m_2(\mathbf{x})\Big\}\mathbf{R}' + \Big\{\mathbf{M}_4(\mathbf{x})\otimes \mathbf{vec}' m_2(\mathbf{x})\Big\}\mathbf{P}' + \mathbf{P}\Big\{\mathbf{M}_4(\mathbf{x})'\otimes \mathbf{vec} m_2(\mathbf{x})\Big\} \\
\Gamma_{(3,3)}(\mathbf{x}) &= \mathbf{R}\Big\{m_3(\mathbf{x})\otimes m_3(\mathbf{x})'\Big\}\mathbf{P}' + \mathbf{vec} m_3(\mathbf{x})\mathbf{vec}' m_3(\mathbf{x}) \\
\Gamma_{(2,2,2)}(\mathbf{x}) &= \mathbf{Q}\Big\{m_2(\mathbf{x})\otimes m_2(\mathbf{x})\otimes m_2(\mathbf{x})\Big\} + \mathbf{P}\Big\{m_2(\mathbf{x})\otimes \mathbf{vec} m_2(\mathbf{x})\mathbf{vec}' m_2(\mathbf{x})\Big\}\mathbf{P}' \\
\mathbf{M}_4(\mathbf{x}) &= \mathbb{E}\left[\mathbf{x}\mathbf{x}'\otimes\mathbf{x}\otimes\mathbf{x}\right].
\end{aligned}$$

3.2. Outline of the proof

The sketch of the proof consists of

(a) injecting Eq. (11) that returns cumulants in terms of moments into Eqs. (18) and (19) which define cumulant matrices;

(b) expressing the terms of the resulting sum with moment matrices.

Let us detail the first step in the case of even orders (the principle is identical for odd orders):

$$\begin{aligned}
c_{2k}(\mathbf{x}) &= \sum_{(j_1,\ldots,j_{2k})\in[\![1,p]\!]^{2k}} \left(\sum_{\pi\in P([\![1,2k]\!])} (-1)^{|\pi|-1}(|\pi|-1)!\prod_{B\in\pi} mom^{(|B|)}(x_{j_l}\ :\ l\in B)\right)\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}}\mathbf{e}'_{j_{2k}} \\
&= \sum_{s\in P(2k)} (-1)^{|s|-1}(|s|-1)!\Gamma_s(\mathbf{x}) \qquad (41)
\end{aligned}$$

where $P(2k)$ denotes the set of all $s=(s_1,\ldots,s_l)$, where $s_1,\ldots,s_l$ are strictly positive integers such that $s_1+\ldots+s_l=2k$ partitions, $|s|$ denotes the length of s – that is the number l of integers in s – and

$$\Gamma_s(\mathbf{x}) = \sum_{\pi\in P_s([\![1,2k]\!])} \underbrace{\left(\sum_{(j_1,\ldots,j_{2k})\in[\![1,p]\!]^{2k}} \prod_{B\in\pi} mom^{(|B|)}(x_{j_l}\ :\ l\in B)\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}}\mathbf{e}'_{j_{2k}}\right)}_{T_\pi}, \qquad (42)$$

with $P_s([\![1,2k]\!])$ the set of partitions of $[\![1,2k]\!]$ of type s. For instance, for $k=2$,

$$P(4) = \{(4);(3,1);(2,2);(2,1,1);(1,1,1,1)\}, \qquad (43)$$

and for $s=(2,2)$, $|s|=2$ and

$$P_s([\![1,4]\!]) = \{\{\{1;2\};\{3;4\}\};\{\{1;3\};\{2;4\}\};\{\{1;4\};\{2;3\}\}\}. \qquad (44)$$

For the second step, it is convenient to browse set $P(2k)$ in the lexicographic order, as suggested in Eq. (43), so that the first term in Eq. (41) will correspond to $s=(2k)$, $|s|=1$, $P_s([\![1,2k]\!])=\{\{\{1;\cdots;2k\}\}\}$; that is partition $\pi=\{\{1;\cdots;2k\}\}$ with one block $B=\{1;\cdots;2k\}$, such that

$$\Gamma_{(2k)}(\mathbf{x}) = T_{\{\{1;\cdots;2k\}\}} = \sum_{(j_1,\ldots,j_{2k})\in[\![1,p]\!]^{2k}} mom^{(2k)}(x_{j_l}\ :\ l\in B)\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}}\mathbf{e}'_{j_{2k}} = m_{2k}(\mathbf{x}). \qquad (45)$$

In addition, in each product $\prod_{B\in\pi}$ of (42), blocks B will be browsed from the smallest to the largest, as suggested by Eq. (43). Therefore, for each partition s of the integer $2k$, there exists a unique type-s partition π of set $[\![1,2k]\!]$ that globally preserves the natural order on $[\![1,2k]\!]$. This will be denoted as the canonical partition associated with s. For instance, the canonical partition of type $s = (2,2)$ is $\pi_s = \{\{1;2\};\{3;4\}\}$. Since the blocks of a canonical partition are consecutive, the corresponding terms are easily found. In general, if the type-s partition is such that the number of odd indices is not greater than the number of even indices plus one, then the corresponding term in (42) is simply

$$T_{\pi_s} = \sum_{(j_1,\ldots,j_{2k})\in[\![1,p]\!]^{2k}} \prod_{B\in\pi_s} mom^{(|B|)}(x_{j_l} \ : \ l\in B)\mathbf{e}_{j_1}\otimes\mathbf{e}'_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}}\otimes\mathbf{e}'_{j_{2k}} = \bigotimes_{B\in\pi_s} m_{|B|}(\mathbf{x}). \tag{46}$$

For non-canonical partitions, $\mathbf{e}_{j_1}\otimes\mathbf{e}'_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}}\otimes\mathbf{e}'_{j_{2k}}$ will be rearranged according to the blocks in partitions π_s. Unfortunately, there is no general matrix formulation in this case. However, since $\mathbf{e}_j\otimes\mathbf{e}'_k = \mathbf{e}'_k\otimes\mathbf{e}_j$, we will make use of the identity

$$\mathbf{e}_{j_1}\otimes\mathbf{e}'_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}}\otimes\mathbf{e}'_{j_{2k}} = \mathbf{e}_{j_1}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}}\otimes\mathbf{e}'_{j_2}\otimes\cdots\otimes\mathbf{e}'_{j_{2k}} = (\mathbf{e}_{j_1}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}})(\mathbf{e}_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_{2k}})'. \tag{47}$$

For orders greater than 4, lemma 1 will be used to rearrange Kronecker products of length 3. For blocks where the number of odd indices is neither equal to nor greater than the number of even indices, the corresponding factor will not be of type $m_{|B|}(\mathbf{x})$. Since orders up to 6 are considered only in this paper, this will be fixed by using the **vec** operator and the moment matrix $\mathbf{M}_4(\mathbf{x}) = \mathbb{E}\left[\mathbf{x}\mathbf{x}'\otimes\mathbf{x}\otimes\mathbf{x}\right]$, and its transpose. Finally, let us note that for a zero-mean random vector, all terms involving a first-order moment – corresponding to partitions with blocks of unit length – will be nil.

3.3. Proof for order 3

The only partition of 3 which does not involve 1 is (3): $P_{(3)}([\![1,3]\!]) = \{\{\{1;2;3\}\}\}$. Thus

$$c_3(\mathbf{x}) = \Gamma_{(3)}(\mathbf{x}) = \sum_{(j_1,j_2,j_3)\in[\![1,p]\!]^3} mom^{(3)}(x_{j_1},x_{j_2},x_{j_3})\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3} = m_3(\mathbf{x}). \tag{48}$$

3.4. Proof for order 4

The only partitions of 4 (see Eq. (43)) which do not involve 1 are (4) and (2, 2):

$$\begin{aligned} P_{(4)}([\![1,4]\!]) &= \{\{\{1;2;3;4\}\}\} \\ P_{(2,2)}([\![1,4]\!]) &= \{\{\{1;2\};\{3;4\}\};\{\{1;3\};\{2;4\}\};\{\{1;4\};\{2;3\}\}\}. \end{aligned}$$

In order to avoid any ambiguity, each term of $\Gamma_{(2,2)}(\mathbf{x})$ will be dentoted by T_π where π is the corresponding partition in (42). Therefore,

$$\begin{aligned} c_4(\mathbf{x}) &= m_4(\mathbf{x}) - \Gamma_{(2,2)}(\mathbf{x}) \\ &= m_4(\mathbf{x}) - \underbrace{m_2(\mathbf{x})\otimes m_2(\mathbf{x})}_{T_{\{\{1;2\};\{3;4\}\}}} \end{aligned} \tag{49}$$

$$
\begin{aligned}
&- \underbrace{\sum_{(j_1,\ldots,j_4)\in[\![1,p]\!]^4} mom^{(2)}(x_{j_1},x_{j_3})mom^{(2)}(x_{j_2},x_{j_4})\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\mathbf{e}'_{j_4}}_{T_{\{\{1;3\};\{2;4\}\}}} \\
&- \underbrace{\sum_{(j_1,\ldots,j_4)\in[\![1,p]\!]^4} mom^{(2)}(x_{j_1},x_{j_4})mom^{(2)}(x_{j_2},x_{j_3})\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\mathbf{e}'_{j_4}}_{T_{\{\{1;4\};\{2;3\}\}}} . \qquad (50)
\end{aligned}
$$

For

$$
\begin{aligned}
\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\mathbf{e}'_{j_4} &= \mathbf{e}_{j_1}\otimes\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}'_{j_4} = \mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}'_{j_2}\otimes\mathbf{e}'_{j_4} = (\mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3})\otimes(\mathbf{e}_{j_2}\otimes\mathbf{e}_{j_4})' \qquad (51)\\
&= K_{pp}(\mathbf{e}_{j_3}\otimes\mathbf{e}_{j_1})(\mathbf{e}_{j_2}\otimes\mathbf{e}_{j_4})' = K_{pp}(\mathbf{e}_{j_3}\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_1}\mathbf{e}'_{j_4}), \qquad (52)
\end{aligned}
$$

it comes

$$
\begin{aligned}
T_{\{\{1;3\};\{2;4\}\}} &= \left(\sum_{(j_1,j_3)\in[\![1,p]\!]^2} mom^{(2)}(x_{j_1},x_{j_3})\mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3}\right)\left(\sum_{(j_2,j_4)\in[\![1,p]\!]^2} mom^{(2)}(x_{j_2},x_{j_4})\mathbf{e}_{j_2}\otimes\mathbf{e}_{j_4}\right)' \\
&= \mathbf{vec}m_2(\mathbf{x})\mathbf{vec}'m_2(\mathbf{x}), \qquad (53)
\end{aligned}
$$

$$
\begin{aligned}
T_{\{\{1;4\};\{2;3\}\}} &= K_{pp}\left(\sum_{(j_2,j_3)\in[\![1,p]\!]^2} mom^{(2)}(x_{j_2},x_{j_3})\mathbf{e}_{j_3}\mathbf{e}'_{j_2}\right)\otimes\left(\sum_{(j_1,j_4)\in[\![1,p]\!]^2} mom^{(2)}(x_{j_1},x_{j_4})\mathbf{e}_{j_1}\mathbf{e}'_{j_4}\right) \\
&= K_{pp}m_2(\mathbf{x})\otimes m_2(\mathbf{x}). \qquad (54)
\end{aligned}
$$

Upon collecting $m_2(\mathbf{x})\otimes m_2(\mathbf{x})$, one finally arrives at

$$
c_4(\mathbf{x}) = m_4(\mathbf{x}) - \mathbf{K}m_2(\mathbf{x})\otimes m_2(\mathbf{x}) - \mathbf{vec}m_2(\mathbf{x})\mathbf{vec}'m_2(\mathbf{x}). \qquad (55)
$$

3.5. Proof for order 5

The only partitions of 5 which do not involve 1 are (5) et (3, 2):

$$
\begin{aligned}
P_{(5)}([\![1,5]\!]) &= \{\{\{1;2;3;4;5\}\}\} \\
P_{(3,2)}([\![1,5]\!]) &= \{\{\{1;2;3\};\{4;5\}\};\{\{1;2;4\};\{3;5\}\};\{\{1;3;4\};\{2;5\}\};\{\{2;3;4\};\{1;5\}\}; \\
&\quad \{\{1;2;5\};\{3;4\}\};\{\{1;3;5\};\{2;4\}\};\{\{2;3;5\};\{1;4\}\};\{\{1;4;5\};\{2;3\}\}; \\
&\quad \{\{2;4;5\};\{1;3\}\};\{\{3;4;5\};\{1;2\}\}\} .
\end{aligned}
$$

One may check that $P_{(3,2)}([\![1,5]\!])$ contains $\frac{5!}{3!2!} = 10$ elements. As before, each term of $\Gamma_{(3,2)}(\mathbf{x})$ will be denoted as T_π where π is the corresponding partition in Eq. (42). Thus

$$
c_5(\mathbf{x}) = m_5(\mathbf{x}) - \Gamma_{(3,2)}(\mathbf{x}), \qquad (56)
$$

where $\Gamma_{(3,2)}(\mathbf{x}) = \sum_{\pi\in P_{(3,2)}([\![1,5]\!])} T_\pi$. Next, the T_π's are computed by factoring terms of order 3 in first positions. The principle is illustrated for instance on $T_{\{\{2;3;5\};\{1;4\}\}}$:

$$\begin{aligned}
T_{\{\{2;3;5\};\{1;4\}\}} &= \sum_{(j_1,\dots,j_5)\in[\![1,p]\!]^5} mom^{(3)}(x_{j_2},x_{j_3},x_{j_5})mom^{(2)}(x_{j_1},x_{j_4})\mathbf{e}_{j_1}\otimes\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}'_{j_4}\otimes\mathbf{e}_{j_5}\\
&= \sum_{(j_1,\dots,j_5)\in[\![1,p]\!]^5} mom^{(3)}(x_{j_2},x_{j_3},x_{j_5})mom^{(2)}(x_{j_1},x_{j_4})(\mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}_{j_5})\otimes(\mathbf{e}'_{j_2}\otimes\mathbf{e}'_{j_4})\\
&= \sum_{(j_1,\dots,j_5)\in[\![1,p]\!]^5} mom^{(3)}(x_{j_2},x_{j_3},x_{j_5})mom^{(2)}(x_{j_1},x_{j_4})\mathcal{P}_{2,1}\mathcal{P}_{3,2}(\mathbf{e}_{j_3}\otimes\mathbf{e}_{j_5}\otimes\mathbf{e}_{j_1})\otimes(\mathbf{e}'_{j_2}\otimes\mathbf{e}'_{j_4})\\
&= \mathcal{P}_{2,1}\mathcal{P}_{3,2}\sum_{(j_1,\dots,j_5)\in[\![1,p]\!]^5}\underbrace{mom^{(3)}(x_{j_2},x_{j_3},x_{j_5})}_{=mom^{(3)}(x_{j_3},x_{j_2},x_{j_5})}mom^{(2)}(x_{j_1},x_{j_4})(\mathbf{e}_{j_3}\otimes\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_5})\otimes(\mathbf{e}_{j_1}\otimes\mathbf{e}'_{j_4})\\
&= \mathcal{P}_{2,1}\mathcal{P}_{3,2}m_3(\mathbf{x})\otimes m_2(\mathbf{x}). \qquad (57)
\end{aligned}$$

Following similar lines,

$$\begin{aligned}
T_{\{\{1;2;3\};\{4;5\}\}} &= m_3(\mathbf{x})\otimes m_2(\mathbf{x}) && (58)\\
T_{\{\{1;2;4\};\{3;5\}\}} &= m_3(\mathbf{x})'\otimes\mathbf{vec}m_2(\mathbf{x}) && (59)\\
T_{\{\{1;3;4\};\{2;5\}\}} &= m_3(\mathbf{x})\otimes m_2(\mathbf{x})\mathbf{K}_{pp} && (60)\\
T_{\{\{2;3;4\};\{1;5\}\}} &= \mathcal{P}_{2,1}m_3(\mathbf{x})'\otimes\mathbf{vec}m_2(\mathbf{x}) && (61)\\
T_{\{\{1;2;5\};\{3;4\}\}} &= \mathcal{P}_{3,2}m_3(\mathbf{x})\otimes m_2(\mathbf{x}) && (62)\\
T_{\{\{1;3;5\};\{2;4\}\}} &= \mathbf{vec}m_3(\mathbf{x})\mathbf{vec}'m_2(\mathbf{x}) && (63)\\
T_{\{\{1;4;5\};\{2;3\}\}} &= \mathcal{P}_{3,2}m_3(\mathbf{x})\otimes m_2(\mathbf{x})\mathbf{K}_{pp} && (64)\\
T_{\{\{2;4;5\};\{1;3\}\}} &= \mathcal{P}_{3,2}\mathcal{P}_{2,1}m_3(\mathbf{x})'\otimes\mathbf{vec}m_2(\mathbf{x}) && (65)\\
T_{\{\{3;4;5\};\{1;2\}\}} &= \mathcal{P}_{2,1}\mathcal{P}_{3,2}m_3(\mathbf{x})\otimes m_2(\mathbf{x}).\mathbf{K}_{pp}. && (66)
\end{aligned}$$

The last step is to collect terms in $m_3(\mathbf{x})\otimes m_2(\mathbf{x})$, $m_3(\mathbf{x})'\otimes\mathbf{vec}m_2(\mathbf{x})$, and $\mathbf{vec}m_3(\mathbf{x})\mathbf{vec}'m_2(\mathbf{x})$, respectively:

$$\begin{aligned}
\Gamma_{(3,2)}(\mathbf{x}) &= \underbrace{\left(\mathbf{I}_{p^3}+\mathcal{P}_{3,2}+\mathcal{P}_{2,1}\mathcal{P}_{3,2}\right)}_{\mathbf{R}}m_3(\mathbf{x})\otimes m_2(\mathbf{x})+\underbrace{\left(\mathbf{I}_{p^3}+\mathcal{P}_{3,2}+\mathcal{P}_{2,1}\mathcal{P}_{3,2}\right)}_{\mathbf{R}}m_3(\mathbf{x})\otimes m_2(\mathbf{x})\mathbf{K}_{pp}\\
&\quad+\underbrace{\left(\mathbf{I}_{p^3}+\mathcal{P}_{2,1}+\mathcal{P}_{3,2}\mathcal{P}_{2,1}\right)}_{\mathbf{P}}m_3(\mathbf{x})'\otimes\mathbf{vec}m_2(\mathbf{x})+\mathbf{vec}m_3(\mathbf{x})\mathbf{vec}'m_2(\mathbf{x})\\
&= \mathbf{R}m_3(\mathbf{x})\otimes m_2(\mathbf{x})\underbrace{\left(\mathbf{I}_{p^2}+\mathbf{K}_{pp}\right)}_{\mathbf{K}}+\mathbf{P}m_3(\mathbf{x})'\otimes\mathbf{vec}m_2(\mathbf{x})+\mathbf{vec}m_3(\mathbf{x})\mathbf{vec}'m_2(\mathbf{x})\\
&= \mathbf{R}m_3(\mathbf{x})\otimes m_2(\mathbf{x})\mathbf{K}+\mathbf{P}m_3(\mathbf{x})'\otimes\mathbf{vec}m_2(\mathbf{x})+\mathbf{vec}m_3(\mathbf{x})\mathbf{vec}'m_2(\mathbf{x}). \qquad (67)
\end{aligned}$$

3.6. Proof for order 6

The 4 partitions of 6 which do not involve 1 are: (6), $(4,2)$, $(3,3)$ et $(2,2,2)$. Therefore $c_6(\mathbf{x})$ is a sum of four terms,

$$c_6(\mathbf{x}) = m_6(\mathbf{x}) - \Gamma_{(4,2)}(\mathbf{x}) - \Gamma_{(3,3)}(\mathbf{x}) + 2\Gamma_{(2,2,2)}(\mathbf{x}), \qquad (68)$$

the last three of which are detailed hereafter.

Expression of $\Gamma_{(4,2)}(\mathbf{x})$:

It is seen that $P_{(4,2)}([\![1,6]\!])$ contains $\frac{6!}{4!2!} = 15$ elements:

$$\begin{aligned} P_{(4,2)}([\![1,6]\!]) &= \{\{\{1;2;3;4\};\{5;6\}\};\{\{1;2;3;5\};\{4;6\}\};\{\{1;2;4;5\};\{3;6\}\}; \\ &\quad \{\{1;3;4;5\};\{2;6\}\};\{\{2;3;4;5\};\{1;6\}\};\{\{1;2;3;6\};\{4;5\}\}; \\ &\quad \{\{1;2;4;6\};\{3;5\}\};\{\{1;3;4;6\};\{2;5\}\};\{\{2;3;4;6\};\{1;5\}\}; \\ &\quad \{\{1;2;5;6\};\{3;4\}\};\{\{1;3;5;6\};\{2;4\}\};\{\{2;3;5;6\};\{1;4\}\}; \\ &\quad \{\{1;4;5;6\};\{2;3\}\};\{\{2;4;5;6\};\{1;3\}\};\{\{3;4;5;6\};\{1;2\}\}\} \end{aligned} \tag{69}$$

$$\Gamma_{(4,2)}(\mathbf{x}) = \sum_{\pi \in P_{(4,2)}([\![1,6]\!])} T_\pi. \tag{70}$$

In the above equation, each T_π is computed so that factors of order 4 come first (e.g. there will be no such term as $m_2(\mathbf{x}) \otimes m_4(\mathbf{x})$). For instance,

$$\begin{aligned} T_{\{\{1;3;5;6\};\{2;4\}\}} &= \sum_{(j_1,\dots,j_6)\in[\![1,p]\!]^6} mom^{(4)}(x_{j_1},x_{j_3},x_{j_5},x_{j_6}) mom^{(2)}(x_{j_2},x_{j_4}) \mathbf{e}_{j_1}\mathbf{e}'_{j_2} \otimes \mathbf{e}_{j_3}\mathbf{e}'_{j_4} \otimes \mathbf{e}_{j_5}\mathbf{e}'_{j_6} \\ &= \sum_{(j_1,\dots,j_6)\in[\![1,p]\!]^6} mom^{(4)}(x_{j_1},x_{j_3},x_{j_5},x_{j_6}) mom^{(2)}(x_{j_2},x_{j_4}) (\mathbf{e}_{j_1} \otimes \mathbf{e}_{j_3} \otimes \mathbf{e}_{j_5})(\mathbf{e}'_{j_2} \otimes \mathbf{e}'_{j_4} \otimes \mathbf{e}'_{j_6}) \\ &= \sum_{(j_1,\dots,j_6)\in[\![1,p]\!]^6} mom^{(4)}(x_{j_1},x_{j_3},x_{j_5},x_{j_6}) mom^{(2)}(x_{j_2},x_{j_4}) (\mathbf{e}_{j_1} \otimes \mathbf{e}_{j_3} \otimes \mathbf{e}_{j_5})(\mathbf{e}'_{j_6} \otimes \mathbf{e}'_{j_2} \otimes \mathbf{e}'_{j_4}) \mathcal{P}_{2,1}\mathcal{P}_{3,2} \\ &= \sum_{(j_1,\dots,j_6)\in[\![1,p]\!]^6} mom^{(4)}(x_{j_1},x_{j_3},x_{j_5},x_{j_6}) mom^{(2)}(x_{j_2},x_{j_4}) (\mathbf{e}_{j_1}\mathbf{e}'_{j_6} \otimes \mathbf{e}_{j_3} \otimes \mathbf{e}_{j_5}) \otimes (\mathbf{e}'_{j_2} \otimes \mathbf{e}'_{j_4}) \mathcal{P}_{2,1}\mathcal{P}_{3,2} \\ &= \mathbf{M}_4(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x}) \mathcal{P}_{2,1}\mathcal{P}_{3,2}. \end{aligned} \tag{71}$$

Following similar lines,

$$\begin{aligned} T_{\{\{1;2;3;4\};\{5;6\}\}} &= m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) && (72) \\ T_{\{\{1;2;3;5\};\{4;6\}\}} &= \mathbf{M}_4(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x}) && (73) \\ T_{\{\{1;2;4;5\};\{3;6\}\}} &= \mathcal{P}_{3,2} m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) && (74) \\ T_{\{\{1;3;4;5\};\{2;6\}\}} &= \mathbf{M}_4(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x}) \mathcal{P}_{2,1} && (75) \\ T_{\{\{2;3;4;5\};\{1;6\}\}} &= \mathcal{P}_{2,1}\mathcal{P}_{3,2} m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) && (76) \\ T_{\{\{1;2;3;6\};\{4;5\}\}} &= m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) \mathcal{P}_{3,2} && (77) \\ T_{\{\{1;2;4;6\};\{3;5\}\}} &= \mathbf{M}_4(\mathbf{x})' \otimes \mathbf{vec} m_2(\mathbf{x}) && (78) \\ T_{\{\{1;3;4;6\};\{2;5\}\}} &= m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) \mathcal{P}_{3,2}\mathcal{P}_{2,1} && (79) \\ T_{\{\{2;3;4;6\};\{1;5\}\}} &= \mathcal{P}_{2,1} \mathbf{M}_4(\mathbf{x})' \otimes \mathbf{vec} m_2(\mathbf{x}) && (80) \\ T_{\{\{1;2;5;6\};\{3;4\}\}} &= \mathcal{P}_{3,2} m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) \mathcal{P}_{3,2} && (81) \\ T_{\{\{2;3;5;6\};\{1;4\}\}} &= \mathcal{P}_{2,1}\mathcal{P}_{3,2} m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) \mathcal{P}_{3,2} && (82) \\ T_{\{\{1;4;5;6\};\{2;3\}\}} &= \mathcal{P}_{3,2} m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) \mathcal{P}_{3,2}\mathcal{P}_{2,1} && (83) \\ T_{\{\{2;4;5;6\};\{1;3\}\}} &= \mathcal{P}_{3,2}\mathcal{P}_{2,1} \mathbf{M}_4(\mathbf{x})' \otimes \mathbf{vec} m_2(\mathbf{x}) && (84) \\ T_{\{\{3;4;5;6\};\{1;2\}\}} &= \mathcal{P}_{2,1}\mathcal{P}_{3,2} m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) \mathcal{P}_{3,2}.\mathcal{P}_{2,1}. && (85) \end{aligned}$$

The last step is to collect terms in $m_4(\mathbf{x}) \otimes m_2(\mathbf{x})$, $\mathbf{M}_4(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x})$, and $\mathbf{M}_4(\mathbf{x})' \otimes \mathbf{vec} m_2(\mathbf{x})$ respectively:

$$\begin{aligned}
\Gamma_{(4,2)}(\mathbf{x}) &= \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}} m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}} m_4(\mathbf{x}) \otimes m_2(\mathbf{x})\mathcal{P}_{3,2}\mathcal{P}_{2,1} \\
&\quad + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}} m_4(\mathbf{x}) \otimes m_2(\mathbf{x})\mathcal{P}_{3,2} \\
&\quad + \mathbf{M}_4(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x}) \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{P}'} \\
&\quad + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{P}} \mathbf{M}_4(\mathbf{x})' \otimes \mathbf{vec} m_2(\mathbf{x}) \\
&= \mathbf{R}\Big\{m_4(\mathbf{x}) \otimes m_2(\mathbf{x})\Big\}\mathbf{R}' + \Big\{\mathbf{M}_4(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x})\Big\}\mathbf{P}' + \mathbf{P}\Big\{\mathbf{M}_4(\mathbf{x})' \otimes \mathbf{vec} m_2(\mathbf{x})\Big\}. \qquad (86)
\end{aligned}$$

<u>Expression of $\Gamma_{(3,3)}(\mathbf{x})$:</u>

It is seen that $P_{(3,3)}([\![1,6]\!])$ contains $\frac{6!}{2!3!3!} = 10$ elements:

$$\begin{aligned}
P_{(3,3)}([\![1,6]\!]) &= \{\{\{1;2;3\};\{4;5;6\}\};\{\{1;2;4\};\{3;5;6\}\};\{\{1;3;4\};\{2;5;6\}\}; \\
&\quad \{\{2;3;4\};\{1;5;6\}\};\{\{1;2;5\};\{3;4;6\}\};\{\{1;3;5\};\{2;4;6\}\}; \\
&\quad \{\{2;3;5\};\{1;4;6\}\};\{\{1;4;5\};\{2;3;6\}\};\{\{2;4;5\};\{1;3;6\}\}; \\
&\quad \{\{3;4;5\};\{1;2;6\}\}\} \qquad (87)
\end{aligned}$$

$$\Gamma_{(3,3)}(\mathbf{x}) = \sum_{\pi \in P_{(3,3)}([\![1,6]\!])} T_\pi. \qquad (88)$$

In the above equation, each T_π is computed so that factor $m_3(\mathbf{x})$ comes first (e.g. there will be no such term as $m_3(\mathbf{x})' \otimes m_3(\mathbf{x})$):

$$\begin{aligned}
T_{\{\{1;2;3\};\{4;5;6\}\}} &= m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' && (89) \\
T_{\{\{1;2;4\};\{3;5;6\}\}} &= \mathcal{P}_{2,1}\mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1}\mathcal{P}_{3,2} && (90) \\
T_{\{\{1;3;4\};\{2;5;6\}\}} &= m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1} && (91) \\
T_{\{\{2;3;4\};\{1;5;6\}\}} &= \mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1}\mathcal{P}_{3,2} && (92) \\
T_{\{\{1;2;5\};\{3;4;6\}\}} &= \mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' && (93) \\
T_{\{\{1;3;5\};\{2;4;6\}\}} &= \mathbf{vec} m_3(\mathbf{x}) \otimes \mathbf{vec}' m_3(\mathbf{x}) && (94) \\
T_{\{\{2;3;5\};\{1;4;6\}\}} &= \mathcal{P}_{2,1}\mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' && (95) \\
T_{\{\{1;4;5\};\{2;3;6\}\}} &= \mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1} && (96) \\
T_{\{\{2;4;5\};\{1;3;6\}\}} &= m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1}\mathcal{P}_{3,2} && (97) \\
T_{\{\{3;4;5\};\{1;2;6\}\}} &= \mathcal{P}_{2,1}\mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1}. && (98)
\end{aligned}$$

Therefore,

$$\Gamma_{(3,3)}(\mathbf{x}) = \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1}$$

$$
\begin{aligned}
&+ \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})'\mathcal{P}_{2,1}\mathcal{P}_{3,2} + \mathbf{vec}m_3(\mathbf{x}) \otimes \mathbf{vec}'m_3 \\
= \quad & \mathbf{R}\Big\{m_3(\mathbf{x}) \otimes m_3(\mathbf{x})'\Big\}\mathbf{P}' + \mathbf{vec}m_3(\mathbf{x}) \otimes \mathbf{vec}'m_3. \qquad (99)
\end{aligned}
$$

<u>Expression of $\Gamma_{(2,2,2)}(\mathbf{x})$:</u>

It is seen that $P_{(2,2,2)}([\![1,6]\!])$ contains $\frac{6!}{2!2!2!2!} = 15$ elements:

$$
\begin{aligned}
P_{(2,2,2)}([\![1,6]\!]) \quad = \quad & \{\{\{1;2\};\{3;4\};\{5;6\}\};\{\{1;3\};\{2;4\};\{5;6\}\};\{\{2;3\};\{1;4\};\{5;6\}\}; \\
& \{\{1;2\};\{3;5\};\{4;6\}\};\{\{1;3\};\{2;5\};\{4;6\}\};\{\{2;3\};\{1;5\};\{4;6\}\}; \\
& \{\{1;2\};\{4;5\};\{3;6\}\};\{\{1;4\};\{2;5\};\{3;6\}\};\{\{2;4\};\{1;5\};\{3;6\}\}; \\
& \{\{1;3\};\{4;5\};\{2;6\}\};\{\{1;4\};\{3;5\};\{2;6\}\};\{\{3;4\};\{1;5\};\{2;6\}\}; \\
& \{\{2;3\};\{4;5\};\{1;6\}\};\{\{2;4\};\{3;5\};\{1;6\}\};\{\{3;4\};\{2;5\};\{1;6\}\}\} \qquad (100)
\end{aligned}
$$

$$
\Gamma_{(2,2,2)}(\mathbf{x}) = \sum_{\pi \in P_{(2,2,2)}([\![1,6]\!])} T_\pi. \qquad (101)
$$

In the above equation, each T_π is computed by imposing the ordering $m_2(\mathbf{x}) < \mathbf{vec}m_2(\mathbf{x}) < \mathbf{vec}'m_2(\mathbf{x})$, which amounts to placing first blocks with polarities {even}{odd}, then {odd}{odd}, and finally {even}{even}:

$$
\begin{aligned}
T_{\{\{1;2\};\{3;4\};\{5;6\}\}} &= m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) && (102)\\
T_{\{\{1;3\};\{2;4\};\{5;6\}\}} &= \mathcal{P}_{3,2}\mathcal{P}_{2,1}m_2(\mathbf{x}) \otimes \mathbf{vec}m_2(\mathbf{x}) \otimes \mathbf{vec}'m_2(\mathbf{x})\mathcal{P}_{2,1}\mathcal{P}_{3,2} && (103)\\
T_{\{\{2;3\};\{1;4\};\{5;6\}\}} &= \mathcal{P}_{2,1}m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) && (104)\\
T_{\{\{1;2\};\{3;5\};\{4;6\}\}} &= m_2(\mathbf{x}) \otimes \mathbf{vec}m_2(\mathbf{x}) \otimes \mathbf{vec}'m_2(\mathbf{x}) && (105)\\
T_{\{\{1;3\};\{2;5\};\{4;6\}\}} &= \mathcal{P}_{3,2}\mathcal{P}_{2,1}m_2(\mathbf{x}) \otimes \mathbf{vec}m_2(\mathbf{x}) \otimes \mathbf{vec}'m_2(\mathbf{x}) && (106)\\
T_{\{\{2;3\};\{1;5\};\{4;6\}\}} &= \mathcal{P}_{2,1}m_2(\mathbf{x}) \otimes \mathbf{vec}m_2(\mathbf{x}) \otimes \mathbf{vec}'m_2(\mathbf{x}) && (107)\\
T_{\{\{1;2\};\{4;5\};\{3;6\}\}} &= \mathcal{P}_{3,2}m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) && (108)\\
T_{\{\{1;4\};\{2;5\};\{3;6\}\}} &= \mathcal{P}_{3,2}\mathcal{P}_{2,1}m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) && (109)\\
T_{\{\{2;4\};\{1;5\};\{3;6\}\}} &= \mathcal{P}_{2,1}m_2(\mathbf{x}) \otimes \mathbf{vec}m_2(\mathbf{x}) \otimes \mathbf{vec}'m_2(\mathbf{x})\mathcal{P}_{2,1}\mathcal{P}_{3,2} && (110)\\
T_{\{\{1;3\};\{4;5\};\{2;6\}\}} &= \mathcal{P}_{3,2}\mathcal{P}_{2,1}m_2(\mathbf{x}) \otimes \mathbf{vec}m_2(\mathbf{x}) \otimes \mathbf{vec}'m_2(\mathbf{x})\mathcal{P}_{2,1} && (111)\\
T_{\{\{1;4\};\{3;5\};\{2;6\}\}} &= m_2(\mathbf{x}) \otimes \mathbf{vec}m_2(\mathbf{x}) \otimes \mathbf{vec}'m_2(\mathbf{x})\mathcal{P}_{2,1} && (112)\\
T_{\{\{3;4\};\{1;5\};\{2;6\}\}} &= \mathcal{P}_{2,1}m_2(\mathbf{x}) \otimes \mathbf{vec}m_2(\mathbf{x}) \otimes \mathbf{vec}'m_2(\mathbf{x})\mathcal{P}_{2,1} && (113)\\
T_{\{\{2;3\};\{4;5\};\{1;6\}\}} &= \mathcal{P}_{2,1}\mathcal{P}_{3,2}m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) && (114)\\
T_{\{\{2;4\};\{3;5\};\{1;6\}\}} &= m_2(\mathbf{x}) \otimes \mathbf{vec}m_2(\mathbf{x}) \otimes \mathbf{vec}'m_2(\mathbf{x})\mathcal{P}_{2,1}\mathcal{P}_{3,2} && (115)\\
T_{\{\{3;4\};\{2;5\};\{1;6\}\}} &= \mathcal{P}_{2,1}\mathcal{P}_{3,2}\mathcal{P}_{2,1}m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}). && (116)
\end{aligned}
$$

Therefore,

$$
\Gamma_{(2,2,2)}(\mathbf{x}) \quad = \quad \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2} + \mathcal{P}_{3,2}\mathcal{P}_{2,1} + \mathcal{P}_{2,1}\mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{Q}} m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x})
$$

$$
\begin{aligned}
&+ \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{P}} m_2(\mathbf{x}) \otimes \mathbf{vec} m_2(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x}) \\
&+ \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{P}} m_2(\mathbf{x}) \otimes \mathbf{vec} m_2(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x}) \mathcal{P}_{2,1} \\
&+ \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{P}} m_2(\mathbf{x}) \otimes \mathbf{vec} m_2(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x}) \mathcal{P}_{2,1}\mathcal{P}_{3,2} \\
= \quad & \mathbf{Q}\Big\{m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x})\Big\} + \mathbf{P}\Big\{m_2(\mathbf{x}) \otimes \mathbf{vec} m_2(\mathbf{x}) \otimes \mathbf{vec}' m_2(\mathbf{x})\Big\}\mathbf{P}'. \qquad (117)
\end{aligned}
$$

This completes the proof of theorem 2.

4. Two immediate consequences

4.1. Upper bounds on the rank of cumulant matrices

Corollary 3. *Let* $\mathbf{x}$ *be a real-valued random vector of dimension* $p \times 1$ *such that* $\mathbb{E}\left[|\mathbf{x}|^n\right] < \infty$ *for a given integer* $n \geq 6$*. Then, its cumulant matrices of orders* 2 *to* 6 *have their rank upper-bounded as follows:*

$$
\begin{aligned}
\mathcal{R}\Big(c_2(\mathbf{x})\Big) &\leq p && (118)\\
\mathcal{R}\Big(c_3(\mathbf{x})\Big) &\leq p && (119)\\
\mathcal{R}\Big(c_4(\mathbf{x})\Big) &\leq \frac{p(p+1)}{2} && (120)\\
\mathcal{R}\Big(c_5(\mathbf{x})\Big) &\leq \frac{p(p+1)}{2} && (121)\\
\mathcal{R}\Big(c_6(\mathbf{x})\Big) &\leq \frac{p(p+1)(p+2)}{6}. && (122)
\end{aligned}
$$

Proof. The first two inequalities are obvious. For the other ones, let us introduce

$$
\widetilde{\mathbf{K}} = \frac{1}{2}\mathbf{K} \qquad \text{and} \quad \widetilde{\mathbf{Q}} = \frac{1}{6}\mathbf{Q}, \qquad (123)
$$

where $\mathbf{K}$ and $\mathbf{Q}$ are matrices of dimension $p^2 \times p^2$ and $p^3 \times p^3$, as given in Eqs. (32) and (35), respectively. By using the symmetry properties of cumulants, it comes that

$$
c_4(\mathbf{x}) = \widetilde{\mathbf{K}} c_4(\mathbf{x}), \quad c_5(\mathbf{x}) = c_5(\mathbf{x})\widetilde{\mathbf{K}}, \quad \text{et} \quad c_6(\mathbf{x}) = \widetilde{\mathbf{Q}} c_6(\mathbf{x}), \qquad (124)
$$

from which the following majorations immediately follows:

$$
\mathcal{R}\Big(c_4(\mathbf{x})\Big) \leq \mathcal{R}\Big(\widetilde{\mathbf{K}}\Big), \quad \mathcal{R}\Big(c_5(\mathbf{x})\Big) \leq \mathcal{R}\Big(\widetilde{\mathbf{K}}\Big), \quad \text{and} \quad \mathcal{R}\Big(c_6(\mathbf{x})\Big) \leq \mathcal{R}\Big(\widetilde{\mathbf{Q}}\Big). \qquad (125)
$$

Since matrices $\widetilde{\mathbf{K}}$ and $\widetilde{\mathbf{Q}}$ are idempotent (see Schott [19, Theorem 1]), their rank is returned by their trace. Thus,

$$
\mathcal{R}\Big(\widetilde{\mathbf{K}}\Big) = \mathrm{Tr}(\widetilde{\mathbf{K}}) = \frac{1}{2}\mathrm{Tr}(\mathbf{I}_{p^2} + \mathbf{K}_{pp}) = \frac{1}{2}\left(p^2 + \mathrm{Tr}(\mathbf{K}_{pp})\right). \qquad (126)
$$

The trace of the commutation matrices is in given Magnus and Neudecker [18] as

$$\mathrm{Tr}(\mathbf{K}_{ab}) = 1 + \gcd(a-1, b-1); \tag{127}$$

where gcd stands for "greatest common divisor". This implies, in particular, $\mathrm{Tr}(\mathbf{K}_{pp}) = p$; therefore,

$$\mathcal{R}\left(\widetilde{\mathbf{K}}\right) = \frac{1}{2}\left(p^2 + p\right) = \frac{p(p+1)}{2}. \tag{128}$$

Similarly,

$$\begin{aligned}
\mathcal{R}\left(\widetilde{\mathbf{Q}}\right) &= \mathrm{Tr}(\widetilde{\mathbf{Q}}) = \frac{1}{6}\mathrm{Tr}(\mathbf{Q}) \\
&= \frac{1}{6}\mathrm{Tr}\left(\mathbf{I}_{p^3} + \mathbf{I}_p \otimes \mathbf{K}_{pp} + \mathbf{K}_{pp} \otimes \mathbf{I}_p + \mathbf{K}_{pp^2} + \mathbf{K}_{p^2p} + (\mathbf{K}_{pp} \otimes \mathbf{I}_p)\mathbf{K}_{p^2p}\right) \\
&= \frac{1}{6}\left(p^3 + p^2 + p^2 + p + p + p^2\right) \\
&= \frac{p^3 + 3p^2 + 2p}{6} = \frac{p(p+1)(p+2)}{6}.
\end{aligned} \tag{129}$$

□

4.2. The sixth-order moment matrix of a Gaussian vector

The expression of the sixth-order moment matrix of a Gaussian vector has been the aim of several research works. It has been provided by Neudecker [11], Magnus and Neudecker [18], Tracy and Sultan [14], and more recently by Schott [19]. It happens here to be a direct corollary of theorem 2.

Corollary 4. *Let* $\mathbf{z} \sim \mathcal{N}(\mathbf{0}_p, \mathbf{\Sigma})$ *be a zero-mean, real-valued, Gaussian vector. Its sixth-order moment matrix reads*

$$m_6(\mathbf{z}) = \mathbb{E}(\mathbf{z}\mathbf{z}' \otimes \mathbf{z}\mathbf{z}' \otimes \mathbf{z}\mathbf{z}') = \mathbf{Q}\left(\mathbf{\Sigma} \otimes \mathbf{\Sigma} \otimes \mathbf{\Sigma}\right) + \mathbf{P}\left(\mathbf{\Sigma} \otimes \mathbf{vec}\mathbf{\Sigma} \otimes \mathbf{vec}'\mathbf{\Sigma}\right)\mathbf{P}' \tag{130}$$

where matrices $\mathbf{Q}$ *and* $\mathbf{P}$ *are defined in theorem 2.*

Proof. Since $\mathbf{z}$ is Gaussian, all its cumulants of order greater than two are nil. Therefore, the only non-zero terms in Eq. (12), with $k = 6$, corresponds to the partition of type $(2, 2, 2)$. Hence,

$$m_6(\mathbf{z}) = \Gamma_{(2,2,2)}(\mathbf{x}), \tag{131}$$

where

$$\Gamma_{(2,2,2)}(\mathbf{x}) = \sum_{\pi \in P_{(2,2,2)}([\![1,6]\!])} \left(\sum_{(j_1,\ldots,j_6) \in [\![1,p]\!]^6} \prod_{B \in \pi} cum^{(2)}(x_{j_l} \ : \ l \in B)\mathbf{e}_{j_1}\mathbf{e}'_{j_2} \otimes \mathbf{e}_{j_3}\mathbf{e}'_{j_4} \otimes \mathbf{e}_{j_5}\mathbf{e}'_{j_6} \right). \tag{132}$$

This quantity is given by Eq. (117), wherein $m_2(\mathbf{x}) = c_2(\mathbf{x}) = \mathbf{\Sigma}$. □

5. Conclusion

This paper has introduced a methodology for deriving explicit formulae for the cumulant matrices of a random vector in terms of moment matrices of lower order. The methodology reproduces known formulae, up to order four, and returns new ones at order five and six, which are all embodied in theorem 2. In principle, it can also be applied to address higher orders. Such formulae of the cumulant matrices have several advantages. First, from the notational point of view, they are quite concise. Second, from the theoretical point of view, they evidence a direct link with moment matrices. As a consequence, the moment matrices of a random Gaussian vector – which have been of some interest in the literature – are easily found be setting the cumulant matrix to zero (e.g. see corollary (4)). In addition, they make possible to upper-bound the rank of the cumulant matrices, as described in corollary (3)). These properties have considerable importance in applications as such independent component analysis (ICA) [2], which strongly rely on the use of the cumulant matrices of order four - the Quadricovariance [4] – or six – the Hexacovariance Albera et al. [6]. In this case, the upper-bound on the rank of the cumulant matrices indicates the maximum number of independent sources that can be separated. Finally, the proposed formulae have a definite advantage when they come to be coded with high-level matrix/array programming language, such as Matlab®. By way of an example, it has been verified by the authors that the numerical computation of the cumulant matrices of orders five and six with dimension $p = 6$ are about 150 times faster than with coded with Leonov and Shiryaev formula (version 7 of Matlab used on a laptop computer with 2.10GHz clock and 4Go RAM).

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