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CONFIGURATIONAL MECHANICS: A TOOL TO INVESTIGATE FRACTURE AND FATIGUE OF RUBBER

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Our objective is twofold: (1) to offer a short state of the art on the framework of configurational mechanics to rubber researchers and engineers, and (2) to show that it is well adapted to rubber mechanics. The state of the art presents the general theory but also highlights the physical meaning of the Eshelby stress tensor and of its path integrals; then, we propose a list of recent applications involving configurational mechanics. Attention is then focused on rubber applications: recent work on the simulation of fracture in inelastic materials and on the development of predictors for fatigue crack initiation is presented.

I. INTRODUCTION

Since the pioneering works of Cadwell et al. for fatigue¹ and Rivlin and Thomas for fracture,² a number of studies have been proposed to investigate the durability of rubber parts. This has led to the definition of various quantities such as the tearing energy, the crack growth rate, and different fatigue life criteria (refs 3–5 and references therein). All these quantities are revealed to be quite efficient to analyze the experimental results and to predict phenomena under simple loading conditions.

Nevertheless, the generalization to three-dimensional situations, that is, 3D parts and multi-axial loading conditions, remains an open problem. It is partly due to the lack of a theoretically well-defined framework, and then to the lack of relevant mechanical quantities to investigate durability for complex problems. In my opinion, such a framework exists; it is called “configurational mechanics” and it is now sufficiently mature to be used by rubber researchers and engineers. The aim of this paper is to present the possibilities given by this theory, especially for elastomer problems.

In the following, a brief state of the art of this theory is proposed, emphasizing the physical significance of the new quantities introduced. Then, the recent applications of this framework to rubber problems are presented.

II. CONFIGURATIONAL MECHANICS

A. INITIAL MOTIVATION

The basic idea underlying the general theory was introduced by Eshelby in 1951.⁶ His objective was to quantify the change in energy in a body due to the change in position of a defect. As this seminal idea is essential to understand the relevance of this theory to investigate fracture and fatigue, I recall here the derivation proposed by Eshelby. In the 1951 paper the derivation was proposed for small strain, and the extension of the theory to large strain was proposed in 1975, simultaneously by Chadwick and Eshelby himself.^{7,8} As shown recently by Maugin⁹ and Steinmann,¹⁰ the notion of body configurations being essential in this theory, the large strain framework is adopted.

The derivation is explained in Figure 1. Consider a body in a given reference configuration as shown in Figure 1a; it contains a defect schematized by a white disk in the figure. Under loading, the body deforms as shown in Figure 1b. We focus on the calculation of the change in energy induced by a change in position of the defect *in the reference configuration*, that is, the change in energy between Figure 1a and d under loading. In this way, consider a contour S_0 that encircles the defect. Similarly, consider a replica of the body with another contour S'_0 as shown in Figure 1a'. First, the difference in energy inside the contour between the body and its replica is

$$\delta E = \int_{V(S'_0)} W dV - \int_{V(S_0)} W dV \quad (1)$$

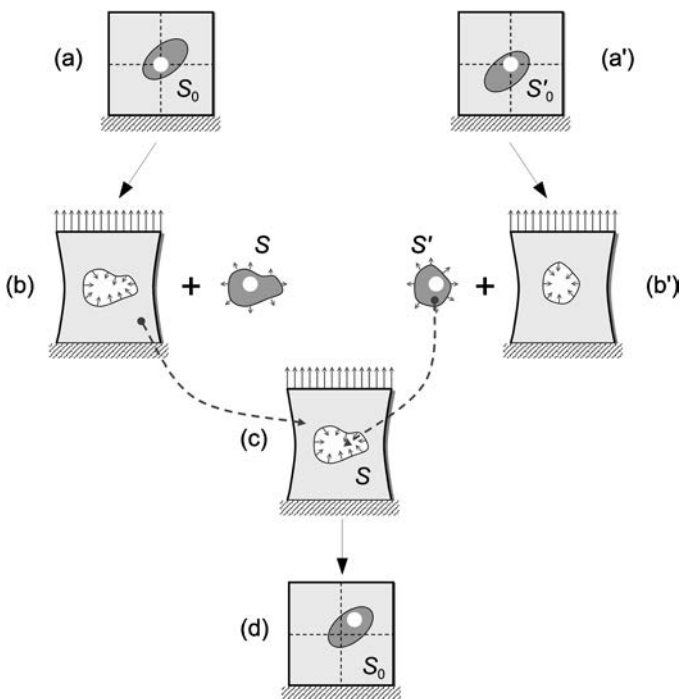


FIG. 1. — The Eshelby problem.

where W is the strain energy density per unit of undeformed volume and $V(S_0)$ [respectively, $V(S'_0)$] is the volume contained in the contour S_0 (respectively, S'_0). Denoting $-d\mathbf{U}$ the homogeneous small change in position of the contour in the reference configuration between S_0 and S'_0 , the change in energy in the contour can be written as

$$\delta E = \frac{d}{d\mathbf{X}} \int_{V(S_0)} W dV \cdot (-d\mathbf{U}) + \dots = -d\mathbf{U} \cdot \int_{S_0} W \mathbf{N} dS + \dots \quad (2)$$

where \mathbf{X} stands for the position in the reference configuration and \mathbf{N} is the outward normal vector to the contour. Second, following the great idea of Eshelby, we consider the following steps:

- in the deformed configuration of Figure 1b, the surface S (deformed counterpart of S_0) is cut out and discarded from the body. Suitable tractions are applied to the surface of the hole to prevent matter relaxation.
- A similar procedure is applied to the replica in the deformed configuration (Figure 1b').
- Finally, the part of the body bounded by S' is fitted into the hole S , as shown in Figure 1c.

Obviously, the deformed surfaces S and S' do not coincide, even if S_0 and S'_0 do so perfectly, by a simple translation in the reference configuration. Indeed, the displacement field $d\mathbf{u}$ that must be applied to the surface of the hole to make S and S' coincident is simply related to $d\mathbf{U}$ by

$$d\mathbf{u} = -\frac{\partial \mathbf{u}}{\partial X} d\mathbf{U} + \dots = -(\mathbf{F} - \mathbf{I}) d\mathbf{U} + \dots \quad (3)$$

in which \mathbf{F} and \mathbf{I} are, respectively, the deformation gradient tensor and the identity tensor. Applying this displacement field on the boundary of the hole leads to the following amount of work:

$$dP = -\int_S d\mathbf{u} \cdot \boldsymbol{\sigma} \mathbf{n} ds = d\mathbf{U} \left(\int_{S_0} \mathbf{F}^T \mathbf{P} \mathbf{N} dS - \int_{S_0} \mathbf{P} \mathbf{N} dS \right) \quad (4)$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor (true stress), \mathbf{n} is the normal vector to the contour S , and \mathbf{P} is the first Piola–Kirchhoff stress tensor (nominal stress). Considering a problem without body force, the integral of the nominal traction over the closed contour S_0 is null due to the equilibrium equation. The part S' can now replace S in the original system (Figure 1c) and the corresponding undeformed body corresponds to the original body in which the position of the defect has changed by $d\mathbf{U}$ as shown in Figure 1d. Finally, the total change in energy $d\Pi$ is simply given by the sum $dE + dP$, that is

$$d\Pi = -d\mathbf{U} \int_{S_0} (W\mathbf{I} - \mathbf{F}^T \mathbf{P}) \mathbf{N} dS \quad (5)$$

So, the change in energy is completely defined by a tensor, denoted $\boldsymbol{\Sigma}$ in the following:

$$\boldsymbol{\Sigma} = W\mathbf{I} - \mathbf{F}^T \mathbf{P} \quad (6)$$

which is referred to as the Eshelby stress tensor or the configurational stress tensor, or also the energy momentum tensor. Moreover, the configurational force (also referred to as the material force in literature) associated with the defect, that is, the thermodynamic force that drives the change in position of the defect, is defined arbitrarily as the negative gradient of the change in

energy

$$\mathbf{G} = \lim_{d\mathbf{U} \rightarrow 0} - \frac{d\Pi}{d\mathbf{U}} = \int_{S_0} \boldsymbol{\Sigma} \mathbf{N} dS \quad (7)$$

Note that this final result is the generalization of the widely known J -integral of Rice.¹¹ More precisely, if the defect considered is a straight crack, the thermodynamic force that drives the displacement of the crack tip in the direction of the crack is simply the component of \mathbf{G} in the direction of the crack and is equal to the J -integral. In the context of rubber mechanics, a similar derivation of the change in energy induced by the growth of a crack was proposed by Rivlin and Thomas in their seminal paper² to calculate the energy release rates of long cracks under different loading conditions. The similarity of the approaches of Eshelby and of Rivlin and Thomas is highlighted in the lecture notes of Eshelby¹² and also by Verron.¹³

B. GENERAL THEORY

In the previous section, the configurational stress tensor was introduced by following the derivation of Eshelby to calculate the change in energy induced by the change of material position in the body. This precursory idea was considered in the 1990s by different authors as the basic concept of a new framework called configurational mechanics. The general theory is now briefly presented. For another point of view of the theory and its applications, the reader can refer to the recent paper of Steinmann et al.¹⁴

What is referred to as configurational mechanics in this paper is also designated as “Eshelbian mechanics” by Maugin,¹⁵ and “mechanics in material space” by Kienzler and Herrmann¹⁶ in contrast to Newtonian mechanics or mechanics in physical space, respectively. Basically, the classical Newtonian mechanics focuses on physical forces induced by displacements in the physical space, that is, the 3D Euclidean space E_3 . The configurational mechanics theory is completely defined with respect to the material space M_3 , which is defined by Truesdell and Noll as the abstract set of material points that constitute the body.¹⁷ So, we deal with a different class of forces, referred to as configurational forces or material forces, which are generated by displacements not in the physical space but in this material space. There are different ways to derive the theory: while Gurtin considers material forces as new basic objects in mechanics,¹⁸ Maugin prefers to directly derive from classical continuum mechanics.⁹ Here the latter viewpoint is adopted.

We consider a hyperelastic body in its reference configuration (C_R) defined in the material space M_3 as shown in Figure 2. Under deformation, the body occupies a series of physical configurations defined in the classical physical space E_3 [from (C_0) to (C) in the figure]. As the body may be inhomogeneous, the strain energy density can depend on both the deformation gradient \mathbf{F} and the material position \mathbf{X} . Neglecting dynamic effects and body forces, balance equations of linear and angular momenta are classically expressed in the Eulerian description as

$$\begin{aligned} \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma} &= 0, \\ \boldsymbol{\sigma}^T &= \boldsymbol{\sigma} \end{aligned} \quad (8)$$

where \mathbf{x} is the position of the particle in the deformed configuration and $\boldsymbol{\sigma}$ is the Cauchy (true) stress tensor. Lagrangian counterparts of the balance laws eq 8 are easily derived by expressing the quantities per unit of surface and volume in the reference configuration of the body; it

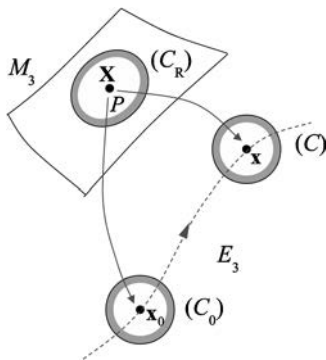


FIG. 2. — Motion in material and physical spaces.

leads to

$$\begin{aligned} \operatorname{div}_{\mathbf{X}} \mathbf{P} &= 0, \\ \mathbf{P} \mathbf{F}^T &= \mathbf{F} \mathbf{P}^T \end{aligned} \quad (9)$$

Generally, deformation induces microstructural changes or rearrangements in the material, for example, growth of microscopic defects, dislocation, or displacement of boundary phases. As argued by Steinmann,¹⁰ describing such rearrangement in the physical space is not an easy task. Thus, to overcome this difficulty, the balance of physical linear momentum has to be completely written onto the material space;⁹ in fact, the pull-back operation that introduced the first Piola–Kirchhoff (nominal) stress tensor \mathbf{P} from the true stress $\boldsymbol{\sigma}$ is only partial because the resulting expression in eq 9 still contains components defined in the current configuration. So, it is necessary to consider an additional pull-back operation in order to canonically project balance equations onto the material space M_3 . For the balance of linear momentum, it can be performed either by applying Noether’s theorem¹⁹ or by multiplying the left-hand side of eq 9₁ by \mathbf{F}^T .⁹ After some algebraic manipulations, this equation reduces to

$$\operatorname{div}_{\mathbf{X}} \boldsymbol{\Sigma} - \left. \frac{\partial W}{\partial \mathbf{X}} \right|_{\text{expl}} = 0 \quad (10)$$

with

$$\boldsymbol{\Sigma} = \mathbf{W} \mathbf{I} - \mathbf{F}^T \mathbf{P} = \mathbf{W} \mathbf{I} - \mathbf{C} \mathbf{S} \quad (11)$$

being the configurational stress tensor introduced above. In eq 10, the second left-hand-side term stands for the explicit differentiation with respect to the material position, that is, the material force vector associated with the inhomogeneity of the material. In eq 11, \mathbf{I} is the identity tensor, \mathbf{S} is the second Piola–Kirchhoff stress tensor, and \mathbf{C} is the right Cauchy–Green strain tensor, that is, $\mathbf{F}^T \mathbf{F}$. Moreover, considering the balance of physical angular momentum, the configurational stress tensor satisfies the following symmetry condition²⁰:

$$\boldsymbol{\Sigma} \mathbf{C} = \mathbf{C} \boldsymbol{\Sigma}^T \quad (12)$$

In the special case of a homogeneous material, the Eshelby stress tensor is divergence-free; moreover, if the material is isotropic, the tensor is symmetric.

To summarize this presentation of the configurational mechanics framework, we emphasize the following points:

- The theory is defined in the material space. It permits the study of motion in this space, that is, the change of reference configuration.
- The central quantity is the configurational stress tensor Σ that can be calculated a posteriori once the strain and the stress fields are known.
- This tensor is divergence-free if the material is homogeneous. As a consequence, the integral of the configurational traction vector over a closed contour is null if the matter enclosed in the contour is homogeneous.
- If the contour contains a flaw (crack tip, inclusion, hole, dislocation, and so on), the contour integral of the traction is equal to the configurational force exerted on the flaw, that is, the thermodynamic force that drives the change in position of the defect in the reference configuration.
- Finally, as argued by Maugin, configurational mechanics can be seen as a “tool to capture singularities.”⁹

C. APPLICATIONS OF THE CONFIGURATIONAL MECHANICS FRAMEWORK

Most of the studies involving configurational mechanics consider the material forces, that is, path integrals around inhomogeneities, because these forces are recognized as the driving forces of defect evolution. So, the configurational stress tensor only appears in the definition of surface tractions, that is, after contraction with the outward normal of the contour. Moreover, the majority of the papers focuses on fracture mechanics (see ref 10 and the references herein). Indeed, as mentioned above, configurational forces which apply on crack tip generalize the classical J -integral, but also the L - and M -integrals.²¹ Nevertheless, the well-established theory of configurational mechanics permits investigation of the fracture in complex frameworks: electromagnetism,²² thermoelasticity,²³ continuum damage,²⁴ and linear viscoelasticity and elastoplasticity.²⁵ Moreover, the configurational framework can also be used to investigate dislocations in metal,^{26,27} movement of interfaces in two-phase bodies,¹⁸ defects in piezoelectric material,¹⁹ and so on. Specific numerical procedures have been proposed recently to calculate the configurational forces in different problems.^{28–30}

Finally, an original application of this framework was recently proposed: the use of configurational forces to optimize finite element meshes in large strain problems. More precisely, the most relevant position of the nodes in a structure can be determined by verifying a posteriori the balance of the material momentum.^{31–33}

Although abundant literature exists regarding the material forces and their applications, few papers are devoted to the peculiar properties of the configurational stress tensor. First, note that the physical significance of the Cartesian components of Σ was first proposed in 1997 by Kienzler and Herrmann.³⁴ Following these authors, at a given material point, “[the ij -component] of the Eshelby tensor is the change in the total energy density at a point of an elastic continuum due to a material unit translation in x_j direction of a unit surface with normal in x_i –direction.” This definition has been extended recently to large strain and to arbitrary directions³⁵: if one considers a unit material surface of outward normal vector \mathbf{N} and a given unit material displacement vector $\boldsymbol{\theta}$, the quantity $\boldsymbol{\theta} \cdot \Sigma \mathbf{N}$ represents the change in total energy due to the change of reference configuration involved by the material displacement of the surface. Finally, local fracture criteria for small strain problems based on the components of the configurational stress tensor have been derived recently³⁶: in two dimensions, principal values and a von

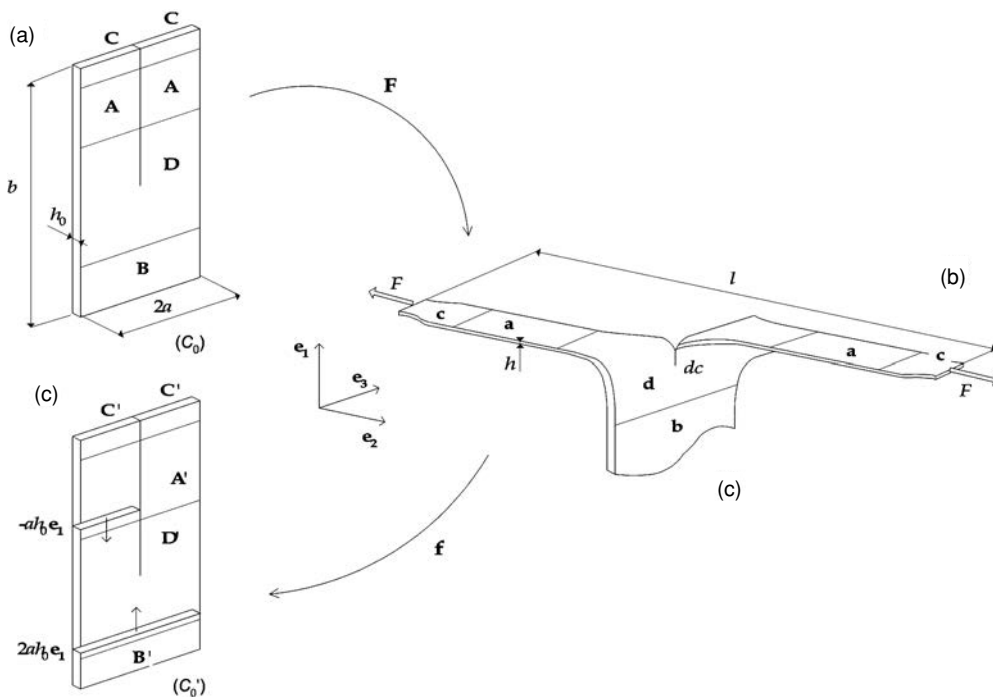


FIG. 3. — The trousers experiment and its corresponding configurations.

Mises-like value of this tensor can be related to the classical stress intensity factors of linear fracture mechanics.

III. APPLICATION TO RUBBER MECHANICS

In the last few years, the theoretical framework of configurational mechanics has been applied to two different problems of rubber mechanics: fracture and fatigue. Before examining these works and in order to illustrate the capabilities of the theory, we first revisit a well-known problem in fracture of rubber.

A. A SIMPLE EXAMPLE

In this section, we propose a new derivation for the calculation of the energy release rate of the trousers test sample. This sample is widely used in rubber mechanics. The result was detailed elsewhere;¹³ here we only recall the essence of the derivation.

Consider a rectangular thin rubber sheet of length b , width $2a$, and uniform thickness h_0 as shown in the reference configuration (C_0) in Figure 3a. A cut of length c ($\gg a$) is made in the sample, and the test consists of spreading the “legs of the trousers” and applying the force F to the legs until the crack length increases by dc as shown in the deformed configuration (C) in Figure 3b. The derivation of the tearing energy of this sample was first proposed by Rivlin and Thomas, who determined the change in energy involved by crack growth.² They established the following expression of the tearing energy:

$$T = -\frac{1}{h_0} \left. \frac{\partial w}{\partial c} \right|_l = 2F \frac{\lambda}{h_0} - 2Wa \quad (13)$$

where w is the strain energy, λ is the stretch ratio in the legs, that is, $l/2c$, and W is the strain energy density per unit of undeformed volume. The suffix $*|_l$ denotes that the differentiation is conducted with constant displacement of the boundaries where forces are applied.

To propose a new derivation founded on configurational mechanics, we consider the second undeformed configuration (C_0'), which is similar to (C_0) but with a crack length $c + dc$. As emphasized above, the Eshelby stress tensor permits determination of the change in energy between the two reference configurations (C_0) and (C_0'). Similar to Rivlin and Thomas, we define the following four regions:

- the regions **A** where deformed states **a** are substantially in uniaxial extension λ ,
- the region **B** where deformed state **b** is substantially undeformed, and
- the regions **C** and **D** where deformed states **c** and **d** are in complex states of stress and strain.

Considering now the change of reference configuration, these regions are transformed as follows (see Figure 3c):

- Regions **A** transform into **A'** by the material translation $-dce_1$ of the material surface $ah_0(-\mathbf{e}_1)$,
- Region **B** transforms into **B'** by the material translation $-dce_1$ of the material surface $2ah_0\mathbf{e}_1'$ and
- Regions **C** and **D** are transformed into regions **C'** and **D'** by a “material rigid body motion”: they are just displaced in the material space.

Recalling the physical significance of the Eshelby stress tensor stated above, the change in energy due to the material displacement $d\mathbf{U}$ of a material surface $dS_0\mathbf{N}_0$ is simply the scalar $d\mathbf{U}\boldsymbol{\Sigma}dS_0\mathbf{N}_0$. Then, we can easily conclude that the total change in energy between (C_0) and (C_0') reduces to the change in energy of regions **A**: with region **B** being undeformed, the Eshelby stress tensor (and then the corresponding change in energy) is null, and regions **C** and **D** being only displaced in the material space, the change in energy to transform them into **C'** and **D'** is also null. Thus

$$dw = dw_{A \rightarrow A'} = 2[-dce_1 \boldsymbol{\Sigma} ah_0(-\mathbf{e}_1)] = 2\Sigma_{11} ah_0 dc \quad (14)$$

where $\boldsymbol{\Sigma}$ is the Eshelby stress tensor in region **A**. As region **a** in Figure 3b is uniaxially stretched and assuming the incompressibility of the material, the corresponding deformation gradient and first Piola–Kirchhoff stress tensor are given by

$$\mathbf{F} = \lambda \mathbf{e}_2 \otimes \mathbf{e}_1 - \frac{1}{\sqrt{\lambda}} \mathbf{e}_1 \otimes \mathbf{e}_2 + \frac{1}{\sqrt{\lambda}} \mathbf{e}_3 \otimes \mathbf{e}_3 \quad (15)$$

and

$$\mathbf{P} = \frac{F}{ah_0} \mathbf{e}_2 \otimes \mathbf{e}_1 \quad (16)$$

And finally the Eshelby stress tensor in region **A** is

$$\begin{aligned} \boldsymbol{\Sigma} &= W(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) - \frac{F\lambda}{ah_0} \mathbf{e}_1 \otimes \mathbf{e}_1, \\ \Sigma_{11} &= W - \frac{F\lambda}{ah_0} \end{aligned} \quad (17)$$

Considering both legs of the trousers, the definition of the tearing energy (first equality in eq 13) and the change in energy eq 14, the expression of the energy release rate for the trousers test sample (right-hand side term in eq 13) is easily recovered.

This simple example demonstrates the ability of the theory to handle rubber problems in which both large strain and definition of configurations are of major importance.

B. FRACTURE OF INELASTIC MATERIALS

The first application concerns the numerical prediction of fracture in elastomers. These works are conducted by the research group of Kaliske from the Technische Universität Dresden (Germany). They consider the general framework of configurational mechanics to carefully compute the J -integral for inelastic materials in large strain.³⁷ The authors calculate the driving forces exerted on the crack. The separation between the material volume force induced by inelasticity and the crack driving force can be easily made with the general theory. Let us consider an inelastic material (viscoelastic or elastoplastic). For large strain problems, the classical multiplicative split of the deformation gradient is classically adopted

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_i \quad (18)$$

where \mathbf{F}_e is the elastic deformation gradient and \mathbf{F}_i the inelastic one. In this case, the balance of the energy momentum eq 10 contains an additional term

$$\operatorname{div}_{\mathbf{X}} \boldsymbol{\Sigma} - \left. \frac{\partial W}{\partial \mathbf{X}} \right|_{\text{expl}} + (\mathbf{F}_e^T \mathbf{P}) : \frac{\partial \mathbf{F}_i}{\partial \mathbf{X}} = 0 \quad (19)$$

This new term can be seen as a material volume force caused by inelasticity. It reflects the fact that due to inelasticity, the material becomes inhomogeneous under loading even if it was homogeneous previously (second term null in eq 19): due to inelastic deformation, material properties are changing inhomogeneously. In the case of plasticity, this term involves the well-known Mandel stress.

Applying this result in fracture mechanics of inelastic materials leads to the definition of three different material forces that can be calculated at the crack tip by integration over a contour surrounding the crack tip. The first one is the total force: it can be determined by integrating the material stress traction as proposed above by eq 7. Nevertheless, this material force is the sum of two different terms: the thermodynamic force that drives the material displacement of the crack tip and the one that drives the material motion in the neighborhood of the crack tip (also called the process zone). The first one corresponds to the classical J -integral

$$J = \lim_{\Gamma \rightarrow 0} \mathbf{e}_1 \int_{\Gamma} \boldsymbol{\Sigma} \mathbf{N} \, d\Gamma \quad (20)$$

where the crack direction is \mathbf{e}_1 . The contour size tends to zero in order not to superimpose the thermodynamic contributions of inelasticity to the crack driving force. The second one characterizes the evolution of the process zone and is obtained by integrating the additional volume force given in eq 19. The general theory of configurational mechanics permits one to precisely separate the respective contributions of dissipative processes and then to estimate well the crack-driving force, which is the relevant fracture mechanics parameter.

This approach has been successfully applied to the prediction of tire durability³⁸ and to the dwell effect.³⁹ Moreover, a recent paper by the same group proposes using the configurational forces to describe the microstructural evolution due to damage, by using a multiscale method.⁴⁰

C. DEVELOPMENT OF NEW PREDICTORS FOR FATIGUE LIFE

The second application recently investigated through the scope of configurational mechanics concerns the prediction of the fatigue life. This work was performed in the research group of the author in the Ecole Centrale de Nantes (France). The aim of the studies is the development of relevant predictors for fatigue crack nucleation that take into account the microstructural phenomena and are theoretically well established. By considering that the occurrence of macroscopic fatigue cracks is the consequence of the growth of pre-existing microscopic defects, the Eshelby stress tensor components can be considered as fatigue life predictors; more precisely, with basic energetic consideration, the smallest eigenvalue of the configurational stress tensor is able to quantify the energy release rate of small flaws.^{41,42} This theory has been thoroughly rationalized by Ait-Bachir in his Ph.D. thesis.⁴³ Coupled with hysteretic viscoelasticity, this approach has been successfully applied to the prediction of the fatigue life improvement of natural rubber under tensile/tensile cyclic loading conditions.⁴⁴ Moreover, the comparison with other recent predictors based on the critical plane method⁴⁵⁻⁴⁷ leads to very good agreement of the approach.⁴⁸ Introducing the polar decomposition of the Eshelby stress tensor, the extension of the method to anisotropic materials has been derived.^{49,50}

Finally, in the same research group and in collaboration with W. V. Mars, the general theory of configurational mechanics has been considered recently to investigate the behavior of small cracks. By using a systematic method of computation,⁴³ a novel interpretation of the energy release rate of a small crack in an infinite hyperelastic medium, under finite, multiaxial straining, has been proposed.^{51,52} The derivation of the energy release rate shows that this quantity can be seen as the unbalance of configurational stresses between the neighborhood of crack faces and the far field. The computation provides a means of bridging between continuum mechanical and fracture mechanical views of defect development in hyperelastic solids, and it is particularly suited to the requirements of critical plane analysis. Moreover, it also demonstrates the systematic proportionality between the energy release rate of a small crack and its length⁴³ as proposed in the past by Rivlin and Thomas² and Yeoh⁵³ for simple loading conditions.

IV. CONCLUDING REMARKS

To conclude, I am convinced that this theory can be a relevant tool to investigate fracture, defects behavior, fatigue, and other complex phenomena in elastomers. Indeed, it presents the following characteristics that are classically adopted in rubber mechanics:

- The difference between the reference and the actual configurations is a key point of the theory. This difference is one of the major prerequisites to investigate large strain problems.
- The theory is well defined in elasticity, and especially in hyperelasticity, that is, when the existence of a strain energy function is postulated. Basics of rubber mechanics are founded on hyperelasticity.
- The theory is sufficiently general to consider inelastic effects. Complex phenomena involved in rubber, that is, viscoelasticity, Mullins effect, and strain-induced crystallization, can be investigated with in this framework.

V. ACKNOWLEDGEMENTS

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