A new model of damage

Claude Stolz, Nicolas Moes

A new model of damage: a moving thick layer approach

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Abstract A new formulation of a damage law is proposed based on a continuous transition between a sound material and a totally or partially broken material. The evolution of damage is then associated with a moving layer. This point of view permits the description of initiation and propagation of defects in an unified framework. The motion of the thick layer is defined in the frame of the moving surface Γ_o separating the sound material and the damaged material. When the damage parameters are continuous functions of the distance to Γ_o, they satisfy the conditions of transport. For particular geometries and loadings the evolution of the system is discussed. Comparison with description of damage with discontinuities and sharp interface is also presented.

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1 Introduction

Fracture mechanics is not sufficient generally to model the full scenario of the degradation of solids under mechanical loading. Crack initiation for instance requires damage description for modelling the gradual loss of stiffness.

When the damage evolves some regularization process are needed to avoid spurious localization. Some constitutive laws based on second gradient description of damage (Bourdin et al. 2000) or phase field approach (Karma et al. 2001; Miehe et al. 2010) have been recently proposed for this purpose. Many other approaches have been proposed to regularize the damage evolution (Pijaudier Cabot and Bazant 1987; Peerlings et al. 1996).

In this paper we propose to model the damage like the propagation of level set surface (Moës et al. 2011). For elastic quasi-brittle material, the evolution of the interface separating the undamaged material $d=0$ from the damaged material $d=1$ have been studied using an energetical description of the propagation (Pradeilles-Duval and Stolz 1991, 1995). In this description the parameter d jumps instantaneously from 0 to 1. The interface has no thickness and some mechanical quantities present discontinuities. Criteria of uniqueness and of stability of the propagation have been also

established. In this approach, nucleation of defects can be considered as a bifurcation of equilibrium solution (Stolz 2007).

Descriptions of moving interfaces and of moving layers are also useful for studying loss of matter. Such a thermodynamical description of thin or thick layers permits to describe complex process of wear between two bodies in relative motion (Stolz 2010).

The initial material and the damaged material are separated by a surface Γ_o. This boundary is a moving interface. A surface is considered as an isopotential or a level-set. This surface is an interface where the material begins to change its properties. Two cases may be considered: the propagation of a sharp interface and the propagation of a moving layer.

Sharp interface. If the transition zone is very thin, the transition from sound material to damaged material can be discontinuous and the interface is called sharp interface. The propagation of such an interface has been studied (Pradeilles-Duval and Stolz 1991, 1995).
The thermodynamical force associated to the propagation of the interface is the local energy release rate $G(s)$. Moreover stability and non-bifurcation conditions for the problem of evolution are given when a normality rule is associated with a generalized Griffith's criterion like $G(s) \leq G_c$.
Moving layer. If the interface is thick, the transition should be more regular. The evolution of damage is then associated to a moving layer. The thickness of the transition zone provides a length scale in the modelization.

The purpose of this article is to define the motion of a layer according to specific damage behaviour. The analysis of dissipation provides the definition of the energy release rate $G(s)$ associated to the motion of the layer. Other expressions of the dissipation are also given in terms of $Eshelby$'s momentum tensor. A normality rule based on a criterion function of $G(s)$ is then proposed. The evolution of damage inside a bar is then studied in this framework after comparison with sharp interface modelization.

2 A level set based damage model

We consider classical constitutive equations for describing the stress–strain relations and we propose a new formulation for the damage evolution.

The main idea of the modelization is to consider that damage occurs in a layer of finite thickness l evolving from 0 to l_c. Damage is described by a parameter d which varies from 0 to 1. The boundary between the sound material and the damaged material is denoted by Γ_o which is moving accordingly to the damage evolution. The surface Γ_o is the level-set $\phi(\mathbf{M}, t) = 0$, where $\phi(\mathbf{M}, t)$ is a signed distance function of point $\mathbf{M}$ to Γ_o. The damage d is considered as a function of the level-set ϕ.

2.1 A simple local damage model

The free energy w per unit of volume depends on the strain $\boldsymbol{\epsilon}$ and on a scalar variable d for the damage: $w = w(\boldsymbol{\epsilon}, d)$. The state law is given by

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\epsilon}}; \quad Y = -\frac{\partial w}{\partial d}, \tag{1}$$

where $\boldsymbol{\sigma}$ is the stress and Y the local energy release rate, they are functions of the state variables $(\boldsymbol{\epsilon}, d)$. For instance, we may use a potential of the form

$$w(\boldsymbol{\epsilon}, d) = \frac{1}{2}\boldsymbol{\epsilon} : \mathcal{C}(d) : \boldsymbol{\epsilon}, \tag{2}$$

then the equations of state are given as

$$\boldsymbol{\sigma} = \mathcal{C}(d) : \boldsymbol{\epsilon}; \quad Y = -\frac{1}{2}\boldsymbol{\epsilon} : \frac{\partial \mathcal{C}}{\partial d} : \boldsymbol{\epsilon}. \tag{3}$$

The thermodynamical force $Y(\boldsymbol{\epsilon}, d)$ must be finite for any $d \in [0, 1]$, this condition of regularity ensures that the dissipation is defined.

The simple form of $\mathcal{C}(d)$ is given by choosing $\mathcal{C} = d\,\mathcal{C}_1 + (1-d)\,\mathcal{C}_o$ or $\mathcal{C}^{-1} = d\,\mathcal{C}_1^{-1} + (1-d)\,\mathcal{C}_o^{-1}$. When d increases monotonically from 0 to 1, this describes the transition between a material of elastic properties $\mathcal{C}_o$ to a partially damaged material of elastic properties $\mathcal{C}_1$. For total damage description $\mathcal{C}_1 = 0$. Such behaviours are generalizations of behaviour considered in Moës et al. (2011).

The evolution of damage is determined by a dissipation potential $\mathrm{D}(\dot{d})$ convex function of $\dot{d}$ such that $\mathrm{D}(0) = 0$. The thermodynamical force Y satisfies the normality rule

$$\mathrm{D}(\dot{d}) + Y(d^* - \dot{d}) \leq \mathrm{D}(d^*), \quad \forall d^*, \tag{4}$$

which is equivalent to the relation $Y = \dfrac{\partial \mathrm{D}}{\partial \dot{d}}$ in the case of a regular potential D. These properties ensure positive dissipation. A dual formulation is given in

terms of dual potential $\mathrm{D}^*(Y) = sup_{\dot{d}}(Y\dot{d} - \mathrm{D}(\dot{d}))$ then $\dot{d} = \dfrac{\partial \mathrm{D}^*}{\partial Y}$.

For a domain of reversibility defined by $Y - Y_c \leq 0$, and a time independent constitutive relation, the normality rule is expressed as

$$\dot{d} \geq 0, \quad Y - Y_c \leq 0, \quad (Y - Y_c)\dot{d} = 0. \tag{5}$$

2.2 A new modelization

A function level set ϕ is introduced. The level set $\phi = 0$ separates the domain Ω into a undamaged and a damaged zone, this surface is denoted Γ_o.

The whole body Ω is then decomposed into three parts, the undamaged body Ω_1, the zone Ω_c of transition, the damaged body Ω_2 when damage is established. In this case, the boundary $\partial\Omega_c$ is decomposed into Γ_o where $d = 0, \phi(\mathbf{M}_o^t, t) = 0$, and Γ_1 where $d = 1, \phi(\mathbf{M}^t, t) = l_c$.

In the damaged zone Ω_c the variable d is an explicit function of the level set. The damage increases as the level set value rises:

$$\begin{cases} d(\phi) = 0, & \phi \leq 0, \quad \mathbf{X} \in \Omega_1, \\ d'(\phi) \geq 0, & 0 \leq \phi \leq l_c, \quad \mathbf{X} \in \Omega_c, \\ d(\phi) = 1, & \phi \geq l_c, \quad \mathbf{X} \in \Omega_2, \end{cases} \tag{6}$$

$d'(\phi)$ is the derivative of d relatively to ϕ. The function d is assumed to be continuous for clarity. Discontinuous function can be chosen in the proposed approach but the obtained expressions must be modified with additional contributions.

As the damage is a given continuous function of the signed distane ϕ to Γ_o, the evolution of the layer is then associated to the motion of the surface Γ_o.

2.3 The problem of equilibrium

Consider a body Ω. The external boundary $\partial\Omega$ is decomposed into two parts: $\partial\Omega_u$ where the displacement $\mathbf{u}$ is prescribed $\mathbf{u}^d(t)$ and $\partial\Omega_T$ where the loading $\mathbf{T}^d(t)$ is applied; $\partial\Omega = \partial\Omega_u \cup \partial\Omega_T, \emptyset = \partial\Omega_u \cap \partial\Omega_T$.

The total potential energy $\mathcal{E}$ is a function of $(\mathbf{u}, \phi)$

$$\mathcal{E}(\mathbf{u}, \phi) = \int_{\Omega} w(\boldsymbol{\epsilon}(\mathbf{u}), d(\phi))\, \mathrm{d}\Omega - \int_{\partial\Omega_T} \mathbf{T}^d.\mathbf{u}\, \mathrm{ds}, \tag{7}$$

$\boldsymbol{\epsilon}(\mathbf{u})$ is the strain, symmetric part of $\nabla\mathbf{u}$. The displacement $\mathbf{u}$ satisfies the boundary conditions on $\partial\Omega_u$: $\mathbf{u} = \mathbf{u}^d$.

Continuity conditions (CC). The displacement $\mathbf{u}$, the stress vectors $\boldsymbol{\sigma}.\mathbf{n}$, the modulus $\mathcal{C}(d)$ are continuous functions accross the surface Γ_o. Then the gradient of the displacement and the free energy are continuous functions along each iso-ϕ surface and inside the layer. These properties imply that the velocity $\dot{\mathbf{u}}$ is also continuous. Therefore virtual variations of the displacement and of the moving surface $\phi = 0$ are uncoupled.

2.4 General motion of a layer

A point $\mathbf{M}_o^t$ on the surface Γ_o has curvilinear coordinates $s^\alpha, \alpha = 1, 2$. The local basis is defined by the tangent vectors $\mathbf{T}_\alpha = \dfrac{\mathrm{d}\mathbf{M}_o^t}{\mathrm{d}s^\alpha}$ and the normal vector $\mathbf{N} = \nabla\phi$ to the surface $\phi(\mathbf{M}_o^t, t) = 0$ (Fig. 1). A point $\mathbf{M}$ at the distance ϕ of the curve Γ_o has coordinates such that

$$\mathbf{M} = \mathbf{M}_o^t(s^\alpha) + \phi\, \mathbf{N}(s^\alpha). \tag{8}$$

In Ω_c an element of volume satisfies

$$\mathrm{d}\Omega = j\, \mathrm{d}\phi\, \mathrm{ds}, \tag{9}$$

where ds is the element area at point $\mathbf{M}_o$. Due to the curvature of the surface the jacobian j is $\det(\boldsymbol{I} - \phi\bar{\boldsymbol{\kappa}})$, where $\bar{\boldsymbol{\kappa}}$ is the curvature tensor of the surface Γ at point $\mathbf{M}_o$.

In 2D, Γ_o is a curve described by the curvilinear coordinate s with tangent vector $\mathbf{T}$ and curvature κ given by

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{ds}} = \kappa\, \mathbf{N}. \tag{10}$$

The local derivatives of the elementary vector are

$$\begin{aligned} \mathrm{d}\mathbf{M} &= \mathrm{d}\mathbf{M}_o + \mathbf{N}\mathrm{d}\phi - \phi\kappa\mathbf{T}\, \mathrm{ds} \\ &= (1 - \kappa\phi)\, \mathbf{T}\, \mathrm{ds} + \mathbf{N}\mathrm{d}\phi. \end{aligned} \tag{11}$$

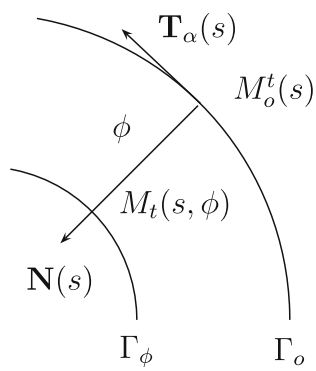

Fig. 1 Local geometry of the layer

these expressions define the local metric and the elementary area is given by $dA = (1-\phi\kappa(s))\mathrm{d}\phi\ \mathrm{d}s$ where $\kappa(s)$ is the curvature at point s of the surface $\phi = 0$.

The motion of the local geometry. When the curve Γ_o is moving, the local basis is changing accordingly. At time $t^+ = t + \Delta t$, the point $\mathbf{M}_o^t$ comes in $\mathbf{M}_o(t^+)$ such that $\phi(\mathbf{M}_o(t^+), t + \Delta t) = 0$ then $\mathbf{M}_o(t^+) = \mathbf{M}_o^t - \Delta t\ a\ (s^\alpha)\mathbf{N}$ where the normal speed a satisfies the relation

$$\frac{\partial \phi}{\partial t} - a\nabla\phi.\mathbf{N} = 0. \tag{12}$$

Any quantity f defined at a point $\mathbf{M}_o^t$ is varying during this motion. Introducing the derivative of f following the motion of Γ_o by

$$D_a f = \lim_{\Delta t \to 0} \frac{f(\mathbf{M}_o(t^+)) - f(\mathbf{M}_o^t)}{\Delta t}. \tag{13}$$

We obtain for the evolution of the geometry

$$D_a\mathbf{T} = -\frac{\mathrm{d}a}{\mathrm{d}s}\mathbf{N}, \quad D_a\mathbf{N} = \frac{\mathrm{d}a}{\mathrm{d}s}\mathbf{T},$$
$$D_a\kappa = -\frac{\mathrm{d}^2a}{\mathrm{d}s^2} - \kappa^2 a \tag{14}$$

In an analogous way, any function $F(\mathbf{M}, t)$ defined at a point of $\mathbf{M}$ of Ω_c is variyng during this motion. Introducing the derivative of F following the motion of Γ_o by

$$D_a F = \lim_{\eta \to 0} \frac{F(\mathbf{M} + \eta D_a\mathbf{M}, t + \eta) - F(\mathbf{M}, t)}{\eta}$$
$$= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \mathbf{M}}.D_a\mathbf{M}$$

where $D_a\mathbf{M} = D_a\mathbf{M}_o + \phi D_a\mathbf{N} = -a\ \mathbf{N} + \phi\ \dfrac{\mathrm{d}a}{\mathrm{d}s}\ \mathbf{T}$.

The position of the point $\mathbf{M}_o^t$ of Γ_o satisfies the equation $\phi(\mathbf{M}_o^t, t) = 0$, then $D_a\phi(\mathbf{M}_o^t, t) = 0$ that is exactly (12). The point $\mathbf{M}_t = \mathbf{M}_o^t + z\ \mathbf{N}$ is on the iso-level set $\phi(\mathbf{M}, t) = z$, the evolution of this surface is given by $D_a\phi(\mathbf{M}_t, t) = 0$ and obviously the normal velocity of this surface is also $a(s)$. As ϕ is a distance function, $||\mathbf{N}|| = 1$, then the variations $\delta\phi$ of ϕ are functions only of s and are denoted δa.

Variations of the potential energy. At time t, the value of the field ϕ is known. The function d is known, and $\dot{d} = \dfrac{\partial d}{\partial \phi}\dfrac{\partial \phi}{\partial t} = -d'(\phi)a(s)$. The variations of d are then $\delta d = -d'(\phi)\delta a(s)$.

The total variations of $\mathcal{E}$ with respect to the displacement and to the function ϕ taking account of the state laws are

$$\frac{\partial \mathcal{E}}{\partial \mathbf{u}}.\delta\mathbf{u} + \frac{\partial \mathcal{E}}{\partial \phi}.\delta a = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\delta\mathbf{u})\,\mathrm{d}\Omega$$
$$- \int_{\partial\Omega_T} \mathbf{T}.\delta\mathbf{u}\ \mathrm{d}s - \int_{\Omega_c} Yd'(\phi)\delta a\ \ \mathrm{d}\Omega.$$

Due to the continuity conditions (CC) the variations of displacement and the variations of the surface $\phi = 0$ are uncoupled.

Equilibrium of the system. For given ϕ, the system is that of a non-homogeneous elastic medium under boundary conditions ($\mathbf{T}^d$ along $\partial\Omega_T$ and $\mathbf{u}^d$ over $\partial\Omega_u$). The solution of the problem of equilibrium is determined by the solution of minimization of the potential energy. Then, for a state of equilibrium, the variations of the total potential with respect to $\mathbf{u}$ are zero

$$\frac{\partial \mathcal{E}}{\partial \mathbf{u}}.\delta\mathbf{u} = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\delta\mathbf{u})\ \mathrm{d}\Omega - \int_{\partial\Omega_T} \mathbf{T}^d.\delta\mathbf{u}\ \ \mathrm{d}s = 0, \tag{15}$$

where the variations $\delta\mathbf{u}$ satisfy the boundary conditions $\delta\mathbf{u} = 0$ over $\partial\Omega_u$. By integration by parts equations of equilibrium are recovered

$$\mathrm{div}\,\boldsymbol{\sigma} = 0, \quad \mathbf{n}.\boldsymbol{\sigma} = \mathbf{T}^d, \quad \text{over } \partial\Omega_T. \tag{16}$$

Evolution of the layer. The variations of $\mathcal{E}$ are then reduced to those with respect to ϕ

$$\frac{\partial \mathcal{E}}{\partial \phi}.\delta a = -\int_{\Omega_c} Yd'(\phi)\ \delta a\ \mathrm{d}\Omega = -\int_{\Gamma_o} G(s)\delta a(s)\ \mathrm{d}s, \tag{17}$$

where the release rate of energy appears associated to the motion of the layer

$$G(s) = \int_o^l Yd'(\phi)\ j\ \mathrm{d}\phi. \tag{18}$$

In 2D situation, this expression becomes

$$G(s) = \int_o^l Yd'(\phi)\ (1 - \phi\kappa(s))\ \mathrm{d}\phi. \tag{19}$$

This defines the variation of the energy when the layer is moving. The thickness at time t is $l(t)$ less than l_c. The description is valid from the initiation of the layer from zero-thickness to a finite thickness l_c.

When l tends to 0, the thermodynamical force $G(s)$ tends to 0, and the dissipation vanishes. Definition of the evolution law must be compatible with this fact.

2.5 Dissipation of the system and evolution law

The dissipation has the classical expression

$$D_m = \int_{\partial\Omega} \mathbf{n}.\boldsymbol{\sigma}.\dot{\mathbf{u}}\ \mathrm{ds} - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} w\ \mathrm{d}\Omega, \tag{20}$$

where the free energy w is a function of the strain $\boldsymbol{\epsilon}$ and of the scalar parameter d as defined (1). The internal parameters evolve only on Ω_c. Using the condition of equilibrium and the motion of the layer the total dissipation is reduced to:

$$D_m = \int_{\Omega_c} Y\dot{d}\ \ \mathrm{d}\Omega + \int_{\partial\Omega_c} ([w] - \boldsymbol{\sigma} : [\boldsymbol{\epsilon}])D_a\mathbf{M}.\mathbf{N}\ \mathrm{ds}. \tag{21}$$

Due to the CC, the free energy is continuous along $\partial\Omega_c$, then the last term vanishes.

The motion of the surface $\phi(\mathbf{M}_t, t) = z$ is given by the normal velocity a, and satisfies the condition $D_a\phi = 0$. The damage parameter d is a function of ϕ, then $D_a d = 0$, and the dissipation is reduced to

$$\begin{aligned} D_m &= \int_{\Omega_c} Y\nabla d.\mathbf{N}\ a\ \mathrm{d}\Omega = \int_{\Gamma_o}\int_{o}^{l} Yd'(\phi)\ a(s)j\ \mathrm{d}\phi\ \mathrm{ds} \\ &= \int_{\Gamma_o} G(s)a(s)\ \mathrm{ds}. \end{aligned} \tag{22}$$

This ensures that

$$D_m = \int_{\Gamma} G(s)a(s)\ \mathrm{ds} = -\frac{\partial\mathcal{E}}{\partial\phi}.a(s). \tag{23}$$

The derivative of the potential energy relative to ϕ is the thermodynamical force associated to the motion of the layer. This is the generalization of the expression of the thermodynamical forces defined previously for moving interface to moving layer.

On the Eshelby momentum tensor. Introducing the Eshelby momentum tensor $\boldsymbol{P}$:

$$\boldsymbol{P} = w\ \boldsymbol{I} - \boldsymbol{\sigma}.\nabla\mathbf{u}, \tag{24}$$

this tensor satisfies the relations

$$\mathrm{div}\,\boldsymbol{P}^T = -Y\nabla d,\ \text{over}\ \Omega_c,\ \mathbf{N}.[\boldsymbol{P}] = 0, \text{along}\ \partial\Omega_c. \tag{25}$$

Using the fact that $\nabla d = d'(\phi)\nabla\phi = d'(\phi)\mathbf{N}$ the dissipation takes the form

$$D_m = -\int_{\Omega_c} \boldsymbol{P} : \nabla D_a\mathbf{M}\ \mathrm{d}\Omega + \int_{\partial\Omega_c} \mathbf{N}.\boldsymbol{P}.D_a\mathbf{M}\ \mathrm{ds}. \tag{26}$$

These expressions can be used in numerical simulations.

Another expression of the dissipation. It can be noticed that from the expression of the local free energy:

$$d_m = \frac{1}{2}(\dot{\boldsymbol{\epsilon}} : \boldsymbol{\sigma} - \boldsymbol{\epsilon} : \dot{\boldsymbol{\sigma}}) = Y\dot{d}. \tag{27}$$

The integration of d_m over the layer is then the global dissipation

$$D_m = \int_{\Omega_c} d_m\ \mathrm{d}\Omega = \int_{\Omega_c} \frac{1}{2}(\dot{\boldsymbol{\epsilon}} : \boldsymbol{\sigma} - \boldsymbol{\epsilon} : \dot{\boldsymbol{\sigma}})\ \mathrm{d}\Omega. \tag{28}$$

Under the hypothesis that the local damage satisfies the condition $D_a d = 0$, the dissipation takes the form

$$\begin{aligned} D_m = &\int_{\partial\Omega_c} \frac{1}{2}(\mathbf{N}.\boldsymbol{\sigma}.\nabla\mathbf{u} - \mathbf{N}.\nabla\boldsymbol{\sigma}.\mathbf{u}).D_a\mathbf{M}\ \mathrm{ds} \\ &- \int_{\Omega_c} \frac{1}{2}(\boldsymbol{\sigma}.\nabla\mathbf{u} - \nabla\boldsymbol{\sigma}.\mathbf{u}) : \nabla D_a\mathbf{M}\ \ \mathrm{d}\Omega \end{aligned}$$

When the thickness l_c vanishes, the volume term vanishes and D_m is reduced to

$$D_m = \lim_{l_c\to 0^+} \int_{\partial\Omega_c} \frac{1}{2}(\mathbf{N}.\boldsymbol{\sigma}.\nabla\mathbf{u} - \mathbf{N}.\nabla\boldsymbol{\sigma}.\mathbf{u}) : D_a\mathbf{M}\ \mathrm{ds}.$$

Using the relation

$$\int_V \boldsymbol{\sigma} : \boldsymbol{\epsilon}\ n_i\ \mathrm{ds} = \int_V \mathbf{n}.\nabla_i\boldsymbol{\sigma}.\mathbf{u} + \mathbf{n}.\boldsymbol{\sigma}.\nabla_i\mathbf{u}\ \mathrm{ds}, \tag{29}$$

at the limit $D_a\mathbf{M} \to -a\mathbf{N}$ the dissipation becomes

$$D_m = \frac{1}{2}\int_{\Gamma_o} [\boldsymbol{\sigma}]_\Gamma : \boldsymbol{\epsilon} - \boldsymbol{\sigma} : [\boldsymbol{\epsilon}]_\Gamma\ \mathrm{ds}\ a, \tag{30}$$

that is the dissipation along a moving interface between two materials with distinct elastic properties. The model of quasi-brittle elastic material (Pradeilles-Duval and Stolz 1995) is recovered.

To define the evolution of the system, we need a link between the thermodynamical force $G(s)$ and the velocity $a(s)$.

On the evolution law. We propose now an evolution law to determine the value of velocity a. Consider the normality rule (5). From integration of $(Y - Y_c)\dot{d} = 0$ over the thickness $l \neq 0$ we get

$$\Big(\int_o^l Yd'(\phi)j\mathrm{d}\phi - Y_c\int_o^l d'(\phi)j\mathrm{d}\phi\Big)a = 0. \tag{31}$$

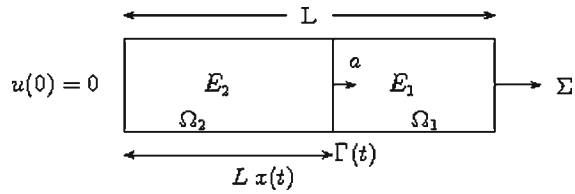

Fig. 2 A sharp interface

This provides a normality law on $G(s)$

$$(G(s) - g_c(s))a = 0, \quad a \geq 0, \quad G(s) - g_c(s) \leq 0. \tag{32}$$

For $l > 0$, $\int_o^l d'(\phi) j \, \mathrm{d}\phi$ is positive, we can introduce the notation $\langle F \rangle$:

$$\langle F \rangle = \frac{\int_o^l F d'(\phi) j \, \mathrm{d}\phi}{\int_o^l d'(\phi) j \, \mathrm{d}\phi}, \tag{33}$$

with this driving force the normality rule is rewritten as

$$(\langle Y \rangle - Y_c)a = 0, \quad a \geq 0, \quad \langle Y \rangle - Y_c \leq 0. \tag{34}$$

Regularity Conditions on Y (RC) . For a regular function $Y(\epsilon, d)$, it can be noticed that when the thickness l vanishes, the value of $\langle Y \rangle$ remains finite and satisfies:

$$\lim_{l \to 0^+} \langle Y \rangle = Y(\epsilon, 0^+), \quad \lim_{l \to 0^+} \langle Y_c \rangle = Y_c \tag{35}$$

Then the critical values for initiation and for evolution of damage are the same.

The thickness l evolves from 0 to l_c. The surface $l = l_c$ is the surface defined by $d(x, t) = 1$. The transition from $d = 0$ to $d = 1$ along the normal vector $\nabla\phi$ is smooth. This approach in essentially non-local, but here the non-locality is restricted to the layer. Any classical constitutive law described by $w(\epsilon, d)$ can be used.

To study the evolution of a system under a loading history, we need two functions $\mathcal{C}(d)$ and $d(\phi)$ or $\mathcal{C}(\phi)$ and $d(\phi)$, because $d(\phi)$ is a continuous increasing function of ϕ.

3 Comparison of sharp and layer model

3.1 A model with sharp interface

A bar of length L is now considered. As depicted on Fig. 2, the bar is made of two linear elastic materials and is submitted to pure tension. The Young modulus of the sound material is E_1 and for the damaged material the Young modulus is E_2.

At point $X = L$, the applied tension is Σ. At point $X = 0$, displacement is $u(0) = 0$. The position of the interface is denoted Γ. We consider that a fraction $x = \Gamma/L$ of the bar is transformed.

The equilibrium and constitutive laws give:

$$\sigma = E_1\epsilon_1 = E_2\epsilon_2 = \Sigma. \tag{36}$$

And the total potential energy is

$$W(\Sigma, x) = -\frac{1}{2} L\Sigma^2 \left(\frac{x}{E_2} + \frac{1-x}{E_1} \right). \tag{37}$$

Differentiation of the energy at equilibrium with respect to the motion of interface is

$$G_\Gamma(\Sigma) = -\frac{\partial W}{\partial \Gamma} = \frac{1}{2}\Sigma^2 \left(\frac{1}{E_2} - \frac{1}{E_1} \right); \tag{38}$$

consequently the dissipation is

$$D_m = G_\Gamma(\Sigma)\dot{\Gamma}. \tag{39}$$

The speed $\dot{\Gamma} = a$ is determined by the normality law

$$a \geq 0, \quad G_\Gamma - G_c \leq 0, \quad (G_\Gamma - G_c)a = 0. \tag{40}$$

The velocity a will be positive if the consistency condition $G_\Gamma - G_c = 0$ is satisfied. This condition determines the critical value of Σ from (49):

$$\Sigma_c = \sqrt{\frac{2G_c E_1 E_2}{E_1 - E_2}}. \tag{41}$$

Initially the bar is composed of material 1 only. Assuming that initial conditions are $x(0^+) = 0$ and $u(L, 0^+) = 0$. Prescribing the displacement at point $X = L$, the total strain of the bar is $\Delta(t)$ and satisfies the global constitutive law

$$\Delta(t) = \frac{u(L, t) - u(0, t)}{L} = \Sigma \left(\frac{x}{E_2} + \frac{1-x}{E_1} \right). \tag{42}$$

The response of the bar is given by

$$\begin{cases} \Delta \leq \Delta_c = \dfrac{\Sigma_c}{E_1}, & \Sigma = E_1\Delta, \quad x = 0, \\ \dfrac{\Sigma_c}{E_2} \geq \Delta \geq \Delta_c, & \Sigma = \Sigma_c, \quad \dot{x} = \dot{\Delta}\dfrac{E_1 E_2}{\Sigma_c(E_1 - E_2)}, \\ \Delta \geq \dfrac{\Sigma_c}{E_2}, & \Sigma = E_2\Delta, \quad x = 1. \end{cases}$$

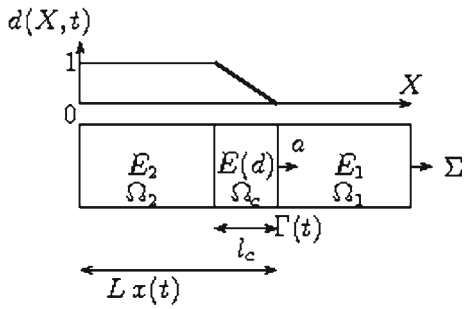

Fig. 3 The propagation of a layer

3.2 A model with a moving layer

The free energy w for unixial response is

$$w(\epsilon, d) = \frac{1}{2}E(d)\epsilon^2, \tag{43}$$

where d varies from 0 to 1, the Young modulus $E(d)$ is a continuous function of d, then there is no discontinuity at $d = 0$. On Fig. 3, $\phi(X, t) = \Gamma(t) - X$ and the value of the damage parameter is given by

$$\begin{cases} \phi(X,t) \le 0, & d(\phi) = 0, \\ 0 \le \phi(X,t) \le l_c, & d(\phi) = \phi/l_c, \\ \phi(X,t) \ge l_c, & d(\phi) = 1. \end{cases} \tag{44}$$

The damage parameter d is an increasing function of the distance ϕ to the boundary Γ separating the sound material to the damaged one. Generally, the function $d(\phi)$ is a given continuous function of ϕ. As the Young modulus E is a continuous function of d and for comparison with the sharp interface, it is natural to consider the matching conditions $E(0) = E_1$ and $E(1) = E_2$.

Initially $\Gamma(t) = 0$ and the propagation of the layer begins at the origin of the bar, so the thickness $\Gamma(t) = l(t)$ is smaller than l_c. The thickness increases to l_c and after this step of initialization, the thickness is kept constant.

For the given constitutive laws, the dissipation is local and only due to damage $d_m = Y\dot{d}$. Integrating over the layer we get the total dissipation due to damage inside the bar:

$$D_m = \int_o^l Y\dot{d}\ \mathrm{d}\phi. \tag{45}$$

Assuming that d is a continuous function of ϕ.

Using the constitutive law, the local driving force Y is

$$Y = -\frac{1}{2}\frac{dE}{dd}\epsilon^2 = -\frac{\Sigma^2}{2E^2(d)}\frac{dE}{dd}. \tag{46}$$

The function $E(d)$ must ensure that Y is finite. Then $\dfrac{dE}{dd}$ is finite. We adopt the notations

$$Y_o(\Sigma) = -\frac{\Sigma^2}{2E_1^2}\frac{dE}{dd}(0^+),\ \ Y_1(\Sigma) = -\frac{\Sigma^2}{2E_2^2}\frac{dE}{dd}(1^-) \tag{47}$$

The motion of the level-set $\phi(X, t) = 0$ is given by the velocity $\dot{\phi} = a(s)$, the total dissipation is finally expressed as

$$D_m = \frac{a}{2}\int_o^l \Big(-\frac{\Sigma^2}{2E^2(d)}\Big)\frac{dE}{dd}d'(\phi)\mathrm{d}\phi = G(l,\Sigma)a, \tag{48}$$

where the release rate of energy $G(l, \Sigma)$ is

$$G(l,\Sigma) = \frac{1}{2}\Sigma^2\left(\frac{1}{E(d(l))} - \frac{1}{E(0)}\right). \tag{49}$$

When $l = l_c$, we recover the expression of the sharp interface.

Using the local normality rule, the value of Y must be smaller than Y_c. This defines the critical value for initiation of damage in a point of the bar. From (35) and (46), the corresponding critical value of Σ is Σ_o such that

$$Y_o(\Sigma_o) = -\frac{1}{2}\frac{\Sigma_o^2}{E_1^2}\ \frac{dE}{dd}(0^+) = Y_c. \tag{50}$$

It can be noticed that the critical value Σ_o depends of the damage law and is generally different from Σ_c. To compare the two models, we assume that the dissipation of the system is the same when the layer moves with the limit thickness l_c.

3.3 The response of the bar

As d is a given function of $\phi(X, t)$, the Young modulus E (function of d) depends on position and time and is perfectly determined as a function of $\phi(X, t)$ denoted now $E(\phi)$.

$$\begin{cases} \phi(X,t) \le 0, & d(\phi) = 0,\ E(\phi) = E_1, \\ 0 \le \phi(X,t) \le l_c, & 0 \le d(\phi) \le 1,\ E(\phi), \\ l_c \le \phi(X,t), & d(\phi) = 1,\ E(\phi) = E_2. \end{cases} \tag{51}$$

On the driving force Y. Inside the layer $0 \le \phi(X, t) \le l_c$, $\dfrac{dE}{dd} = E'(\phi)/d'(\phi)$ then we have for the local force Y defined by (46)

$$\begin{cases} \phi(X,t) \le 0, & Y(\phi,\Sigma) = Y_o(\Sigma), \\ 0 \le \phi(X,t) \le l_c, & Y(\phi,\Sigma) = -\dfrac{\Sigma^2}{E^2(\phi)}\dfrac{E'(\phi)}{d'(\phi)}, \\ l_c \le \phi(X,t), & Y(\phi,\Sigma) = Y_1(\Sigma). \end{cases} \tag{52}$$

The function Y is a continuous function of X.

Global answer for a given $\phi(X,t)$. Due to equilibrium $\sigma = \Sigma$, and the constitutive behaviour defines the strain

$$\epsilon(X,t) = \frac{\mathrm{d}u}{\mathrm{d}X}(X,t) = \frac{\Sigma(t)}{E(\phi)},$$

and by integration over the length of bar, the global answer of the bar is obtained as

$$\Delta = \frac{u(L,t) - u(0,t)}{L} = \frac{1}{L}\int_o^L \boldsymbol{\epsilon}(X,t)\mathrm{d}X.$$

As previously we use the notation $x = \Gamma/L$ and $x_c = l_c/L$.

State 1. Initially, the bar is elastic with modulus E_1:

$$\Delta = \Sigma/E_1, \quad Y(X) \le Y_c.$$

State 2. A damaged zone exists with a thickness l smaller than l_c, then $\Gamma = l < l_c$ or $x \le x_c$, $\phi(X,t) = \Gamma - X$:

$$\Delta = \Sigma \int_o^\Gamma \frac{\mathrm{d}X}{E(\phi)L} + \frac{\Sigma}{E_1}(1-x), \ \langle Y\rangle \le Y_c.$$

State 3. The layer is established, $l_c \le \Gamma \le L, x_c = l_c/L, \phi(X,t) = \Gamma - X$:

$$\Delta = \frac{\Sigma}{E_2}(x - x_c) + \int_{\Gamma - l_c}^{\Gamma} \frac{\Sigma}{E(\phi)}\frac{\mathrm{d}X}{L} + \frac{\Sigma}{E_1}(1-x), \langle Y\rangle \le Y_c.$$

State 4. The layer arrives at point $\Gamma = L, x = 1$. $d = 1$ for $X = L - l, l \le l_c, \phi(X,t) = L + l_c - l - X$:

$$\Delta = \frac{\Sigma}{E_2}(1 - l/L) + \Sigma \int_{L-l}^{L} \frac{\mathrm{d}X}{E(\phi)L}.$$

State 5. All the bar is damaged:

$$\Delta = \frac{\Sigma}{E_2}.$$

The constitutive law (state 1–5) can be written as

$$\Delta = S(x)\ \Sigma, \tag{53}$$

where $0 \le x \le 1$. The modulus $S(x)$ evolves with the motion of the layer, then $\dot{S} = \dfrac{\mathrm{d}S}{\mathrm{d}x}\dot{x} = S'(x)\dot{x}$.

3.4 The problem of evolution

During state 1–3, the mechanical state $(l, \Gamma, \Delta, \Sigma)$ must be compatible with the domain of reversibility $G(l,\Sigma) \le Y_c d(l)$. For each value of $l \le l_c$ the critical value Σ_l is defined by

$$G(l,\Sigma_l) = Y_c d(l), \quad \text{or} \quad \langle Y\rangle = Y_c. \tag{54}$$

For state 4, these conditions must be reconsidered, $\Gamma = L$ and $d = 1$ for $X = L - l$; the modulus of elasticity at $X = L$ is $E(l_c - l)$, using (49) Eq. (54) becomes:

$$G(l,\Sigma) = \frac{1}{2}\Sigma^2\left(\frac{1}{E_2} - \frac{1}{E(l_c - l)}\right) = Y_c(1 - d(l_c - l)).$$

Now, we consider only the evolution during state 1–3. It can be noticed that for $\Sigma \le \Sigma_l$ the layer cannot evolve, because $G(l,\Sigma)$ is an increasing function of Σ. The condition $F(l,\Sigma) = G(l,\Sigma) - Y_c d(l) \le 0$ implies that for $F = 0$ the evolution of F satisfies

$$\dot{F} = \left(\frac{\partial G}{\partial l}\dot{l} + \frac{\partial G}{\partial \Sigma_l}\dot{\Sigma} - Y_c\frac{dd}{dl}\dot{l}\right) \le 0. \tag{55}$$

At any state for which $F = G(l,\Sigma_l) - Y_c d(l) = 0$, the evolution of the bar is determined by

$$\dot{\Delta} = S'(x)\dot{x}\Sigma_l + S(x)\dot{\Sigma}, \tag{56}$$

$$\frac{\partial G}{\partial l}\dot{l} + \frac{\partial G}{\partial \Sigma_l}\dot{\Sigma} - Y_c\frac{dd}{dl}\dot{l} \le 0, \tag{57}$$

$$\dot{l} = L\ \dot{x} \ge 0. \tag{58}$$

For increasing x, the global Young modulus (S^{-1}) is decreasing, then $S'(x) \ge 0$.

Discussion for each possible state. The evolution depends on the state.

State 1: The solution is purely elastic.

State 2: The thickness of the layer is $l = xL$ and the consistency condition gives the value of $\dot{x}$ by solving the inequalities

$$\left(\frac{\partial G}{\partial l} - Y_c\frac{dd}{dl}\right)L\dot{x} + \frac{\partial G}{\partial \Sigma}\frac{\dot{\Delta} - S'(x)\Sigma_l\dot{x}}{S} \le 0, \quad \dot{x} \ge 0.$$

A unique solution $\dot{x} > 0$ exists for $\dot{\Delta} > 0$ and $\dfrac{\partial G}{\partial \Sigma} \geq 0$ if

$$\left(\frac{\partial G}{\partial l} - G_c \frac{dd}{dl}\right) L - \frac{S'(x)}{S} \frac{\partial G}{\partial \Sigma} \leq 0. \quad (59)$$

State 3: The energy release rate G does not depend on x and $d(l_c) = 1$ then the unique solution is given by $\dot{\Sigma} = 0$ and

$$\dot{\Delta} = S'(x)\ \Sigma_c^o\ \dot{x}.$$

When we apply an increasing global strain $\Delta(t)$ the bar follows successively the different states.

3.5 Matching conditions between the two models for a damaged bar

After the step of initialization, $l = l_c$. The thermodynamical forces for the two descriptions, using the sharp interface or the moving layer, are identical, because $E(0) = E_1$, $E(l_c) = E_2$

$$G(l_c, \Sigma) = G_\Gamma(\Sigma). \quad (60)$$

The motion of the layer is given by the normality rule

$$G(l, \Sigma) - Y_c d(l) \leq 0, \quad a \geq 0, \quad (61)$$

$$(G(l, \Sigma) - Y_c d(l))a = 0. \quad (62)$$

When the interface is established ($l = l_c$), the equality of the two dissipations, obtained for a sharp interface ($G_c a$) and for a moving layer ($Y_c d(l) a$), provides the value of Y_c:

$$D_m = G_c a = Y_c d(l_c) a \quad \Rightarrow Y_c = G_c\ . \quad (63)$$

Using the normality rule (62), the velocity a is positive if and only if $G(l, \Sigma) = Y_c d(l)$. This defines the critical value Σ_l of Σ for which the layers evolves. For $l = 0$ the critical value of Σ is Σ_o and for l_c the critical value is Σ_c.

As G is an increasing function of Σ at fixed l, a unique solution with a layer is obtained if $\Sigma_l \leq \Sigma_o$, otherwise all points of the elastic domain (where $d = 0$) are critical with respect to initiation.

At point $\phi(X, t) = l_c$, we obtain $Y(l_c, \Sigma) = Y_1(\Sigma)$, this ensures the continuity of Y along the bar (Figs. 4, 5).

4 Particular choice of constitutive laws

We study the reponse of the bar for two particular constitutive laws defined by specific function d and E (Figs. 4, 5).

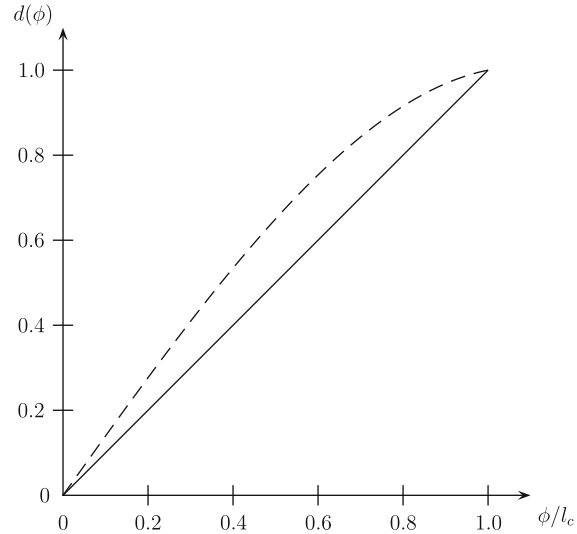

Fig. 4 Damage $d(\phi)$, $d_1 = 0$, *solid line*, $d_1 \neq 0$: *dashed line*

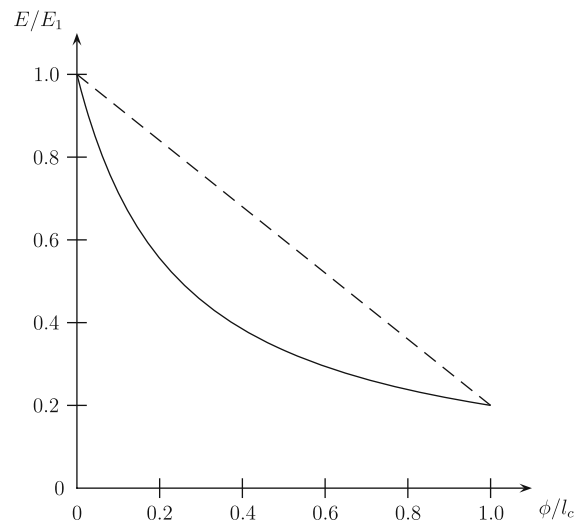

Fig. 5 Young modulus evolution $E(\phi)$, C1: *solid line*, C2: *dashed line*

– Case 1 (C1). The functions $E(\phi)$,$d(\phi)$ are given by

$$\begin{cases} \phi(X,t) \leq 0, & E(\phi) = E_1, \quad d(\phi) = 0, \\ l_c \geq \phi(X,t) \geq 0, & \begin{cases} \dfrac{1}{E(\phi)} = \dfrac{1 - \phi/l_c}{E_1} + \dfrac{\phi/l_c}{E_2} \\ d(\phi) = \dfrac{\phi}{l_c}\left(1 + d_1 - d_1 \left(\dfrac{\phi}{l_c}\right)^\beta\right) \end{cases} \\ \phi(X,t) \geq l_c, & E(\phi) = E_2, \quad d(\phi) = 1. \end{cases}$$

The definition of finite Y and $\langle Y \rangle$ implies restrictions on parameters d_1 and β: $d_1 \geq 0$ and $\beta \geq 0$, the exponent β is chosen such d' is finite $\forall \phi \in [0, l_c]$.

$$d'(\phi) = \frac{1}{l_c}\left(1 + d_1 - d_1(\beta+1)\left(\frac{\phi}{l_c}\right)^{\beta}\right) < \infty. \quad (64)$$

then d_1 and β satisfy: $d_1 \geq 0$ and $0 < \beta < \dfrac{1}{d_1}$ or $d_1 = 0$. The value Y is obtained in the layer as

$$Y(\phi, \Sigma) = -\frac{1}{2}\frac{\Sigma^2}{E^2(\phi)}\frac{E'(\phi)}{d'(\phi)}, \quad (65)$$

then $Y_o(\Sigma)$ and $Y_1(\Sigma)$ are

$$Y_o(\Sigma) = \frac{\Sigma^2}{2(1+d_1)}\frac{E_1 - E_2}{E_1 E_2},$$

$$Y_1(\Sigma) = \frac{\Sigma^2}{2(1-\beta d_1)}\frac{E_1 - E_2}{E_1 E_2}.$$

A positive velocity a is obtained if the consistency condition is satisfied

$$G(l, \Sigma_l) = Y_c d(l). \quad (66)$$

In this case, we get the relation

$$\langle Y \rangle = \frac{1}{2}\Sigma^2\left(\frac{1}{E_2} - \frac{1}{E_1}\right) = Y_c\left(1 + d_1\left(1 - \left(\frac{l}{l_c}\right)^{\beta}\right)\right).$$

This implies that for $d_1 = 0$, the layer initiates and after moves at $\Sigma = \Sigma_c$

$$\Sigma_c = \sqrt{\frac{2Y_c E_1 E_2}{E_1 - E_2}}. \quad (67)$$

For $d_1 > 0$

$$\Sigma_l = \Sigma_c\sqrt{\left(1 + d_1\left(1 - \left(\frac{l}{l_c}\right)^{\beta}\right)\right)}. \quad (68)$$

Initiation. The thermodynamical force Y for $d = 0$ is

$$Y(0, \Sigma) = Y_o(\Sigma). \quad (69)$$

At initiation the critical value Σ_o satisfies $Y_o(\Sigma_o) = Y_c$. As $E'(0) = \dfrac{1}{l_c}\left(\dfrac{1}{E_2} - \dfrac{1}{E_1}\right)$ and $d'(0) = (1+d_1)/l_c$, we get

$$\Sigma_o = \Sigma_c\sqrt{1+d_1} \geq \Sigma_c. \quad (70)$$

State 2. The local value of Y in the layer is

$$Y(\phi, \Sigma_l) = \frac{1}{2}\frac{\Sigma_l^2}{E^2(\phi)}\frac{E'(\phi)}{d'(\phi)}, \quad (71)$$

that is

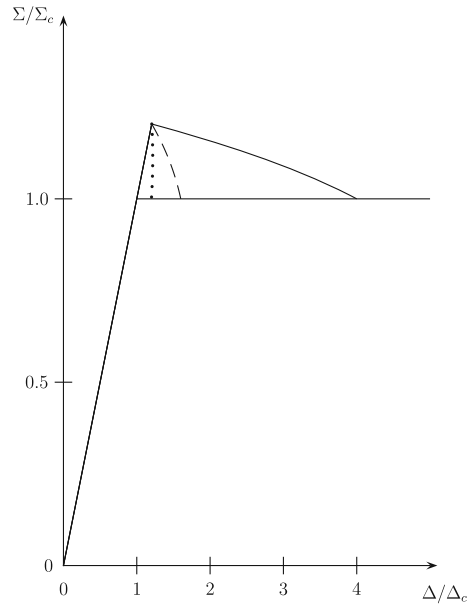

Fig. 6 Case C1, Global response, $E2/E1 = .2$, $d_1 = .45$, $\beta = 2$, $l_c = 5$, $L = 10$, *solid line*, $L = 50$, *dashed line*, $L = 250$, *dotted line*

$$Y(\phi, \Sigma_l) = \frac{1}{2}\left(\frac{1}{E_2} - \frac{1}{E_1}\right)\Sigma_l^2\frac{1}{d'(\phi)}, \quad (72)$$

after reduction, for $0 < X < l$, $\phi(X, t) = l - X$:

$$Y(\phi, \Sigma_l) = Y_c\frac{1 + d_1 - d_1\left(\dfrac{l}{l_c}\right)^{\beta}}{1 + d_1 - d_1(\beta+1)\left(\dfrac{\phi}{l_c}\right)^{\beta}}, \quad (73)$$

then along the bar Y is a decreasing function of X. It can be noticed that Y is continuous in $X = l$. Then in Ω_1 the initiation of new defect is not possible. The local value $Y(\phi, \Sigma)/Y_c$ is plotted in Fig. 7 for different value of l/l_c. This shows the evolution of the local force Y during the state 2.

On Fig. 6, the global response of the bar is plotted for $\beta = 2$, $d_1 = .45$, $l_c = 5$ and different value of ratio l_c/L, in space $(\Delta/\Delta_c, \Sigma/\Sigma_c)$, where $\Delta_c = E_1\Sigma_c$. The strain - stress curve is determined for given l:

$$\Sigma_l = \Sigma_c\sqrt{1 + d_1 - d_1\left(\frac{l}{l_c}\right)^{\beta}}, \quad (74)$$

$$\Delta = \Sigma_l\left(\frac{1}{E_1} + \frac{3}{2}\left(\frac{1}{E_2} - \frac{1}{E_1}\right)\left(\frac{l}{l_c}\right)\frac{l_c}{L}\right). \quad (75)$$

For this case, it is easy to show that when l_c/L is small, snap-back occurs and the condition of uniqueness (59) is not satisfied. The evolution of the local force Y during stet 2 is plotted in Fig. 7.

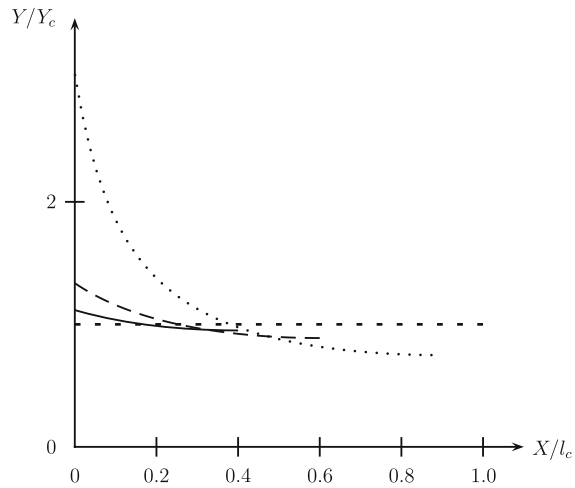

Fig. 7 Evolution of $Y(X, \Sigma_l)/Y_c$ in the layer in case C1 for $l/l_c = .3$, *solid line*; .6, *dashed line*; .9, *dotted line*

State 3. $\Sigma = \Sigma_c$, the local force $Y(\phi, \Sigma_c)$ is given by

$$\begin{cases} \phi(X,t) \le 0, & Y(\phi, \Sigma_c) = Y_o(\Sigma_c) \le Y_o(\Sigma_o), \\ 0 \le \phi(X,t) \le l_c, & Y(\phi, \Sigma_c) \\ & \quad = \dfrac{Y_c}{1 + d_1 - d_1(\beta+1)\left(\dfrac{\phi}{l_c}\right)^{\beta}}, \\ l_c \le \phi(X,t), & Y(\phi, \Sigma_c) = Y_1(\Sigma_c). \end{cases}$$

– Case 2 (C2). For $0 \le \phi(X,t) \le l_c$, consider now the expressions:

$$E(\phi) = (1 - \phi/l_c)E_1 + \phi/l_c E_2, \tag{76}$$

$$d(\phi) = \phi/l_c. \tag{77}$$

As previously, $Y(\phi, \Sigma)$ is non uniform and is a decreasing function of X, for $\phi(X,t) = l(t) - X$, with $X \le l$:

$$Y(\phi, \Sigma) = \frac{\Sigma^2}{2} \frac{E_1 - E_2}{\left(\left(1 - \dfrac{\phi}{l_c}\right) E_1 + \dfrac{\phi}{l_c} E_2\right)^2}, \tag{78}$$

and for $d = 0$ and $d = 1$ we have:

$$Y_o(\Sigma) = \frac{1}{2}\frac{\Sigma^2}{E_1^2}(E_1 - E_2),$$

$$Y_1(\Sigma) = \frac{1}{2}\frac{\Sigma^2}{E_2^2}(E_1 - E_2).$$

The consistency condition gives the critical loading Σ_l

$$\Sigma_l^2 = \frac{2Y_c E_1}{E_1 - E_2}\left(\frac{l}{l_c}E_2 + \left(1 - \frac{l}{l_c}\right)E_1\right), \tag{79}$$

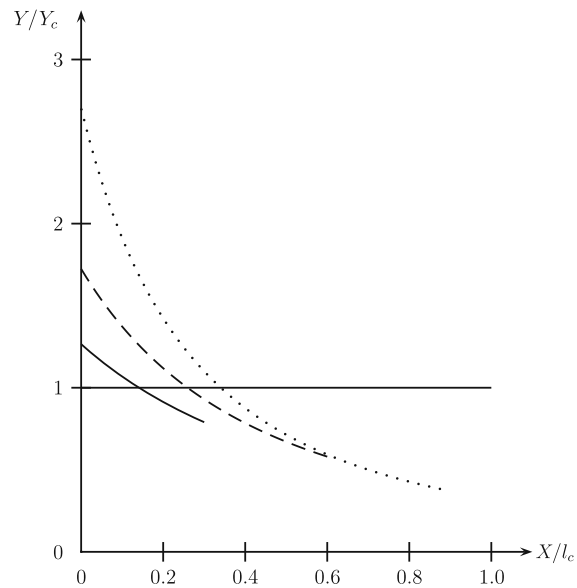

Fig. 8 Evolution of $Y(X, \Sigma_l)/Y_c$ in the layer Case C2, $E_2/E_1 = .2$, $l/l_c = .3$, *solid line*; .6, *dashed line*; .9, *dotted line*

then

$$\Sigma_0^2 = \frac{2Y_c E_1^2}{E_1 - E_2} > \Sigma_{l_c}^2 = \frac{2Y_c E_1 E_2}{E_1 - E_2} = \Sigma_c^2. \tag{80}$$

The value Σ_o of initiation of the layer is greater than the value Σ_c. When $\Sigma = \Sigma_l$

$$Y(\phi, \Sigma_l) = Y_c E_1 \frac{\left(1 - \dfrac{l}{l_c}\right) E_1 + \dfrac{l}{l_c} E_2}{\left(\left(1 - \dfrac{\phi}{l_c}\right) E_1 + \dfrac{\phi}{l_c} E_2\right)^2}, \tag{81}$$

then

$$Y(l, \Sigma_l) \ge Y_c, \quad Y(0, \Sigma_l) = Y_c. \tag{82}$$

Figure 8 shows the local force $Y(X, \Sigma_l)$ during the state 2 for different value of l/l_c along the layer. The strain–stress curve is easy to obtained:

$$\Sigma_l = \Sigma_c \sqrt{\frac{l}{l_c} + \left(1 - \frac{l}{l_c}\right)\frac{E_1}{E_2}},$$

$$\Delta = -\frac{l_c \Sigma_l}{L(E_1 - E_2)}\left(\ln \frac{l}{l_c} + \left(1 - \frac{l}{l_c}\right)\frac{E_1}{E_2}\right) + \frac{\Sigma_l}{E_1}\left(1 - \frac{l}{L}\right).$$

In this case, there is a snap-back during the growth of the layer. Typical global responses of the bar are plotted on Fig. 9 for different values of the ratio l_c/L.

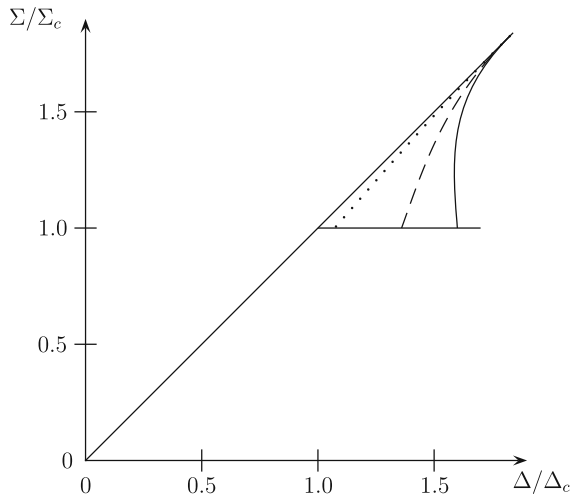

Fig. 9 Case C2. Global response. $E_2/E_1 = .2$, $l_c = 5$, $L = 10$, *solid line*, 50, *dashed line*, 250, *dotted line*

State 3. Now $\Sigma = \Sigma_c$ and Y is defined along the bar as

$$\begin{cases} \phi(X,t) \le 0, & Y(\phi, \Sigma_c) = Y_o(\Sigma_c) \le Y_o(\Sigma_o), \\ 0 \le \phi(X,t) \le l_c, & Y(\phi, \Sigma_c) \\ & \quad = \dfrac{Y_c E_1 E_2}{\left(\left(1 - \dfrac{\phi}{l_c}\right) E_1 + \dfrac{\phi}{l_c} E_2\right)^2}, \\ l_c \le \phi(X,t), & Y(\phi, \Sigma_c) = Y_1(\Sigma_c), \end{cases}$$

where $Y_o(\Sigma_c) = Y_c E_2/E_1$ and $Y_1(\Sigma_c) = Y_c E_1/E_2$.

These two examples show the role of the constitutive law on the phase of initiation of defect. The strain–stress curve for the bar depends on the ratio E_2/E_1 and l_c/L. The solution is not necessary unique, taking account of the remark on the consistency solution.

5 Conclusion

We have studied a new approach on damage evolution based on the notion of level set and moving layer. The initiation of damage and its evolution are presented in the same framework.

Comparisons with the evolution of a sharp interface are made. The critical loading value at initiation is greater than the critical value of loading for the propagation of the moving layer. When the limit thickness l_c is reached, the critical value of the loading to obtain propagation is the same for the sharp interface and the moving layer.

Generally the dissipation and the thermodynamical forces for motion of the layer depend on the local shape of the layer, in particular on the curvature of the moving surface $d = 0$. It is kwown that the presence of curvature is a factor of stabilization of the propagation of an interface (Son et al. 1989). This will be studied in a forthcoming paper.

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