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Adaptive control for finite element analysis in plasticity

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An error on the constitutive law, called dissipation error, is used to measure and to enhance the quality of finite element computations of plastic structures whose behaviour is described by internal variables. This measure takes into account all the classical sources of error involved in the computation: the space discretization (the mesh), the time discretization and the iterative technique used to solve the nonlinear discrete problem. To quantify more specifically the contribution of these three error sources to the error, three error indicators are calculated: a space indicator, a time indicator and an iteration indicator. Then, the error and the three indicators are used in an adaptive strategy: the space indicator allows us to adapt the mesh, the time indicator to adapt the time discretization and the iteration indicator makes it possible to limit the number of iterations of the global iterative algorithm.

1. Introduction

The importance of numerical simulation in the industrial activity is growing from day to day and the problems treated are becoming more and more complex (three-dimensional problems with large deformations, impacts on composites, etc.). The crucial issue is now to be able to control the computation, i.e. to estimate the error and optimize the calculation parameters (mesh size, time discretization, number of iterations for the iterative techniques, etc.).

In addition to the developments of the computational techniques, important advances have been made in the materials behaviour modelization. For instance, the description of material models using a set of internal variables (plastic strain, hardening parameters, etc.) is becoming more and more popular.

Here, we are concerned with error estimation and adaptive control for finite element computations of plastic structures whose behaviour is described by internal variables. For instance, the Prandtl–Reuss plastic model falls into this category. The assumptions of small strains and displacements, as well as isothermal and quasi-static loading, are made.

Many papers deal with error estimation for linear problems. Three main approaches must be distinguished. The first one, chronologically speaking, is based on the concept of error on the constitutive law [1] and has been applied, among other areas, to thermal [1] and elastic problems [2]. The second one, introduced by Babūška and Rheinboldt [3,4], then developed by Zienkiewicz, Gago and Kelly [5,6] and more recently by Oden et al. [7], uses the equilibrium residuals through local problems to estimate the error. The last one, developed by Zienkiewicz and Zhu [8–10], consists of comparing the finite element solution to a smoother one obtained by special averaging techniques. A validation of these a posteriori error estimators can be found in [11,12]. Finally, let us mention

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the dual analysis approach based on upper and lower bounds for the energy [13,14].

In comparison with the linear case, much less work has been devoted to the nonlinear case. For nonlinear time-independent problems, Babuška and Rheinboldt [15] and Johnson and Hansbo [16] have designed error estimates for nonlinear elasticity and Hencky-type plasticity, respectively. For nonlinear time-dependent problems, techniques devised for linear problems or time-independent nonlinear problems are used at each time step [17–19]. Unfortunately, the error estimates so built do not take into account the errors due to the time discretization. In a time-dependent nonlinear problem, the quality of the finite element solution at time t depends indeed not only on the quality of the mesh, but also on the two following influences: the quality of the time discretization performed since the beginning of the loading, and the defect of convergence of the global iterative algorithm at each computed time (Newton’s algorithm, for instance).

An error on the constitutive law taking into account the three sources of error described previously has been proposed by Ladevèze [20]. This measure, based on Drucker’s inequality, was first applied in [21] for plane stress problems and three-node triangles; a procedure to adapt the mesh is also described. The Drucker error has recently been reused and enhanced in order to conduct a simultaneous adaptive control of the space and time discretization for three- and six-node triangles in plane and axisymmetric problems [22]. The Drucker error is based on a sufficient condition that ensures the stability of the material.

For materials described by a set of internal variables, a new a posteriori error estimate, called dissipation error, has been elaborated in [23,24]. The key idea is to divide the equations of the problem into two groups:

- A group of equations related to the free energy including the equation of equilibrium, the kinematic constraints and the state equations.
- A group of equations related to the dissipation, i.e. the laws describing the evolution of the material state.

The dissipation error characterizes the quality of an approximate admissible solution, i.e. a solution satisfying the first group of equations. This quality is quantified by the non-verification of the second group of equations. This fact naturally leads to the terminology of dissipation error. The usefulness of the dissipation error has been evaluated in the framework of the classical incremental finite element method in [25]. As for the Drucker error, the dissipation error does take into account all the error sources involved in the computation. To quantify more specifically the contributions

of the space and time discretizations, two indicators have been introduced in [25].

In the present paper, we introduce a third indicator to quantify the contribution of the iterative technique to the error. Then, an adaptive strategy is proposed. The error and the three indicators makes it possible to adapt simultaneously all the parameters of the computation: the mesh, the time discretization and the stopping criterion of the iterative algorithm. The strategy is tested for two plane stress problems, with the Prandtl–Reuss plastic model.

The plan of the paper is as follows. The reference problem and the dissipation error are described in Sections 2 and 3, respectively. The implementation of the dissipation error in the framework of the classical incremental finite element is detailed in Section 4. In Section 5, the time and space error indicator are recalled and the iteration indicator is introduced. The adaptive control strategy is proposed in the final section and two complete examples are treated.

2. The continuous reference problem

Concerning the notations, vectors will be underlined (\underline{U} , \underline{U}^* , ...) and second-order tensors in outline font. For instance, strains and stresses will be denoted by $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$, respectively. This notation will also be used to denote the additional internal variables. More complex operators will be in bold (e.g. \mathbf{K} for the Hooke’s tensor). Finally, the first derivative of a function f with scalar argument will be denoted by f' .

The solid medium under study occupies a domain Ω bounded by $\partial\Omega$, which is independent of t (small strain and displacement assumption). The environment of the medium is schematized for all $t \in [0, T]$, with an imposed displacement \underline{U}_d on a part $\partial_1\Omega$ of the boundary, a traction \underline{F}_d on $\partial_2\Omega$ (complementary to $\partial_1\Omega$), and a volumic load \underline{f}_d on the domain Ω . The partition of $\partial\Omega$ in $\partial_1\Omega$ and $\partial_2\Omega$ is taken as constant in time.

The desired solution must fulfill the kinematic constraints, the equilibrium equation, the state equations, the evolution laws and the initial conditions. We will work in the usual bounded energy frame, \mathcal{U} denotes the space of the displacement field \underline{U} defined on Ω , and \mathcal{S} the space of the stress field, also defined on Ω . The extensions of these two spaces to the entire time range $[0, T]$ will be denoted $\mathcal{U}^{[0, T]}$ and $\mathcal{S}^{[0, T]}$. Finally, Tr stands for the trace.

The *kinematic constraints* read

$$\underline{U} \in \mathcal{U}_{\text{ad}}^{[0, T]} \quad (1)$$

where

$$\mathcal{U}_{\text{ad}}^{[0,T]} = \{ \underline{U} \in \mathcal{U}^{[0,T]} \text{ such that } \underline{U} = \underline{U}_d \text{ on } [0,T] \times \partial_1 \Omega \} \quad (2)$$

The *quasi-static equilibrium equation* takes the form

$$\sigma \in \mathcal{S}_{\text{ad}}^{[0,T]} \quad (3)$$

where $\mathcal{S}_{\text{ad}}^{[0,T]} = \{ \sigma \in \mathcal{S}^{[0,T]} \text{ satisfying Eq. (4)} \forall \underline{U}^* \in \mathcal{U}_0, \forall t \in [0,T] \}$,

$$\int_{\Omega} \text{Tr}[\sigma \epsilon(\underline{U}^*)] d\Omega - \int_{\Omega} f_d \underline{U}^* d\Omega - \int_{\partial_2 \Omega} E_d \underline{U}^* dS = 0 \quad (4)$$

In the above,

$$\mathcal{U}_0 = \{ \underline{U} \in \mathcal{U} \text{ such that } \underline{U} = 0 \text{ on } \partial_1 \Omega \} \quad (5)$$

The behaviour is described by internal variables. The state of the material is characterized at each point by the total strain ϵ , the inelastic strain ϵ^P and a set of internal variables denoted by \mathbb{X} . The associated variables are the stress σ for ϵ and ϵ^P , and the quantity \mathbb{Y} for \mathbb{X} . Thus, the expression of the dissipation is

$$\text{Tr}[\sigma \dot{\epsilon}^P - \mathbb{Y} \circ \dot{\mathbb{X}}]. \quad (6)$$

The second term specifies the contribution of (\mathbb{X}, \mathbb{Y}) to the dissipation. If \mathbb{X} denotes a column of \mathbb{R}^p , then \mathbb{Y} is also a column of \mathbb{R}^p and

$$\mathbb{Y} \circ \mathbb{X} = \mathbb{Y}^t \mathbb{X},$$

where t stands for the usual transposition. More precisely, two spaces \mathbf{e} and \mathbf{f} are placed in duality by the following bilinear form:

$$\left[\begin{array}{c} \dot{\epsilon}^P \\ -\dot{\mathbb{X}} \end{array} \right], \left[\begin{array}{c} \sigma \\ \mathbb{Y} \end{array} \right] \rightarrow \text{Tr}[\sigma \dot{\epsilon}^P] - \mathbb{Y} \circ \dot{\mathbb{X}},$$

$$\mathbf{e} \times \mathbf{f} \rightarrow \mathbb{R}$$

With the small strain assumption, the total strain is the symmetric part of the gradient of \underline{U} ; in an orthonormal basis, it can be written

$$[\epsilon(\underline{U})]_{ij} = \frac{1}{2} (U_{i,j} + U_{j,i})$$

This assumption also yields the additivity relation for the rate of elastic $\dot{\epsilon}^e$ and inelastic $\dot{\epsilon}^P$ strains, $\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^P$.

For the *state equations*, according to the first principle of thermodynamics, a free energy ψ , depending only on the state variables ϵ , ϵ^P and \mathbb{X} can be introduced. The following classical assumptions are made:

- ψ depends only on the elastic strain ϵ^e and the internal variables \mathbb{X} ;
- $\psi(\epsilon^e, \mathbb{X}) = \psi_e(\epsilon^e) + \psi_p(\mathbb{X})$;

- linear elasticity: $\psi_e(\epsilon^e) = \frac{1}{2} \text{Tr}[\mathbf{K} \epsilon^e \epsilon^e]$, where \mathbf{K} is Hooke's tensor.

The derivation of ψ yields the state equations $\sigma = \mathbf{K} \epsilon^e$ and $\mathbb{Y} = \mathbf{G}(\mathbb{X})$, where $\mathbf{G}(\mathbb{X})$ is the derivative of ψ_p with respect to \mathbb{X} .

For the *evolution laws*, the second principle of thermodynamics, written as

$$\text{Tr}[\sigma \dot{\epsilon}^P - \mathbb{Y} \circ \dot{\mathbb{X}}] \geq 0$$

imposes a constraint on the evolution laws relating $(\dot{\epsilon}^P, -\dot{\mathbb{X}})$ and (σ, \mathbb{Y}) . This law can be written:

$$\left[\begin{array}{c} \dot{\epsilon}^P \\ -\dot{\mathbb{X}} \end{array} \right] = \mathbf{B} \left(\left[\begin{array}{c} \sigma \\ \mathbb{Y} \end{array} \right] \right), \quad \epsilon^P = 0, \mathbb{X} = 0 \quad \text{for } t = 0$$

\mathbf{B} is an operator relevant to the material. It must be positive to respect the second principle of thermodynamics. The material is supposed to be initially virgin.

A typical way to define the operator B is to give a scalar function $\varphi^*(\sigma, \mathbb{Y})$, generally convex, called the potential of dissipation, and to write

$$\left(\begin{array}{c} \epsilon^P \\ -\dot{\mathbb{X}} \end{array} \right) \in \left(\begin{array}{c} \partial_{\sigma} \varphi^*(\sigma, \mathbb{Y}) \\ \partial_{\mathbb{Y}} \varphi^*(\sigma, \mathbb{Y}) \end{array} \right) \quad (7)$$

where $(\partial_{\sigma} \varphi^*, \partial_{\mathbb{Y}} \varphi^*)$ denotes the subdifferential of φ^* at (σ, \mathbb{Y}) . This defines a *standard material*. When the potential is differentiable, the subdifferential becomes a classical gradient and the belonging an equality. The interest of a standard model lies in the following classical property. The second principle of thermodynamics is fulfilled if the potential satisfies:

$$\varphi^* \text{ convex}, \quad \varphi^*(0,0) = 0, \quad \varphi^*(\bullet, \bullet) \geq 0 \quad (8)$$

As an example, consider the Prandtl–Reuss plastic model. In addition to the plastic strain, the model involves another scalar internal variable p that can be interpreted as the cumulative plastic strain. The free energy is of the form:

$$\psi(\epsilon^e, p) = \frac{1}{2} \text{Tr}[\mathbf{K} \epsilon^e \epsilon^e] + g(p)$$

where g is a function characterizing the hardening law. So,

$$\sigma = \mathbf{K} \epsilon^e, \quad \mathbb{Y} \equiv R = g'(p) \quad (9)$$

Classical hardening laws are $R = \lambda p$ (linear hardening), $R = k_y p^{1/m}$ (power hardening) and $R = R_M (1 - \exp(-\gamma p))$ (exponential hardening) where λ , k_y , m , R_M and γ are constant material parameters. Prandtl–Reuss materials are standard with a dissipation potential given by

$$\varphi^*(\sigma, R) = 0 \quad \text{if } f \leq 0$$

$$\varphi^*(\sigma, R) = +\infty \text{ if } f > 0$$

where

$$f(\sigma, R) = \|\sigma^D\| - (R + R_0)$$

σ^D is the deviator of the stresses, R_0 is the initial yield stress, and

$$\|\sigma^D\| = (\text{Tr}[\sigma^D \sigma^D])^{1/2}$$

One may easily check that φ^* fulfills the sufficient conditions (8) and therefore the second thermodynamics principle.

3. The dissipation error

The notion of error on the constitutive law has been introduced in [26]. It relies on splitting the equations of the problem into two groups. When the behaviour is formulated by a functional law, the first group combines both the equilibrium equations and the kinematic constraints, and the second group contains the constitutive law. The quality of an approximate solution satisfying the first group (i.e. an admissible solution) is quantified by the non-fulfilment of the second group of equations (constitutive law). If Drucker's stability inequality holds for the material, a natural way to measure the error can be obtained [21,22]. The error on the constitutive relation has a strong mechanical basis since less confidence is given to the behaviour of the material.

When the state of the material is described by internal variables, the notion of admissibility must be revised. Indeed, the state equations—associated with the free energy—and the evolution laws—associated with the dissipative phenomena—must be distinguished in the formulation. In [24], the state equations are included in the definition of admissibility, the error being measured on the evolution laws alone. The problem is divided precisely into two groups:

- The first group defines the admissibility of a solution. It combines the equations related to the free energy: the equilibrium equation, the kinematic constraints and the state equations: $\sigma = \mathbf{K}(\epsilon - \epsilon^P)$ and $\mathbb{Y} = \mathbf{G}(\mathbb{X})$. We also add to this group the initial conditions (defining the initial state of the material, here taken as virgin) $(\epsilon^P, \mathbb{X}) = 0$ at $t = 0$ on Ω .
- The second group, related to the dissipation, only includes the evolution laws.

Two questions must be answered to actually utilize the dissipation error concept. The first one deals with the admissibility. In general, the finite element solution obtained from the computation is not admissible. For instance, the stresses are not exactly statically admissi-

ble. The construction of an admissible solution from the finite element one will be addressed in Section 4. The second question concerns the definition of the error measurement.

For standard material, the error measurement can be defined using the properties of the Legendre–Fenchel transform. This transform associates a dual potential $\varphi(\dot{\epsilon}^P, -\dot{\mathbb{X}})$ to $\varphi^*(\sigma, \mathbb{Y})$:

$$\varphi(\dot{\epsilon}^P, -\dot{\mathbb{X}}) \equiv \sup_{(\sigma, \mathbb{Y}) \in \mathbf{f}} (\text{Tr}[\sigma \dot{\epsilon}^P] - \mathbb{Y} \circ \mathbb{X} - \varphi^*(\sigma, \mathbb{Y}))$$

The Legendre–Fenchel transform possesses two interesting classical properties. Using the condensed notation

$$\eta(\dot{\epsilon}^P, \dot{\mathbb{X}}, \sigma, \mathbb{Y}) \equiv \varphi(\dot{\epsilon}^P, -\dot{\mathbb{X}}) + \varphi^*(\sigma, \mathbb{Y}) - \text{Tr}[\sigma \dot{\epsilon}^P] + \mathbb{Y} \circ \dot{\mathbb{X}}$$

they take the form

$$\eta(\dot{\epsilon}^P, \dot{\mathbb{X}}, \sigma, \mathbb{Y}) \geq 0, \quad \forall (\dot{\epsilon}^P, \dot{\mathbb{X}}, \sigma, \mathbb{Y}) \in \mathbf{e} \times \mathbf{f}$$

$$\eta(\dot{\epsilon}^P, \dot{\mathbb{X}}, \sigma, \mathbb{Y}) = 0 \Leftrightarrow \begin{pmatrix} \dot{\epsilon}^P \\ -\dot{\mathbb{X}} \end{pmatrix} \in \begin{pmatrix} \partial_\sigma \varphi^*(\sigma, \mathbb{Y}) \\ \partial_\mathbb{Y} \varphi^*(\sigma, \mathbb{Y}) \end{pmatrix}$$

The first relation is usually called the Legendre–Fenchel inequality, and the second relation simply means that η is zero if and only if the evolution laws are satisfied.

The quantity η allows one to measure the quality of verification of the evolution laws at each time and each place. The *absolute error* e may thus be defined as follows

$$e \equiv \int_0^T \int_\Omega \eta(\dot{\epsilon}^P, \dot{\mathbb{X}}, \sigma, \mathbb{Y}) d\Omega dt \quad (10)$$

The absolute error is zero if and only if the admissible solution and the exact solution both coincide on $[0, T]$. The *relative error* ϵ is defined by $\epsilon = e/D$, where

$$D \equiv 4 \sup_{t \in [0, T]} d_t$$

$$d_t \equiv (1 - \gamma) \int_0^t \int_\Omega \sup \left[\left(\varphi^*(\sigma, \mathbb{Y}) + \varphi(\dot{\epsilon}^P, \dot{\mathbb{X}}), \frac{R_0}{\|\sigma^D\|} \left| \dot{\psi}_\epsilon(\epsilon^e) \right| \right) \right] d\Omega dt \\ + \gamma \int_\Omega \frac{1}{2} \text{Tr}[\mathbf{K}^{-1} \sigma \sigma] + \psi_p(\mathbb{X}) d\Omega_t$$

The γ parameter ($0 \leq \gamma \leq 1$) gives the relative weight of the dissipation and the free energy in the denominator. We set $\gamma = \frac{1}{2}$. The minimum dissipation value

$$\frac{R_0}{\|\mathbb{D}\|} \left| \dot{\psi}_e(\epsilon^e) \right|$$

where $\dot{\psi}_e$ is the derivative of the elastic free energy with respect to time, has been introduced in [25] to obtain reasonable error in elasticity in comparison with the classical error on the constitutive law for such problems [2].

Thanks to the simple norm chosen, it is easy to express the global error ϵ in terms of local contributions. The *local contribution* to the error of the space–time domain ω_i ($\omega_i \subset [0, T] \times \Omega$) is defined by

$$\epsilon_{\omega_i} \equiv \frac{\int_{\omega_i} \eta \, d\omega_i}{D} \quad (11)$$

If $\cup_{i=1}^I \omega_i = [0, T] \times \Omega$ and $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$,

$$\epsilon = \sum_{i=1}^I \epsilon_{\omega_i} \quad (12)$$

In the finite element computation, the time–space domain is naturally divided into elements E and time-steps Δt . If ω denotes $\Delta t \times E$, $\Delta t \times \Omega$ or $[0, T] \times E$, ϵ_ω represents the contribution to the error of a time-step over an element, a time step over the whole structure or an element over the whole time period, respectively. If ω denotes $[0, t] \times \Omega$, ϵ_ω represents the contribution of the time interval $[0, t]$ over the whole structure:

$$\epsilon_{[0, t] \times \Omega} \equiv \frac{\int_0^t \int_{\Omega} \eta \, d\Omega dt}{D} \quad (13)$$

For the applications, we will consider the Prandtl–Reuss plastic model with linear hardening. The expressions of the *dual potentials* for this model are given by

$$\varphi^*(\sigma, R) = \Psi_{C^*}, \quad \varphi(\dot{\epsilon}^P, -\dot{p}) = R_0 \dot{p} + \Psi_C \quad (14)$$

where

$$C^* = \{(\sigma, R) \in \mathbf{f} \mid f \leq 0, R \geq 0\}$$

$$C = \{(\dot{\epsilon}^P, -\dot{p}) \in \mathbf{e} \mid \text{Tr}[\dot{\epsilon}^P] = 0, \|\dot{\epsilon}^P\| - \dot{p} \leq 0\}$$

Ψ_A is the indicator function associated with the convex domain A (i.e. $\Psi_A = 0$ inside A , and $\Psi_A = +\infty$ outside). The condition $R \geq 0$ is introduced in the definition of the potential φ^* for reasons detailed in [25].

4. Finite element and admissible solutions

The dissipation error cannot be directly measured on the finite element solution because it is not gener-

ally admissible. We will now see how it is possible to easily get an admissible solution from the finite element one in the case of the classical incremental finite element computation obtained through the displacement method. First, we detail the discrete problem.

4.1. The discrete problem

In an incremental method, the problem to be solved on $[0, T]$ is divided into a succession of resolutions over $[t_n, t_{n+1}]$ ($n = 0, \dots, N-1$; $t_0 = 0$, $t_N = T$). Assuming the solution is known until t_n , one must then build the solution over $[t_n, t_{n+1}]$. First, a time discretization must be carried out. Usually, one only seeks the solution at the final time t_{n+1} . So, the kinematic constraints and the equilibrium equation are written at t_{n+1} , and the behaviour becomes a nonlinear algebraic relation linking the fields at t_{n+1} . Formally,

$$\mathbb{T}_{n+1} - \mathbb{T}_n = \mathcal{A}_n(\epsilon_{n+1} - \epsilon_n) \quad (15)$$

where \mathbb{T}_n, ϵ_n ($\mathbb{T}_{n+1}, \epsilon_{n+1}$) are the stresses and strains at t_n (t_{n+1}). The \mathcal{A}_n notation emphasizes that the stresses at t_{n+1} are no longer expressed as a functional law of the strain rate history, but now as a nonlinear algebraic law of the increase of strain $\epsilon_{n+1} - \epsilon_n$ over the time step. Note that the internal variables generally explicitly appear in the numerical scheme.

Concerning the space discretization, the domain is divided into elements on which the displacement field is interpolated between nodal values using shape functions. The displacement at \underline{M} reads

$$\underline{U}_h(\underline{M}) = \mathbf{N}(\underline{M})\mathbf{q} \quad (16)$$

where $\mathbf{N}(\underline{M})$ is the shape function matrix and \mathbf{q} the nodal displacement set. The discrete problem to be solved at t_{n+1} read as given below.

4.1.1. Kinematic constraints

$$\underline{U}_{h,n+1} \in \mathcal{U}_{h,\text{ad}}^{t_{n+1}} \quad (17)$$

where

$$\mathcal{U}_{h,\text{ad}}^{t_{n+1}} = \{ \underline{U} \in \mathcal{U}_h \text{ such that } \underline{U} = \underline{U}_d \text{ on } \partial_1 \Omega \text{ at } t_{n+1} \}$$

and \mathcal{U}_h denotes the finite element displacement space.

4.1.2. Equilibrium equations

$$\mathbb{T}_{h,n+1} \in \mathcal{S}_{h,\text{ad}}^{t_{n+1}} \quad (18)$$

where

$$\begin{aligned} \mathcal{S}_{h,ad}^{l_{n+1}} &= \{ \mathfrak{P} \in \mathcal{S}_h \text{ satisfying Eq. (19) at } t_{n+1} \forall \underline{U}^* \in \mathcal{U}_{h,0} \} \\ \int_{\Omega_h} \text{Tr}[\mathfrak{P}_{h,n+1} \epsilon(\underline{U}^*)] d\Omega_h &- \int_{\Omega} f_d \circ \underline{U}^* d\Omega \\ - \int_{\partial_2 \Omega} F_d \circ \underline{U}^* dS &= 0 \end{aligned} \quad (19)$$

In the above,

$$\mathcal{U}_{h,0} = \{ \underline{U} \in \mathcal{U}_h \text{ such that } \underline{U} = 0 \text{ on } \partial_1 \Omega \}$$

and \mathcal{S}_h denotes the space of stresses defined at the integration points of the domain Ω . The integration $\int_{\Omega_h} d\Omega_h$ in Eq. (19) is performed classically using a discrete integration technique. Thus, the stresses given by the computation (as well as the plastic strains and internal variables) are only known at the integration points.

4.1.3. Constitutive relations

$$\mathfrak{P}_{h,n+1} - \mathfrak{P}_{h,n} = \mathcal{A}_n(\epsilon(\underline{U}_{h,n+1}) - \epsilon(\underline{U}_{h,n}))$$

The discrete problem, which is a nonlinear algebraic system, is solved in an iterative way, typically with Newton's method. Two types of solutions always exist as the iterations proceed: a solution $(\check{\underline{U}}_{h,n+1}, \check{\mathfrak{P}}_{h,n+1})$ satisfying both the kinematic constraints and the equilibrium in a finite element sense

$$\check{\underline{U}}_{h,n+1} \in \mathcal{U}_{h,ad}^{l_{n+1}}, \quad \check{\mathfrak{P}}_{h,n+1} \in \mathcal{S}_{h,ad}^{l_{n+1}}$$

and a solution $(\tilde{\underline{U}}_{h,n+1}, \tilde{\mathfrak{P}}_{h,n+1})$ satisfying both the kinematic constraints and the "discrete" behaviour

$$\tilde{\underline{U}}_{h,n+1} \in \mathcal{U}_{h,ad}^{l_{n+1}}$$

$$\tilde{\mathfrak{P}}_{h,n+1} - \tilde{\mathfrak{P}}_{h,n} = \mathcal{A}_n(\epsilon(\tilde{\underline{U}}_{h,n+1}) - \epsilon(\tilde{\underline{U}}_{h,n}))$$

Deriving a solution $(\check{\underline{U}}_{h,n+1}, \check{\mathfrak{P}}_{h,n+1})$ from the solution $(\tilde{\underline{U}}_{h,n+1}, \tilde{\mathfrak{P}}_{h,n+1})$ is performed by integrating the behaviour at each integration point of the structure with $\check{\underline{U}}_{h,n+1} = \tilde{\underline{U}}_{h,n+1}$. Deriving a solution $(\tilde{\underline{U}}_{h,n+1}, \tilde{\mathfrak{P}}_{h,n+1})$ from $(\check{\underline{U}}_{h,n+1}, \check{\mathfrak{P}}_{h,n+1})$ (equilibrium phase) is performed by a linear global resolution. When the two solutions are close, the iterative process stops and the finite element code generally gives the solution satisfying the "discrete" behaviour.

4.2. The construction of an admissible solution

From the finite element solution, we must construct

an admissible solution (denoted by hats), i.e. a solution satisfying

$$\begin{aligned} \hat{\underline{U}} &\in \mathcal{U}_{ad}^{[0,T]}, \quad \hat{\mathfrak{P}} \in \mathcal{S}_{ad}^{[0,T]} \\ \hat{\mathfrak{P}} &= \mathbf{K}(\epsilon(\hat{\underline{U}}) - \epsilon^p), \quad \hat{\mathfrak{Y}} = \mathbf{G}(\hat{\mathfrak{X}}), \quad \text{on } [0,T] \times \Omega \\ (\hat{\epsilon}^p, \hat{\mathfrak{X}}) &= 0 \text{ at } t = 0 \text{ on } \Omega \end{aligned}$$

The rather unrestrictive assumption of linear evolution for imposed loads and displacements between consecutive computed time is applied. We also assume that the prescribed displacements and applied loads are zero at $t = 0$.

4.2.1. Construction of $\hat{\mathfrak{P}} \in \mathcal{S}_{ad}^{[0,T]}$

The finite element stresses are not statically admissible at the time steps (they are not even known at every point, but only at the integration points). Let us rigorously construct equilibrated stresses at each time step. Suppose we do have stresses that satisfy the equilibrium equation in the finite element sense $\check{\mathfrak{P}}_{h,n+1}$ (if this is not the case, we just need to apply the "equilibrium phase" of the iterative method). From these stresses, we then construct the rigorously equilibrated stresses $\hat{\mathfrak{P}}_{n+1}$ using techniques developed for the elasticity [2]. These techniques consist first of building load densities in equilibrium with the applied loading on the boundary of each element. Then, stresses are built on each element separately. The density building only requires the resolution of small local linear problems, and the stress building is conducted analytically element by element.

With equilibrated stresses at each time step, and owing to the linear evolution of the loading, we interpolate $\hat{\mathfrak{P}}_{n+1}$ over each time step to obtain $\hat{\mathfrak{P}} \in \mathcal{S}_{ad}^{[0,T]}$. Let a_n (a_{n+1}) be the values of a at t_n (t_{n+1}). Linear interpolation of a means:

$$\text{on } [t_n, t_{n+1}], \quad a = \frac{t_{n+1} - t}{t_{n+1} - t_n} a_n + \frac{t - t_n}{t_{n+1} - t_n} a_{n+1}$$

At $t = 0$, we take $\hat{\mathfrak{P}}_0 = 0$, which is statically admissible with the initial null loading.

4.2.2. Construction of $\hat{\underline{U}} \in \mathcal{U}_{ad}^{[0,T]}$

Since the prescribed displacements display linear evolution on each time step, and since the finite element displacement field satisfies the kinematic constraints at each computed time, an element of $\mathcal{U}_{ad}^{[0,T]}$ is simply obtained by interpolating the finite element displacement field between the computed times. At $t = 0$, we take $\hat{\underline{U}} = 0$. For the Prandtl-Reuss plastic model, the inelastic strains take place at constant volume. In order to avoid an infinite error, the finite element field

is modified at each computed time before the interpolation in order to satisfy the incompressibility condition $\text{Tr}[\epsilon(\hat{U}) - \mathbf{K}^{-1}\hat{\phi}] = 0$. For the plane stress problems treated in the application, this condition is easily achieved by adequately choosing the strain in the thickness $\hat{\epsilon}_{33}$.

4.2.3. Construction of the inelastic strains

The following expression must be used to compute the inelastic strains on $[0, T] \times \Omega$ in order to respect Hooke's law:

$$\epsilon^p = \epsilon(\hat{U}) - \mathbf{K}^{-1}\hat{\phi} \quad (20)$$

As $\hat{U} = 0$ and $\hat{\phi} = 0$ at $t = 0$ on Ω , we have $\epsilon^p = 0$ at $t = 0$ on Ω . The initial condition on $\hat{\epsilon}^p$ is thus satisfied.

4.2.4. Determination of the internal variables values

The computed internal variables are seldom given in classical finite element code. So, herein we do not consider them as given at the computed time, and we construct them. Several choices are possible. Here, for the Prandtl–Reuss plastic model with linear hardening, we take:

$$\hat{p}_{n+1} = \sup(\hat{p}_{1,n+1}, \hat{p}_{2,n+1})$$

where

$$\hat{p}_{1,n+1} = \hat{p}_n + (t_{n+1} - t_n) \|\hat{\epsilon}_{n+1}^p\|$$

$$\hat{p}_{2,n+1} = \frac{1}{\lambda} \langle \|\hat{\phi}_{n+1}^D\| - R_0 \rangle_+$$

and \hat{p} is obtained by a linear interpolation of the \hat{p}_{n+1} . This choice results from a step by step minimization of the error [25].

4.3. Interpretation of the error

The interpretation of the dissipation error in terms of more classical norms like the L^2 error in the stresses or the energy (H^1) norm in the displacements is not straightforward and further work needs to be done in this direction. As a matter of fact, to our knowledge, the regularity of the exact solution (i.e. whether the stress field belongs to L^2 and the displacement field to H^1) is still not clear from the mathematical point of view for a general constitutive relation described by internal variables. In Ref. [25], the dissipation error has been compared to the exact free energy error and we obtained on an example that the dissipation error was overestimating the energy error by a factor of 1.6, the exact solution being obtained through an “overkill” (very fine discretizations in time and space).

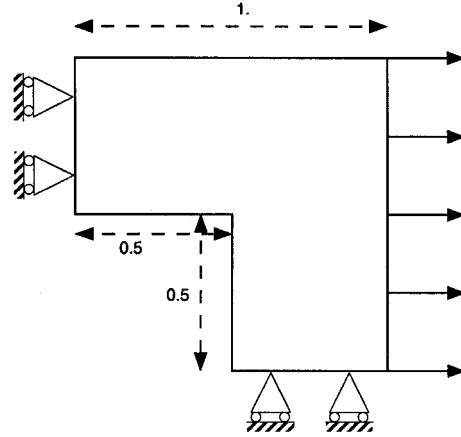


Fig. 1. The holed plate problem.

5. The error indicators

The dissipation error takes into account all the unavoidable error sources entering into the computation of a time-dependent nonlinear problem: the space discretization, the time discretization and the iterative technique used to solve the nonlinear discrete problem. To demonstrate this fact, we observe the behaviour of the error when the space or time discretizations are refined. Two plane stress problems are considered: the holed plate problem and the frame problem. The geometry and the loading are given in Figs. 1–4. For the two examples, the Prandtl–Reuss plastic model is used with the following dimensionless parameters

$$R_0 = 1, \quad \lambda = 8.16, \quad E = 244.95, \quad \nu = 0.3$$

E and ν denote Young's modulus and Poisson's ratio, respectively. The finite element results are obtained with the CASTEM2000 code and three- and six-nodes triangular elements are used.

Table 1 (first line) presents the evolution of the error

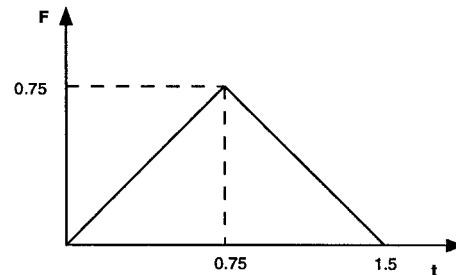


Fig. 2. The loading.

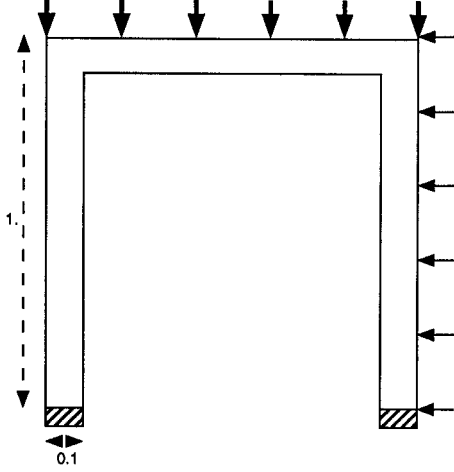


Fig. 3. The frame problem.

ϵ with an increasing number of time increments for a given mesh and Table 2 (first line) presents the evolution of the error for an increasing quality of the mesh and a given time discretization.

We can conclude that the error decreases as the space or time discretizations are refined and tends to stabilize under a given level of refinement. This result proves that the error depends on both the space and time discretizations. Under a certain refinement in space (time), the error is stable because it is mainly due to the time (space) discretization.

Let us define the time error as the limit of the error when the number of elements tends to “infinity” (thus, no error due to the space exists any more):

$$\epsilon_{\text{time}} \equiv \lim_{h \rightarrow 0} \epsilon$$

Similarly, we define the space error as the limit of the error when the number of time steps tends to “infinity”:

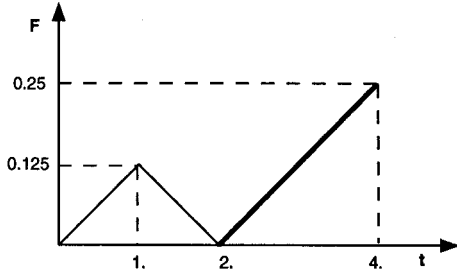


Fig. 4. The loading.

$$\epsilon_{\text{space}} \equiv \lim_{\Delta t \rightarrow 0} \epsilon$$

When both the number of elements and time steps become large, the error consists only of the convergence default of the global iterative algorithm. This error, called the iteration error, reads

$$\epsilon_{\text{ite}} \equiv \lim_{h \rightarrow 0, \Delta t \rightarrow 0} \epsilon$$

With the adaptive control of the computational parameters in mind, it is important to calculate reliable estimates of the quantities ϵ_{time} , ϵ_{space} and ϵ_{ite} , as direct calculation of these quantities is not realistic.

The dissipation error is associated to the continuous reference problem described in Section 2. There exist three discrete problems besides the continuous one: the space discrete problem, the time discrete problem and the time–space discrete problem. Applying the dissipation error concept to the three “new” discrete problems provides reliable estimates of ϵ_{time} , ϵ_{space} and ϵ_{ite} . In the next section, we give the precise writing of the three discrete problems. Then, we define the three estimates, called indicators: the space indicator, the time indicator and the iteration indicator. Note that the time and space error indicators concept has already been studied in the past [22,27].

5.1. The discrete problems

5.1.1. The space discrete problem

$\mathcal{U}_h^{[0,T]}$ denotes the set of finite element displacements defined on $[0,T] \times \Omega$ and $\mathcal{S}_h^{[0,T]}$ denote the set of stresses defined on $[0,T] \times \Omega_h$ where Ω_h is the set of the integration points of the structure. The quantities $\epsilon_h^p, \mathbb{X}_h, \mathbb{Y}_h$ are also to be found on $[0,T] \times \Omega_h$. One must find $(\underline{U}_h, \sigma_h, \mathbb{Y}_h, \epsilon_h^p, \mathbb{X}_h)$ satisfying:

- the kinematic constraints

$$\underline{U}_h \in \mathcal{U}_{h,\text{ad}}^{[0,T]}$$

where $\mathcal{U}_{h,\text{ad}}^{[0,T]} = \{\underline{U} \in \mathcal{U}_h^{[0,T]} \text{ such that } \underline{U} = \underline{U}_d \text{ on } [0,T] \times \partial_1 \Omega\}$;

- the equilibrium equation

$$\sigma_h \in \mathcal{S}_{h,\text{ad}}^{[0,T]}$$

where $\mathcal{S}_{h,\text{ad}}^{[0,T]} = \{\sigma \in \mathcal{S}_h^{[0,T]} \text{ satisfying Eq. (19) } \forall \underline{U}^* \in \mathcal{U}_{h,0}, \forall t \in [0,T]\}$, $\mathcal{U}_{h,0} = \{\underline{U} \in \mathcal{U}_h \text{ such that } \underline{U} = 0 \text{ on } \partial_1 \Omega\}$;

- the state equations

on $[0,T] \times \Omega_h$,

$$\sigma_h = \mathbf{K}(\epsilon(\underline{U}_h) - \epsilon_h^p)$$

Table 1
Error and indicators (%) for a growing number of time steps

The holed plate problem					
Number of steps	2	4	8	16	32
ϵ	39.58	12.96	7.93	7.30	7.66
i_{space}	5.20	5.40	6.04	7.08	7.78
i_{time}	37.61	8.15	2.05	0.51	0.17
$i_{\text{ite}} \times 10^2$	1.09	3.02	3.73	7.11	6.56
The frame problem					
Number of steps	3	6	12	24	48
ϵ	25.75	16.37	13.83	13.76	14.01
i_{space}	10.90	11.67	12.47	13.29	13.87
i_{time}	17.68	4.82	0.92	0.24	0.08
$i_{\text{ite}} \times 10^2$	0.28	1.39	0.92	1.19	1.73

$$\mathbb{Y}_h = \mathbf{G}(\mathbb{X}_h)$$

- the evolution laws

$$\text{on } [0, T] \times \Omega_h, \quad \begin{pmatrix} \dot{\epsilon}_h^p \\ -\dot{\mathbb{X}}_h \end{pmatrix} \in \begin{pmatrix} \partial_{\sigma_h} \varphi^*(\sigma_h, \mathbb{Y}_h) \\ \partial_{\mathbb{Y}_h} \varphi^*(\sigma_h, \mathbb{Y}_h) \end{pmatrix}$$

and at $t = 0$, $\epsilon_h^p = 0$, $\mathbb{X}_h = 0$, on Ω_h .

scheme is chosen:

$$\begin{pmatrix} \dot{\epsilon}_{n+1}^p \\ -\dot{\mathbb{X}}_{n+1} \end{pmatrix} \in \begin{pmatrix} \partial_{\sigma_{n+1}} \varphi^*(\sigma_{n+1}, \mathbb{Y}_{n+1}) \\ \partial_{\mathbb{Y}_{n+1}} \varphi^*(\sigma_{n+1}, \mathbb{Y}_{n+1}) \end{pmatrix}$$

where $\dot{\epsilon}_{n+1}^p$ and $\dot{\mathbb{X}}_{n+1}$ follow the notation

$$\dot{x}_{n+1} = \frac{x_{n+1} - x_n}{t_{n+1} - t_n} \quad (21)$$

5.1.2. The time discrete problem

The solution $(\underline{U}_{n+1}, \sigma_{n+1}, \mathbb{Y}_{n+1}, \epsilon_{n+1}^p, \mathbb{X}_{n+1})$ is to be found at every computed time and over the whole domain. The equilibrium equation is continuous and the evolution laws are discretized. The implicit

5.1.3. The time-space discrete problem

Both the equilibrium equation and the evolutions laws are discretized. The solution is denoted by $(\underline{U}_{h,n+1}, \sigma_{h,n+1}, \mathbb{Y}_{h,n+1}, \epsilon_{h,n+1}^p, \mathbb{X}_{h,n+1})$.

Table 2
Error and indicators (%) for a growing quality of the mesh

The holed plate problem					
Mesh	mesh1	mesh2	mesh3	mesh4	mesh5
ϵ	14.71	7.93	4.35	3.70	2.59
i_{time}	2.15	2.04	2.08	2.08	2.11
i_{space}	12.23	6.04	2.33	1.60	0.52
$i_{\text{ite}} \times 10^2$	3.86	3.73	6.84	4.48	7.01
The frame problem					
Mesh	mesh1	mesh2	mesh3	mesh4	mesh5
ϵ	40.94	35.89	21.79	19.47	18.64
i_{time}	19.29	17.93	17.66	17.65	17.65
i_{space}	34.08	26.50	5.44	2.35	1.25
$i_{\text{ite}} \times 10^2$	0.38	0.26	0.21	0.19	0.60

5.2. Definition of the indicators

Let us now apply the dissipation concept to the three discrete problems. The admissible solution definition still corresponds to the kinematic constraints, the equilibrium equation and the state equations. So, an admissible solution for the space discrete problem is a solution defined on the entire interval of time that satisfies the kinematic constraints, the equilibrium equation in the finite element sense and the state equations at each integration point of the structure. An admissible solution for the time discrete problem is a solution defined on the whole domain and satisfying rigorously the kinematic constraints, the equilibrium equation and the state equations at each computed time. Finally, an admissible solution for the time–space discrete problem is a solution satisfying the kinematic constraints, the equilibrium equation in the finite element sense and the state equations at each integration points and each computed time. The construction of the admissible solutions partly uses the technique described in Section 4.2. The three admissible solutions will be denoted by hats.

The error associated to an admissible solution is measured by the non-verification of the evolution laws. This leads to the following absolute error indicators:

$$I_{\text{time}} \equiv \int_0^T \int_{\Omega_h} \eta(\hat{\epsilon}^p)_h, \hat{\mathbb{X}}_h, \hat{\mathbb{G}}_h, \hat{\mathbb{Y}}_h d\Omega_h dt$$

$$I_{\text{space}} \equiv \sum_{n=0}^{N-1} (t_{n+1} - t_n) \int_{\Omega} \eta(\hat{\epsilon}^p)_{n+1}, \hat{\mathbb{X}}_{n+1}, \hat{\mathbb{G}}_{n+1}, \hat{\mathbb{Y}}_{n+1} d\Omega$$

$$I_{\text{ite}} \equiv \sum_{n=0}^{N-1} (t_{n+1} - t_n) \int_{\Omega_h} \eta(\hat{\epsilon}^p)_{h,n+1}, \hat{\mathbb{X}}_{h,n+1}, \hat{\mathbb{G}}_{h,n+1}, \hat{\mathbb{Y}}_{h,n+1} d\Omega_h$$

The integration over Ω_h indicates that the integrations points are those of the finite element computation. Note also that I_{space} really measures the error associated to the time discrete problem since

$$\eta(\hat{\epsilon}^p_{n+1}, \hat{\mathbb{X}}_{n+1}, \mathbb{G}_{n+1}, \mathbb{Y}_{n+1}) = 0$$

$$\begin{pmatrix} \hat{\epsilon}^p_{n+1} \\ -\hat{\mathbb{X}}_{n+1} \end{pmatrix} \in \begin{pmatrix} \partial_{\mathbb{G}_{n+1}} \varphi^*(\mathbb{G}_{n+1}, \mathbb{Y}_{n+1}) \\ \partial_{\mathbb{Y}_{n+1}} \varphi^*(\mathbb{G}_{n+1}, \mathbb{Y}_{n+1}) \end{pmatrix}$$

The same remark holds for the iteration indicator.

The corresponding relative error indicators are defined as

$$i_{\text{time}} = \frac{I_{\text{time}}}{D_{\text{time}}} \quad i_{\text{space}} = \frac{I_{\text{space}}}{D_{\text{space}}} \quad i_{\text{ite}} = \frac{I_{\text{ite}}}{D_{\text{ite}}}$$

where

$$D_{\text{time}} \equiv 2 \sup_{t \in [0, T]} d_t$$

$$d_t \equiv \int_0^t \int_{\Omega_h} b_h d\Omega_h dt + \int_{\Omega_h} \psi^*(\hat{\mathbb{G}}_h, \hat{\mathbb{X}}_h) d\Omega_h |_t$$

$$b_h = \sup \left[\varphi^*(\hat{\mathbb{G}}_h, \hat{\mathbb{X}}_h) + \varphi(\hat{\epsilon}^p_h, \hat{\mathbb{X}}_h), \frac{R_0}{\|\hat{\mathbb{G}}_h^D\|} |\dot{\psi}_e(\hat{\epsilon}^e_h)| \right]$$

$$\psi^*(\hat{\mathbb{G}}_h, \hat{\mathbb{X}}_h) = \frac{1}{2} \text{Tr}[\mathbf{K}^{-1} \hat{\mathbb{G}}_h \hat{\mathbb{G}}_h] + \psi_p(\hat{\mathbb{X}}_h)$$

$$D_{\text{space}} \equiv 2 \sup_{k \in [0, \dots, N-1]} d_k$$

$$d_k \equiv \sum_{n=0}^k (t_{n+1} - t_n) \int_{\Omega} b_{n+1} d\Omega + \int_{\Omega} \psi^*(\hat{\mathbb{G}}_{k+1}, \hat{\mathbb{X}}_{k+1}) d\Omega \\ + \sum_{n=0}^k (t_{n+1} - t_n)^2 \int_{\Omega} \psi^*(\hat{\mathbb{G}}_{n+1}, \hat{\mathbb{X}}_{n+1}) d\Omega$$

$$b_{n+1} = \sup \left[\varphi^*(\hat{\mathbb{G}}_{n+1}, \hat{\mathbb{Y}}_{n+1}) \\ + \varphi(\hat{\epsilon}^p_{n+1}, \hat{\mathbb{X}}_{n+1}), \frac{R_0}{\|\hat{\mathbb{G}}_{n+1}^D\|} |\text{Tr}[\mathbf{K} \hat{\epsilon}^e_{n+1} \hat{\epsilon}^e_{n+1}]| \right]$$

$$D_{\text{ite}} \equiv 2 \sup_{k \in [0, \dots, N-1]} d_{h,k}$$

$d_{h,k}$ has the same expression as d_k except that Ω is replaced by Ω_h and $d_{h,k}$ is evaluated with the fields $(\underline{U}_{h,n+1}, \mathbb{G}_{h,n+1}, \mathbb{Y}_{h,n+1}, \mathbb{R}_{h,n+1}, \mathbb{X}_{h,n+1})$.

5.3. Numerical results

Tables 1 and 2 give the computed indicators. We note that:

- The time (space) indicator $i_{\text{time}}(i_{\text{space}})$ is almost insensitive to the number of elements (time steps) and approximates very well the time (space) error $\epsilon_{\text{time}}(\epsilon_{\text{space}})$.
- The space and time indicators decrease monotonically with respect to the number of elements and time steps, respectively.
- The error is close to the sum of the space and time indicators.
- The iteration indicator i_{ite} is very small in comparison with the space and time indicators.

Table 3

Influence of the tolerance on the error and on the indicators for the holed plate problem. “Number of iterations” is the total number of iterations of the finite element computation

θ	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
<i>The holed plate problem (coarse discretization)</i>						
ϵ	25.24	13.89	12.96	12.96	12.96	12.96
i_{space}	28.89	7.73	5.41	5.40	5.40	5.40
i_{time}	23.09	9.94	8.22	8.15	8.14	8.14
i_{ite}/θ	27.38	36.21	13.82	30.23	29.71	38.53
Number of iterations	4	17	31	50	64	78
<i>The holed plate problem (fine discretization)</i>						
ϵ	20.69	6.25	2.71	2.55	2.57	2.57
i_{space}	20.54	6.50	2.12	1.88	1.89	1.89
i_{time}	20.09	5.21	1.07	0.60	0.53	0.52
i_{ite}/θ	20.04	55.54	63.51	85.02	115.1	124.2
Number of iterations	7	18	56	97	137	169

The last point shows that the error due to the iterative technique is much smaller than the time and space discretization errors. During the finite element computation, the quality of the discrete problem resolution is governed by a tolerance θ on the equilibrium equation residuals. The convergence criterion is based on the energy norm:

$$\frac{\|\tilde{\Phi}_{h,n+1} - \tilde{\Phi}_{h,n}\|_{L_2(\Omega_n)}}{\|\tilde{\Phi}_{h,n+1}\|_{L_2(\Omega_n)}} \leq \theta \quad (22)$$

The finite element results for Tables 1 and 2 were obtained with $\theta = 10^{-3}$.

Tables 3 and 4 give the influence of a tolerance modification on the error and on the indicators. A coarse and a fine discretization are considered for both examples. We note that:

- under a certain critical tolerance θ_{crit} , the error and the space and time indicators are insensitive to the tolerance;
- the iteration indicator is more or less linear in the tolerance;
- the tolerance θ decreases exponentially with respect to the number of iterations.

θ_{crit} , which depends both on the quality of the space and time discretization, is smaller when the discretization is finer. In other words, when the discretization of the problem is coarse, only a few iterations are needed; otherwise a larger number is needed.

The critical tolerance concept is important in the framework of adaptive control of the computation. Indeed, if the finite element computation is carried out with the critical tolerance, no useless global iteration is

Table 4

Influence of the tolerance on the error and on the indicators for the frame problem

θ	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
<i>The frame problem (coarse discretization)</i>						
ϵ	26.60	16.42	16.36	16.37	16.37	16.37
i_{space}	22.74	11.75	11.67	11.67	11.67	11.67
i_{time}	21.62	5.32	4.87	4.82	4.82	4.82
i_{ite}/θ	17.78	5.59	6.48	4.48	5.08	5.17
Number of iterations	28	72	106	130	154	180
<i>The frame problem (fine discretization)</i>						
ϵ	12.40	6.05	5.37	5.36	5.36	5.36
i_{space}	12.59	5.13	4.47	4.47	4.47	4.47
time	10.74	2.39	0.64	0.55	0.55	0.55
i_{ite}/θ	11.09	17.01	8.23	6.20	7.54	8.04
Number of iterations	40	112	194	262	322	386

Table 5

Space indicator (%) as a function of the number of degrees of freedom for three- and six-node triangles

Number of degrees of freedom	184	662	2506	9746
i_{space} (three-node)	32.90	28.05	14.63	5.42
i_{space} (six-node)	13.38	5.37	1.82	0.62

done and the influence of the iterative technique is avoided. For the examples treated, the following relation

$$i_{\text{ite}} = \frac{1}{10} \min(i_{\text{time}}; i_{\text{space}})$$

is more or less satisfied when $\theta \approx \theta_{\text{crit}}$. This relation is independent from the discretization of the problem and will be taken into account in the next section.

6. Adaptive control

The adaptive control of the computation is taken in the following sense: given a first coarse computation leading to an error ϵ , one wishes to make a new calculation (as cheap as possible) leading to a required error ϵ_0 . We propose the following strategy.

Initial parameters choice: the initial mesh results from a mesh optimization in elasticity for a required error of, for instance, $2\epsilon_0$, with the loading applied corresponding to the maximum loading in the real problem. The time discretization needs simply to represent the loading. The initial tolerance is set to $\theta = 10^{-2}$.

Determination of the optimized parameters: the error behaves almost as the sum of the space and time indicators. In order to share efficiently the two indicators in the required error, we minimize a cost function. An h -type mesh adaptation is used (the size of the elements is modified but their degree is kept constant). We suppose that the cost function is proportional to the number of elements at a certain power α and the number of time steps at a certain power β . The unknowns involved in the cost function are the size modifications of the elements and the time steps, r_E and $r_{\Delta t}$, respectively. Let h_E be the size of an element in the initial mesh and h_E^* the size of the corresponding element in the optimized mesh. r_E is the quotient

$$r_E = \frac{h_E^*}{h_E}$$

Similarly, $r_{\Delta t}$ is the quotient of the time step size Δt in the optimized time discretization to the size Δt^* in the initial one

$$r_{\Delta t} = \frac{\Delta t^*}{\Delta t}$$

The cost function thus reads

$$\left(\sum_E \frac{1}{r_E^2} \right)^\alpha \left(\sum_{\Delta t} \frac{1}{r_{\Delta t}} \right)^\beta$$

Note that the r_E ($r_{\Delta t}$) are defined for each element (time-step) in the initial mesh (time discretization).

We still need to express the required error in terms of the r_E and $r_{\Delta t}$. The element contributions to the space indicator for the optimized mesh i_{space, E^*} are supposed to be linked to the contribution in the initial mesh $i_{\text{space}, E}$ by

$$i_{\text{space}, E^*} = r_E^p i_{\text{space}, E}$$

The same supposition is made for the time contributions:

$$i_{\text{time}, \Delta t^*} = r_{\Delta t}^q i_{\text{time}, \Delta t}$$

The choice of the parameters p and q will be discussed further. The minimization problem reads:

Problem 1.

Minimize $\left(\sum_E \frac{1}{r_E^2} \right)^\alpha \left(\sum_{\Delta t} \frac{1}{r_{\Delta t}} \right)^\beta$ under the restrictions:

$$\epsilon_0 = i_{\text{space}, 0} + i_{\text{time}, 0}, \quad i_{\text{space}, 0} = \sum_E r_E^p i_{\text{space}, E},$$

$$i_{\text{time}, 0} = \sum_{\Delta t} r_{\Delta t}^q i_{\text{time}, \Delta t}$$

When solving this minimization problem, one observes that the goal indicators $i_{\text{space}, 0}$ and $i_{\text{time}, 0}$ depend only upon the goal error ϵ_0 and the four parameters α , β , p and q :

$$i_{\text{space}, 0} = \frac{2\alpha/p}{2\alpha/p + \beta/q} \epsilon_0, \quad i_{\text{time}, 0} = \frac{\beta/q}{2\alpha/p + \beta/q} \epsilon_0$$

Table 6
Summary of the adaptive control for the holed plate problem

	ϵ	i_{space}	i_{time}	i_{ite}	Number of elements	Number of steps	θ
Initially	43.91	13.40	37.75	0.09	37	2	1×10^{-2}
Asked	10	8	2	0.20			
Obtained	11.80	9.6	3.1	1.20	69	5	$2 \times 12 \times 10^{-2}$
Asked	5	4	1	0.10			
Obtained	5.04	4.31	0.62	0.11	172	7	1.72×10^{-3}

It can also be observed that the Problem 1 is decoupled in Problems 2 and 3.

Problem 2.

$$\text{Minimize } \sum_E \frac{1}{r_E^2} \quad \text{with } i_{\text{space},0} = \sum_E r_E^p i_{\text{space},E}$$

Problem 3.

$$\text{Minimize } \sum_{\Delta t} \frac{1}{r_{\Delta t}} \quad \text{with } i_{\text{time},0} = \sum_{\Delta t} r_{\Delta t}^q i_{\text{time},\Delta t}$$

Owing to the decreasing behaviour of the time indicator in Table 1, we have taken $q = 2$. Concerning the space, Table 5 gives the space indicator evolution for a growing number of degrees of freedom for the holed plate problem. One can see that the six-node triangles are much more efficient than the three-node triangles. So, we will use only six-node triangles and we take $p = 2$. With the value $q = 2$, $p = 2$ and the choice $\alpha = 2$, $\beta = 1$ the space and time indicators contribute to the goal error in the ratios $4/5$, $1/5$, respectively.

Minimization problems 2 and 3 give the size modifications for the mesh and the time steps. These modifications are used to remesh the domain and to define a new time discretization. The time interval is “remeshed” on the basis of $r_{\Delta t}$.

Finally, the new tolerance θ^* is chosen so that

$$i_{\text{ite},0} = \frac{1}{10} \min(i_{\text{time},0}; i_{\text{space},0})$$

Since i_{ite} is proportional to the tolerance, we take

$$\theta^* = \frac{i_{\text{ite},0}}{i_{\text{ite}}}$$

If the quotient of the initial error to the goal error is too important (≥ 3), one or several intermediate steps are performed in the adaptive control.

Examples: Tables 6 and 7 and Figs. 5–6 summarize two adaptive control examples. In both cases, the goal error is set to 5% and one intermediate step is needed. The initial meshes were optimized in elasticity for an error of 10%. We observe that the mesh gets finer in the re-entrant corners and that the time discretization has been refined during the loadings and has not been modified during the unloadings. All the meshes were obtained with the ARAIGNEE [28] software.

7. Conclusion

A new a posteriori error estimation, called dissipation error, has been implemented to control time-dependent nonlinear finite element analysis. The dissipation error, combined with efficient time, space and iteration indicators made it possible to adapt simultaneously the mesh, the time discretization and the stopping criterion of the iterative algorithm. The reliability of the adaptive strategy was shown for two plane stress problems with the plastic Prandtl–Reuss model.

Table 7
Summary of the adaptive control for the frame problem

	ϵ	i_{space}	i_{time}	i_{ite}	Number of elements	Number of steps	θ
Initially	28.15	14.33	17.90	0.03	194	3	1×10^{-2}
Asked	10	8	2	0.20			
Obtained	12.30	10.03	2.18	0.38	253	7	7.41×10^{-2}
Asked	5	4	1	0.10			
Obtained	5.37	4.48	0.74	0.17	661	11	1.91×10^{-2}

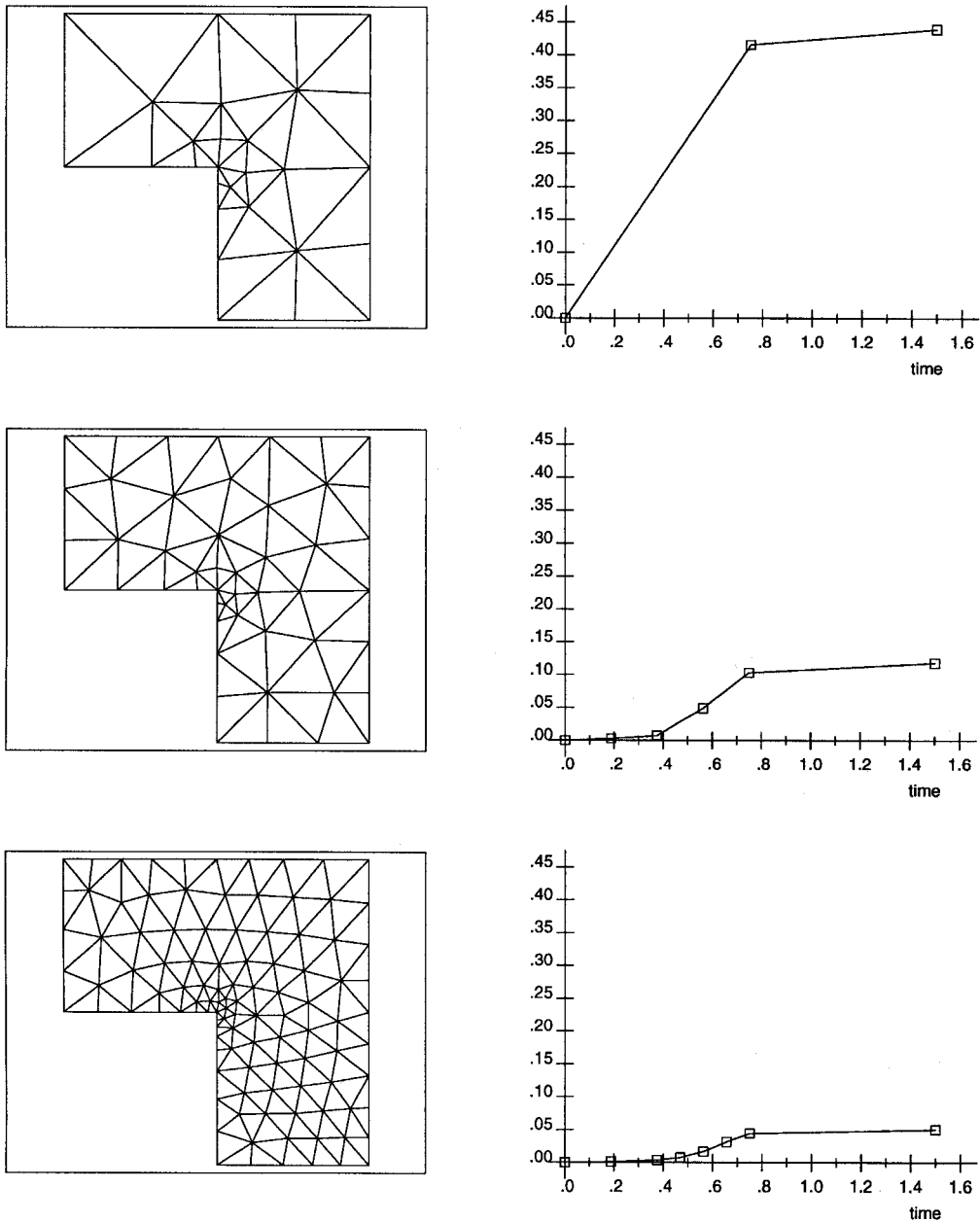


Fig. 5. Initial, intermediate and final mesh and error evolution for the holed plate problem.

References

- [1] Ladevèze P. The basis of the error on the constitutive law method for the control of finite element computations: the work of the mid 70', Internal report 163, LMT, Ecole Normale Supérieure de Cachan, 1995 (in French).
- [2] Ladevèze P, Pelle JP, Rougeot P. Error estimation and mesh optimization for classical finite elements. *Engng Comput* 1991;8:69–80.
- [3] Babūška I, Rheinboldt WC. A posteriori estimates for the finite element method. *Int J Numer Meth Engng* 1978;12:1597–615.
- [4] Babūška I, Rheinboldt WC. Adaptive approaches and re-

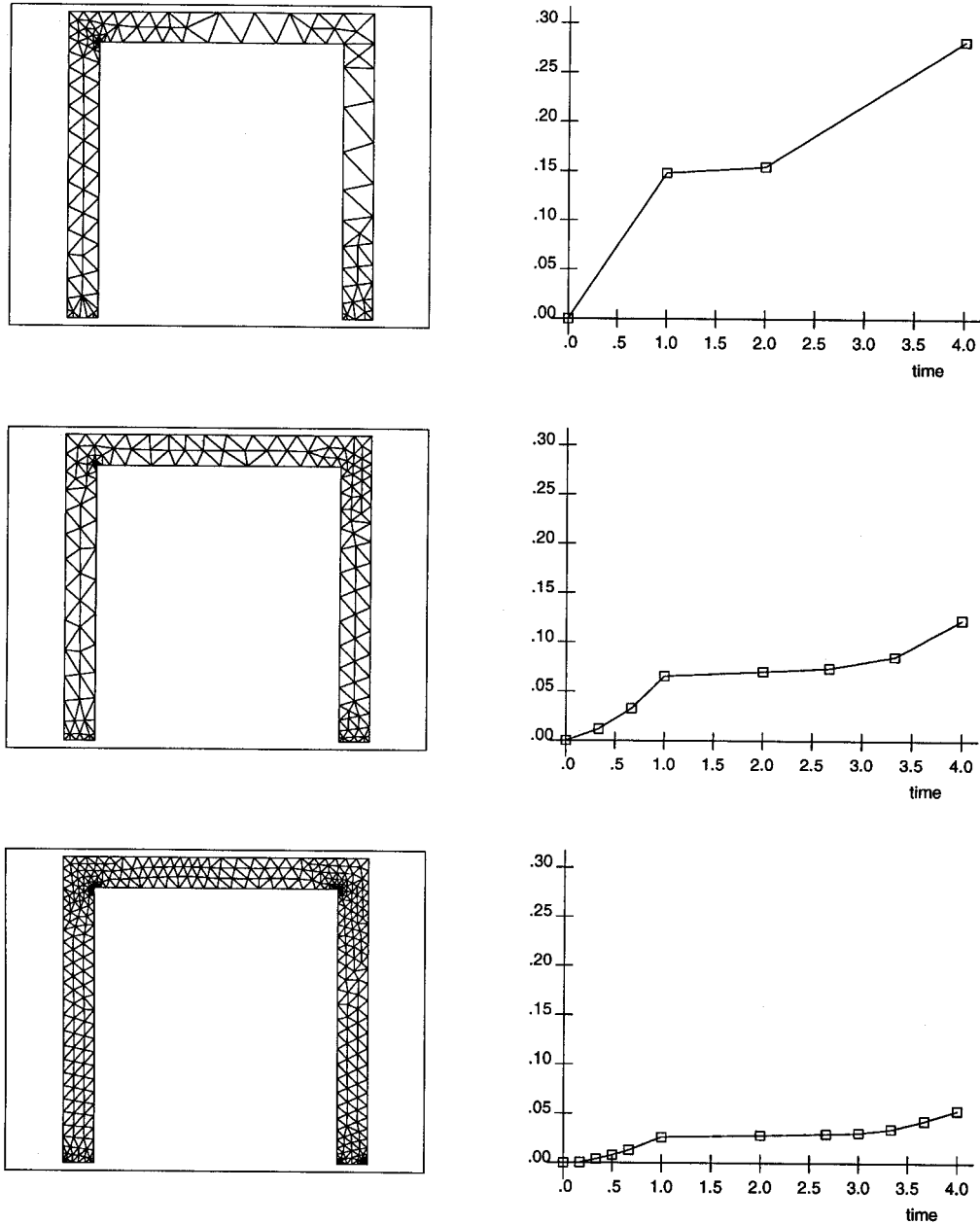


Fig. 6. Initial, intermediate and final mesh and error evolution for the frame problem.

liability estimators in finite element analysis. *Comput Meth Appl Mech Engng* 1979;17/18:519–40.

- [5] Kelly DW, Gago J, Zienkiewicz OC, Babüska I. A posteriori error analysis and adaptative processes in finite element method. Part 1: error analysis. *Int J Numer Meth Engng* 1983;19:1593–619.
- [6] Gago J, Kelly DW, Zienkiewicz OC, Babüska I. A posteriori error analysis and adaptative processes in finite el-

ement method. Part 2: adaptative mesh refinement. *Int J Numer Meth Engng* 1983;19:1621–56.

- [7] Oden JT, Demkowicz L, Rachowicz W, Westermann TA. Toward a universal h - p adaptative finite element strategy. Part 2: a posteriori error estimation. *Comput MethAppl Mech Engng* 1989;77:113–80.
- [8] Zienkiewicz OC, Zhu JZ. A simple error estimator and

- adaptive procedure for practical engineering analysis. *Int J Numer Meth Engng* 1987;24:337–57.
- [9] Zienkiewicz OC, Zhu JZ. The superconvergent patch recovery and a posteriori error estimates. Part 1: the recovery technique. *Int J Numer Meth Engng* 1992;33:1331–64.
- [10] Zienkiewicz OC, Zhu JZ. The superconvergent patch recovery and a posteriori error estimates. Part 2: error estimates and adaptativity. *Int J Numer Meth Engng* 1992;33:1365–82.
- [11] Babuska I, Strouboulis T, Upadhyay CS, Gangaraj SK, Copps K. Validation of a posteriori error estimators by numerical approach. *Int J Numer Meth Engng* 1994;37:1073–123.
- [12] Babuska I, Strouboulis T, Gangaraj SK, Upadhyay CS. Validation of recipes for the recovery of stresses and derivatives by a computer-based approach. *Math Comput Modell* 1994;20:45–89.
- [13] Fraeijs de Veubeke B. Displacement and equilibrium models in the finite element method. In: Zienkiewicz OC, editor. *Stress analysis*. New York: Wiley, 1965 Chapter 9.
- [14] Debongnie JF, Zhong HG, Beckers P. Dual analysis with general boundary conditions. *Comput Meth Appl Mech Engng* 1995;122:183–92.
- [15] Babuska I, Rheinboldt WC. Computational error estimates and adaptive processes for some nonlinear structural problems. *Comput Meth Appl Mech Engng* 1982;34:895–937.
- [16] Johnson C, Hansbo P. Adaptive finite element methods in computational mechanics. *Comput Meth Appl Mech Engng* 1992;101:143–81.
- [17] Bass JM, Oden JT. Adaptive finite element methods for a class of evolution problems in viscoplasticity. *Int J Engng Sci* 1987;25(6):623–53.
- [18] Zienkiewicz OC, Liu YC, Huang GC. Error estimation and adaptativity in flow formulation for forming problems. *Int J Numer Meth Engng* 1988;25:23–42.
- [19] Tie B, Aubry D. Error estimates, h adaptive strategy and hierarchical concept for non linear finite element method. In: Hirsch Ch, et al., editors. *Numerical methods in engineering '92*. Oxford: Elsevier, 1992. p. 17–24.
- [20] Ladevèze P. Sur une famille d'algorithmes en mécanique des structures. *C-R Acad Sci, Sér II* 1985;300:41–4.
- [21] Ladevèze P, Coffignal G, Pelle JP. Accuracy of elastoplastic and dynamic analysis. In: Babuska I, Gago J, Oliveira, Zienkiewicz OC, editors. *Accuracy estimates and adaptive refinements in Finite Element computations*. New York: Wiley, 1986. p. 181–203.
- [22] Gallimard L, Ladevèze P, Pelle JP. Error estimation and adaptativity in elastoplasticity. *Int J Numer Meth Engng* 1996;39:189–217.
- [23] Ladevèze P. La méthode à grand incrément de temps pour l'analyse de structures à comportement non linéaires décrit par variables internes. *C-R Acad Sci, Sér II* 1989;309(11):1095–9.
- [24] Ladevèze P. *Mécanique non linéaire des structures. Nouvelle approche et méthodes de calcul non incrémentales*. Hermes, 1996.
- [25] Ladevèze P, Moës N. A new a posteriori error estimation for nonlinear time-dependent finite element analysis. *Comput Meth Appl Mech Engng* 1998;157:45–68.
- [26] Ladevèze P. Nouvelles procédures d'estimations d'erreur relative à la méthode des éléments finis. In: *Proceedings, Journées Eléments finis, Rennes, 1977*.
- [27] Coorevits P. *Maillage adaptatif anisotrope: application aux problèmes de dynamiques*. Thèse, Ecole Normale Supérieure de Cachan, 1993.
- [28] Coffignal G. *Optimisation et fiabilité des calculs éléments finis en élastoplasticité*. Thèse d'état, Université de Pierre et Marie Curie, 1987.