



# On the wave operator for dissipative potentials with small imaginary part

Xue Ping Wang, Lu Zhu

## ► To cite this version:

Xue Ping Wang, Lu Zhu. On the wave operator for dissipative potentials with small imaginary part. Asymptotic Analysis, 2014, 86, pp.49-57. hal-01005797

**HAL Id: hal-01005797**

**<https://hal.science/hal-01005797>**

Submitted on 13 Jun 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON THE WAVE OPERATOR FOR DISSIPATIVE POTENTIALS WITH SMALL IMAGINARY PART

XUE PING WANG AND LU ZHU

**ABSTRACT.** We determine the range of the incoming wave operator for the pair of operators  $(-\Delta, -\Delta + V_1(x) - i\epsilon V_2(x))$  on  $L^2(\mathbb{R}^n)$  under the conditions  $n \geq 3$  and 0 is a regular point of  $-\Delta + V_1$ ,  $V_2 \geq 0$  and  $\epsilon > 0$  is small enough. This implies that the dissipative scattering operator is bijective.

## 1. INTRODUCTION

The quantum scattering for non-selfadjoint operators appears in many physical situations such as optical model of nuclear scattering ([6]). Its Hilbert space theory is studied in [8, 9] and [3, 4, 5, 12]. See also [1, 2, 7]. In particular, one can construct the scattering operator for a pair of operators  $(H, H_0)$  where  $H_0$  is selfadjoint and  $H$  is maximally dissipative, if the perturbation is of short-range in Enss' sense. Several equivalent conditions for the asymptotic completeness of dissipative quantum scattering are discussed in [4]. However, to our knowledge, there is still no result on the asymptotic completeness itself in this framework. The purpose of this work is to give a result in this direction under some conditions.

Let  $H = -\Delta + V(x)$  be the Schrödinger operator with a dissipative potential  $V$ , which means that  $V = V_1 - iV_2$ , where  $V_1$  and  $V_2$  are real functions satisfying  $V_2(x) \geq 0$  and  $V_2(x) > 0$  on some non-trivial open set. Suppose that

$$|V_j(x)| \leq C\langle x \rangle^{-\rho_0}, \quad x \in \mathbb{R}^n, \quad (1.1)$$

for some  $\rho_0 > 1$ . Here  $\langle x \rangle = (1 + |x|^2)^{1/2}$ . Mild local singularities can be included with little additional effort. Denote  $H_0 = -\Delta$  and  $H_1 = -\Delta + V_1$ .  $H$  defined on  $D(-\Delta)$  is maximally dissipative and the

---

2000 *Mathematics Subject Classification.* 35J10, 35P15, 47A55, 81Q10.

*Key words and phrases.* Dissipative quantum scattering, asymptotic completeness, non-selfadjoint Schrödinger operators.

Research supported in part by French National Research Project NOSEVOL and by Qian Ren programme at Nanjing University.

numerical range of  $H$  is contained in  $\{z; \Re z \geq -R, -R \leq \Im z \leq 0\}$  for some  $R > 0$ . The wave operators

$$W_-(H, H_0) = \text{s-}\lim_{t \rightarrow -\infty} e^{itH} e^{-itH_0} \quad (1.2)$$

$$W_+(H_0, H) = \text{s-}\lim_{t \rightarrow +\infty} e^{itH_0} e^{-itH} \quad (1.3)$$

exist on  $L^2(\mathbb{R}^n)$  and on  $\mathcal{H}_{ac}$ , respectively, where  $\mathcal{H}_{ac}$  is the closure of the subspace

$$\mathcal{M}(H) = \{f \in L^2; \exists C_f \text{ s.t. } \int_0^\infty |\langle e^{-itH} f, g \rangle|^2 dt \leq C_f \|g\|^2, \forall g \in L^2\}.$$

See [4, 12]. It is known that  $\text{Ran } W_-(H, H_0) \subset \mathcal{H}_{ac}$  (see Lemma 2 of [4]). The dissipative scattering operator  $S(H, H_0)$  for the pair  $(H, H_0)$  is then defined as

$$S(H, H_0) = W_+(H_0, H) W_-(H, H_0). \quad (1.4)$$

$W_+(H_0, H)$  should be compared with the adjoint of the outgoing wave operator in selfadjoint cases, because for the pair of selfadjoint operators  $(H_1, H_0)$ , the scattering operator  $\tilde{S}(H_1, H_0)$  is defined as

$$\tilde{S}(H_1, H_0) = W_+(H_1, H_0)^* W_-(H_1, H_0).$$

A fundamental question for quantum scattering for a pair of selfadjoint operators is to study the asymptotic completeness of wave operators which implies that the scattering operator is unitary. In dissipative quantum scattering, the scattering operator  $S(H, H_0)$  is a contraction:  $\|S(H, H_0)\| \leq 1$ . The completeness of dissipative scattering can be interpreted as the bijectivity of  $S(H, H_0)$ . The equivalence of the following two conditions is due to E. B. Davies (Theorem 7, [4]):

- (1) The range of  $W_-(H, H_0)$  is closed;
- (2) The scattering operator  $S(H, H_0)$  is bijective on  $L^2$ .

In fact, E.B. Davies proves more general results in an abstract setting which can be applied to our case under the assumption (1.1) with  $\rho_0 > 1$ .

In this work, we study the dissipative quantum scattering under the assumption that the imaginary part of the potential is small. Let  $H(\epsilon) = -\Delta + V_1(x) - i\epsilon V_2(x)$ , where  $\epsilon > 0$  is a small parameter and  $V_j$  satisfies the condition (1.1) with  $V_2 \geq 0$  and  $V_2 \neq 0$ . Denote

$$W_-(\epsilon) = W_-(H(\epsilon), H_0) \text{ and } S(\epsilon) = S(H(\epsilon), H_0)$$

the wave and scattering operators defined as above with  $H = H(\epsilon)$ .

**Theorem 1.1.** *Assume the condition (1.1) with  $\rho_0 > 2$  and  $n \geq 3$ . Suppose that 0 is neither an eigenvalue nor a resonance of  $H_1$ . Then one for some  $\epsilon_0 > 0$*

$$\text{Ran } W_-(\epsilon) = \text{Ran } \Pi'(\epsilon), \quad 0 < \epsilon \leq \epsilon_0, \quad (1.5)$$

where  $\Pi'(\epsilon) = 1 - \Pi(\epsilon)$  and  $\Pi(\epsilon)$  is the Riesz projection associated with discrete spectrum of  $H(\epsilon)$ .

Theorem 1.1 can be compared with the asymptotic completeness of wave operators in the selfadjoint case which says that

$$\text{Ran } W_{\pm}(H_1, H_0) = \text{Ran } \Pi_{ac},$$

where  $\Pi_{ac}$  is the projection onto the absolutely continuous spectra subspace of  $H_1$ . Under the condition  $\rho_0 > 2$ ,  $\Pi(\epsilon)$  is of finite rank and  $\text{Ran } \Pi'(\epsilon) = \text{Ker } \Pi(\epsilon)$  is closed. As consequence of Theorem 1.1 and Theorem 7 of [4], the scattering operator  $S(\epsilon)$  is bijective for  $\epsilon > 0$  small enough. Consequently, the dynamics of the semigroup of contractions can be described explicitly as follows. For any  $f \in L^2$ , one can decompose it as  $f = f_1 + f_2$  with  $f_1 \in \text{Ran } \Pi(\epsilon)$  and  $f_2 \in \text{Ran } \Pi'(\epsilon)$ . Since  $H(\epsilon)$  has a finite number of eigenvalues, all with negative imaginary part,  $e^{-itH(\epsilon)}f_1$  decreases exponentially as  $t \rightarrow +\infty$ . The existence of the scattering operator  $S(\epsilon)$  implies that there exists  $f_{\infty} \in L^2$  such that

$$\lim_{t \rightarrow +\infty} \|e^{-itH(\epsilon)}f_2 - e^{-itH_0}f_{\infty}\| = 0 \quad (1.6)$$

and the asymptotic completeness of the wave operator  $W_-(\epsilon)$  ensures that  $f_{\infty} \neq 0$  if  $f_2 \neq 0$ . For any  $f \in L^2$ ,  $t \rightarrow \|e^{-itH(\epsilon)}f\|$  is decreasing in  $t > 0$ . Theorem 1.1 shows that either  $\|e^{-itH(\epsilon)}f\|$  decreases exponentially (when  $f \in \text{Ran } \Pi(\epsilon)$ ) or it tends to some non-zero limit as  $t$  goes to the infinity (when  $f \notin \text{Ran } \Pi(\epsilon)$ ).

The proof of Theorem 1.1 is based on a uniform global limiting absorption principle for the resolvent of  $H(\epsilon)$  on the range of  $\Pi'(\epsilon)$  which is proved in Section 2. By the technique of selfadjoint dilation for dissipative operators, this gives a uniform Kato smoothness estimate for the semigroup  $e^{-itH(\epsilon)}$ . The condition that 0 is neither an eigenvalue nor a resonance of  $H_1$  is necessary for such uniform estimates. In Section 3, we identify the range of  $W_-(\epsilon)$  for  $\epsilon > 0$  small, making use of the asymptotic completeness of the wave operators for the selfadjoint pair  $(H_1, H_0)$ .

## 2. SOME RESOLVENT ESTIMATES

Let  $R(z, \epsilon) = (H(\epsilon) - z)^{-1}$ ,  $z \notin \sigma(H(\epsilon))$  and  $R_j(z) = (H_j - z)^{-1}$ ,  $j = 0, 1$ . Denote  $L^{2,s} = L^2(\mathbb{R}^n; \langle x \rangle^s dx)$  the weighted  $L^2$ -space and  $\|f\|_s = \|f\|_{L^{2,s}}$ . If no confusion is possible in the context, we denote  $\|\cdot\|$  both the norm of functions in  $L^2$  and the operator norm for bounded operators on  $L^2$ .

It is well known that for  $V_1$  satisfying (1.1) with  $\rho_0 > 2$ ,  $H_1 = -\Delta + V_1(x)$  has only a finite number of eigenvalues. Assume in addition that  $n \geq 3$  and 0 is neither an eigenvalue and nor a resonance of  $H_1$ , one can show that the number of eigenvalues (counted according to their algebraic multiplicities) of  $H(\epsilon)$  is equal to that of  $H_1$  when  $\epsilon > 0$  is small enough. More precisely, let  $\lambda_1 < \lambda_2 < \cdots < \lambda_l < 0$  be the distinct eigenvalues of  $H_1$ ,  $\lambda_j$  being of multiplicity  $n_j$ . Let  $N_1 = \sum_{j=1}^l n_j$  be the number of eigenvalues of  $H$ . Then the number of eigenvalues of  $H(\epsilon)$  is equal to  $N_1$  and are located inside

$$\mathcal{F} = \cup_{j=1}^l \mathcal{F}_j$$

where  $\mathcal{F}_j = \{z; |z - \lambda_j| < C\epsilon, -C\epsilon \leq \Im z \leq -c\epsilon\}$  for some  $C, c > 0$  ([15]). Denote by  $\Pi_j$  the spectral projection of  $H_1$  associated with  $\lambda_j$  and  $\Pi_j(\epsilon)$  the Riesz projection of  $H(\epsilon)$  associated with eigenvalues near  $\lambda_j$ :

$$\Pi_j(\epsilon) = \frac{1}{2\pi i} \int_{|z - \lambda_j| = \delta} (z - H(\epsilon))^{-1} dz,$$

where  $\delta > 0$  is small enough and is fixed.

Recall that for each fixed  $\epsilon > 0$ , there are no real eigenvalues of  $H(\epsilon)$  and 0 is not a resonance if  $n \geq 3$  ([15]). The limit

$$R(\lambda + i0, \epsilon) = \lim_{\delta \rightarrow 0_+} (H(\epsilon) - (\lambda + i\delta))^{-1}$$

is well defined for any  $\lambda \in \mathbb{R}$  as operators from  $L^{2,s}$  to  $L^{2,-s}$  with  $s > 1$ . See [11] for  $\lambda > 0$  and [15] for  $\lambda$  near 0. The purpose of this Section is to prove the following

**Theorem 2.1.** *Under the assumptions of Theorem 1.1, one has the uniform global resolvent estimate*

$$\|\langle x \rangle^{-s} \Pi'(\epsilon) R(\lambda + i0, \epsilon) \Pi'(\epsilon) \langle x \rangle^{-s}\| \leq C_s \langle \lambda \rangle^{-1/2}, \lambda \in \mathbb{R} \quad (2.7)$$

uniformly in  $\epsilon$ . Here  $\Pi'(\epsilon) = 1 - \Pi(\epsilon)$ ,  $\Pi(\epsilon) = \sum_j \Pi_j(\epsilon)$  being the Riesz projection of  $H(\epsilon)$  associated to  $\sigma_{\text{disc}}(H(\epsilon))$ .

Remark that this uniform estimate is not true if 0 is an eigenvalue or a resonance of  $H_1$ , because then  $H(\epsilon)$  will have complex eigenvalues

near 0 with imaginary part of the order  $O(\epsilon)$ . The resolvent  $R(z, \epsilon)$  still blows up even if one projects out the range of the associated Riesz projection. The proof of Theorem 2.1 is divided into several steps.

**Lemma 2.2.** *For  $s, s' \in \mathbb{R}$  with  $s + s' \leq \rho_0$ , one has*

$$\|\langle x \rangle^s (\Pi_j(\epsilon) - \Pi_j) \langle x \rangle^{s'}\| \leq C\epsilon, 0 < \epsilon \leq \epsilon_0 \quad (2.8)$$

**Proof.** On the circle  $|z - \lambda_j| = \delta$  with  $\delta > 0$  small enough, one has  $\|R_1(z)\| \leq \delta^{-1}$ . From the equation  $R(z, \epsilon) = R_1(z)(1 - i\epsilon V_2 R_1(z))^{-1}$ , one deduces that

$$\|R(z, \epsilon)\| \leq C\delta^{-1}$$

uniformly for  $|z - \lambda_j| = \delta$  and  $0 < \epsilon \leq \epsilon_0$ . By a successive commutator technique, one deduces that for any  $s \in \mathbb{R}$ ,

$$\|\langle x \rangle^s R(z, \epsilon) \langle x \rangle^{-s}\| \leq C_{s, \delta} \quad (2.9)$$

uniformly for  $|z - \lambda_j| = \delta$  and  $0 < \epsilon \leq \epsilon_0$ . (2.8) follows from the relation

$$\Pi_j(\epsilon) - \Pi_j = -\frac{\epsilon}{2\pi} \int_{|z - \lambda_j| = \delta} R(z, \epsilon) V_2 R_1(z) dz \quad (2.10)$$

and the decay assumption on  $V_2$ . ■

**Lemma 2.3.** *Let  $0 < c_0 < -\lambda_l$ . Under the assumption of Theorem 1.1, one has for any  $s > 1$*

$$\|\langle x \rangle^{-s} R(\lambda + i0, \epsilon) \langle x \rangle^{-s}\| \leq C_s \langle \lambda \rangle^{-\frac{1}{2}}, \quad \lambda > -c_0, \quad (2.11)$$

uniformly in  $\epsilon \in ]0, \epsilon_0]$ . Here

$$R(\lambda + i0, \epsilon) = \lim_{\delta \rightarrow 0_+} (H(\epsilon) - (\lambda + i\delta))^{-1}$$

**Proof.** Since 0 is a regular point of  $H_1$ , it is well known that

$$\|\langle x \rangle^{-s} R_1(\lambda + i0) \langle x \rangle^{-s}\| \leq C \langle \lambda \rangle^{-\frac{1}{2}}, \quad \lambda > -c_0, \quad (2.12)$$

for any  $s > 1$ , where

$$R_1(\lambda + i0) = \lim_{\delta \rightarrow 0_+} (H_1 - (\lambda + i\delta))^{-1}$$

Since  $\rho_0 > 2$ ,  $V_2 R_1(\lambda + i0)$  is uniformly bounded on  $L^{2,s}$  if  $1 < s < \rho_0/2$ . (2.11) follows from the equation

$$R(\lambda + i0, \epsilon) = R_1(\lambda + i0)(1 - i\epsilon V_2 R_1(\lambda + i0, \epsilon))^{-1}$$

for  $\epsilon_0 > 0$  small enough. ■

**Lemma 2.4.** *One has*

$$\|R(\lambda, \epsilon)\| \leq C\epsilon^{-1} \langle \lambda \rangle^{-1} \quad (2.13)$$

for  $\lambda \leq -c_0$  and  $\epsilon \in ]0, \epsilon_0]$ .

**Proof.** To prove (2.13), it suffices to prove the estimate for  $\lambda$  near some eigenvalue  $\lambda_j$  of  $H_1$ . Let  $\Pi_j$  denote the spectral projection of  $H_1$  associated with the eigenvalue  $\lambda_j$ .  $(1 - \Pi_j)R_1(z)(1 - \Pi_j)$  is holomorphic in the region  $|\Re z - \lambda_j| \leq \delta$  for  $\delta > 0$  small enough. The resolvent equation shows that

$$E(z, \epsilon) = (1 - \Pi_j)R(z, \epsilon)(1 - \Pi_j) \quad (2.14)$$

defined for  $\Im z \geq 0$  and  $\Re z \leq -c_0$  has a holomorphic extension into the region  $|\Re z - \lambda_j| \leq \delta$  and  $\Im z > -1$  and is uniformly bounded in  $z$  and  $0 < \epsilon \leq \epsilon_0$ . One can check the following Feshbach-Grushin identity:

$$R(z, \epsilon) = E(z, \epsilon) - (1 + i\epsilon E(z, \epsilon)V_2)\Pi_j(E_{-+}(z, \epsilon))^{-1}\Pi_j(1 + i\epsilon V_2 E(z, \epsilon)) \quad (2.15)$$

for  $z \notin \sigma(H(\epsilon))$ , where

$$E_{-+}(z, \epsilon) = \Pi_j(z - \lambda_j + i\epsilon V_2 - \epsilon^2 V_2 E(z, \epsilon)V_2)\Pi_j \quad (2.16)$$

Since  $V_2 \geq 0$  and  $V_2(x) > 0$  in a nontrivial open set,  $\Pi_j V_2 \Pi_j$  is positively definite on  $\text{Ran } \Pi_j$ . Let  $\mu_1$  denote the smallest eigenvalue of  $\Pi_j V_2 \Pi_j$  on  $\text{Ran } \Pi_j$ . One has  $\mu_1 > 0$ . From the estimate

$$\Im E_{-+}(z, \epsilon) = \Pi_j(\Im z + \epsilon V_2 + O(\epsilon^2))\Pi_j, \quad \lambda \in \mathbb{R},$$

one deduces that for any  $c_1 < \mu_1$ , one has for some  $c > 0$

$$\Im \langle E_{-+}(z, \epsilon)f, f \rangle \geq c\epsilon \|f\|^2, \quad f \in \text{Ran } \Pi_j,$$

uniformly in  $\Im z \geq -c_1\epsilon$ ,  $|\Re z - \lambda_j| \leq \delta$  and  $\epsilon \in ]0, \epsilon_0]$ . It follows that for  $z$  in this region,

$$\|(E_{-+}(z, \epsilon))^{-1}\Pi_j\| \leq C\epsilon^{-1} \quad (2.17)$$

Estimate (2.13) follows from (2.15) when  $\lambda$  is near some  $\lambda_j$ . When  $\lambda < -c_0$  and  $|\lambda - \lambda_j| \geq \delta > 0$  for all  $1 \leq j \leq l$ , estimate (2.13) comes from the trivial resolvent estimate for  $H_1$

$$\|R_1(\lambda)\| \leq C\langle \lambda \rangle^{-1}$$

in this region and an argument of perturbation. ■

**Proof of Theorem 2.1.** Denote

$$\Pi_j(\epsilon) - \Pi_j = \epsilon S_j(\epsilon) \quad (2.18)$$

Then Lemma 2.2 says that for any  $s \in \mathbb{R}$ ,  $S_j(\epsilon) : L^{2,s} \rightarrow L^{2,s+\rho_0}$  is uniformly bounded. Let  $\Pi_{ac}$  denote the spectral projection of  $H_1$  onto its absolutely continuous subspace and  $\Pi_d = \sum_{j=1}^l \Pi_j$ . Since the

singular continuous spectrum of  $H_1$  is absent, one has  $\Pi_{ac} = 1 - \Pi_d$ . In addition,

$$\Pi'(\epsilon) - \Pi_{ac} = -\epsilon S(\epsilon), \quad \text{where } S(\epsilon) = \sum_{j=1}^l S_j(\epsilon). \quad (2.19)$$

With these notations, one has

$$R(\lambda, \epsilon)\Pi'(\epsilon) = -\epsilon R(\lambda, \epsilon)S(\epsilon) + (R_1(\lambda) + i\epsilon R(\lambda, \epsilon)V_2 R_1(\lambda))\Pi_{ac}. \quad (2.20)$$

The Spectral Theorem for the selfadjoint operator  $H_1$  gives

$$\|\Pi_{ac}R_1(\lambda)\| \leq C\langle\lambda\rangle^{-1}, \quad \lambda \leq -c_0.$$

Here  $c_0 > 0$  is fixed as in Lemma 2.3. Lemma 2.4 and Equation (2.20) show that

$$\|R(\lambda, \epsilon)\Pi'(\epsilon)\| \leq C\langle\lambda\rangle^{-1}, \quad \lambda \leq -c_0, \quad (2.21)$$

uniformly in  $\epsilon > 0$ . By the exponential decay of the eigenfunctions of  $H_1$  associated with the eigenvalues  $\lambda_j < -c_0$ ,  $\Pi_d$  is continuous from  $L^{2,-s}$  to  $L^{2,s}$  for any  $s \in \mathbb{R}$ . Lemma 2.2 shows that for any  $s, s' \in \mathbb{R}$  with  $s + s' \leq \rho_0$  one has

$$\|\langle x \rangle^{s'} \Pi(\epsilon) \langle x \rangle^s\| \leq C_{s,s'} \quad (2.22)$$

Using Lemma 2.3, one obtains for  $1 < s \leq \rho_0/2$

$$\|\langle x \rangle^{-s} \Pi'(\epsilon) R(\lambda + i0, \epsilon) \Pi'(\epsilon) \langle x \rangle^{-s}\| \leq C_s \langle \lambda \rangle^{-\frac{1}{2}}, \quad \lambda > -c_0, \quad (2.23)$$

uniformly in  $\epsilon \in ]0, \epsilon_0]$ . Estimate (2.7) follows from (2.21) and (2.23). ■

The following Kato smoothness estimate for the semigroup of contractions is the main ingredient to prove Theorem 1.1.

**Corollary 2.5.** *Under the conditions of Theorem 2.1, one has, for  $s > 1$ ,*

$$\int_0^\infty \|\langle x \rangle^{-s} \Pi'(\epsilon) e^{-itH(\epsilon)} f\|^2 dt \leq C \|f\|^2, \quad \forall f \in L^2, \quad (2.24)$$

*uniformly in  $\epsilon \in ]0, \epsilon_0]$ .*

(2.24) follows from (2.7) and Proposition 2.2 of [16]. See also [11] for some special case. In fact, using the high energy estimate in (2.7) and Proposition 2.2 of [16], one can obtain a slightly better smoothness estimate:  $\forall s > 1, \exists C_s$  such that

$$\int_0^\infty \|\langle x \rangle^{-s} \langle D_x \rangle^{1/2} \Pi'(\epsilon) e^{-itH(\epsilon)} f\|^2 dt \leq C_s \|f\|^2, \quad \forall f \in L^2, \quad (2.25)$$



uniformly in  $0 < \epsilon \leq \epsilon_0$ . Since 0 is a regular point of  $H_1$ , an estimate similar to (2.24) also holds for  $H_1$ :

$$\int_{-\infty}^{\infty} \|\langle x \rangle^{-s} \Pi_{ac} e^{-itH_1} f\|^2 dt \leq C_s \|f\|^2, \quad \forall f \in L^2, s > 1. \quad (2.26)$$

### 3. PROOF OF THEOREM 1.1

**Proof of Theorem 1.1.** To prove Theorem 1.1, one remarks firstly that since the eigenvalues of  $H(\epsilon)$  are all of negative imaginary part, one has

$$\text{s-}\lim_{t \rightarrow +\infty} \Pi(\epsilon) e^{-itH(\epsilon)} = 0,$$

which implies that  $\Pi(\epsilon)W_-(H(\epsilon), H_0) = 0$ . Therefore one has

$$\Pi'(\epsilon)W_-(H(\epsilon), H_0) = W_-(H(\epsilon), H_0)$$

Decompose  $W_-(H(\epsilon), H_0)$  as

$$W_-(H(\epsilon), H_0) = \Pi'(\epsilon)(1 - \epsilon K(\epsilon))W_-(H_1, H_0),$$

where

$$K(\epsilon) = \int_0^{\infty} e^{-itH(\epsilon)} \Pi'(\epsilon) V_2 e^{itH_1} \Pi_{ac} dt.$$

By the asymptotic completeness of  $W_-(H_1, H_0)$ , one has

$$\text{Ran } W_-(H_1, H_0) = \text{Ran } \Pi_{ac}. \quad (3.27)$$

According to (2.24) and (2.26), one has for  $s = \rho_0/2 > 1$

$$\begin{aligned} & |\langle K(\epsilon)f, g \rangle| \\ & \leq C \left\{ \int_0^{\infty} \|\langle x \rangle^{-s} \Pi'(\epsilon) e^{-itH(\epsilon)} g\|^2 dt \right\}^{1/2} \left\{ \int_0^{\infty} \|\langle x \rangle^{-s} \Pi_{ac} e^{-itH_1} f\|^2 dt \right\}^{1/2} \\ & \leq C' \|f\| \|g\| \end{aligned}$$

uniformly in  $\epsilon > 0$  small. This proves that  $K(\epsilon)$  is uniformly bounded on  $L^2$ . Recall that  $\Pi'(\epsilon) - \Pi_{ac} = -\epsilon S(\epsilon)$ . Since

$$\Pi'(\epsilon)\Pi_{ac} = \Pi'(\epsilon)(1 + \Pi_{ac} - \Pi'(\epsilon)) = (1 + \Pi'(\epsilon) - \Pi_{ac})\Pi_{ac},$$

one has

$$\begin{aligned} & \Pi'(\epsilon)(1 - \epsilon K(\epsilon))\Pi_{ac} \\ & = \Pi'(\epsilon)(1 + \epsilon S(\epsilon) - \epsilon K(\epsilon)\Pi_{ac}) \\ & = (1 - \epsilon S(\epsilon) - \epsilon \Pi'(\epsilon)K(\epsilon))\Pi_{ac}. \end{aligned}$$

$1 + \epsilon S(\epsilon) - \epsilon K(\epsilon)\Pi_{ac}$  and  $1 - \epsilon S(\epsilon) - \epsilon \Pi'(\epsilon)K(\epsilon)$  are invertible on  $L^2$  with bounded inverse for  $\epsilon > 0$  small enough. We claim that

$$\Pi'(\epsilon)(1 - \epsilon K(\epsilon))\Pi_{ac} : \text{Ran } \Pi_{ac} \rightarrow \text{Ran } \Pi'(\epsilon) \quad (3.28)$$

is bijective for  $\epsilon > 0$  small enough. In fact, if  $g \in \text{Ran } \Pi_{ac}$  such that  $\Pi'(\epsilon)(1 - \epsilon K(\epsilon))\Pi_{ac}g = 0$ , then  $(1 - \epsilon S(\epsilon) - \epsilon \Pi'(\epsilon)K(\epsilon))g = 0$ , therefore  $g = 0$ . This proves  $\Pi'(\epsilon)(1 - \epsilon K(\epsilon))\Pi_{ac}$  is injective for  $\epsilon > 0$  small enough. For  $f \in \text{Ran } \Pi'(\epsilon)$ , take

$$g = (1 + \epsilon S(\epsilon) - \epsilon K(\epsilon)\Pi_{ac})^{-1}f.$$

Then

$$\Pi'(\epsilon)(1 - \epsilon K(\epsilon))\Pi_{ac}g = \Pi'(\epsilon)f = f.$$

This proves that

$$\Pi'(\epsilon)(1 - \epsilon K(\epsilon))\Pi_{ac} : \text{Ran } \Pi_{ac} \rightarrow \text{Ran } \Pi'(\epsilon)$$

is surjective, hence bijective for  $\epsilon > 0$  small enough. Since

$$\text{Ran } W_-(H_1, H_0) = \text{Ran } \Pi_{ac},$$

it follows that  $\text{Ran } W_-(H(\epsilon), H_0) = \text{Ran } \Pi'(\epsilon)$  for  $\epsilon > 0$  small enough.  $\blacksquare$

According to [4],  $S(\epsilon)$  is bijective if and only if  $\text{Ran } W_-(H(\epsilon), H_0)$  is closed. In our case,  $\Pi'(\epsilon) = 1 - \Pi(\epsilon)$  and  $\Pi(\epsilon)$  is a projection of finite rank. So  $\text{Ran } \Pi'(\epsilon) = \text{Ker } \Pi(\epsilon)$  is closed. Consequently, under the conditions of Theorem 1.1, the dissipative scattering operator  $S(\epsilon)$  is invertible on  $L^2$  for  $\epsilon > 0$  small enough.

**Remark.** Since  $S(\epsilon)$  is bijective,  $W_-(\epsilon)$  is bijective from  $L^2$  onto  $\text{Ran } \Pi'(\epsilon)$ , therefore invertible with bounded inverse. The intertwining relation  $W_-(\epsilon)H_0 = H(\epsilon)W_-(\epsilon)$  gives a relation between the characteristic functions of  $H_0$  and  $H(\epsilon)$  ([14]). Since the spectrum of  $H_0$  is purely absolutely continuous, one may guess that under the conditions of Theorem 1.1, the range of  $\Pi'(\epsilon)$  coincides with the absolute continuous spectral subspace as defined in [14]. This would mean that the singular continuous spectrum of  $H(\epsilon)$  is absent. See [13] for a result in one dimensional case without the smallness condition on the imaginary part of the potential, but under the assumption of absence of spectral singularities.

## REFERENCES

- [1] R. Alicki, On the scattering theory for quantum dynamical semigroups. Ann. Inst. H. Poincaré Sect. A (N.S.) 35 (1981), no. 2, 97–103.
- [2] R. Alicki, A. Frigerio, Scattering theory for quantum dynamical semigroups. II. Ann. Inst. H. Poincaré Sect. A (N.S.) 38 (1983), no. 2, 187–197.
- [3] E. B. Davies, Two-channel Hamiltonians and the optical model for nuclear scattering, Ann. Inst. H. Poincaré, Vol. XXIX(4) (1978), 395–413.
- [4] E. B. Davies, Non-unitary scattering and capture. I. Hilbert space theory, Ann. Inst. H. Poincaré, Commun. Math. Phys., 71(1980), 277–288.

- [5] E. B. Davies, Non-unitary scattering and capture. II. Quantum dynamical semigroup theory, *Ann. Inst. H. Poincaré*, Vol. XXXII(4) (1980), 361-375.
- [6] H. Feshbach, Unified theory of nuclear reactions. I., *Ann. Phys.*, 5(1958), 357-390.
- [7] M. Kadowaki, On a framework of scattering for dissipative systems. *Osaka J. Math.* 40 (2003), no. 1, 245-270.
- [8] T. Kato, Wave operators and similarity for some non-selfadjoint operators, *Math. Annalen*, 162(1966), 258-279.
- [9] Ph. A. Martin, Scattering theory with dissipative interactions and time delay, *Nuovo Cim.*, (30)B (1975), 217-238.
- [10] B. S. Pavlov, Selfadjoint dilation of the dissipative Schrödinger operator and its resolution in terms of eigenfunctions, *Math. URSS Sbornik*, 31(4)(1977), 457-478.
- [11] J. Royer, Limiting absorption principle for the dissipative Helmholtz equation, *Commun. in PDE*, 35(2010), 1458-1489.
- [12] B. Simon, Phase space analysis of simple scattering systems: extensions of some work of Enss, *Duke Math. J.*, 40(1)(1980), 119-168.
- [13] S. A. Stepin, Dissipative Schrödinger operator without a singular continuous spectrum, *Mat. Sbornik* 195(6) (2004), 137-156
- [14] B. Sz. Nagy, C. Foiaş, Harmonic analysis of operators on Hilbert space, North Holland, Amsterdam, 1970.
- [15] X. P. Wang, Number of eigenvalues for dissipative Schrödinger operators under perturbation, *J. Math. Pures Appl.*, 96(2011), 409-422.
- [16] X. P. Wang, Time-decay of semigroups generated by dissipative Schrödinger operators, *J. Diff. Equations*, 253(2012), 3523-3542.

DÉPARTEMENT DE MATHÉMATIQUES, UMR 6629 CNRS, UNIVERSITÉ DE NANTES, 44322 NANTES CEDEX 3 FRANCE, E-MAIL: XUE-PING.WANG@UNIV-NANTES.FR