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# Mathematical analysis of a inf-convolution model for image processing

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**Abstract** We deal with a second order image decomposition model to perform denoising and texture extraction that was previously presented in [10]. We look for the decomposition  $f = u + v + w$  where  $u$  is a first order term,  $v$  a second order term and  $w$  the remainder term (0 order). For highly textured images the model gives a two-scale texture decomposition:  $u$  can be viewed as a *macro-texture* (larger scale) which oscillations are not too large and  $w$  is the *micro-texture* (very oscillating) that contains the noise. Here, we perform mathematical analysis of the model and give qualitative properties of solutions using the dual problem and inf-convolution formulation.

**Keywords** Second order total variation · image decomposition · variational method · inf-convolution

**Mathematics Subject Classification (2000)** 65D18 · 68U10 · 65K10

## 1 Introduction

The most famous variational denoising model is the Rudin-Osher-Fatemi one ([1, 29]). This model involves a regularization term that preserves discontinuities, what a classical  $H^1$ -Tychonov regularization method does not. The observed image to recover is split in two parts  $u_d = w + u$  where  $w$  represents the oscillating component (noise or texture) and  $u$  is the *smooth* part. So we look for a solution  $u$  such that  $u_d = w + u$  with  $u \in BV(\Omega)$  and  $w \in L^2(\Omega)$ , where  $BV(\Omega)$  is the functions of bounded variation space defined on an open subset  $\Omega \subset \mathbb{R}^d$  ([3, 4, 22]). The regularization term involves only the so-called *cartoon* component  $u$ , while the remainder term  $w := u_d - u$  represents the noise to be minimized.

A lot of people have investigated such decomposition models based on variational formulation, considering that an image can be decomposed into many

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components, each component describing a particular property of the image ([5, 7, 25–27, 30] and references therein for example).

In [10, 11] we have presented second order models where the (first order) classical total variation term has been replaced by a second order total variation term with the appropriate functional framework, namely the space of functions with bounded hessian introduced as  $BH(\Omega)$  in [20] (and denoted  $BV^2(\Omega)$  in [8, 11, 10]). The use of such a model allows to get rid of the *staircasing effect* that appears with the ROF model in denoising processes. To achieve this goal K. Bredies and al. have recently introduced a generalized total variation definition (TGV) [13, 14, 12] that is a nice compromise/mixture between the first and second order derivatives. It is in some sense an extension of the inf-convolution (we recall the definition later) of the first and second order derivatives. The model we present here is not more efficient than the TGV- based one for denoising. However, it provides a decomposition of the image at different scales what the TGV -model does not a priori. This paper is focused on this decomposition that provides a multiscale description of textured images.

Second order models have also been investigated in the context of segmentation and inpainting problems with Mumford-Shah types functionals (see [2, 16, 17] for example). The functional framework is the so called  $GSBV$  space composed of functions  $u$  whose truncated forms  $(\min(-N, \max(u, N)))$  belong to  $SBV_{loc}$  for every  $N \in \mathbb{N}$ . The definition of  $GSBV^2$  is slightly different from the one we consider since

$$GSBV^2(\Omega) = \{u \in GSBV(\Omega) \mid \nabla u \in [GSBV(\Omega)]^d\} .$$

The aim of this paper is to give an existence result without any additional penalization term as in [10] and to perform a qualitative analysis of the model. Uniqueness et regularity issues will also be addressed.

More precisely, we assume that an image (in  $L^2(\Omega)$ ) can be split in three components: a smooth (continuous) part  $v$ , a *cartoon* (piecewise constant) part  $u$  and an oscillating part  $w$  that should involve noise and/or fine textures. Such decompositions have already been investigated by Aujol and al. [5, 7]. These authors use the Meyer space of oscillating functions [24] rather than the  $BH(\Omega)$  space (we shall present these spaces in the sequel). The model we propose here is different: the oscillating part of the image is not penalized but a priori included in the remainder term  $w = u_d - u - v$ , while  $v$  is the smooth part (in  $BH(\Omega)$ ) and  $u$  belongs to  $BV(\Omega)$ : we hope  $u$  to be piecewise constant so that its jump set gives the image contours. For highly textured images, the model provides a two-scale texture decomposition:  $u$  can be viewed as a *macro-texture* (large scale) whose oscillations are not too large and  $w$  is the *micro-texture* (much more oscillating) that contains the noise.

Therefore, we look for components  $u$ ,  $v$  and  $w$  that belong to different spaces:  $u$  belongs to  $BV(\Omega)$  (and if possible not to  $W^{1,1}(\Omega)$ ),  $v \in BH(\Omega)$  and  $w \in L^2(\Omega)$ . This last component  $w = u_d - u - v$  lies in the same space as the observed image  $u_d$ .

The paper is organized as follows. We first present the functional framework and perform a quick comparison between the second-order total variation we use and the one defined by Bredies et al. in [13]. In section 3, we present the variational model, give existence result and an equivalent formulation with inf-convolution.

This allows to compute the dual problem. Next section is devoted to giving qualitative properties of the solutions .

## 2 Functional framework for second order variational analysis

### 2.1 Spaces $BV(\Omega)$ and $BH(\Omega)$

In the whole paper,  $\Omega$  is an open bounded subset of  $\mathbb{R}^d$  (practically  $d = 2$ ) smooth enough (with the cone property and Lipschitz for example). More precisely, if  $d = 2$ ,  $\Omega$  may satisfy next assumption

$$\left\{ \begin{array}{l} \Omega \text{ is a bounded connected open set, strongly Lipschitz such that} \\ \partial\Omega \text{ is the union of finitely many } \mathcal{C}^2 \text{ curves} \end{array} \right. \quad (2.1)$$

Following [3,4,6] and [11,20], we recall the definitions and main properties of the spaces of functions of first and second order bounded variation. The space  $BV(\Omega)$  is the classical Banach space of functions of bounded variation defined by

$$BV(\Omega) = \{u \in L^1(\Omega) \mid TV(u) < +\infty\},$$

where  $TV(u)$  is the total variation of  $u$

$$TV(u) := \sup \left\{ \int_{\Omega} u(x) \operatorname{div} \xi(x) dx \mid \xi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^d), \|\xi\|_{\infty} \leq 1 \right\}, \quad (2.2)$$

endowed with the norm  $\|u\|_{BV(\Omega)} = \|u\|_{L^1} + TV(u)$ .

We say that a sequence  $(u_n)_{n \in \mathbb{N}}$  of  $BV(\Omega)$  converges to some  $u \in BV(\Omega)$  for the *intermediate* (or *strict*) convergence if  $u_n$  strongly converges to  $u$  for the  $L^1(\Omega)$  topology and  $TV(u_n)$  converges to  $TV(u)$  (in  $\mathbb{R}$ ) (see [3,4,31] ).

The space of functions with bounded hessian has been introduced by Demengel [20] (where it was denoted  $BH(\Omega)$ ). It is the space of  $W^{1,1}(\Omega)$  functions such that  $TV^2(u) < +\infty$ , where

$$W^{1,1}(\Omega) = \{ u \in L^1(\Omega) \mid \nabla u \in L^1(\Omega) \},$$

$\nabla u$  stands for the first order derivative of  $u$  in the sense of distributions and

$$TV^2(u) := \sup \left\{ \int_{\Omega} \langle \nabla u, \operatorname{div}(\xi) \rangle_{\mathbb{R}^d} \mid \xi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{d \times d}), \|\xi\|_{\infty} \leq 1 \right\} < \infty, \quad (2.3)$$

is the second order total variation of  $u$ . Here,  $\operatorname{div}(\xi) = (\operatorname{div}(\xi_1), \operatorname{div}(\xi_2), \dots, \operatorname{div}(\xi_d))$ , and

$$\forall i, \xi_i = (\xi_{i,1}, \xi_{i,2}, \dots, \xi_{i,d}) \in \mathbb{R}^d, \quad \operatorname{div}(\xi_i) = \sum_{j=1}^d \frac{\partial \xi_{i,j}}{\partial x_j}.$$

The space  $BH(\Omega)$  endowed with the following norm

$$\|f\|_{BH(\Omega)} := \|f\|_{W^{1,1}(\Omega)} + TV^2(f) = \|f\|_{L^1} + \|\nabla f\|_{L^1} + TV^2(f), \quad (2.4)$$

where  $TV^2$  is given by (2.3) is a Banach space. Note that a function  $u$  belongs to  $BH(\Omega)$  if and only if  $u \in W^{1,1}(\Omega)$  and  $\frac{\partial u}{\partial x_i} \in BV(\Omega)$  for  $i \in \{1, \dots, d\}$ . In particular

$$TV^2(u) \leq \sum_{i=1}^d TV \left( \frac{\partial u}{\partial x_i} \right) \leq d TV^2(u).$$

We give thereafter important properties of these spaces which proofs can be found in [3, 4, 11, 15, 20] for example.

**Theorem 1** [*Semi-continuity of total variation*]

- i. The mapping  $u \mapsto TV(u)$  is lower semi-continuous (denoted in short *lsc*) from  $BV(\Omega)$  to  $\mathbb{R}^+$  for the  $L^1(\Omega)$  topology.
- ii. The operator  $TV^2$  is lower semi-continuous from  $BH(\Omega)$  endowed with the strong topology of  $W^{1,1}(\Omega)$  to  $\mathbb{R}$ .

**Theorem 2** [*Embedding results*] Assume  $d \geq 2$ . Then

- i.  $BH(\Omega) \hookrightarrow W^{1,q}(\Omega)$  with  $q \leq \frac{d}{d-1}$ , with continuous embedding. Moreover the embedding is compact if  $q < \frac{n}{n-1}$ . In particular

$$BH(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [1, \infty[, \quad \text{if } d = 2.$$

- ii. If  $d = 2$ 
  - $BV(\Omega) \subset L^2(\Omega)$  with continuous embedding.
  - $BV(\Omega) \subset L^p(\Omega)$  with compact embedding, for every  $p \in [1, 2)$ .
- iii. If  $d = 2$  and if  $\Omega$  satisfies assumption (2.1) then  $BH(\Omega) \subset C^0(\bar{\Omega})$ .

So  $BH(\Omega) \subset H^1(\Omega)$  with continuous embedding and  $BH(\Omega) \subset W^{1,1}(\Omega)$  with compact embedding. Let us define the space  $BV_0(\Omega)$  as the space of functions of bounded variation that vanish on the boundary  $\partial\Omega$  of  $\Omega$ . More precisely as  $\Omega$  is bounded and  $\partial\Omega$  is Lipschitz, functions of  $BV(\Omega)$  have a trace of class  $L^1$  on  $\partial\Omega$  [3, 4, 31], and the trace mapping  $T : BV(\Omega) \rightarrow L^1(\partial\Omega)$  is linear, continuous from  $BV(\Omega)$  equipped with the intermediate convergence to  $L^1(\partial\Omega)$  endowed with the strong topology ([4] Theorem 10.2.2 p 386). The space  $BV_0(\Omega)$  is then defined as the kernel of  $T$ . It is a Banach space, endowed with the induced norm:

$$BV_0(\Omega) := \{u \in BV(\Omega) \mid u|_{\partial\Omega} = 0\}.$$

In addition, if  $u \in BH(\Omega)$ ,  $\nabla u \in BV(\Omega)^n$  ( as a consequence of the definition of  $BH(\Omega)$  ) and we may define the normal derivative trace operator  $\nu : BH(\Omega) \rightarrow L^1(\partial\Omega)$  (with  $\nu(u) = \nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial n}$ ). This operator is linear, continuous from  $BH(\Omega)$  equipped with the strong convergence in  $W^{1,1}$  and the convergence of  $TV^2$  ( intermediate convergence in  $BH$ ) to  $L^1(\partial\Omega)$  endowed with the strong topology. So we may define similarly

$$BH_0(\Omega) := \{u \in BH(\Omega) \mid \frac{\partial u}{\partial x_i} = 0 \text{ on } \partial\Omega, \quad i = 1, \dots, d\}.$$

We set also

$$BV_m(\Omega) := \{u \in BV(\Omega) \mid \int_{\Omega} u(x) dx = 0 \quad i = 1, \dots, n\},$$

and

$$BH_m(\Omega) := \{u \in BH(\Omega) \mid \int_{\Omega} \frac{\partial u}{\partial x_i} dx = 0 \ i = 1, \dots, d\} .$$

The Ostrograski formula gives

$$\int_{\Omega} \frac{\partial u}{\partial x_i} dx = - \int_{\partial\Omega} u_i n_i ,$$

where  $u_i$  is the  $i^{th}$  partial function with respect to the  $i^{th}$  coordinate and  $n = (n_1, \dots, n_d)$  is the outer normal vector. In particular, if  $u = 0$  on  $\partial\Omega$ , then  $u \in BH_m(\Omega)$ . At last we shall use the following result of [8]:

**Lemma 1** *Let  $\Omega \subset \mathbb{R}^n$  be an open Lipschitz bounded set. There exist generic constants only depending on  $\Omega$ ,  $C_1, C_2 > 0$  such that*

$$\forall u \in BV_m(\Omega) \quad \|u\|_{L^1(\Omega)} \leq C_1 TV(u), \quad (2.5)$$

$$\forall u \in BH_0(\Omega) \cup BH_m(\Omega) \quad TV(u) \leq C_2 TV^2(u) \quad (2.6)$$

## 2.2 Comparison with BGV<sup>2</sup>

As already mentionned, a different approach of second-order total variation spaces has been set in [13]. The main difference lies in the choice of the test functions for the weak variational formulation. The authors define the *Total Generalized Variation*  $TGV^2(u)$  as the supremum of the duality product between  $u$  and symmetric tests functions that are bounded together with their derivative. First, we note that we may define  $TV^2(u)$  in a equivalent way as following: for any  $\xi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{d \times d})$  recall that

$$\forall i, \xi_i = (\xi_{i,1}, \xi_{i,2}, \dots, \xi_{i,d}) \in \mathbb{R}^d, \quad \text{div}(\xi_i) = \sum_{j=1}^d \frac{\partial \xi_{i,j}}{\partial x_j}$$

and define as in [13]

$$\text{div}^2 \xi := \sum_{i,j=1}^d \frac{\partial^2 \xi_{i,j}}{\partial x_i \partial x_j} .$$

Let us call :  $\mathcal{B} := \left\{ \xi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{d \times d}), \|\xi\|_{\infty} \leq 1 \right\}$ . Then,

$$\forall u \in W^{1,1}(\Omega) \quad TV^2(u) := \sup \left\{ \int_{\Omega} u \text{div}^2 \xi dx, \xi \in \mathcal{B} \right\}, \quad (2.7)$$

Indeed, an integration by parts gives

$$\int_{\Omega} u \text{div}^2 \xi dx = - \int_{\Omega} (\nabla u, \text{div} \xi)_{\mathbb{R}^d} dx .$$

Let be  $\alpha = (\alpha_0, \alpha_1) > 0$ , we call

$$TGV_{\alpha}^2(u) = \sup \left\{ \int_{\Omega} u \text{div}^2 \xi dx, \xi \in \mathcal{B}_{\alpha} \right\},$$

where  $\mathcal{B}_{\alpha} := \left\{ \xi \in \mathcal{K}, \xi_{ij} = \xi_{ji} \ \forall i, j, \|\xi\|_{\infty} \leq \alpha_0, \|\text{div} \xi\|_{\infty} \leq \alpha_1 \right\}$ . We may define ([13])

$$BGV_{\alpha}^2(\Omega) = \left\{ u \in L^1(\Omega), TGV_{\alpha}^2(u) < +\infty \right\}. \quad (2.8)$$

**Proposition 1** Let be  $\alpha = (\alpha_0, \alpha_1) > 0$ . For every function  $u$  in  $W^{1,1}(\Omega)$  we get

$$TGV_\alpha^2(u) \leq \alpha_0 TV^2(u) .$$

Therefore

$$\forall \alpha > 0 \quad BH(\Omega) \subset BGV_\alpha^2(\Omega)$$

with continuous embedding.

*Proof.* As  $\mathcal{B}_\alpha \subset \alpha_0 \mathcal{B}$  the first relation is obvious. Moreover if  $u \in BH(\Omega)$ , then  $u \in W^{1,1}(\Omega)$  and  $TGV_\alpha^2(u) < +\infty$ . In addition

$$\|u\|_{BVG_\alpha^2} = \|u\|_{L^1} + TGV_\alpha^2(u) \leq \|u\|_{W^{1,1}} + \alpha_0 TV^2(u) \leq \max(1, \alpha_0) \|u\|_{BH} ,$$

which gives the continuity of the embedding mapping.  $\square$

**Corollary 1** For any  $u \in BH(\Omega)$ ,  $TV^2(u) = 0$  if and only if  $u$  is a polynomial function of order 1.

*Proof.* For any  $u \in BH(\Omega)$ ,  $TV^2(u) = 0 \implies TGV_\alpha^2(u) = 0$ . Then we use Proposition 3.3 of [13].  $\square$

The main difference between the two approaches concerns the functions regularity. The  $BGV^2(\Omega)$  functions do not necessarily belong to  $L^1(\Omega)$ . In particular, the indicator function of smooth open sets belong to  $BGV^2(\Omega)$  and not to  $BH(\Omega)$ . On the other hand, we cannot have Sobolev-type embeddings for  $BGV^2(\Omega)$ .

### 3 A second-order variational model for image decomposition

#### 3.1 Presentation of the model

We have already presented this model in [10] so that we do no detail so much. However we provide here an existence result that was expected but only proved in the finite dimensional case and give a inf-convolution formulation. We now assume that  $u_d$  belongs to  $L^2(\Omega)$  and that the image we want to recover can be decomposed as  $u_d = w + u + v$  where  $u$ ,  $v$  and  $w$  are functions that characterize different parts of  $u_d$ . Components belong to different functional spaces:  $v$  is the (smooth) second order part and belongs to  $BH(\Omega)$ ,  $u$  is a  $BV(\Omega)$  component and  $w \in L^2(\Omega)$  is the remainder term. We consider the following cost functional defined on  $BV(\Omega) \times BH(\Omega)$ :

$$\mathcal{F}_{\lambda,\mu}(u, v) = \frac{1}{2} \|u_d - u - v\|_{L^2(\Omega)}^2 + \lambda TV(u) + \mu TV^2(v), \quad (3.9)$$

where  $\lambda, \mu > 0$ . We are looking for a solution to the optimization problem

$$\inf \{ \mathcal{F}_{\lambda,\mu}(u, v) \mid (u, v) \in BV(\Omega) \times BH_0(\Omega) \} \quad (\mathcal{P}_{\lambda,\mu})$$

**Remark 1** We decide to look for the minima of  $\mathcal{F}_{\lambda,\mu}$  on  $BV(\Omega) \times BH_0(\Omega)$  and not  $BV(\Omega) \times BH(\Omega)$  to get an existence result. This will cause troubles to set the dual problem because of the computation of Legendre-Fenchel conjugate functions. Nevertheless, the constraint  $v \in BH_0(\Omega)$  (that is  $\frac{\partial v}{\partial n} = 0$  on  $\partial\Omega$ ) is a usual one in image processing and the difficulty will be overcome in the discrete setting.

We expect  $v$  to be the smooth *colored* part of the image,  $u$  to be a  $BV(\Omega) \setminus BH(\Omega)$  function which derivative is a measure supported by the contours and  $w := u_d - u - v \in L^2$  is the noise and/or small textures (we shall detail this point later). Problem  $\mathcal{P}_{\lambda,\mu}$  can be (formally) viewed as a minimization problem where the regularization term is an inf-convolution. Recall that the inf-convolution ([4] p 324 ) is defined as

$$(f \# g)(v) = \inf \{ f(u) + g(v - u), u \in V \},$$

where  $f, g : V \rightarrow \mathbb{R} \cup \{+\infty\}$ . If we set  $V = L^2(\Omega)$ ,

$$\Phi_\lambda^1(u) = \begin{cases} \lambda TV(u) & \text{if } u \in BV(\Omega) \\ +\infty & \text{else.} \end{cases} \quad \text{and} \quad \Phi_\mu^2(v) = \begin{cases} \mu TV^2(v) & \text{if } v \in BH_0(\Omega) \\ +\infty & \text{else.} \end{cases} \quad (3.10)$$

then

$$(\Phi_\lambda^1 \# \Phi_\mu^2)(\varpi) = \inf_{u \in BV(\Omega), v \in BH_0(\Omega), u+v=\varpi} \lambda TV(u) + \mu TV^2(v),$$

and problem  $\mathcal{P}_{\lambda,\mu}$  may be written as

$$\inf_{\varpi \in L^2(\Omega)} \frac{1}{2} \|u_d - \varpi\|_{L^2(\Omega)}^2 + (\Phi_\lambda^1 \# \Phi_\mu^2)(\varpi).$$

This formulation is to compare to the one by Bredies and al. [13,14,12]

$$\inf_{\varpi \in L^2(\Omega)} \frac{1}{2} \|u_d - \varpi\|_{L^2(\Omega)}^2 + TGV_\alpha^2(\varpi),$$

which seems to be more efficient for denoising. However, we are interested in the decomposition components  $u$  and  $v$ . First, we give an existence result for problem  $(\mathcal{P}_{\lambda,\mu})$ .

**Theorem 3 (Existence)** *The problem  $(\mathcal{P}_{\lambda,\mu})$  has at least an optimal solution  $(u^*, v^*)$  in  $BV(\Omega) \times BH_0(\Omega)$ .*

*Proof.* We first prove that the auxiliary problem

$$\inf \{ \mathcal{F}_{\lambda,\mu}(u, v) \mid (u, v) \in BV_m(\Omega) \times BH_0(\Omega) \} \quad (3.11)$$

has an optimal solution. Let  $(u_n, v_n) \in BV_m(\Omega) \times BH_0(\Omega)$  be a minimizing sequence, i.e.

$$\lim_{n \rightarrow +\infty} \mathcal{F}_{\lambda,\mu}(u_n, v_n) = \inf \{ \mathcal{F}_{\lambda,\mu}(u, v) \mid (u, v) \in BV_m(\Omega) \times BH_0(\Omega) \} < +\infty.$$

Therefore

- $TV^2(v_n)$  is bounded and with lemma 1,  $\|\nabla v_n\|_{L^1}$  is bounded as well.
- $TV(u_n)$  is bounded. Using once again lemma 1 this yields that  $u_n$  is bounded in  $L^1(\Omega)$ . Therefore the sequence  $u_n$  is bounded in  $BV(\Omega)$ .
- As  $u_n + v_n$  is  $L^2$ -bounded, it is  $L^1$ -bounded as well so that  $v_n$  is  $L^1$  bounded. As  $\|\nabla v_n\|_{L^1}$  and  $TV^2(v_n)$  are bounded this means that the sequence  $v_n$  is bounded in  $BH(\Omega)$ .



With the compactness result of Theorem 2, this yields that  $(v_n)_{n \in \mathbb{N}}$  strongly converges (up to a subsequence) in  $W^{1,1}(\Omega)$  to  $v^* \in BH(\Omega)$ . Moreover,  $v^* \in BH_0(\Omega)$  because the normal derivative operator is continuous (as mentioned before). Similarly  $(u_n)_{n \in \mathbb{N}}$  strongly converges (up to a subsequence) in  $L^1(\Omega)$  to  $u^* \in BV_m(\Omega)$ . Moreover  $u_n + v_n$  weakly converges to  $u^* + v^*$  in  $L^2(\Omega)$ . With theorem 1 we get

$$TV(u^*) \leq \liminf_{n \rightarrow +\infty} TV(u_n), \quad TV^2(v^*) \leq \liminf_{n \rightarrow +\infty} TV^2(v_n).$$

So

$$\mathcal{F}_{\lambda,\mu}(u^*, v^*) \leq \liminf_{n \rightarrow +\infty} \mathcal{F}_{\lambda,\mu}(u_n, v_n) = \min_{(u,v) \in BV_m(\Omega) \times BH_0(\Omega)} \mathcal{F}_{\lambda,\mu}(u, v),$$

and  $(u^*, v^*)$  is a solution to (3.11).

For every  $(u, v) \in BV(\Omega) \times BH_0(\Omega)$ , we have  $(u - \bar{u}, v + \bar{u}) \in BV_m(\Omega) \times BH_0(\Omega)$  where  $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u$  is the mean value of  $u$ . Moreover

$$\mathcal{F}_{\lambda,\mu}(u, v) = \mathcal{F}_{\lambda,\mu}(u - \bar{u}, v + \bar{u}) \geq \mathcal{F}_{\lambda,\mu}(u^*, v^*).$$

Therefore  $(u^*, v^*)$  is an optimal solution to  $(\mathcal{P}_{\lambda,\mu})$ .  $\square$

**Remark 2** *Uniqueness of the solution is challenging. We shall prove partial results in section 4.2.*

### 3.2 Optimality conditions

In what follows, we fix  $\lambda > 0$  and  $\mu > 0$  and set for any  $u \in L^2(\Omega) : \mathcal{N}(u) = \frac{1}{2} \|u\|_2^2$ .

Functionals  $\Phi_{\lambda}^1$  and  $\Phi_{\mu}^2$  have been defined in (3.10).

It is easy to see that  $(\bar{u}, \bar{v})$  is a solution to  $(\mathcal{P}_{\lambda,\mu})$  if and only if

$$\bar{u} = \operatorname{argmin} \left\{ \frac{1}{2} \|u_d - \bar{v} - u\|^2 + \Phi_{\lambda}^1(u), u \in L^2(\Omega) \right\}, \quad (3.12)$$

$$\bar{v} = \operatorname{argmin} \left\{ \frac{1}{2} \|u_d - \bar{u} - v\|^2 + \Phi_{\mu}^2(v), v \in L^2(\Omega) \right\}. \quad (3.13)$$

and we may derive optimality conditions in a standard way :

**Theorem 4**  $(\bar{u}, \bar{v})$  is a solution to  $(\mathcal{P}_{\lambda,\mu})$  if and only if

$$\bar{w} := u_d - \bar{u} - \bar{v} \in \partial\Phi_{\lambda}^1(\bar{u}) \cap \partial\Phi_{\mu}^2(\bar{v}). \quad (3.14)$$

The proof is obvious. Here  $\partial f(u)$  stands for the subdifferential of  $f$  at  $u$  where  $f : V \rightarrow \mathbb{R}$ :

$$\partial f(u) = \{u^* \in V' \mid \forall v \in V \quad f(v) - f(u) \geq \langle u^*, v - u \rangle\},$$

and  $\langle \cdot, \cdot \rangle$  is the duality product between  $V$  et  $V'$ .

### 3.3 Inf-convolution formulation

We have already noticed that the penalization term in  $(\mathcal{P}_{\lambda,\mu})$  is an infimal convolution term. In addition,  $(\mathcal{P}_{\lambda,\mu})$  can be viewed as successive inf-convolution processes.

**Lemma 2** *The functionals  $\mathcal{N}\#\Phi_\lambda^1$  and  $\mathcal{N}\#\Phi_\mu^2$  are convex, continuous from  $L^2(\Omega)$  to  $L^2(\Omega)$ .*

*Proof.* In the sequel we set  $\Phi = \Phi_\lambda^1$  or  $\Phi_\mu^2$  indifferently. As  $\Phi$  and  $\mathcal{N}$  are convex so is  $\mathcal{N}\#\Phi$  (see [23] for example). Let be  $u \in L^2(\Omega)$ :

$$(\mathcal{N}\#\Phi)(u) = \inf_{v \in L^2(\Omega)} \frac{1}{2} \|u - v\|_2^2 + \Phi(v) \leq \frac{1}{2} \|u\|_2^2 + \Phi(0) = \frac{1}{2} \|u\|_2^2 .$$

As  $(\mathcal{N}\#\Phi)(0) = 0$  this gives the  $\mathcal{N}\#\Phi$  continuity at 0 and its boundedness in a neighborhood of 0. As it is convex, it continuous on its whole domain  $L^2(\Omega)$  (see [21] for example).  $\square$

Note that problem (3.12) is equivalent to  $\bar{u} \in \mathcal{N}\#\Phi_\lambda^1(u_d - \bar{v})$  and (3.13) is equivalent to  $\bar{v} \in \mathcal{N}\#\Phi_\mu^2(u_d - \bar{u})$ . In fact, problem  $(\mathcal{P}_{\lambda,\mu})$  can be written as successive inf-convolution processes. More precisely we have

**Theorem 5** *Let  $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$  be a solution to  $(\mathcal{P}_{\lambda,\mu})$  and  $\bar{m} := \inf(\mathcal{P}_{\lambda,\mu})$ . Then*

$$\begin{aligned} \bar{m} &= \mathcal{N}(u_d - \bar{u} - \bar{v}) + \Phi_\lambda^1(\bar{u}) + \Phi_\mu^2(\bar{v}) \\ &= (\mathcal{N}\#\Phi_\lambda^1)(u_d - \bar{v}) + \Phi_\mu^2(\bar{v}) = (\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u}) + \Phi_\lambda^1(\bar{u}) \\ &= (\Phi_\lambda^1\#\mathcal{N}\#\Phi_\mu^2)(u_d) = (\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d). \end{aligned}$$

*Proof.* Let  $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$  be a solution to  $(\mathcal{P}_{\lambda,\mu})$ . Then, for every  $(u, v) \in BV(\Omega) \times BH_0(\Omega)$ , we get

$$\bar{m} = \mathcal{N}(u_d - \bar{u} - \bar{v}) + \Phi_\lambda^1(\bar{u}) + \Phi_\mu^2(\bar{v}) \leq \mathcal{N}(u_d - u - v) + \Phi_\lambda^1(\bar{u}) + \Phi_\mu^2(v) . \quad (3.15)$$

This gives, for every  $v \in BH_0(\Omega)$

$$\bar{m} \leq \inf_{u \in L^2(\Omega)} \mathcal{N}(u_d - u - v) + \Phi_\lambda^1(\bar{u}) + \Phi_\mu^2(v) = (\mathcal{N}\#\Phi_\lambda^1)(u_d - v) + \Phi_\mu^2(v)$$

so that

$$\bar{m} \leq \inf_{v \in BH_0(\Omega)} (\mathcal{N}\#\Phi_\lambda^1)(u_d - v) + \Phi_\mu^2(v) \leq (\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d) .$$

Similarly

$$\bar{m} \leq (\Phi_\lambda^1\#\mathcal{N}\#\Phi_\mu^2)(u_d) .$$

Conversely, by definition of inf-convolution, we get for every  $(u, v) \in BV(\Omega) \times BH_0(\Omega)$

$$(\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d) \leq (\mathcal{N}\#\Phi_\lambda^1)(u_d - v) + \Phi_\mu^2(v) \leq \mathcal{N}(u_d - v - u) + \Phi_\lambda^1(u) + \Phi_\mu^2(v) ,$$

so that  $(\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d) \leq \bar{m}$ .

We finally obtain  $\bar{m} = (\Phi_\lambda^1\#\mathcal{N}\#\Phi_\mu^2)(u_d) = (\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d)$ .  $\square$

### 3.4 Computing Fenchel conjugate function

We are going to write the dual problem of  $(\mathcal{P}_{\lambda,\mu})$  and we need to compute the conjugate functions of  $\Phi_\lambda^1$  and  $\Phi_\mu^2$  and  $\tilde{f} : u \mapsto f(u_d + u)$ . We recall that if  $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ , the Legendre-Fenchel conjugate  $f^*$  is defined on  $V'$  as

$$\forall u^* \in V' \quad f^*(u^*) = \sup_{u \in V} \langle u^*, u \rangle - f(u) .$$

We obviously have

$$\forall \lambda > 0, \forall u^* \in V' \quad (\lambda f)^*(u^*) = \lambda f^*\left(\frac{u^*}{\lambda}\right) ,$$

and the following useful result:

**Proposition 2** [4] *Let  $V$  be a normed space and  $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$  a closed, convex, proper function. then*

$$u^* \in \partial f(u) \iff u \in \partial f^*(u^*) \iff f(u) + f^*(u^*) = \langle u^*, u \rangle ,$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality  $V - V'$  product.

**Lemma 3** *Let be  $f : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $\tilde{f}$  such that  $\tilde{f}(u) = f(u_d + u)$ . Then  $\tilde{f}$  conjugate function writes*

$$\forall u^* \in L^2(\Omega) \quad (\tilde{f})^*(u^*) = f^*(u^*) - (u^*, u_d)_2 ,$$

where  $(\cdot, \cdot)_2$  denotes the  $L^2(\Omega)$  inner product.

*Proof.* Let be  $u^* \in L^2(\Omega)$ . We have

$$\begin{aligned} (\tilde{f})^*(u^*) &= \sup_{u \in L^2(\Omega)} (u, u^*)_2 - f(u_d + u) = \sup_{w \in L^2(\Omega)} (w - u_d, u^*)_2 - f(w) \\ &= \sup_{w \in L^2(\Omega)} (w, u^*)_2 - f(w) - (u_d, u^*)_2 = f^*(u^*) - (u^*, u_d)_2 . \end{aligned}$$

□

In the sequel  $\mathbf{1}_C$  denotes the indicator function of the set  $C$  :

$$\mathbf{1}_C(u) = \begin{cases} 0 & \text{if } u \in C \\ +\infty & \text{else.} \end{cases}$$

**Lemma 4** *The conjugate function of  $\Phi_\lambda^1$  is  $(\Phi_\lambda^1)^* = \lambda \mathbf{1}_{\lambda \mathcal{K}_1}$ , where  $\mathcal{K}_1 = \overline{\mathbf{K}_1}$  is the  $L^2$ -closure of*

$$\mathbf{K}_1 := \left\{ \xi = \operatorname{div} \varphi \mid \varphi \in \mathcal{C}_c^1(\Omega), \|\varphi\|_\infty \leq 1 \right\} . \quad (3.16)$$

*The conjugate function of  $\Phi_\mu^2$  is  $(\Phi_\mu^2)^* = \mu \mathbf{1}_{\mu \mathcal{K}_2}$ , where  $\mathcal{K}_2 \supset \overline{\mathbf{K}_2}$  and  $\overline{\mathbf{K}_2}$  is the  $L^2$ -closure of*

$$\mathbf{K}_2 := \left\{ \xi = \operatorname{div}^2 \psi \mid \psi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{d \times d}), \|\psi\|_\infty \leq 1 \right\} . \quad (3.17)$$

*Proof.* It is known that the conjugate  $TV^*$  of  $TV$  is the indicator function of  $\overline{\mathcal{K}_1}$  (see [6,19] for example). As  $\Phi_\lambda^1 = \lambda TV$  (or  $+\infty$  outside  $BV(\Omega)$ ) then  $(\Phi_\lambda^1)^*(u^*) = \lambda TV^* \left( \frac{u^*}{\lambda} \right)$ . This gives the result.

The result is not exactly the same since  $\Phi_\mu^2$  is equal to  $\mu TV^2$  on  $BH_0(\Omega)$  and  $+\infty$  outside (and in particular on  $BH(\Omega) \setminus BH_0(\Omega)$ ). Therefore the conjugate of  $\Phi_\mu^2$  is not the same as the conjugate of  $\mu TV^2$ . We know that the conjugate function of  $TV^2$  is  $\mathbf{1}_{\overline{\mathcal{K}_2}}$  (see [11]); as  $\Phi_1^2$  is positively homogeneous ( $\mu = 1$ ), it is the indicator function of a closed subset  $\mathcal{K}_2$  of  $L^2(\Omega)$ . Moreover

$$\mathbf{1}_{\mathcal{K}_2}(v^*) = (\Phi_1^2)^*(v^*) = \sup_{v \in BH_0} \langle v^*, v \rangle - TV^2(v) \leq \sup_{v \in BH} \langle v^*, v \rangle - TV^2(v) = \mathbf{1}_{\overline{\mathcal{K}_2}}(v^*).$$

This implies that  $\overline{\mathcal{K}_2} \subset \mathcal{K}_2$  but we cannot prove the converse inclusion (for example). We end the proof with the same argument as in the  $BV$  case.  $\square$   
Eventually it is easy to see that  $\mathcal{N}^* = \mathcal{N}$ .

### 3.5 Dual problem to $(\mathcal{P}_{\lambda,\mu})$

In the present subsection we use convex duality tools that we recall thereafter (see [4] for example).

**Theorem 6** [ [4] p 366] *Let  $V$  be a banach space,  $f, g : V \rightarrow \mathbb{R} \cup \{+\infty\}$  lower semi-continuous convex functions and  $A$  a linear continuous operator from  $V$  to  $V$ . Assume there exists  $u_o \in \text{dom } g$  and  $f$  continuous at  $Au_o$ . Then*

$$\inf_{u \in V} (f(Au) + g(u)) = \max_{u^* \in V^*} (-f^*(u^*) - g^*(-A^*u^*)).$$

Moreover, if  $\bar{u}$  is a solution to the primal problem and  $\bar{u}^*$  is a solution to the dual one then

$$\bar{u}^* \in \partial f(A\bar{u}) \text{ and } -A^*\bar{u}^* \in \partial g(\bar{u}),$$

where  $\partial f(u)$  stands for the subdifferential of  $f$  at  $u$ .

**Theorem 7** [ [4] p 328] *Let  $V$  be a banach space and  $f, g : V \rightarrow \mathbb{R} \cup \{+\infty\}$  proper functions. Then*

$$(f \# g)^* = f^* + g^*.$$

In addition if  $f$  and  $g$  satisfy the assumptions of Theorem 6, then

$$(f + g)^* = f^* \# g^*.$$

Now we may compute the dual problem to  $(\mathcal{P}_{\lambda,\mu})$  and get the following

**Theorem 8** *The dual problem to  $(\mathcal{P}_{\lambda,\mu})$  writes*

$$\inf_{w \in \lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2} \frac{1}{2} \|u_d - w\|_2^2. \quad (3.18)$$

The unique solution  $w^*$  is the  $L^2$ -projection of  $u_d$  on the closed convex set  $\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2$ :

$$w^* = \Pi_{\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2}(u_d).$$

*Proof.* Solving problem  $(\mathcal{P}_{\lambda,\mu})$  is equivalent to solving

$$\inf_{u \in L^2(\Omega)} (\mathcal{N} \# \Phi_\mu^2)(u_d - u) + \Phi_\lambda^1(u) = \inf_{u \in L^2(\Omega)} (\widetilde{\mathcal{N} \# \Phi_\mu^2})(Au) + \Phi_\lambda^1(u)$$

with  $Au = -u$ . It clear that  $A^* = A$ . Moreover,  $\Phi_\lambda^1$  is lsc with respect to the  $L^1$ -topology and thus for  $L^2$ -topology since  $\Omega$  is bounded. As  $\mathcal{N} \# \Phi_\mu^2$ ,  $\widetilde{\mathcal{N} \# \Phi_\mu^2}$ ,  $A$  and  $\Phi_\lambda^1$  fulfill assumptions of Theorem 6, the dual problem of  $(\mathcal{P}_{\lambda,\mu})$  writes

$$(\mathcal{P}^*) \quad \max_{w \in L^2(\Omega)} -(\widetilde{\mathcal{N} \# \Phi_\mu^2})^*(w) - (\Phi_\lambda^1)^*(w),$$

where  $(\widetilde{\mathcal{N} \# \Phi_\mu^2})(w) = (\mathcal{N} \# \Phi_\mu^2)(u_d + w)$ . Using Lemma 3 and Theorem 7 it easy to see that

$$\begin{aligned} (\widetilde{\mathcal{N} \# \Phi_\mu^2})^*(w) &= -(u_d, w)_2 + (\mathcal{N} \# \Phi_\mu^2)^*(w) = -(u_d, w)_2 + \mathcal{N}^*(w) + (\Phi_\mu^2)^*(w) \\ &= -(u_d, w)_2 + \mathcal{N}(w) + (\Phi_\mu^2)^*(w). \end{aligned}$$

Therefore,  $(\mathcal{P}^*)$  writes

$$\begin{aligned} &\max_{w \in L^2(\Omega)} (u_d, w)_2 - \mathcal{N}(w) - (\Phi_\lambda^1)^*(w) - (\Phi_\mu^2)^*(w), \\ \Leftrightarrow & - \min_{w \in L^2(\Omega)} -(u_d, w)_2 + \mathcal{N}(w) + (\Phi_\lambda^1)^*(w) + (\Phi_\mu^2)^*(w). \end{aligned}$$

Finally,  $(\mathcal{P}^*)$  is equivalent to

$$\min_{w \in \lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2} \frac{1}{2} \|u_d - w\|_2^2, \quad (3.19)$$

The dual problem has obviously a unique solution  $w^*$  which is the  $L^2$  projection of  $u_d$  on the closed convex set  $\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2$ .  $\square$

Next we have a relation between the solutions to  $(\mathcal{P}_{\lambda,\mu})$  and the (unique) solution of the dual problem.

**Theorem 9** 1. Let  $w^*$  be the (unique) solution to the dual problem  $(\mathcal{P}_{\lambda,\mu})^*$ :

$$w^* = \Pi_{\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2}(u_d).$$

Then there exists  $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$  an optimal solution to  $(\mathcal{P}_{\lambda,\mu})$  such that

$$w^* = u_d - \bar{u} - \bar{v} \text{ and } w^* \in \partial\Phi_\mu^2(\bar{v}) \cap \partial\Phi_\lambda^1(\bar{u}).$$

2. Conversely, if  $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$  is any solution to  $(\mathcal{P}_{\lambda,\mu})$  then

$$\bar{w} = u_d - \bar{u} - \bar{v} = \Pi_{\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2}(u_d). \quad (3.20)$$

*Proof.* Let  $(\bar{u}, \bar{v})$  be a solution to  $(\mathcal{P}_{\lambda, \mu})$ . A direct consequence of Theorem 6 is

$$w^* \in \partial\Phi_\lambda^1(\bar{u}) \quad \text{and} \quad w^* \in \partial(\widetilde{\mathcal{N}\#\Phi_\mu^2})(-\bar{u}).$$

A simple calculus shows that

$$\partial(\widetilde{\mathcal{N}\#\Phi_\mu^2})(-\bar{u}) = \partial(\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u})$$

so that

$$w^* \in \partial\Phi_\lambda^1(\bar{u}) \cap \partial(\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u}).$$

As

$$(\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u}) = \mathcal{N}(u_d - \bar{u} - \bar{v}) + \Phi_\mu^2(\bar{v}) = \operatorname{argmin}_{v \in L^2(\Omega)} \frac{1}{2} \|v + \bar{u} - u_d\|^2 + \Phi_\mu^2(v),$$

then

$$u_d - \bar{v} - \bar{u} \in \partial\Phi_\mu^2(\bar{v});$$

so, the inf-convolution is exact and we get ([23])

$$\partial(\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u}) = \bigcup_{v \in L^2(\Omega)} \partial\mathcal{N}(u_d - \bar{u} - v) \cap \partial\Phi_\mu^2(v).$$

As  $\partial\mathcal{N}(u_d - \bar{u} - v) = \{u_d - \bar{u} - v\}$  this means that there exists  $\bar{v} \in L^2(\Omega)$  such that

$$w^* = u_d - \bar{u} - \bar{v} \in \partial\Phi_\mu^2(\bar{v}).$$

So

$$w^* \in \partial\Phi_\mu^2(\bar{v}) \cap \partial\Phi_\lambda^1(\bar{u}),$$

with  $\bar{v} = u_d - \bar{u} - w^*$ . This prove that  $(\bar{u}, \bar{v})$  is a solution to  $(\mathcal{P}_{\lambda, \mu})$  as well: we use Theorem 4 with  $\bar{w} = w^*$  to conclude.

Let us prove the converse property. Let  $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$  be a solution to  $(\mathcal{P}_{\lambda, \mu})$  and  $\bar{w} = u_d - \bar{u} - \bar{v}$ . Theorem 4 yields

$$\bar{w} \in \partial\Phi_\lambda^1(\bar{u}) \cap \partial\Phi_\mu^2(\bar{v}),$$

that is

$$\bar{u} \in \partial(\Phi_\lambda^1)^*(\bar{w}) \quad \text{and} \quad \bar{v} \in \partial(\Phi_\mu^2)^*(\bar{w}).$$

With the previous computations this gives

$$\bar{u} \in \partial\lambda_{\lambda\mathcal{K}_1}(\bar{w}) \quad \text{and} \quad \bar{v} \in \partial\lambda_{\mu\mathcal{K}_2}(\bar{w}).$$

Therefore

$$\forall w \in \lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2 \quad \langle \bar{u}, w - \bar{w} \rangle \leq 0 \quad \text{and} \quad \langle \bar{v}, w - \bar{w} \rangle \leq 0.$$

Adding the above inequalities gives

$$\forall w \in \lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2 \quad \langle \bar{u} + \bar{v}, w - \bar{w} \rangle = \langle u_d - \bar{w}, w - \bar{w} \rangle \leq 0.$$

This is equivalent to (3.20).

**Corollary 2** *If  $(\bar{u}, \bar{v})$  is a solution to  $(\mathcal{P}_{\lambda, \mu})$ , then  $\bar{w} = \bar{u} + \bar{v}$  is unique. In particular, there is a unique solution to  $(\mathcal{P}_{\lambda, \mu})$  such that  $\bar{u} = 0$  almost everywhere.*

**Remark 3** *We cannot permute the role of  $\Phi_\lambda^1$  and  $\Phi_\mu^2$  in the previous proof because  $\Phi_\mu^2$  is not lower semi-continuous with respect to the  $L^2$  topology. Indeed  $L^2(\Omega)$  is not embedded in  $W^{1,1}(\Omega)$ .*

## 4 Solution properties ( $d \leq 2$ )

### 4.1 Structure of the solutions

Recall (see [5, 24]) that the Meyer space  $G(\Omega)$  is defined as

$$G(\Omega) := \{ f \in L^2(\Omega) \mid \exists \varphi = (\varphi_1, \varphi_2) \in L^\infty(\Omega, \mathbb{R}^2) \ f = \operatorname{div} \varphi \text{ and } \varphi \cdot n = 0 \text{ on } \partial\Omega \}$$

where  $n$  is the outer normal vector to  $\partial\Omega$ . The space  $G$  is endowed with a norm denoted by  $\|\cdot\|_G$  and defined as

$$\|f\|_G = \inf\{ \|\sqrt{\varphi_1^2 + \varphi_2^2}\|_\infty \mid f = \operatorname{div} \varphi, \varphi \cdot n = 0 \text{ on } \partial\Omega \}.$$

We shall need the

**Lemma 5** ( [5], **Lemma 2.1**) *For every  $u \in BV(\Omega)$  and  $g \in G(\Omega)$  then*

$$\left| \int_{\Omega} u(x) g(x) dx \right| \leq TV(u) \|g\|_G.$$

We may now precise the structure of a generic solution.

**Theorem 10** *Let us denote by  $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$  a solution to problem  $(\mathcal{P}_{\lambda, \mu})$  (for any fixed  $\lambda$  and  $\mu$ ) and set  $\bar{w} = u_d - \bar{u} - \bar{v}$ .*

- i.  $\bar{w} = u_d - \bar{u} - \bar{v} \in G(\Omega)$*
- ii. If  $d = 2$  and  $\Omega$  satisfies assumption (2.1),  $\bar{v}$  is continuous on  $\bar{\Omega}$ .*
- iii. If  $d = 2$ ,  $\Omega$  satisfies assumption (2.1) and  $u_d \in BV(\Omega) \cap L^\infty(\Omega)$ , then the jump set of  $\bar{u}$  is included in the jump set of  $u_d$ .*

*Proof.* (i) This is a direct consequence of Theorem 9. Indeed  $\bar{w} \in \lambda\mathcal{K}_1$ . Therefore, there exists a sequence  $\varphi_n \in C_c^1(\Omega, \mathbb{R}^2)$  with  $\|\varphi_n\|_\infty \leq 1$  such that  $w_n = \lambda \operatorname{div}(\varphi_n)$   $L^2$ -converges to  $\bar{w}$ . As  $\|\varphi_n\|_\infty \leq 1$  on may extract a weak-star subsequence that converges to  $\bar{\varphi}$  in  $L^\infty(\Omega)$ . Therefore  $\bar{\varphi} \in L^\infty(\Omega)$  and  $\bar{\varphi} \cdot n = 0$  on  $\partial\Omega$ . So, we get :

$$\forall u \in \mathcal{D}(\Omega) \quad (w_n, u)_{L^2} = \lambda \int_{\Omega} \operatorname{div} \varphi_n u = -\lambda \int_{\Omega} \varphi_n \nabla u \rightarrow -\lambda \int_{\Omega} \bar{\varphi} \nabla u.$$

As  $(w_n, u)_{L^2} \rightarrow (\bar{w}, u)_{L^2}$  this gives

$$(\bar{w}, u) = -\lambda \langle \bar{\varphi}, \nabla u \rangle = \lambda \langle \operatorname{div} \bar{\varphi}, u \rangle,$$

in the distributional sense. Therefore  $\bar{w} = \operatorname{div}(\lambda \bar{\varphi})$ . Moreover,  $\bar{\varphi} \cdot n = 0$  on  $\partial\Omega$  since  $\varphi_n$  as compact support. This proves that  $\bar{w} \in G(\Omega)$ .

(ii) Assumption (2.1) yields that  $\bar{v} \in BH(\Omega)$  is continuous (Theorem 2, (iii)).

(iii) With (ii), the jump discontinuity set of  $u_d$  is the same as the one of  $u_d - \bar{v}$ . Moreover  $\bar{u}$  is a solution to

$$\min_{u \in BV(\Omega)} \frac{1}{2} \|u_d - \bar{v} - u\|^2 + \lambda TV(u),$$

Therefore, following [[18], Theorem 3.3 ] we get the result.

**Remark 4** The point (i) means that  $\bar{w}$  is an oscillating function: this is consistent with the fact that we expect  $\bar{w}$  to be the noise and/or micro-textures. The continuity of  $\bar{v}$  still hold if  $d \geq 2$ . Assumptions on  $\Omega$  are slightly different (see [15, 20]).

**Corollary 3** Let us denote by  $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$  a solution to problem  $(\mathcal{P}_{\lambda, \mu})$  (for any fixed  $\lambda$  and  $\mu$ ) and set  $\bar{w} = u_d - \bar{u} - \bar{v}$ . Then

$$\int_{\Omega} \bar{w}(x) dx = 0 .$$

*Proof.* This is a direct consequence of proposition 2.1 of [5].

The previous theorem deals with the case where  $u_d \in BV(\Omega)$ . This is not the case if  $u_d$  is noisy for example. In the case where  $u_d \notin BV(\Omega)$  we have the following results due to W. Ring [28].

We first consider the 1D case where  $\Omega = (a, b)$ . Following Proposition 4 of [28], if we assume that

$$\forall \mathcal{U} \text{ open subset of } (a, b) \text{ with positive Lebesgue measure} \quad (\mathcal{H}_1)$$

$$u_d \text{ does not coincide on } \mathcal{U} \text{ with some function } u \in BV(a, b).$$

then  $u_d - \bar{v}$  satisfies  $\mathcal{H}_1$  and we get  $D_a \bar{u} = 0$  where  $D_a u$  is the absolutely continuous part of the measure  $Du$ . Let  $\Gamma$  be the support of the singular part of  $D\bar{u}$ . Therefore  $\bar{u}$  is piecewise constant on  $(a, b) \setminus \Gamma$ .

We have also a similar result for the 2D-case . Assume that

$$\forall \mathcal{U} \text{ open subset of } \Omega, u_d|_{\mathcal{U}} \text{ is not equal not a } BV(\Omega) \text{ function.} \quad (\mathcal{H}_2)$$

then  $u_d - \bar{v}$  satisfies  $(\mathcal{H}_2)$  as well (since  $\bar{v} \in W^{1,1}(\Omega)$ ). Following Proposition 6 of [28], there is no open subset  $\omega$  of  $\Omega$  on which both components  $\frac{\partial \bar{u}}{\partial x_i}, i = 1, 2$  have constant, non-zero sign.

## 4.2 Uniqueness

The functional  $\mathcal{F}_{\lambda, \mu}$  is convex but not strictly convex, because of the degenerating direction  $u + v = 0$ . It is obvious that if  $(u^*, v^*)$  is a solution then  $(u^* + c, v^* - c)$ , where  $c$  is constant, is a solution as well. Let us call

$$\mathbf{C}(\Omega) := \{(u, v) \in BV(\Omega) \times BH_0(\Omega) \mid \exists c \in \mathbb{R} \ u = c \text{ and } v = -c \text{ a.e on } \Omega \}. \quad (4.21)$$

The question of uniqueness reduces to uniqueness up to  $\mathbf{C}(\Omega)$  functions. In other words, if  $(u_1, v_1)$  and  $(u_2, v_2)$  are two optimal solutions of  $(\mathcal{P}_{\lambda, \mu})$  can we show that  $u_2 = u_1 + c$  and  $v_2 = v_1 - c$  where  $c$  is a constant function? It is still an open problem for the 2D case. We shall discuss this point more precisely in the numerical section. Nevertheless we may give partial results:

**Proposition 3** Assume  $(u_1, v_1)$  and  $(u_2, v_2)$  are two optimal solutions of  $(\mathcal{P}_{\lambda, \mu})$ . Then there exists  $\varphi \in BV(\Omega) \cap BH_0(\Omega)$  such that  $u_2 = u_1 - \varphi$  and  $v_2 = v_1 + \varphi$ .

*Proof.* Set  $u = u_2 - u_1 (\in BV(\Omega))$  and  $v = v_2 - v_1 (\in BH_0(\Omega))$ . As  $u_d - u_1 - v_1 = u_d - u_2 - v_2$  (this is the unique solution of the dual problem), then  $u + v = 0$ . This yields that  $u = -v \in BV(\Omega) \cap BH_0(\Omega)$  and we get the result.



**Lemma 6** *The only solutions  $(\bar{u}, \bar{v})$  to  $(\mathcal{P}_{\lambda, \mu})$  that satisfies  $\bar{u} + \bar{v} = 0$  are functions of  $\mathbf{C}(\Omega)$ .*

*Proof.* Assume that  $\bar{u} + \bar{v} = 0$  then  $\bar{u} \in BH_0(\Omega)$  and  $\Phi^2(\bar{v}) = \Phi^2(-\bar{u}) = \Phi^2(\bar{u})$ . As  $\mathcal{F}_{\lambda, \mu}(\bar{u}, \bar{v}) \leq \mathcal{F}_{\lambda, \mu}(u, v)$ , for every  $(u, v) \in BV(\Omega) \times BH_0(\Omega)$  this yields

$$\|u_d\|_{L^2(\Omega)}^2 + 2\lambda TV(\bar{u}) + 2\mu TV^2(\bar{u}) \leq \|u_d - u - v\|_{L^2(\Omega)}^2 + 2\lambda TV(u) + 2\mu TV^2(v) .$$

Taking  $u = v = 0$  gives

$$\|u_d\|_{L^2(\Omega)}^2 + 2\lambda TV(\bar{u}) + 2\mu TV^2(\bar{u}) \leq \|u_d\|_{L^2(\Omega)}^2 .$$

So we get  $\lambda TV(\bar{u}) + \mu TV^2(\bar{u}) = 0$ . This implies that  $TV(\bar{u}) = 0$  and that  $\bar{u}$  is a constant function.

**Theorem 11** *Let be  $(\lambda, \mu)$  nonnegative real numbers such that  $\lambda \geq \|u_d\|_G$  and  $\mu \geq C_2\lambda$  where  $C_2$  is the constant of Lemma 1. Then the  $\mathbf{C}(\Omega)$  functions are the only solutions to  $(\mathcal{P}_{\lambda, \mu})$ .*

*Proof.* Let us assume that  $\lambda \geq \|u_d\|_G$  and  $\mu \geq C_2\lambda$  where  $C_2$  is the constant of Lemma 1. Lemma 5 gives

$$\forall (u, v) \in BV(\Omega) \times BH_0(\Omega) \quad |(u_d, u + v)_2| \leq \lambda TV(u + v)$$

since  $u_d \in L^2(\Omega)$  and  $BH_0(\Omega) \subset BV(\Omega)$ . Then

$$|(u_d, u + v)_2| \leq \lambda TV(u) + \lambda TV(v) .$$

Lemma 1 gives a constant  $C_2$  (only depending on  $\Omega$ ) such that

$$\forall v \in BH_0(\Omega) \quad TV(v) \leq C_2 TV^2(v) ,$$

so that  $\forall (u, v) \in BV(\Omega) \times BH_0(\Omega)$

$$|(u_d, u + v)_2| \leq \lambda TV(u) + C_2 \lambda TV^2(v) \leq \lambda TV(u) + \mu TV^2(v). \quad (4.22)$$

Finally, we get for every  $(u, v) \in BV(\Omega) \times BH_0(\Omega)$

$$\frac{1}{2} \|u_d\|^2 = \frac{1}{2} \|u_d - u - v\|^2 - \frac{1}{2} \|u + v\|^2 + (u_d, u + v)_2 \leq \frac{1}{2} \|u_d - u - v\|^2 + \lambda TV(u) + \mu TV^2(v).$$

This means that  $\mathcal{F}_{\lambda, \mu}(0, 0) \leq \mathcal{F}_{\lambda, \mu}(u, v)$  : so  $(0, 0)$  is a solution to  $(\mathcal{P}_{\lambda, \mu})$ . Let  $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$  be another solution to  $\mathcal{P}_{\lambda, \mu}$ . With proposition 3, we get  $\bar{u} + \bar{v} = 0$  and lemma 6 gives  $(\bar{u}, \bar{v}) \in \mathbf{C}(\Omega)$ . This ends the proof.

**Remark 5** *The previous theorem tells that if  $\frac{\mu}{\lambda}$  and  $\lambda$  are large enough then the set of solutions is  $\mathbf{C}(\Omega)$  . In addition, if we impose (for example) that  $u \in G$  (that is  $u$  has a null mean value), then the unique solution is  $(0, 0)$  since  $\mathbf{C}(\Omega) \cap (G \times BH_0(\Omega)) = \{(0, 0)\}$ .*

Eventually, we have a uniqueness result for the 1D case:

**Theorem 12** *Assume  $n = 1$ ,  $\Omega = ]a, b[$  and that  $u_d$  satisfies assumption  $(\mathcal{H}_1)$ . Then, for every  $\lambda > 0, \mu > 0$  problem  $(\mathcal{P}_{\lambda, \mu})$  has a unique solution up to a  $\mathbf{C}(\Omega)$  function. More precisely, if  $(u_1, v_1)$  and  $(u_2, v_2)$  are two optimal solutions of  $(\mathcal{P}_{\lambda, \mu})$  then  $\varphi := u_2 - u_1 = v_2 - v_1$  is a constant function. In particular, problem  $(\mathcal{P}_{\lambda, \mu})$  has a unique solution  $(u^*, v^*)$  such that  $u^*$  has a null mean value.*

*Proof.* Let  $(u_1, v_1)$  and  $(u_2, v_2)$  be two optimal solutions of  $(\mathcal{P}_{\lambda, \mu})$ . Then, with proposition 3, there exists  $\varphi \in BV(\Omega) \cap BH_0(\Omega)$  such that  $\varphi = u_2 - u_1 = v_2 - v_1$ . If  $u_d$  satisfies  $(\mathcal{H}_1)$  then  $u_d - v_i, i = 1, 2$  obviously satisfies this assumption as well. As  $u_i, i = 1, 2$  is solution to the ROF problem

$$u_i = \operatorname{argmin} \left\{ \frac{1}{2} \|u_d - v_i - u\|^2 + \Phi_\lambda^1(u), u \in L^2(\Omega) \right\}, i = 1, 2.$$

then,  $u_1, u_2$  and  $\varphi$  are piecewise constant on  $\Omega$ . In addition  $\varphi = v_2 - v_1 \in BH(\Omega) \subset W^{1,1}(\Omega)$ . This implies that  $\varphi$  is continuous and proves that  $\varphi$  is constant.

## 5 Conclusion

The mathematical analysis of this problem has been completed by numerical experiments ([9]). The model is well adapted to texture extraction. In the case, where the data is noiseless and/or is not too much textured, the decomposition given par  $\lambda \lesssim \mu$  and initialization  $u_0 = v_0 = 0$ , gives a cartoon part which is piecewise constant as expected. This means that  $u = \sum_i u_i \mathbf{1}_{\Gamma_i}$  where  $\bigcup_i \Gamma_i$  is the contour set. In this case, the remainder  $L^2$  term is the texture and/or noise. The decomposition is robust with respect to quantification, sampling and is always the same for any  $\mu \gg \lambda$ , once  $\lambda$  has been chosen.

In the case where the image is highly textured the model provides a two-scale decomposition. The  $TV$  part represents the macro-texture and the  $L^2$  part the micro-texture and/or noise. The scaling is tuned via the ratio  $\rho = \frac{\lambda}{\mu}$ .

From numerical computations, we infer that problem  $(\mathcal{P}_{\lambda, \mu})$  has a unique solution (up to  $\mathbf{C}(\Omega)$  functions) but the question is still open.

## References

1. R. Acar and C.R. Vogel. Analysis of bounded variation penalty methods for ill-posed problems. *Inverse Problems* 10, (6:1217–1229), 1994.
2. L. Ambrosio, L. Faina, and R. March. Variational approximation of a second order free discontinuity problem in computer vision. *SIAM J. Math. Anal.*, 32(6):1171–1197, 2001.
3. L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.
4. H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces*, volume 6 of *MPS/SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2006. Applications to PDEs and optimization.
5. G. Aubert and J.-F. Aujol. Modeling very oscillating signals. Application to image processing. *Appl. Math. Optim.*, 51(2):163–182, 2005.
6. G. Aubert and P. Kornprobst. *Mathematical Problems in Image Processing, Partial Differential Equations and the Calculus of Variations*, volume 147 of *Applied Mathematical Sciences*. Springer Verlag, 2006.

7. J.-F. Aujol, G. Aubert, L. Blanc-Féraud, and A. Chambolle. Image decomposition into a bounded variation component and an oscillating component. *J. Math. Imaging Vision*, 22(1):71–88, 2005.
8. M. Bergounioux. On Poincaré-wirtinger inequalities in BV - spaces. *Control & Cybernetics*, 4(40):921–929, 2011.
9. M. Bergounioux. Inf-convolution model : numerical experiment. Technical report, hal.archives-ouvertes.fr/hal-01002958v1, 2014.
10. M. Bergounioux and L. Piffet. A second-order model for image denoising. *Set-Valued Var. Anal.*, 18(3-4):277–306, 2010.
11. M. Bergounioux and L. Piffet. A full second order variational model for multiscale texture analysis. *Computational Optimization and Applications*, 54:215–237, 2013.
12. K. Bredies and M Holler. Regularization of linear inverse problems with total generalized variation. *Journal of Inverse and Ill-posed Problems*, 68, 2014.
13. K. Bredies, K. Kunisch, and T. Pock. Total generalized variation. *SIAM J. Imaging Sci.*, 3(3):492–526, 2010.
14. K. Bredies, K. Kunisch, and T. Valkonen. Properties of  $l^1$ -tgv<sup>2</sup>: the one-dimensional case. *J. Math. Anal. Appl.*, 398(1):438–454, 2013.
15. M. Carriero, A. Leaci, and F. Tomarelli. Special bounded hessian and elastics-plastic plate. *Rend. Ac. Naz delle Scienze*, XVI(13):223–258, 1992.
16. M. Carriero, A. Leaci, and F. Tomarelli. Uniform density estimates for the blake & zisserman functional. *Discrete and Continuous Dynamical Systems*, 31(4):1129–1150, 2011.
17. M. Carriero, A. Leaci, and F. Tomarelli. Free gradient discontinuity and image inpainting. *J. Math. Sciences*, 181(6):805–819, 2012.
18. V. Caselles, A. Chambolle, and M. Novaga. The discontinuity set of solutions of the tv denoising problem and some extensions. *SIAM Multiscale Model. Simul.*, 6(3):879–894, 2007.
19. A. Chambolle. An algorithm for total variation minimization and applications. *Journal of Mathematical Imaging and Vision*, 20:89–97, 2004.
20. F. Demengel. Fonctions à hessien borné. *Annales de l'institut Fourier*, 34(2):155–190, 1984.
21. I. Ekeland and R. Temam. *Convex Analysis and Variational problems*. SIAM Classic in Applied Mathematics, 28, 1999.
22. L.C. Evans and R. Gariepy. *Measure theory and fine properties of functions*. CRC Press, 1992.
23. J.-B. Hiriart-Urruty and R.R. Phelps. Subdifferential calculus using  $\varepsilon$ -subdifferentials. *Journal of Functional Analysis*, 118:150–166, 1993.
24. Y. Meyer. *Oscillating Patterns in Image Processing and Nonlinear Evolution Equations*, volume 22 of *University Lecture Series*. AMS, 2001.
25. S. Osher, A. Sole, and Vese L. Image decomposition and restoration using total variation minimization and the  $h^1$  norm. *SIAM Journal on Multiscale Modeling and Simulation*, 1-3(349-370), 2003.
26. S. Osher and L. Vese. Modeling textures with total variation minimization and oscillating patterns in image processing. *Journal of Scientific Computing*, 19(1-3):553–572, 2003.
27. S. Osher and L. Vese. Image denoising and decomposition with total variation minimization and oscillatory functions. special issue on mathematics and image analysis. *J. Math. Imaging Vision*, 20(1-2):7–18, 2004.
28. W. Ring. Structural properties of solutions of total variation regularization problems. *ESAIM, Math Modelling and Numerical Analysis*, 34:799–840, 2000.
29. L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.
30. W. Yin, D. Goldfarb, and S. Osher. A comparison of three total variation based texture extraction models. *J. Vis. Commun. Image*, 18:240–252, 2007.
31. W.P. Ziemer. *Weakly Differentiable Functions - Sobolev Space and Functions of Bounded Variation*. Springer, 1980.