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# HOW ROUND-OFF ERRORS HELP TO COMPUTE THE ROTATION SET OF TORUS HOMEOMORPHISMS

PIERRE-ANTOINE GUIHÉNEUF

**ABSTRACT.** The goals of this paper are to obtain theoretical models of what happens when a computer calculates the rotation set of a homeomorphism, and to find a good algorithm to perform simulations of this rotation set. To do that we introduce the notion of observable rotation set, which takes into account the fact that we can only detect phenomenon appearing on positive Lebesgue measure sets; we also define the asymptotic discretized rotation set which in addition takes into account the fact that the computer calculates with a finite number of digits.

It appears that both theoretical results and simulations suggest that the asymptotic discretized rotation set is a much better approximation of the rotation set than the observable rotation set, in other words we need to do coarse round-off errors to obtain numerically the rotation set.

## 1. INTRODUCTION

The concept of rotation number for circle homeomorphisms was introduced by H. Poincaré in 1885. In [Poi85], he states the theorem of classification of orientation-preserving circle homeomorphisms: if a homeomorphism has a rational rotation number, then it possesses a periodic point and all its periodic points have the same period; moreover the  $\omega$ -limit set of every point is a periodic orbit (the dynamics is asymptotically periodic). On the contrary, if a homeomorphism has an irrational rotation number  $\alpha$ , then it is semi-conjugated to the rigid rotation of angle  $\alpha$  (the dynamics contains that of the irrational rotation). Ever since, the rotation number has been the fundamental tool in the study of the dynamics of circle homeomorphisms (see for example [Her79]).

About a century after was introduced a generalisation to dimension 2 of this rotation number, the rotation set for homeomorphisms of the torus which are homotopic to the identity. Due to the loss of natural cyclic order on the phase space, there is no longer a single speed of rotation for orbits; informally the rotation set is defined as the set of all possible rotation speeds of all possible orbits. Like in dimension 1, this topological invariant gives precious informations about the dynamics of the homeomorphism; for example, depending of the shape of this set, we can ensure the existence of periodic points of a given period ([Fra88], [Fra89]). Moreover, the size of this convex set gives lower bounds on the topological entropy of the homeomorphism ([LM91] and [Kwa93] for an explicit estimation)...

The aim of this paper is to tackle the question of numerical approximation of the rotation set: given a homeomorphism of the torus homotopic to the identity, is it possible to compute numerically its rotation set? In particular, is it possible to detect its dimension? Is it possible to approximate it in Hausdorff topology? And what algorithm shall we use to compute it?

First of all, we build a theoretical model of what happens when we try to calculate the rotation set of a homeomorphism with a computer. To do that, we first take into account the fact that the computer can calculate only a finite number of orbits; in particular it will detect only phenomenon that occur on Lebesgue-positive measure sets. This leads us to the notion of *observable rotation set*: a rotation vector is called observable if it is the rotation vector of an observable measure in the sense given by E. Catsigeras and H. Enrich in [CE11]; more precisely, a measure  $\mu$  is observable if for every  $\varepsilon > 0$ , the set of points which have a Birkhoff limit whose distance to  $\mu$  is smaller than  $\varepsilon$  has Leb-positive measure (see Definition 5).

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However, this notion of observable measure does not take into account the fact that the computer uses finite precision numbers and can calculate only finite length orbits; this observation leads to the definition of the *asymptotic discretized rotation set* in the following way. We fix a sequence of finite grids on the torus with precision going to 0; the discretized rotation set on one of these grids is the collection of rotation vectors of periodic orbits of the discretization of the homeomorphism on this grid (see Section 4); the asymptotic discretized rotation set is then the upper limit of these discretized rotation sets on the grids. We focus mainly on the generic behaviour of both observable and asymptotic discretized rotation sets. We recall that a result of A. Passeggi states that for a generic dissipative homeomorphism of the torus the rotation set is a polygon with rational vertices, possibly degenerated<sup>1</sup> [Pas14]. In this chapter we will prove the following result about generic dissipative homeomorphisms.

**Theorem.** *For a generic dissipative homeomorphism,*

- (1) *the observable rotation set is the closure of the set of rotation vectors corresponding to Lyapunov stable periodic points (Lemma 18);*
- (2) *the convex hull of the observable rotation set, the convex hull of the asymptotic discretized rotation set and the rotation set are equal;*
- (3) *if the rotation set has non-empty interior, there is no need to take convex hulls, i.e. both observable and asymptotic discretized rotation sets coincide with the rotation set (Propositions 17 and 24).*

Thus, it is possible to obtain the rotation set of a generic dissipative homeomorphism from the observable or the asymptotic discretized rotation set. In other words, from the theoretical point of view, it is possible to recover numerically the rotation set of a generic homeomorphism. The generic conservative setting is quite different.

**Theorem.** *For a generic conservative homeomorphism,*

- (1) *the rotation set has non-empty interior (Proposition 2);*
- (2) *the observable rotation set consists in a single vector: the mean rotation vector (Proposition 21). On the other hand, the asymptotic discretized rotation set coincides with the rotation set (Theorem 25).*

Moreover, we can obtain similar results for generic conservative  $C^1$ -diffeomorphisms.

**Theorem.** *For a generic conservative  $C^1$ -diffeomorphism,*

- (1) *the rotation set has non-empty interior (Proposition 2);*
- (2) *if it is ergodic (see the discussion page 9), then the observable rotation set consists in a single point: the mean rotation vector. On the other hand, the asymptotic discretized rotation set coincides with the rotation set (Proposition 27).*

These results suggest the quite surprising moral that to recover the rotation set of a conservative homeomorphisms, it is better to do coarse roundoff errors at each iteration. More precisely, if we compute a finite number of orbits with arbitrarily good precision and long length, we will find only the mean rotation vector of the homeomorphism; but if we make roundoff errors while computing, we will be able to retrieve the whole rotation set.

We have performed numerical simulations to see whether these behaviours can be observed in practice or not. To obtain numerically an approximation of the observable rotation set, we have calculated rotation vectors of long segments of orbits for a lot of starting points with high precision (these points being chosen randomly). For the numerical approximation of the asymptotic discretized rotation set we have chosen a fine enough grid on the torus and have calculated the rotation vectors of periodic orbits of the discretization of the homeomorphism on this grid.

We have first made simulations on some examples where the rotation set is known to be the square  $[0, 1]^2$ . On the one hand, it informs us on the shape of the rotation set we should obtain numerically, on the other hand, it limits a bit the “genericity” of the examples we can produce. We also produced simulations for a homeomorphism for which we do not know the shape of the rotation set.

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<sup>1</sup>Namely it can be a segment or a singleton. However there are open sets of homeomorphisms where the rotation set has non-empty interior.

In the dissipative case, we made attractive the periodic points which realize the vertex of the rotation set  $[0, 1]^2$ . It is obvious that these rotation vectors, which are realized by attractive periodic points with basin of attraction of reasonable size, will be detected by the simulations of both observable and asymptotic discretized rotation sets; that is we observe in practice: we can recover quickly the rotation set in both cases (Figures 2 and 3).

In the conservative setting, we observe the surprising behaviour predicted by the theory: when we compute the rotation vectors of long segments of orbits we obtain mainly rotation vectors which are quite close to the mean rotation vector, in particular we do not recover the initial rotation set. More precisely, when we perform simulations with three hours of calculation we only obtain rotation vectors close to the mean rotation vector (Figure 8). On the other hand, when we calculate the union of the discretized rotation sets over several grids to obtain a simulation of the asymptotic discretized rotation set, the rotation set is detected very quickly by the convex hulls of discretized rotation sets (less than one minute of calculation) and when we compute more orders of discretizations, we obtain a set which is quite close to  $[0, 1]^2$  for Hausdorff distance (Figure 10). Moreover, when we compute the observable rotation set of a homeomorphism whose rotation set is unknown, we obtain a single rotation vector (Figure 11); but when we simulate the asymptotic discretized rotation set, then we obtain a sequence of “thick” sets whose convex hulls seem to converge (Figure 13). As for theoretical results, this suggests the following lesson:

*When we compute segments of orbits with very good precision it is very difficult to recover the rotation set. However, when we decrease the number of digits used in computations we can obtain quickly a very good approximation of the rotation set.*

This phenomenon can be explained by the fact that each grid of the torus is stabilized by the corresponding discretization of the homeomorphism. Thus, there exists an infinite number of grids such that every periodic point of the homeomorphism is shadowed by some periodic orbits of the discretizations on these grids.

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## 2. NOTATIONS AND PRELIMINARIES

**2.1. Notations.** The set of homeomorphisms of  $\mathbf{T}^2$  will be denoted by  $\text{Homeo}(\mathbf{T}^2)$  and the subset of  $\text{Homeo}(\mathbf{T}^2)$  consisting in homeomorphisms preserving Lebesgue measure will be denoted by  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$ . Elements of  $\text{Homeo}(\mathbf{T}^2)$  will be called *dissipative* and elements of  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$  will be called *conservative*. As usual, these two spaces are equipped with the metric of uniform convergence. We also define the set  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$  of conservative  $C^1$ -diffeomorphisms, which is equipped with the metric of uniform convergence of both the maps and their derivatives.

We denote by  $\mathcal{P}$  the set of probability measures on  $\mathbf{T}^2$ , equipped with a distance  $\text{dist}$  compatible with the weak-\* topology; by Banach-Alaoglu-Bourbaki theorem  $\mathcal{P}$  is compact. Let  $f$  be a homeomorphism of the torus  $\mathbf{T}^2$  homotopic to the identity<sup>2</sup>. For  $x \in \mathbf{T}^2$ , we denote by  $p\omega(x)$  the set of limit points of the sequence

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \right\}_{n \in \mathbf{N}^*}.$$

It is a compact subset of the set  $\mathcal{M}(f)$  of  $f$ -invariant Borel probability measures.

For  $K \subset \mathbf{T}^2$  we will denote by  $\text{diam}_{\text{int}}(K)$  the diameter of the biggest euclidean ball included in  $K$ . By  $K' \subset\subset K$  we mean that there exists an open set  $O$  such that  $K' \subset O \subset K$ . In the sequel the set  $K$  will be called *strictly periodic* if there exists an integer  $i > 0$  such that  $f^i(K) \subset\subset K$ .

<sup>2</sup>From now every homeomorphism will be supposed homotopic to the identity.

**2.2. Generic properties.** The topological spaces  $\text{Homeo}(\mathbf{T}^2)$ ,  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$  and  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$  are Baire spaces (see [Gui12]), *i.e.* in these spaces the intersection of every countable collection of dense open sets is dense. We call  $G_\delta$  a countable intersection of open sets; a property satisfied on at least a  $G_\delta$  dense set is called *generic*. Note that in a Baire space generic properties are stable under intersection. Sometimes we will use the phrase “for a generic homeomorphism  $f \in \mathcal{U}$ , we have the property  $(P)$ ”. By that we will mean that “there exists a  $G_\delta$  dense subset  $G$  of  $\mathcal{U}$ , such that every  $f \in G$  satisfy the property  $(P)$ ”.

**2.3. Rotation sets.** The definition of the rotation set is made to mimic the rotation number for homeomorphisms of the circle. At first sight the natural generalisation to dimension 2 of this notion is the point rotation set, defined as follows. For every homeomorphism  $f$  of the torus  $\mathbf{T}^2$  homotopic to the identity we take a lift  $F : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  of  $f$  to the universal cover  $\mathbf{R}^2$  of  $\mathbf{T}^2$ . The difference with the one dimensional case is that as we lose the existence of a total order on our space, the sequence  $\frac{F^n(\tilde{x}) - \tilde{x}}{n}$  no longer need to converge. Thus, we have to consider all the possible limits of such sequences, called *rotation vectors*; the set of rotation vectors associated to  $\tilde{x} \in \mathbf{R}^2$  will be denoted by  $\bar{\rho}(\tilde{x})$ :

$$\bar{\rho}(\tilde{x}) = \bigcap_{N_0 \in \mathbf{N}} \bigcup_{n \geq N_0} \overline{\left\{ \frac{F^n(\tilde{x}) - \tilde{x}}{n} \right\}}.$$

Then, the *point rotation set* is defined as  $\rho_{pts}(F) = \bigcup_{\tilde{x} \in \mathbf{R}^2} \bar{\rho}(\tilde{x})$ . Unfortunately this definition is not very convenient and it turns out that when we interchange the limits in the previous definition, we obtain the *rotation set*

$$\rho(F) = \bigcap_{M \in \mathbf{N}} \bigcup_{m \geq M} \overline{\left\{ \frac{F^m(\tilde{x}) - \tilde{x}}{m} \mid \tilde{x} \in \mathbf{R}^2 \right\}}$$

which has much better properties and is easier to manipulate. In particular, it is compact and convex (see [MZ89]), and it is the convex hull of  $\rho_{pts}(F)$ . Moreover, it coincides with the *measure rotation set*: if we denote by  $D(F)$  the *displacement function*, defined on  $\mathbf{T}^2$  by  $D(F)(x) = F(\tilde{x}) - \tilde{x}$ , where  $\tilde{x}$  is a lift of  $x$  to  $\mathbf{R}^2$  (we easily check that this quantity does not depend of the lift), then (recall that  $\mathcal{M}(f)$  is the set of  $f$ -invariant probability measures)

$$\rho(F) = \left\{ \int_{\mathbf{T}^2} D(F)(x) \, d\mu \mid \mu \in \mathcal{M}(f) \right\}.$$

Finally, for a homeomorphism  $f$  preserving  $\text{Leb}$ , we denote by  $\rho_{mean}(F)$  the *mean rotation vector* of  $F$ :

$$\rho_{mean}(F) = \int_{\mathbf{T}^2} D(F)(x) \, d\text{Leb}(x).$$

The geometry of the rotation set of a generic dissipative homeomorphism is given by a recent result published by A. Passeggi:

**Theorem 1** (Passeggi, [Pas14]). *On an open and dense set of homeomorphisms  $f \in \text{Homeo}(\mathbf{T}^2)$ , the rotation set is locally constant around  $f$  and is equal to a rational polygon.*

We end this paragraph by giving a proof that if  $f$  is a generic conservative homeomorphism of the torus, then  $\rho(F)$  has non-empty interior.

**Proposition 2.** *On a open dense<sup>3</sup> subset of  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$  or  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$ ,  $\rho(F)$  has non-empty interior.*

*Remark 3.* We do not know the shape of the boundary of the rotation set of a generic conservative homeomorphism. In particular we do not know if it is a polygon or not.

*Proof of Proposition 2.* We use an argument due to S. Crovisier.

Let us begin by giving an elementary proof for the case of homeomorphisms. If  $\rho(F)$  consists in a single point, we use classical perturbation techniques for conservative homeomorphisms to create a persistent periodic point  $x$  for  $f$ . Then, by composing by a small rotation of the torus, we can move a little the mean rotation vector; in particular as the rotation set still contains the rotation vector of the persistent periodic point  $x$ , it is not reduced to a single point. Now if the rotation set is a segment, by a  $C^0$  ergodic closing lemma, we can create a persistent

<sup>3</sup>In particular, if  $f$  is generic.

periodic point whose rotation vector is close to the mean rotation vector in the following way. A small perturbation<sup>4</sup> allows us to suppose that the homeomorphism we obtained, still denoted by  $f$ , is ergodic (it is the Oxtoby-Ulam theorem, see [OU41]). We then choose a recurrent point  $y \in \mathbf{T}^2$  which verifies the conclusion of Birkhoff's theorem: for  $N$  large enough, the measure  $\frac{1}{N} \sum_{k=0}^{N-1} \delta_{f^k(y)}$  is close to the measure  $\text{Leb}$ . As this point is recurrent, by making a little perturbation, we can make it periodic and even persistent (see for example [DF00] or [Gui12, Part 3.2]); by construction  $\bar{p}(y)$  is close to the mean rotation vector. We now have two persistent periodic points, say  $x$  and  $y$ , whose rotation vectors are different. It then suffices to compose by an appropriate rotation such that the mean rotation vector goes outside of the line generated by these two rotation vectors, and to repeat the construction to find a persistent periodic point whose rotation vector is close to this new mean rotation vector. Thus, we obtain a homeomorphism  $g$  which is arbitrarily close to  $f$  and possesses three periodic points  $x$ ,  $y$  and  $z$  whose rotation vectors are non-aligned; therefore the rotation set of this homeomorphism has nonempty interior. Moreover, as the periodic points  $x$ ,  $y$  and  $z$  are persistent, this property remains true on a neighbourhood of  $g$ , which concludes the proof for  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$ .

For the case of  $C^1$ -diffeomorphisms, it suffices to replace the  $C^0$ -ergodic closing lemma by the  $C^1$ -ergodic closing lemma.

**Theorem 4** (R. Mañé, [Mañ82]). *Let  $f \in \text{Diff}^1(\mathbf{T}^2, \text{Leb})$ ,  $\mathcal{U}$  a neighbourhood of  $f$  in  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$ ,  $\delta > 0$  and  $\mu$  an  $f$ -invariant Borel probability measure. Then for  $\mu$ -almost every  $x \in \mathbf{T}^2$ , there exists  $g \in \mathcal{U}$  and  $\tau \geq 1$  such that  $x$  is  $\tau$ -periodic for  $g$  and for all  $1 \leq k \leq \tau$ ,  $d(f^k(x), g^k(x)) \leq \delta$ .*

In other words, it is possible to perturb  $f$  by  $g$  such that the measure  $\mu$  is close to a periodic measure of  $g$ . Then, Franks lemma [Fra71] allows us to perturb the differential of  $g$  on the periodic orbit to avoid having the eigenvalue 1, so that the periodic point becomes persistent (see [KH95, page 319]). The rest of the proof is identical to the  $C^0$  case.  $\square$

**2.4. Observable measures.** From the ergodic viewpoint, we could be tempted to define the observable rotation set to be the set of rotation vectors associated to physical measures (or SRB measures, see [You02]), which are defined to express which measures can be observed in practice. However, such measures do not need to exist for every dynamical system, in this case the associated observable rotation set would be empty. To solve this problem of non existence of physical measures, E. Catsigeras and H. Enrich have defined in [CE11] the weaker notion of *observable measure*:

**Definition 5.** A probability measure  $\mu$  is *observable* for  $f$  if, for every  $\varepsilon > 0$ , the set

$$A_\varepsilon(\mu) = \{x \in \mathbf{T}^2 \mid \exists \nu \in p\omega(x) : \text{dist}(\nu, \mu) < \varepsilon\} \quad (1)$$

has Leb-positive measure. The set of observable measures is denoted by  $\text{Obs}(f)$ .

The interesting property of these measures is that, unlike physical measures, they always exist. More precisely, the set  $\text{Obs}(f)$  is a non-empty compact subset of the set of invariant measures of  $f$  containing the set of physical measures (see [CE11]).

*Remark 6.* The behaviour of observable measures is compatible with topological conjugacy in the following sense: if  $\mu$  is observable for  $f$  and  $h$  is a homeomorphism which preserves null sets, then  $h^*\mu$  is observable for  $hfh^{-1}$ .

*Example 7.* (1) If  $f = \text{Id}$ , then  $\text{Obs}(f) = \{\delta_x \mid x \in X\}$ , but  $f$  has no physical measure.  
 (2) If a dynamical system possesses a collection of physical measures whose basins of attraction cover almost all the phase space  $X$  (for example if it is ergodic with respect to a smooth measure), then the set of physical measures coincides with the set of observable measures.

**Proposition 8.** *If  $f$  is generic among  $\text{Homeo}(\mathbf{T}^2)$ , then*

$$\text{Obs}(f) = \text{Cl}\{\delta_\omega \mid \omega \text{ is a Lyapunov stable periodic orbit}\},$$

where  $\text{Cl}$  denotes the closure.

<sup>4</sup>Small enough to ensure that the persistent periodic point still exists.

Thus, a generic homeomorphism  $f$  has a lot of observable measures<sup>5</sup>, but no physical measure (it is a direct consequence of the shredding lemma, see [AA13]). To prove this proposition we will need the following lemma.

**Lemma 9.** *For a generic homeomorphism  $f \in \text{Homeo}(X)$ , for every strictly periodic topological ball  $O$  (i.e. there exists  $i > 0$  such that  $f^i(O) \subset\subset O$ ), there exists a Lyapunov stable periodic point  $x \in O$ .*

*Proof of Lemma 9.* We begin by choosing a countable basis of closed sets of  $X$ : for example we can take  $\mathcal{K}_N$  the set of unions of the closures of the cubes of order  $N$ . We also denote by  $\mathcal{B}$  the set of all closed topological balls of  $X$ . We define  $\mathcal{U}_{k,\varepsilon,N}$  as the set of homeomorphisms such that each large enough strictly periodic ball contains a smaller strictly periodic ball with the same period<sup>6</sup>:

$$\mathcal{U}_{k,\varepsilon,N} = \left\{ f \in \text{Homeo}(X) \left| \begin{array}{l} \forall K \in \mathcal{K}_N \cap \mathcal{B} \text{ s.t. } \exists i \leq k \text{ s.t.} \\ f^i(K) \subset\subset K \text{ and } \text{diam}_{\text{int}}(K) > \varepsilon, \\ \exists K' \subset K, K' \in \mathcal{B} \text{ s.t. } \text{diam}(K') < \varepsilon/2 \\ \text{and } f^i(K') \subset\subset K' \end{array} \right. \right\}.$$

Then for every  $k, \varepsilon, N$ , it is straightforward that the set  $\mathcal{U}_{k,\varepsilon,N}$  is an open subset of  $\text{Homeo}(X)$ . To show that it is dense it suffices to apply Brouwer's theorem to each  $K$  such that  $f^i(K) \subset\subset K$  and to make the obtained periodic point attractive.

We now prove that every  $f \in \bigcap_{k,\varepsilon,N} \mathcal{U}_{k,\varepsilon,N}$  satisfies the conclusions of the lemma. First of all, remark that for every topological ball  $K$  with non-empty interior which is strictly  $i$ -periodic, there exists  $N \in \mathbf{N}$  and a smaller topological ball  $\tilde{K} \subset K$  which is strictly  $i$  periodic such that  $\tilde{K} \in \mathcal{K}_N$ . It implies that if  $f$  belongs to the  $G_\delta$  dense set  $\bigcap_{k,\varepsilon,N} \mathcal{U}_{k,\varepsilon,N}$ , then for every topological ball  $K$  with non-empty interior which is strictly  $i$ -periodic, there exists  $N \in \mathbf{N}$  and a topological ball  $K' \subset \tilde{K} \subset K$  which is strictly  $i$  periodic and at least twice smaller. Taking the intersection of such balls, we obtain a periodic point with period  $i$  which is Lyapunov stable by construction.  $\square$

*Proof of Proposition 8.* The first inclusion is easy: it suffices to remark that every stable measure supported by a Lyapunov stable periodic orbit is observable.

For the other inclusion, let  $f$  be a generic dissipative homeomorphism,  $\mu \in \text{Obs}(f)$  and  $\varepsilon > 0$ . By hypothesis  $\text{Leb}(A_\varepsilon(\mu)) > 0$  (see Equation (1)), then  $\varepsilon' = \frac{1}{2} \min(\varepsilon, \text{Leb}(A_\varepsilon(\mu))) > 0$ . As  $f$  is generic, it satisfies the conclusions of the shredding lemma (see [AA13]) applied to  $f$  and  $\varepsilon'$ , in particular there exists a Borel set  $B \subset A_\varepsilon(\mu)$  and an open set  $O \subset \mathbf{T}^2$  such that:

- $\text{Leb}(B) > 0$ ;
- $O$  is strictly periodic:  $\exists i > 0 : f^i(O) \subset\subset O$ ;
- $\text{diam}(O) < \varepsilon'$ ,
- every orbit of every point of  $B$  belongs to  $O$  eventually.

By Lemma 9,  $O$  contains a Lyapunov stable periodic point whose orbit is denoted by  $\omega$ ; thus for every  $x \in B$  and every  $\nu \in p\omega(x)$ , we have  $\text{dist}(\nu, \delta_\omega) < \varepsilon'$ . But by hypothesis  $\text{dist}(\nu, \mu) < \varepsilon$ , then  $\text{dist}(\mu, \delta_\omega) < 2\varepsilon$ , with  $\omega$  a Lyapunov stable periodic orbit.  $\square$

**Lemma 10.** *If  $f$  is generic among  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$ , then  $\text{Obs}(f) = \{\text{Leb}\}$  coincide with the set of physical measures.*

*Proof of Lemma 10.* A classical theorem of J. Oxtoby and S. Ulam [OU41] states that a generic conservative homeomorphism  $f \in \text{Homeo}(\mathbf{T}^2, \text{Leb})$  is ergodic with respect to the measure  $\text{Leb}$ . But Remark 1.8 of [CE11] states that if the measure  $\text{Leb}$  is ergodic, then  $\text{Obs}(f) = \{\text{Leb}\}$ .  $\square$

### 3. OBSERVABLE ROTATION SETS

**3.1. Definitions.** As said before, from the notion of observable measure, it is easy to define a notion of observable ergodic rotation set. Another definition, more topologic, seemed reasonable to us for observable rotation sets:

<sup>5</sup>The set of Lyapunov stable periodic orbits is a Cantor set.

<sup>6</sup>For a compact set  $K$ ,  $\text{diam}_{\text{int}}(K)$  denotes the diameter of the biggest euclidean ball included in  $K$ .

**Definition 11.**

$$\rho^{obs}(F) = \left\{ v \in \mathbf{R}^2 \mid \forall \varepsilon > 0, \text{Leb}\{x \mid \exists u \in \bar{\rho}(x) : d(u, v) < \varepsilon\} > 0 \right\}.$$

$$\rho_{mes}^{obs}(F) = \left\{ \int_{\mathbf{T}^2} D(F)(x) d\mu(x) \mid \mu \in \text{Obs}(f) \right\}.$$

These two sets are non-empty compact subsets of the classical rotation set, and the first one is even a subset of  $\rho_{pts}(F)$ . The next lemma states that these two definitions coincide:

**Lemma 12.**  $\rho_{mes}^{obs}(F) = \rho^{obs}(F)$ .

*Proof of Lemma 12.* We first prove that  $\rho_{mes}^{obs}(F) \subset \rho^{obs}(F)$ . Let  $v \in \rho_{mes}^{obs}(F)$  and  $\varepsilon > 0$ . Then there exists  $\mu \in \text{Obs}(f)$  such that  $v = \int_{\mathbf{T}^2} D(F) d\mu$ , in particular  $\text{Leb}(A_{\varepsilon/2}(\mu)) > 0$ . But if  $x \in A_{\varepsilon/2}(\mu)$ , then there exists a strictly increasing sequence of integers  $(n_i(x))_i$  such that for every  $i \geq 0$ ,

$$\text{dist} \left( \frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} \delta_{f^k(x)}, \mu \right) < \varepsilon.$$

Thus,

$$\left| \frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} D(F)(f^k(x)) - \int_{\mathbf{T}^2} D(F) d\mu \right| < \varepsilon,$$

in other words the inequality

$$\left| \frac{F^{n_i(x)}(x) - x}{n_i(x)} - v \right| < \varepsilon$$

holds for every  $i$  and on a Leb-positive measure set of points  $x$ .

For the other inclusion, let  $v \in \rho^{obs}(F)$  and set

$$\tilde{A}_\varepsilon(v) = \{x \in \mathbf{T}^2 \mid \exists u \in \bar{\rho}(x) : d(u, v) < \varepsilon\}.$$

By hypothesis,  $\text{Leb}(\tilde{A}_\varepsilon(v)) > 0$  for every  $\varepsilon > 0$ . To each  $x \in \tilde{A}_\varepsilon(v)$  we associate the set  $p\omega_\varepsilon^v(x)$  of limit points of the sequence of measures

$$\frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} \delta_{f^k(x)},$$

where  $(n_i(x))_i$  is a strictly increasing sequence such that

$$\left| \frac{F^{n_i(x)}(x) - x}{n_i(x)} - v \right| < \varepsilon.$$

By compactness of  $\mathcal{P}$ , the set  $p\omega_\varepsilon^v(x)$  is non-empty and compact. In the sequel we will use the following easy remark: if  $0 < \varepsilon < \varepsilon'$  and  $x \in \tilde{A}_\varepsilon$ , then  $p\omega_\varepsilon^v(x) \subset p\omega_{\varepsilon'}^v(x)$ .

By contradiction, suppose that for every  $\mu \in \mathcal{P}$ , there exists  $\varepsilon_\mu > 0$  such that

$$\text{Leb}\{x \in \tilde{A}_{\varepsilon_\mu}(v) \mid \exists \nu \in p\omega_{\varepsilon_\mu}^v(x) : \text{dist}(\nu, \mu) < \varepsilon_\mu\} = 0.$$

By compactness,  $\mathcal{P}$  is covered by a finite number of balls  $B(\mu_j, \varepsilon_{\mu_j})$ . Taking  $\varepsilon = \min \varepsilon_{\mu_j}$ , for every  $j$  we have

$$\text{Leb}\{x \in \tilde{A}_\varepsilon(v) \mid \exists \nu \in p\omega_\varepsilon^v(x) : \text{dist}(\nu, \mu) < \varepsilon_{\mu_j}\} = 0,$$

thus, as balls  $B(\mu_j, \varepsilon_{\mu_j})$  cover  $\mathcal{P}$ ,

$$\text{Leb}\{x \in \tilde{A}_\varepsilon(v) \mid p\omega_\varepsilon^v(x) \cap \mathcal{P} \neq \emptyset\} = 0,$$

which is a contradiction.

Therefore, there exists  $\mu_0 \in \mathcal{P}$  such that for every  $\varepsilon > 0$ ,

$$\text{Leb}\{x \in \tilde{A}_\varepsilon(v) \mid \exists \nu \in p\omega_\varepsilon^v(x) : \text{dist}(\nu, \mu_0) < \varepsilon\} > 0,$$

in particular  $\mu_0 \in \text{Obs}(f)$ . Furthermore, for  $\varepsilon > 0$ , there exists  $x \in \tilde{A}_\varepsilon(v)$  and  $\mu_x \in p\omega_\varepsilon^v(x)$  such that  $\text{dist}(\mu_x, \mu_0) < \varepsilon$ . As  $\mu_x \in p\omega_\varepsilon^v(x)$ , there exists a sequence  $(n_i(x))_i$  such that

$$\text{dist} \left( \mu_x, \frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} \delta_{f^k(x)} \right) < \varepsilon \quad \text{and} \quad \left| \frac{F^{n_i(x)}(x) - x}{n_i(x)} - v \right| < \varepsilon.$$

Thus,

$$\text{dist} \left( \mu_0, \frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} \delta_{f^k(x)} \right) < 2\varepsilon.$$

Integrating this estimation according to the function  $D(F)$ , we obtain:

$$\left| \int_{\mathbf{T}^2} D(F) d\mu_0 - \frac{F^{n_i(x)}(x) - x}{n_i(x)} \right| < 2\varepsilon,$$

so

$$\left| \int_{\mathbf{T}^2} D(F) d\mu_0 - v \right| < 3\varepsilon,$$

for every  $\varepsilon > 0$ , in other words,

$$v = \int_{\mathbf{T}^2} D(F) d\mu_0.$$

□

**3.2. Properties of the observable rotation set.** We begin by giving two lemmas which state the dynamical behaviour of the observable rotation sets.

**Lemma 13.** *For every  $q \in \mathbf{N}$ ,  $\rho^{obs}(F^q) = q\rho^{obs}(F)$ .*

*Proof of Lemma 13.* It suffices to remark that  $\bar{\rho}_{F^q}(x) = q\bar{\rho}_F(x)$  (one inclusion is trivial and the other is easily obtained by Euclidean division). □

*Remark 14.* In general  $\rho^{obs}(F^{-1}) \neq -\rho^{obs}(F)$ : see for instance the point 3 of Example 16.

**Lemma 15.** *If  $H$  is a homeomorphism of  $\mathbf{R}^2$  commuting with integral translations and preserving null sets, then  $\rho^{obs}(H \circ F \circ H^{-1}) = \rho^{obs}(F)$ .*

*Proof of Lemma 15.* It follows easily from the fact that the notion of observable measure is stable by conjugacy (see Remark 6). □

We now give a few simple examples of calculation of observable rotation sets.

*Example 16.* (1) If  $f = \text{Id}$ , then  $\rho^{obs}(F) = \{(0, 0)\}$ .

(2) If

$$F(x, y) = (x + \cos(2\pi y), y),$$

then  $\rho_{pts}(F) = \rho^{obs}(F) = [-1, 1] \times \{0\}$ .

(3) If

$$F(x, y) = \left( x + \cos(2\pi y), y + \frac{1}{100} \sin(2\pi y) \right),$$

then  $\rho_{pts}(F) = \{(0, -1), (0, 1)\}$ , but  $\rho^{obs}(F) = \{(0, -1)\}$  and  $\rho^{obs}(F^{-1}) = \{(0, 1)\}$ .

(4) Let

$$P \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \frac{1}{2} \cos(2\pi y) + 1 \\ y \end{pmatrix} \quad \text{and} \quad Q \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \frac{1}{2} \cos(2\pi x) + 1 \end{pmatrix}.$$

Then the rotation set of the (conservative) homeomorphism  $F = P \circ Q$  is equal to  $[0, 1]^2$ . Moreover, we can perturb  $F$  into a (conservative) homeomorphism  $\tilde{F}$  such that  $\tilde{F}$  is the identity on the neighbourhoods of the points whose coordinates belong to  $1/2\mathbf{Z}$  (applying for example the technique of local modification [DF00]). Then, the vertices of the square  $[0, 1]^2$  belong to the observable rotation set of  $\tilde{F}$ .

(5) Let  $P$  be a convex polygon with rational vertices. In [Kwa92], J. Kwapisz has constructed an axiom A diffeomorphism  $f$  of  $\mathbf{T}^2$  whose rotation set is the polygon  $P$ . It is possible to modify slightly Kwapisz's construction so that all the sinks of  $f$  are fixed points, and so that the union of the basins of these sinks have Leb-full measure. Hence, the observable rotation set of  $f_P$  is reduced to  $\{(0, 0)\}$ .

We now give the results about the link between the rotation set and the observable rotation set in the generic setting. We begin by the dissipative case.

**Proposition 17.** *If  $f$  is generic among  $\text{Homeo}(\mathbf{T}^2)$ , then  $\rho(F) = \text{conv}(\rho^{obs}(F))$ . If moreover  $f$  is generic with a non-empty interior rotation set, then  $\rho(F) = \rho^{obs}(F)$ .*

To prove this proposition, we will use the following lemma, which is a direct consequence of Proposition 8.

**Lemma 18.** *If  $f$  is generic among  $\text{Homeo}(\mathbf{T}^2)$ , then*

$$\rho^{obs}(F) = \text{Cl}\{\bar{\rho}(\tilde{x}) \mid x \text{ is a Lyapunov stable periodic point}\}.$$

We will also need a theorem of realization of rotation vectors by periodic points.

**Theorem 19** (J. Franks, Theorem 3.2 of [Fra89]). *For every  $f \in \text{Homeo}(\mathbf{T}^2)$ , every rational point of the interior of  $\rho(F)$  is realized as the rotation vector of a periodic point of the homeomorphism  $f$ .*

*Proof of Proposition 17.* Theorem 1 states that for an open dense set of homeomorphisms, the rotation set is a rational polygon. Then, a theorem of realization of J. Franks [Fra88, Theorem 3.5] implies that every vertex of this polygon is realized as the rotation vector of a periodic point of the homeomorphism, which can be made attractive by a little perturbation of the homeomorphism. Then generically we can find a Lyapunov stable periodic point which shadows the previous periodic point (by Lemma 9), in particular it has the same rotation vector. Thus every vertex of  $\rho(F)$  belongs to  $\rho^{obs}(F)$  and  $\rho(F) = \text{conv}(\rho^{obs}(F))$ .

For  $\varepsilon > 0$ , we can find a finite  $\varepsilon$ -dense subset  $R_\varepsilon$  of  $\rho(F)$  made of rational points. Thus, Theorem 19 associates to each of these rational vectors a periodic point of the homeomorphism which realizes this rotation vector; we can even make these periodic points of the homeomorphism attractive. Thus, for every  $\varepsilon > 0$ , the set  $O_\varepsilon$  made of the homeomorphisms such that every vector of  $R_\varepsilon$  is realized by a strictly periodic open subset of  $\mathbf{T}^2$  is open and dense in the set of homeomorphisms with non-empty interior rotation set. Applying Lemma 9, we find a  $G_\delta$  dense subset of  $O_\varepsilon$  on which every strictly periodic open subset of  $\mathbf{T}^2$  contains a Lyapunov stable periodic point; on this set the Hausdorff distance between  $\rho(F) = \rho^{obs}(F)$  is smaller than  $\varepsilon$ . The conclusion of the proposition then easily follows from Baire theorem.  $\square$

*Remark 20.* It is not true that  $\rho(F) = \rho^{obs}(F)$  holds for a generic homeomorphism: see for instance the point 3 of Example 16, where on a neighbourhood of  $f$  the set  $\rho^{obs}$  is contained in a neighbourhood of the points  $(0, -1)$  and  $(0, 1)$ .

For the conservative case, we recall the result of Proposition 2: the rotation set of a generic conservative homeomorphism has non-empty interior. The following result states that in this case the observable rotation set is much smaller, more precisely it consists in a single vector, namely the mean rotation vector.

**Proposition 21.** *If  $f$  is generic among  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$ , then  $\rho^{obs}(F) = \{\rho_{\text{Leb}}(F)\}$ , where  $\rho_{\text{Leb}}(F)$  is the mean rotation vector with respect to the measure  $\text{Leb}$ .*

Thus, for almost every  $x \in \mathbf{T}^2$  (with respect to the measure  $\text{Leb}$ ), the set  $\bar{\rho}(x)$  is reduced to a single point which is the mean rotation vector.

*Proof of Proposition 21.* It is easily implied by the fact that the measure  $\text{Leb}$  is the only observable measure (Lemma 10, which easily follows from Oxtoby-Ulam theorem).  $\square$

A famous open problem asks whether a generic element of  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$  is ergodic or not<sup>7</sup>. A lot has been done to tackle this question, see for example the introduction of [ACW14] for a short survey. If we assume that a generic element of  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$  is ergodic, then we obtain the following result.

**Conjecture 22.** *If  $f$  is generic among  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$ , then  $\rho^{obs}(F) = \{\rho_{\text{Leb}}(F)\}$ .*

#### 4. DISCRETIZED ROTATION SETS

We now take into account the fact that the computer has a finite digital precision. For  $N \in \mathbf{N}^*$ , we equip the torus  $\mathbf{T}^2$  with a *grid of discretization*

$$E_N = \left\{ \left( \frac{i}{2^N}, \frac{j}{2^N} \right) \mid 0 \leq i, j \leq 2^N - 1 \right\}.$$

<sup>7</sup>Or more generally, for any compact manifold of dimension  $\geq 2$  and any good measure.

We then define the projection  $P_N = \mathbf{T}^2 \rightarrow E_N$  by: for  $x \in \mathbf{T}^2$ ,  $P_N(x)$  is (one of) point of  $E_N$  which is the nearest to  $x$ . The *discretization* of a homeomorphism  $f : \mathbf{T}^2 \rightarrow \mathbf{T}^2$  with respect to the grid  $E_N$  is defined as the map  $f_N = P_N \circ f : E_N \rightarrow E_N$ . For each  $N$  the map  $f_N$  is finite, thus it has a finite number of periodic orbits.

The *discretized rotation set* is defined as follows. Consider a lift  $F : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  of  $f$  and a lift  $\tilde{E}_N$  of the grid  $E_N$  to  $\mathbf{R}^2$ . Then

$$\rho(F_N) = \bigcap_{M \in \mathbf{N}} \overline{\bigcup_{m \geq M} \left\{ \frac{F_N^m(\tilde{x}) - \tilde{x}}{m} \mid \tilde{x} \in \mathbf{R}^2 \right\}}.$$

Remark that this set coincides with the set of rotation vectors of the periodic orbits of  $f_N$ . Then the *asymptotic discretized rotation set* is the upper limit of the sets  $\rho(F_N)$ :

$$\rho^{discr}(F) = \bigcap_{M \in \mathbf{N}} \bigcup_{N \geq M} \rho(F_N).$$

The first result is that for every homeomorphism  $f$ , the discretized rotation set  $\rho(F_N)$  is almost included in the rotation set  $\rho(F)$  when  $N$  is large enough. This property follows easily with a compactness argument from the convergence of the sequence  $f_N$  to the homeomorphism  $f$  (for example for the Hausdorff distance on the graphs of these maps).

**Proposition 23.** *For every homeomorphism  $f$  and every  $\varepsilon > 0$ , it exists  $N_0 \in \mathbf{N}$  such that for every  $N \geq N_0$ , we have  $\rho(F_N) \subset B(\rho(F), \varepsilon)$ , where  $B(\rho(F), \varepsilon)$  denotes the set of points whose distance to  $\rho(F)$  is smaller than  $\varepsilon$ . In particular  $\rho^{discr}(F) \subset \rho(F)$ .*

*Proof of Proposition 23.* By definition of the rotation set, for  $\varepsilon > 0$  there exists  $m \in \mathbf{N}$  such that

$$\left\{ \frac{F^m(\tilde{x}) - \tilde{x}}{m} \mid \tilde{x} \in \mathbf{R}^2 \right\} \subset B(\rho(F), \varepsilon).$$

Then there exists  $N_0 \in \mathbf{N}$  such that for every  $N \geq N_0$ ,

$$\left| \frac{F^m(\tilde{x}) - \tilde{x}}{m} - \frac{F_N^m(\tilde{x}_N) - \tilde{x}_N}{m} \right| \leq \varepsilon.$$

This allows us to handle the case of long periodic orbits of the discretizations: by euclidean division, each periodic orbit of  $f_N$  of length bigger than  $m/\varepsilon$  will be in the  $\varepsilon$  neighbourhood of the convex hull of the set

$$\frac{F_N^m(\tilde{x}_N) - \tilde{x}_N}{m},$$

so in the  $3\varepsilon$ -neighbourhood of the rotation set  $\rho(f)$ .

For short orbits we argue by contradiction: suppose that there exist  $\varepsilon > 0$  such that for every  $N_0 \in \mathbf{N}$  there exists  $N \geq N_0$  and  $x_N \in E_N$  which is periodic under  $f_N$  with period smaller than  $m/\varepsilon$  and whose associated rotation vector is not in  $B(\rho(F), \varepsilon)$ . Then up to take subsequences these periodic points  $x_N$  have the same period and converge to a periodic point  $x \in \mathbf{T}^2$  whose associated rotation vector (for  $F$ ) is not in  $B(\rho(F), \varepsilon)$ , which is impossible.  $\square$

The other inclusion depends on the properties of the map  $f$ . We begin by the dissipative case.

**Proposition 24.** *If  $f$  is generic among  $\text{Homeo}(\mathbf{T}^2)$ , then  $\rho(F_N)$  tends to  $\rho(F)$  for the Hausdorff topology. In particular  $\rho^{discr}(F) = \rho(F)$ .*

*Proof of Proposition 24.* The fact that the upper limit of  $\rho(F_N)$  is included in  $\rho(F)$  follows directly from Lemma 23.

It remains to prove that the lower limit of  $\rho(F_N)$  contains  $\rho(F)$ . First of all the rotation set is the closure of rotation vectors of Lyapunov stable periodic points (Proposition 8). To each one of these points we can associate a periodic closed set  $K$  with non-empty interior and with period  $\tau$  which has the same rotation vector. Then there exists an open set  $O \subset K$  such that for  $N$  large enough and  $x \in K$  we also have  $f_N^\tau(x_N) \in O \subset K$ . Thus there exists  $i \in \mathbf{N}^*$  such that  $f_N^{\tau i}(x_N) = f_N^{2\tau i}(x_N)$  and  $f_N^{\tau i}(x_N)$  has the same rotation vector as  $K$ , thus the same rotation vector as the initial Lyapunov stable periodic point.  $\square$

For the conservative case, with the same techniques as in [Gui13], we can prove the following result.

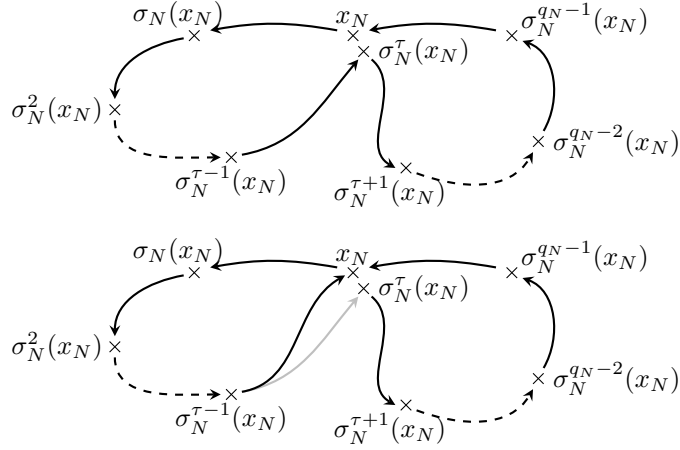


FIGURE 1. Modification of a cyclic permutation in the proof of Lemma 26

**Theorem 25.** *If  $f$  is generic among  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$ , then for every compact subset  $K$  of the rotation set of  $F$  there exists a subsequence  $f_{N_i}$  of discretizations such that  $\rho_{N_i}(F)$  tends to  $K$  for the Hausdorff topology. In particular  $\rho^{\text{discr}}(F) = \rho(F)$ .*

The combination of the realisation theorem of J. Franks (Theorem 19), the fact that for a generic conservative homeomorphism the rotation set has non-empty interior (Proposition 2), and the fact that every periodic point can be made persistent by a small perturbation, reduces the proof to that of the following lemma.

**Lemma 26.** *If  $f$  is generic among  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$ , then for every finite collection of rotation vectors  $\{v_1, \dots, v_n\}$ , each one realized by a persistent periodic orbit of  $f$ , there exists a subsequence  $f_{N_i}$  of discretizations such that for every  $i$ ,  $\rho_{N_i}(f) = \{v_1, \dots, v_n\}$ .*

*Proof of Lemma 26.* We denote by  $\mathcal{D}_q$  the set of subsets of  $\mathbf{Q}^2$  made of elements whose coordinates are of the type  $p'/q'$ , with  $0 < q' < q$  and  $-q^2 < p' < q^2$ . Consider the set

$$\bigcap_{q, N_0} \bigcap_{D \in \mathcal{D}_q} \bigcup_{N \geq N_0} \left\{ f \in \text{Homeo}(\mathbf{T}^2, \text{Leb}) \mid (\forall v \in D, v \text{ is realised by a } \right. \\ \left. \text{persistent periodic point of } f) \implies \rho(F_N) = D \right\}. \quad (2)$$

To prove the lemma it suffices to prove that this set contains a  $G_\delta$  dense. It is obtained in combining the arguments of Proposition 25 and Proposition 39 of [Gui13]. We present the main arguments.

Let  $f \in \text{Homeo}(\mathbf{T}^2, \text{Leb})$ ,  $\varepsilon > 0$ ,  $q, N_0 \in \mathbf{N}$  and  $D \in \mathcal{D}_q$ . We suppose that for all  $v \in D$ ,  $v$  is realizable by a persistent periodic orbit  $\omega_i$  of  $f$ . For all of these orbits  $\omega_1, \dots, \omega_\ell$ , we denote by  $p_i$  the length of the orbit  $\omega_i$  and choose a point  $x_i$  belonging to  $\omega_i$ . We then apply Lax's theorem (see [Lax71] and [Alp76], see also Theorem 20 of [Gui13]): if  $N$  is large enough, then there exists a cyclic permutation  $\sigma_N$  of  $E_N$  such that  $d_N(f, \sigma_N) < \varepsilon$ . If  $N$  is large enough, then the families

$$\{(x_1)_N, \dots, \sigma_N^{p_1-1}((x_1)_N)\}, \dots, \{(x_\ell)_N, \dots, \sigma_N^{p_\ell-1}((x_\ell)_N)\}$$

are disjoint and satisfy  $d((x_i)_N, \sigma_N^{p_i-1}((x_i)_N)) < \varepsilon$  for all  $i$ . We then use the same technique as in the proof of Proposition 25 of [Gui13] (see also Figure 1) to close each orbit  $\{(x_i)_N, \dots, \sigma_N^{p_i-1}((x_i)_N)\}$ . The discrete map  $\sigma'_N$  we obtain has then exactly  $\ell$  periodic orbits, and each of them has the same rotation vector as the corresponding real periodic orbit of  $f$ . We then use the proposition of finite maps extension (in other words a  $C^0$  closing lemma, see for example [Gui12, Section 2.2]) to build a homeomorphism  $g$  which is  $\varepsilon$ -close to  $f$  and whose discretization  $g_N$  satisfies  $\rho(g_N) = D$ ; moreover we can suppose that this occurs on a whole neighbourhood of  $g$ . This proves that the set of (2) contains a  $G_\delta$  dense subset of  $\text{Homeo}(\mathbf{T}^2, \text{Leb})$ .  $\square$

In the  $C^1$ -case, applying an elementary perturbation lemma, it is possible to obtain a weaker result about discretized rotation sets (in this case, we can not control what happens on the whole grid  $E_N$ , but only on a subgrid of  $E_N$ ).

**Proposition 27.** *If  $f$  is generic among  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$ , then there exists a subsequence  $f_{N_i}$  of discretizations such that  $\rho_{N_i}(F)$  tends to  $\rho(F)$  for the Hausdorff topology.*

As for Theorem 25, this proposition is easily deduced from the following lemma.

**Lemma 28.** *If  $f$  is generic among  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$ , then for every finite collection of rotation vectors  $\{v_1, \dots, v_n\}$ , each one realized by a periodic orbit of  $f$ , there exists a subsequence  $f_{N_i}$  of discretizations such that for every  $i$ ,  $\{v_1, \dots, v_n\} \subset \rho_{N_i}(f)$ .*

*Proof of Lemma 28.* The proof of this lemma is very similar to that of Lemma 26 (we take the same notations).

Consider the set

$$\bigcap_{q, N_0} \bigcap_{D \in \mathcal{D}_q} \bigcup_{N \geq N_0} \left\{ f \in \text{Diff}^1(\mathbf{T}^2, \text{Leb}) \mid (\forall v \in D, v \text{ is realised by a } \omega'_i \text{ of } f) \implies D \subset \rho(F_N) \right\}. \quad (3)$$

To prove the lemma, it suffices to prove that this set contains a  $G_\delta$  dense subset of  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$ .

Let  $f \in \text{Homeo}(\mathbf{T}^2, \text{Leb})$ ,  $\varepsilon > 0$ ,  $q, N_0 \in \mathbf{N}$  and  $D \in \mathcal{D}_q$ . We suppose that for all  $v \in D$ ,  $v$  is realizable by a persistent periodic orbit  $\omega_i$  of  $f$ . Then, by an elementary perturbation lemma (see for example [Cro06, Proposition II.1]), it is possible to perturb  $f$  into a diffeomorphism  $g$  such that  $d_{C^1}(f, g) < \varepsilon$  and that there exists  $N \geq N_0$  such that for every  $i$ , there exists a periodic orbit  $\omega'_i$  of  $g$  which is close to  $\omega_i$  (in particular, it has the same rotation vector) and such that  $\omega'_i \subset E_N$ . Moreover, perturbing a little  $g$  if necessary (as in the proof of Proposition 2), we can suppose that the periodic orbits  $\omega'_i$  are persistent. This proves that the set of (3) contains a  $G_\delta$  dense subset of  $\text{Diff}^1(\mathbf{T}^2, \text{Leb})$ .  $\square$

## 5. NUMERICAL SIMULATIONS

We have conducted numerical simulations of the rotation sets associated to both dissipative and conservative homeomorphisms. For the first examples we treated, we have made the deliberate choice to choose homeomorphisms whose rotation set is known to be the square  $[0, 1]^2$ . Of course these homeomorphisms are not the best candidates for “generic” homeomorphisms, but at least we are sure of what is the shape of the rotation set we want to obtain.

As an example of dissipative homeomorphism we have taken  $f_1 = R_1 \circ Q_1 \circ P_1$ , with

$$\begin{aligned} P_1(x, y) &= \left( x, y + \frac{1}{2}(\cos(2\pi(x + \alpha)) + 1) \right. \\ &\quad \left. + 0.0234 \sin^2(4\pi(x + \alpha))(\sin(6\pi(x + \alpha)) + 0.3754 \cos(26\pi(x + \alpha))) \right), \\ Q_1(x, y) &= \left( x + \frac{1}{2}(\cos(2\pi(y + \beta)) + 1) \right. \\ &\quad \left. + 0.0213 \sin^2(4\pi(y + \beta))(\sin(6\pi(y + \beta)) + 0.4243 \cos(22\pi(y + \beta))) \right), \\ R_1(x, y) &= (x - 0.0127 \sin(8\pi(x + \alpha)) + 0.000324 \sin(33\pi(x + \alpha)), \\ &\quad y - 0.0176 \sin(12\pi(y + \beta)) + 0.000231 \sin(41\pi y)), \end{aligned}$$

$\alpha = 0.00137$  and  $\beta = 0.00159$ .

The homeomorphisms  $P_1$  and  $Q_1$  are close to the homeomorphisms

$$\tilde{P}(x, y) = \left( x, y + \frac{1}{2}(\cos(2\pi(x + \alpha)) + 1) \right) \quad \text{and} \quad \tilde{Q}(x, y) = \left( x + \frac{1}{2}(\cos(2\pi(y + \beta)) + 1), y \right);$$

it can easily be seen that the rotation set of the homeomorphism  $\tilde{Q} \circ \tilde{P}$  is the square  $[0, 1]^2$ , whose vertices are realized by the points  $(0, 0)$ ,  $(0, 1/2)$ ,  $(1/2, 0)$  and  $(1/2, 1/2)$ . The perturbations  $P_1$  and  $Q_1$  of  $\tilde{P}$  and  $\tilde{Q}$  are small enough (in  $C^2$  topology) to ensure that the rotation set is still the square  $[0, 1]^2$ ; these perturbations are made in order to make  $f_1$  “more generic” (in particular, the periodic orbits whose rotation vectors realize the vertices of the square do not belong to the grids). The key property of the homeomorphism  $R_1$  is that it has the fixed points of  $Q_1 \circ P_1$  which realize the vertices of  $[0, 1]^2$  as fixed attractive points; this creates fixed attractive points which realize the vertices of the rotation set.

We have chosen  $R_1$  to be very close to the identity in  $C^1$ -topology to ensure that the basins of the sinks and sources are large enough. Indeed, J.-M. Gambaudo and C. Tresser have shown

in [GT83] that, even for dissipative diffeomorphisms defined by very simple formulas, sinks and sources are often undetectable in practice because the size of their basins are too small.

For the first example of conservative homeomorphism, we have chosen the map  $g_1 = Q_1 \circ P_1$ , whose expression is very similar to those of  $f_1$ . Again, a simple calculus shows that the rotation set of  $g_1$  is the same as that of  $\tilde{Q} \circ \tilde{P}$ , which is  $[0, 1]^2$ .

We have conducted other series of simulations for two other examples of conservative homeomorphisms. The first one has an expression which is very similar to that of  $g_1$ , but the cosines are replaced by a piecewise affine map with the same following properties:  $s$  is 1-periodic,  $s(0) = 1$ ,  $s(1/2) = 0$  and  $s$  is affine between 0 and 1/2 and between 1/2 and 1. More precisely, we set  $g_2 = Q_2 \circ P_2$ , with

$$P_2(x, y) = (x, y + 2s(x + \alpha) + 0.0234s(2(x + \alpha)) + 0.0167s(10(x + \alpha)));$$

$$Q_2(x, y) = (x + 2s(y + \beta) + 0.0213s(2(y + \beta)) + 0.0101s(6(y + \beta)), y).$$

The properties of  $s$  imply that the rotation set of  $g_2$  is also the square  $[0, 1]^2$ ; the difference with  $g_1$  is that the vertices of this rotation set are no longer realized by elliptic periodic points, which makes them harder to detect.

For the last conservative homeomorphism we tested, we made “random” choices of the coefficients; we do not know *a priori* what is its rotation set. More precisely, we took  $g_3 = Q_3 \circ P_3$ , with

$$P_3(x, y) = (x, y + 0.3 \sin(2\pi(x + 0.34137)) + 0.2 \sin(3\pi(x + 0.21346)) + 0.578675));$$

$$Q_3(x, y) = (x + 0.25 \sin(2\pi(y + 0.9734)) + 0.35 \sin(3\pi(y - 0.20159)) + 0.551256, y).$$

We will test on simulations whether the computed rotation sets seem to converge or not. If so, it could be a good indication that the rotation set we obtained is close to the actual rotation set.

We have made two kinds of simulations of the rotation set.

- In the first one we have computed the rotation vectors of segments of orbits of length 1 000 with good precision (52 binary digits); in other words for  $N$  random starting points  $x \in \mathbf{T}^2$ , we have computed  $\frac{F^{1000}(x) - x}{1000}$ . We have made these tests for  $N = 100$ , which takes around 1s of calculation,  $N = 10\,000$ , which takes about 2min of calculation, and  $N = 1\,000\,000$ , which takes about 4h of calculation. This is maybe the most simple process that can be used to find numerically the rotation set. It should lead to a good approximation of the observable rotation set; in particular, Proposition 17 suggests that, for the dissipative homeomorphism  $f_1$ , we should obtain a set which is close (for Hausdorff distance) to the square  $[0, 1]^2$ , and if not at least a set whose convex hull is this square. On the other hand, for the conservative homeomorphisms  $g_i$ , Proposition 21 suggests that we should only obtain the mean rotation vector, which is close to  $(1/2, 1/2)$ .
- In the second kind of simulations we have computed the rotation vectors of the periodic orbits of the discretization  $(f_i)_N$  on a grid  $N \times N$ ; these simulations calculate the discretized rotation sets. For each homeomorphism we have represented these sets for  $N = 499$ ,  $N = 500$  and  $N = 501$ , each calculation taking about 2s of calculation. We have also computed the union of the discretized rotation sets for  $100 \leq N \leq M$ , which represents the asymptotic discretized rotation set. We represent these sets for  $M = 100$  ( $\simeq 0.5$ s of calculation),  $M = 150$  ( $\simeq 15$ s of calculation),  $M = 200$  ( $\simeq 45$ s of calculation),  $M = 500$  ( $\simeq 13$ min of calculation),  $M = 1\,000$  ( $\simeq 1$ h 45min of calculation),  $M = 2\,000$  ( $\simeq 14$ h of calculation) and for  $g_3$ ,  $M = 3950$  ( $\simeq 100$ h of calculation). The theory tells us that in both conservative and dissipative cases, for some  $N$ , the discretized rotation set should be close (for Hausdorff distance) to the square  $[0, 1]^2$ ; a weaker property would be that its convex hull should be close to this square. Moreover this should also be true for the asymptotic discretized rotation sets.

Notice that these two methods are formally the same: making simulations on a grid  $N \times N$  is equivalent to calculate with  $-\log_2 N$  binary digits (for example about 10 for  $N = 1\,000$ ). The only difference is that for the second method we use deliberately a very bad numerical precision, which allows us to detect the actual dynamics of the discretizations.

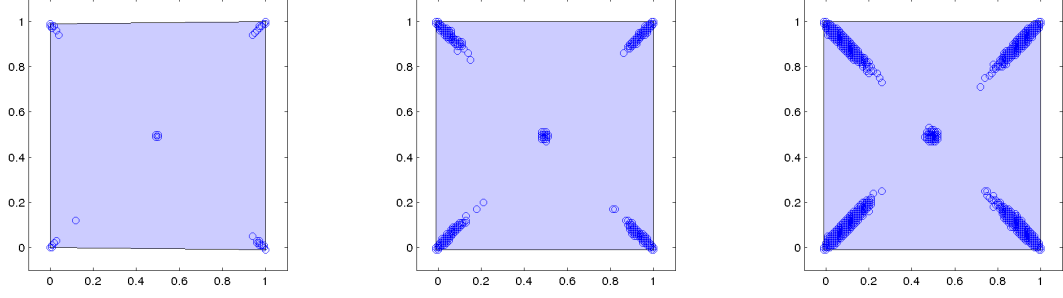


FIGURE 2. Observable rotation set of  $f_1$ ,  $k$  orbits of length 1 000 with random starting points with  $k = 100$  (left), 10 000 (middle) and 1 000 000 (right)

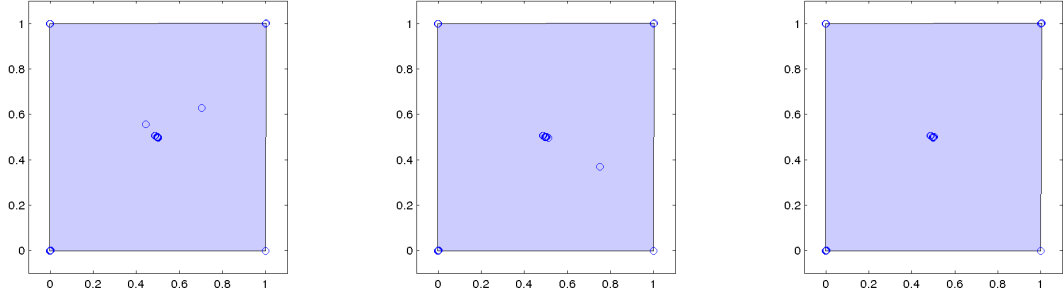


FIGURE 3. Discretized rotation set of  $f_1$  on grids  $E_N$ , with  $N = 499$  (left),  $N = 500$  (middle) and  $N = 501$  (right)

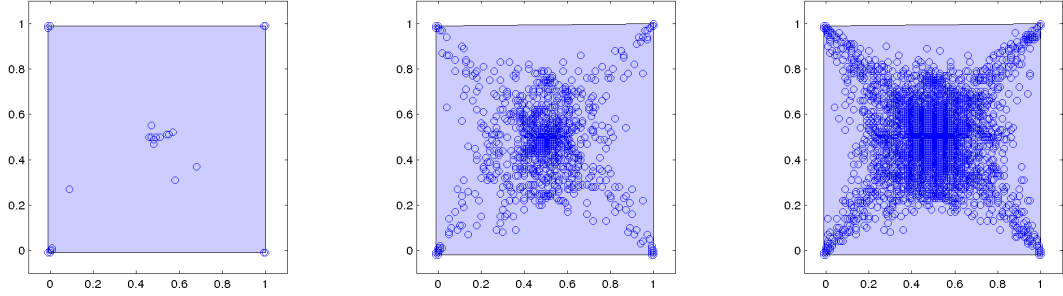


FIGURE 4. Asymptotic discretized rotation set of  $f_1$  as the union of the discretized rotation sets on grids  $E_N$  with  $100 \leq N \leq M$  with  $M = 100$  (left),  $M = 500$  (middle),  $M = 2000$  (right)

Moreover, in practice, for a given calculation time, the calculation of the rotation set by discretization (*i.e.* by the second method) allows to compute much more orbits than the other method. More precisely, the algorithm we have used to compute the asymptotic discretized rotation set visits each point of the grid  $N \times N$  once. Thus, for  $N^2$  starting points we only have to compute  $N^2$  images of the discretization of the homeomorphism on the grid; the number of rotation vectors we obtain is simply the number of periodic orbits of the discretization. So in a certain sense this second algorithm is much faster than the naive algorithm consisting in computing long segments of orbits. All the simulations have been performed on a computer equipped with a processor Intel Core I5 2.40GHz.

In the dissipative case, a lot of the obtained rotation vectors are close to one of the vertices of the real rotation set  $[0, 1]^2$  of  $f_1$ , the others being located around  $(1/2, 1/2)$  (see Figure 2).

This is what is predicted by the theory, in particular by Lemma 18: we detect rotation vectors realized by Lyapunov stable periodic points. The fact that the rotation vectors are not located exactly on the vertices of  $[0, 1]^2$  can be explained by the slow convergence of the orbits to the attractive points: it may take a while until the orbit become close to one of the Lyapunov stable periodic points. We will see that this behaviour is very different from the one in the conservative case, even if the homeomorphism  $f_1$  is very close to  $g_1$  (approximately  $10^{-2}$  close).

For the discretized rotation sets of  $f_1$ , the vertices of  $[0, 1]^2$  are also detected, and we only have a few points in the interior of the square (see Figure 3). However, when we compute the asymptotic discretized rotation set (see Figure 4), we observe that the computed rotation vectors fill a great proportion of the square  $[0, 1]^2$ , as predicted by the theory.

In the conservative case, the rotation vectors of the observable rotation set are mainly quite close to the mean rotation vector of  $g_1$ , as predicted by Proposition 21. In particular in Figure 5, left, all the 100 rotation vectors of the computed observable rotation set are in the neighbourhood of  $(1/2, 1/2)$ . Thus, the behaviour of these vectors is governed by Birkhoff's ergodic theorem with respect to the ergodic measure  $\text{Leb}$ ; *a priori* this behaviour is quite chaotic and converges slowly: a typical orbit will visit every measurable subset with a frequency proportional to the measure of this set, so the rotation vectors will take time to converge. When the number of computed orbits increases (Figure 5, middle and right), we observe that a few rotation vectors are not close to the mean rotation vector; for  $10^6$  different orbits we even detect three of the vertices of the actual rotation set. Anyway, even after 4 hours of calculation, we are unable to recover completely the initial rotation set of the homeomorphism.

On the other hand, the convex hull of the discretized rotation set gives quickly a very good approximation of the rotation set. For example on a grid  $500 \times 500$  (Figure 6), with 2s of calculation (and even on a grid  $100 \times 100$  and 0.2s of calculation), we obtain a rotation set whose convex hull is already very close to  $[0, 1]^2$ . However, for a single size of grid, we do not obtain exactly the conclusions of Theorem 25 which states that for some integers  $N$  the

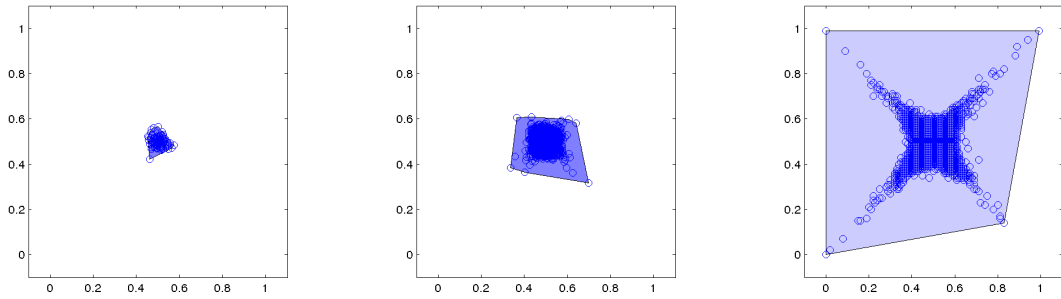


FIGURE 5. Observable rotation set of  $g_1$ ,  $k$  orbits of length 1 000 with random starting points with  $k = 100$  (left), 10 000 (middle) and 1 000 000 (right)

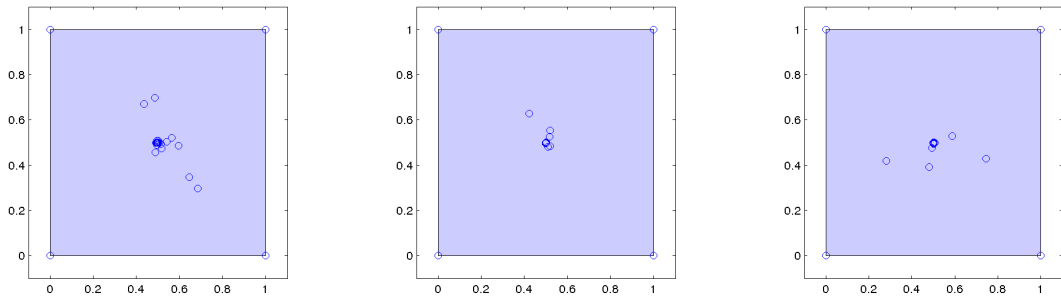


FIGURE 6. Discretized rotation set of  $g_1$  on grids  $E_N$ , with  $N = 499$  (left),  $N = 500$  (middle) and  $N = 501$  (right)

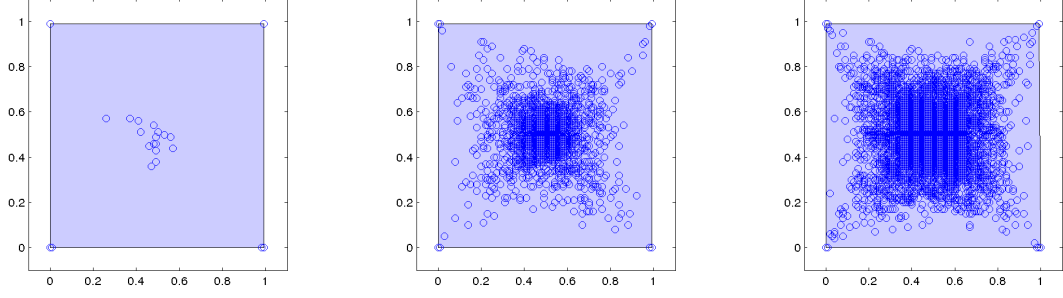


FIGURE 7. Asymptotic discretized rotation set of  $g_1$  as the union of the discretized rotation sets on grids  $E_N$  with  $100 \leq N \leq M$  with  $M = 100$  (left),  $M = 500$  (middle),  $M = 2000$  (right)

discretized rotation set should be close to the rotation set for Hausdorff distance; here for each  $N$  we only have a few points in the interior of  $[0, 1]^2$ . However, when we represent the union of the discretized rotation sets on grids  $N \times N$  with  $100 \leq N \leq 2000$  (Figure 7), we recover almost all the rotation set of  $g_1$ , except from the points which are close to one edge of the square but far from its vertices. The fact that we can obtain very easily the vertices of the rotation set can be due to the fact that in our example  $f_1$  these vertices are realized by elliptic periodic points of the homeomorphism (in fact the derivative on this points is the identity). That is why we also conducted simulations of the homeomorphism  $g_2$  whose rotation set is also the square  $[0, 1]^2$  whose vertices are realized by non-elliptic periodic points.

In fact, when we compute the observable rotation set for  $g_2$  (Figure 8), we only find rotation vectors which are close to the mean rotation vector, even after 4h of calculation. As the periodic

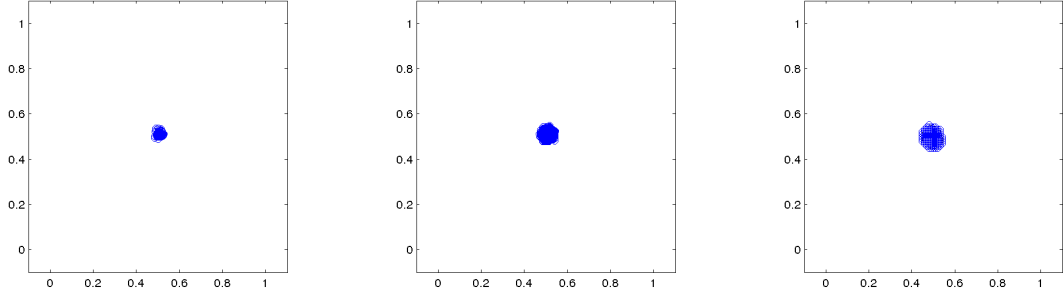


FIGURE 8. Observable rotation set of  $g_2$ ,  $k$  orbits of length 1000 with random starting points with  $k = 100$  (left), 10 000 (middle) and 1 000 000 (right)

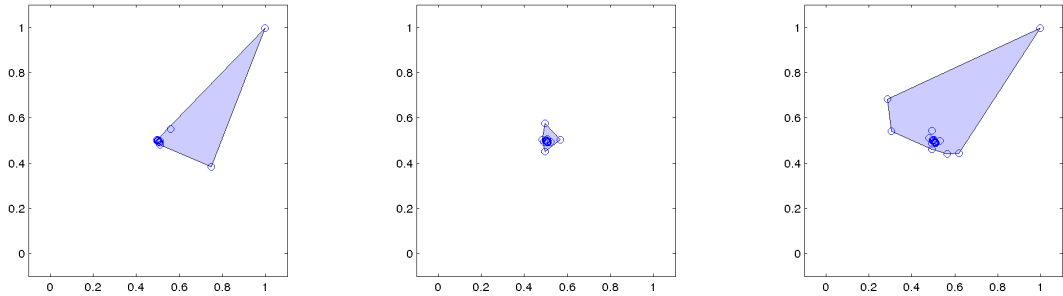


FIGURE 9. Discretized rotation set of  $g_2$  on grids  $E_N$ , with  $N = 499$  (left),  $N = 500$  (middle) and  $N = 501$  (right)

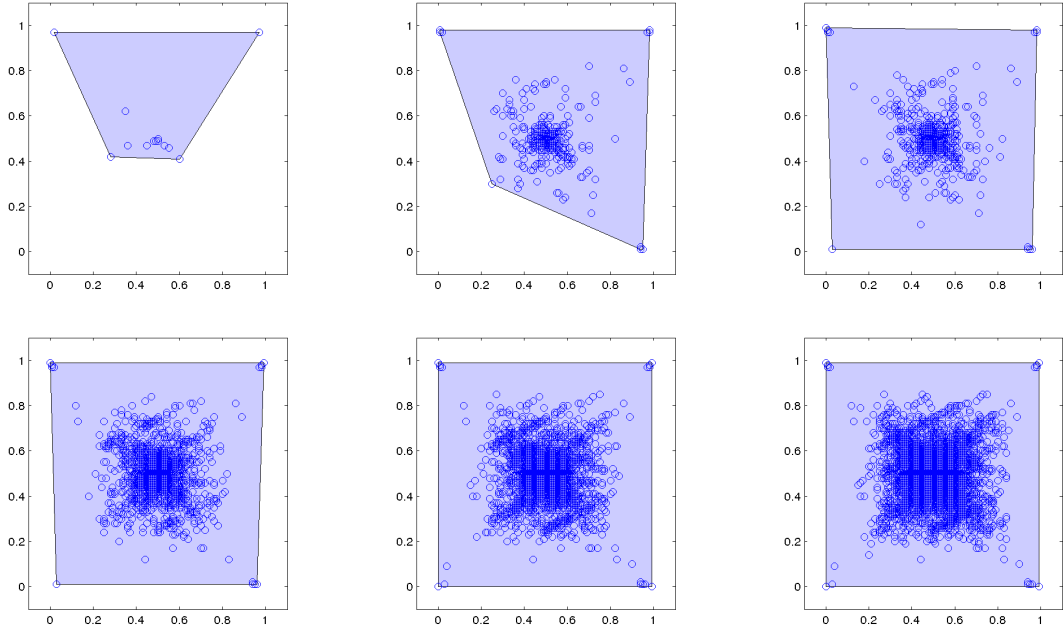


FIGURE 10. Asymptotic discretized rotation set of  $g_2$  as the union of the discretized rotation sets on grids  $E_N$  with  $100 \leq N \leq M$  with  $M = 100$  (top left),  $M = 150$  (top middle),  $M = 200$  (top right),  $M = 500$  (bottom left),  $M = 1000$  (bottom middle) and  $M = 2000$  (bottom right)

points which realize the vertices of the rotation set are no longer elliptic, they are much more unstable and thus they are not detected by these simulations.

The sets detected by the discretized rotation sets of order 499, 500 and 501 (Figure 9) are quite bigger than those detected by the simulations of the observable rotation set, even if the time of calculation is much smaller. However, we do not recover the whole rotation set of the homeomorphism (we conducted simulations for higher orders around  $N = 1000$  and  $N = 2000$  and the behaviour is similar). By contrast, the simulations of the asymptotic discretized rotation set (Figure 10) allows us to see the actual rotation set of the homeomorphism: when we represent all the rotation vectors of the discretizations of order  $100 \leq N \leq M$  with  $M = 200$  (which takes about 45s of calculation) we obtain a set which is very close to the square  $[0, 1]^2$ ; for  $M = 100$  ( $\simeq 1h 45min$  of calculation) we recover almost exactly the initial rotation set.

Finally, the behaviour of the observable rotation set of  $g_3$  is very similar to that of  $g_2$  (see Figure 11): even when we compute 1000 000 different orbits with random starting points, we only obtain rotation vectors which are close to  $(0.55, 0.5)$ , which should be a good approximation of the mean rotation vector.

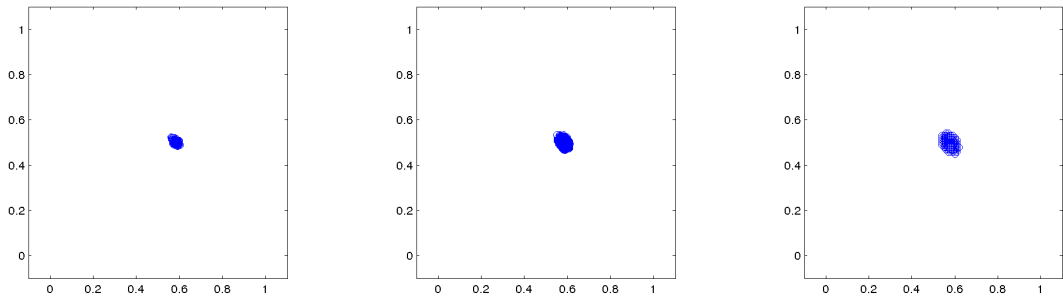


FIGURE 11. Observable rotation set of  $g_3$ ,  $k$  orbits of length 1000 with random starting points with  $k = 100$  (left), 10 000 (middle) and 1 000 000 (right)

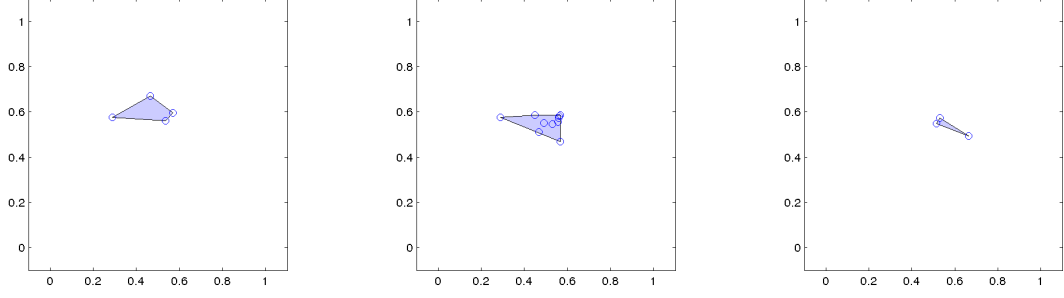


FIGURE 12. Discretized rotation set of  $g_3$  on grids  $E_N$ , with  $N = 499$  (left),  $N = 500$  (middle) and  $N = 501$  (right)

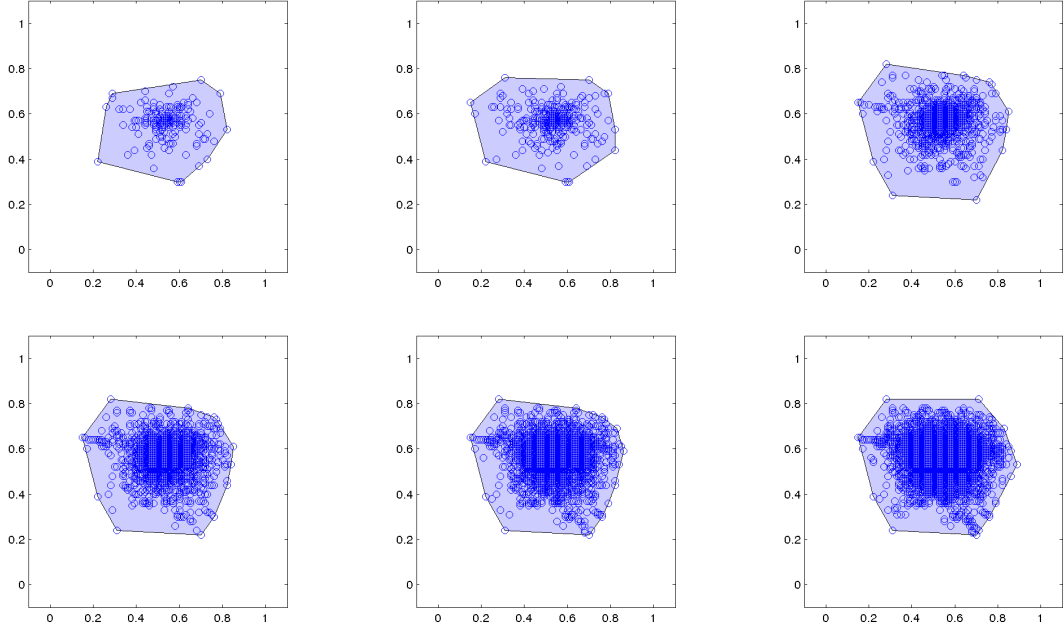


FIGURE 13. Asymptotic discretized rotation set of  $g_3$  as the union of the discretized rotation sets on grids  $E_N$  with  $100 \leq N \leq M$  with  $M = 150$  (top left),  $M = 200$  (top middle),  $M = 500$  (top right),  $M = 1000$  (bottom left),  $M = 2000$  (bottom middle) and  $M = 3950$  (bottom right)

Like for  $g_2$ , the simulations of the discretized rotation sets for the grids  $E_N$  with  $N \in \{499, 500, 501\}$  (Figure 12) are not very convincing: the sets do not seem to converge to anything. We have to compute the asymptotic discretized rotation sets (Figure 13) to see something that looks like a convergence for the Hausdorff topology of the computed rotation sets. Yet, this convergence in practical is not an evidence that the set we compute is close to the actual rotation set of  $g_3$ . To our knowledge, it is impossible to ensure that for a given order of discretization, the asymptotic rotation set computed to this order is close to the rotation set of  $g_3$ .

Considering the results of the numerical experiments, we can formulate the following moral.

*Simulations on practical examples confirm what the theory predicts for generic conservative homeomorphisms and diffeomorphisms: to compute the rotation set of such maps, the algorithm which calculates the “asymptotic discretized rotation set” gives (quickly) a quite good approximation of the rotation set, while the naive algorithm that calculates the “observable rotation set” does not work at all (and moreover is very slow).*

## REFERENCES

- [AA13] Flavio Abdenur and Martin Andersson, *Ergodic theory of generic continuous maps*, Comm. Math. Phys. **318** (2013), no. 3, 831–855. MR 3027586
- [ACW14] Artur Avila, Sylvain Crovisier, and Amie Wilkinson, *Diffeomorphisms with positive metric entropy*, arXiv:1408.4252 [math.DS], 2014.
- [Alp76] Steve Alpern, *New proofs that weak mixing is generic*, Invent. Math. **32** (1976), no. 3, 263–278.
- [CE11] Eleonora Catsigeras and Heber Enrich, *SRB-like measures for  $C^0$  dynamics*, Bull. Pol. Acad. Sci. Math. **59** (2011), no. 2, 151–164.
- [Cro06] Sylvain Crovisier, *Perturbation of  $C^1$ -diffeomorphisms and generic conservative dynamics on surfaces*, Dynamique des difféomorphismes conservatifs des surfaces: un point de vue topologique, Panor. Synthèses, vol. 21, Soc. Math. France, Paris, 2006, pp. 1–33. MR 2288283 (2008b:37072)
- [DF00] Fons Daalderop and Robbert Fokink, *Chaotic homeomorphisms are generic*, Topology Appl. **102** (2000), no. 3, 297–302.
- [Fra71] John Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. **158** (1971), 301–308. MR 0283812 (44 #1042)
- [Fra88] ———, *Recurrence and fixed points of surface homeomorphisms*, Ergodic Theory Dynam. Systems **8\*** (1988), no. Charles Conley Memorial Issue, 99–107.
- [Fra89] ———, *Realizing rotation vectors for torus homeomorphisms*, Trans. Amer. Math. Soc. **311** (1989), no. 1, 107–115.
- [GT83] Jean-Marc Gambaudo and Charles Tresser, *Some difficulties generated by small sinks in the numerical study of dynamical systems: two examples*, Phys. Lett. A **94** (1983), no. 9, 412–414.
- [Gui12] Pierre-Antoine Guihéneuf, *Propriétés dynamiques génériques des homéomorphismes conservatifs*, Ensaïos Matemáticos [Mathematical Surveys], vol. 22, Sociedade Brasileira de Matemática, Rio de Janeiro, 2012.
- [Gui13] ———, *Dynamical properties of spatial discretizations of a generic homeomorphism*, to appear in Ergodic Theory and Dynamical Systems, 2013.
- [Her79] Michael Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 49, 5–233. MR 538680 (81h:58039)
- [KH95] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza. MR 1326374 (96c:58055)
- [Kwa92] Jarosław Kwapisz, *Every convex polygon with rational vertices is a rotation set*, Ergodic Theory Dynam. Systems **12** (1992), no. 2, 333–339.
- [Kwa93] ———, *An estimate of entropy for toroidal chaos*, Ergodic Theory Dynam. Systems **13** (1993), no. 1, 123–129.
- [Lax71] Peter Lax, *Approximation of measure preserving transformations*, Comm. Pure Appl. Math. **24** (1971), 133–135.
- [LM91] Jaume Llibre and Robert S. MacKay, *Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity*, Ergodic Theory Dynam. Systems **11** (1991), no. 1, 115–128.
- [Mañ82] Ricardo Mañé, *An ergodic closing lemma*, Ann. of Math. (2) **116** (1982), no. 3, 503–540. MR 678479 (84f:58070)
- [MZ89] Michał Misiurewicz and Krystyna Ziemian, *Rotation sets for maps of tori*, J. London Math. Soc. (2) **40** (1989), no. 3, 490–506. MR 1053617 (91f:58052)
- [OU41] John Oxtoby and Stanislaw Ulam, *Measure-preserving homeomorphisms and metrical transitivity*, Ann. of Math. **42** (1941), no. 2, 874–920.
- [Pas14] Alejandro Passeggi, *Rational polygons as rotation sets of generic homeomorphisms of the two torus*, J. Lond. Math. Soc. (2) **89** (2014), no. 1, 235–254. MR 3174742
- [Poi85] Henri Poincaré, *Mémoire sur les courbes définies par une équation différentielle*, J. de Math Pures Appl (1885), no. 1, 167–244.
- [You02] Lai-Sang Young, *What are SRB measures, and which dynamical systems have them?*, J. Statist. Phys. **108** (2002), no. 5-6, 733–754, Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

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