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Mathematical analysis of a inf-convolution model for image processing

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Abstract We deal with a second order image decomposition model to perform denoising and texture extraction that was previously presented in [10]. We look for the decomposition $f = u + v + w$ where u is a first order term, v a second order term and w the remainder term (0 order). For highly textured images the model gives a two-scale texture decomposition: u can be viewed as a *macro-texture* (larger scale) which oscillations are not too large and w is the *micro-texture* (very oscillating) that contains the noise. Here, we perform mathematical analysis of the model and give qualitative properties of solutions using the dual problem and inf-convolution formulation.

Keywords Second order total variation · image decomposition · variational method · inf-convolution

Mathematics Subject Classification (2000) 65D18 · 68U10 · 65K10

1 Introduction

The most famous variational denoising model is the Rudin-Osher-Fatemi one ([1, 29]). This model involves a regularization term that preserves discontinuities, what a classical H^1 -Tychonov regularization method does not. The observed image to recover is split in two parts $u_d = w + u$ where w represents the oscillating component (noise or texture) and u is the *smooth* part. So we look for a solution u such that $u_d = w + u$ with $u \in BV(\Omega)$ and $w \in L^2(\Omega)$, where $BV(\Omega)$ is the functions of bounded variation space defined on an open subset $\Omega \subset \mathbb{R}^d$ ([3, 4, 22]). The regularization term involves only the so-called *cartoon* component u , while the remainder term $w := u_d - u$ represents the noise to be minimized.

A lot of people have investigated such decomposition models based on variational formulation, considering that an image can be decomposed into many

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components, each component describing a particular property of the image ([5, 7, 25–27, 30] and references therein for example).

In [10, 11] we have presented second order models where the (first order) classical total variation term has been replaced by a second order total variation term with the appropriate functional framework, namely the space of functions with bounded hessian introduced as $BH(\Omega)$ in [20] (and denoted $BV^2(\Omega)$ in [8, 11, 10]). The use of such a model allows to get rid of the *staircasing effect* that appears with the ROF model in denoising processes. To achieve this goal K. Bredies and al. have recently introduced a generalized total variation definition (TGV) [13, 14, 12] that is a nice compromise/mixture between the first and second order derivatives. It is in some sense an extension of the inf-convolution (we recall the definition later) of the first and second order derivatives. The model we present here is not more efficient than the TGV- based one for denoising. However, it provides a decomposition of the image at different scales what the TGV -model does not a priori. This paper is focused on this decomposition that provides a multiscale description of textured images.

Second order models have also been investigated in the context of segmentation and inpainting problems with Mumford-Shah types functionals (see [2, 16, 17] for example). The functional framework is the so called $GSBV$ space composed of functions u whose truncated forms $(\min(-N, \max(u, N)))$ belong to SBV_{loc} for every $N \in \mathbb{N}$. The definition of $GSBV^2$ is slightly different from the one we consider since

$$GSBV^2(\Omega) = \{u \in GSBV(\Omega) \mid \nabla u \in [GSBV(\Omega)]^d\} .$$

The aim of this paper is to give an existence result without any additional penalization term as in [10] and to perform a qualitative analysis of the model. Uniqueness et regularity issues will also be addressed.

More precisely, we assume that an image (in $L^2(\Omega)$) can be split in three components: a smooth (continuous) part v , a *cartoon* (piecewise constant) part u and an oscillating part w that should involve noise and/or fine textures. Such decompositions have already been investigated by Aujol and al. [5, 7]. These authors use the Meyer space of oscillating functions [24] rather than the $BH(\Omega)$ space (we shall present these spaces in the sequel). The model we propose here is different: the oscillating part of the image is not penalized but a priori included in the remainder term $w = u_d - u - v$, while v is the smooth part (in $BH(\Omega)$) and u belongs to $BV(\Omega)$: we hope u to be piecewise constant so that its jump set gives the image contours. For highly textured images, the model provides a two-scale texture decomposition: u can be viewed as a *macro-texture* (large scale) whose oscillations are not too large and w is the *micro-texture* (much more oscillating) that contains the noise.

Therefore, we look for components u , v and w that belong to different spaces: u belongs to $BV(\Omega)$ (and if possible not to $W^{1,1}(\Omega)$), $v \in BH(\Omega)$ and $w \in L^2(\Omega)$. This last component $w = u_d - u - v$ lies in the same space as the observed image u_d .

The paper is organized as follows. We first present the functional framework and perform a quick comparison between the second-order total variation we use and the one defined by Bredies et al. in [13]. In section 3, we present the variational model, give existence result and an equivalent formulation with inf-convolution.

This allows to compute the dual problem. Next section is devoted to giving qualitative properties of the solutions .

2 Functional framework for second order variational analysis

2.1 Spaces $BV(\Omega)$ and $BH(\Omega)$

In the whole paper, Ω is an open bounded subset of \mathbb{R}^d (practically $d = 2$) smooth enough (with the cone property and Lipschitz for example). More precisely, if $d = 2$, Ω may satisfy next assumption

$$\left\{ \begin{array}{l} \Omega \text{ is a bounded connected open set, strongly Lipschitz such that} \\ \partial\Omega \text{ is the union of finitely many } \mathcal{C}^2 \text{ curves} \end{array} \right. \quad (2.1)$$

Following [3,4,6] and [11,20], we recall the definitions and main properties of the spaces of functions of first and second order bounded variation. The space $BV(\Omega)$ is the classical Banach space of functions of bounded variation defined by

$$BV(\Omega) = \{u \in L^1(\Omega) \mid TV(u) < +\infty\},$$

where $TV(u)$ is the total variation of u

$$TV(u) := \sup \left\{ \int_{\Omega} u(x) \operatorname{div} \xi(x) dx \mid \xi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^d), \|\xi\|_{\infty} \leq 1 \right\}, \quad (2.2)$$

endowed with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1} + TV(u)$.

We say that a sequence $(u_n)_{n \in \mathbb{N}}$ of $BV(\Omega)$ converges to some $u \in BV(\Omega)$ for the *intermediate* (or *strict*) convergence if u_n strongly converges to u for the $L^1(\Omega)$ topology and $TV(u_n)$ converges to $TV(u)$ (in \mathbb{R}) (see [3,4,31]).

The space of functions with bounded hessian has been introduced by Demengel [20] (where it was denoted $BH(\Omega)$). It is the space of $W^{1,1}(\Omega)$ functions such that $TV^2(u) < +\infty$, where

$$W^{1,1}(\Omega) = \{ u \in L^1(\Omega) \mid \nabla u \in L^1(\Omega) \},$$

∇u stands for the first order derivative of u in the sense of distributions and

$$TV^2(u) := \sup \left\{ \int_{\Omega} \langle \nabla u, \operatorname{div}(\xi) \rangle_{\mathbb{R}^d} \mid \xi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{d \times d}), \|\xi\|_{\infty} \leq 1 \right\} < \infty, \quad (2.3)$$

is the second order total variation of u . Here, $\operatorname{div}(\xi) = (\operatorname{div}(\xi_1), \operatorname{div}(\xi_2), \dots, \operatorname{div}(\xi_d))$, and

$$\forall i, \xi_i = (\xi_{i,1}, \xi_{i,2}, \dots, \xi_{i,d}) \in \mathbb{R}^d, \quad \operatorname{div}(\xi_i) = \sum_{j=1}^d \frac{\partial \xi_{i,j}}{\partial x_j}.$$

The space $BH(\Omega)$ endowed with the following norm

$$\|f\|_{BH(\Omega)} := \|f\|_{W^{1,1}(\Omega)} + TV^2(f) = \|f\|_{L^1} + \|\nabla f\|_{L^1} + TV^2(f), \quad (2.4)$$

where TV^2 is given by (2.3) is a Banach space. Note that a function u belongs to $BH(\Omega)$ if and only if $u \in W^{1,1}(\Omega)$ and $\frac{\partial u}{\partial x_i} \in BV(\Omega)$ for $i \in \{1, \dots, d\}$. In particular

$$TV^2(u) \leq \sum_{i=1}^d TV \left(\frac{\partial u}{\partial x_i} \right) \leq d TV^2(u).$$

We give thereafter important properties of these spaces which proofs can be found in [3, 4, 11, 15, 20] for example.

Theorem 1 [*Semi-continuity of total variation*]

- i. The mapping $u \mapsto TV(u)$ is lower semi-continuous (denoted in short *lsc*) from $BV(\Omega)$ to \mathbb{R}^+ for the $L^1(\Omega)$ topology.
- ii. The operator TV^2 is lower semi-continuous from $BH(\Omega)$ endowed with the strong topology of $W^{1,1}(\Omega)$ to \mathbb{R} .

Theorem 2 [*Embedding results*] Assume $d \geq 2$. Then

- i. $BH(\Omega) \hookrightarrow W^{1,q}(\Omega)$ with $q \leq \frac{d}{d-1}$, with continuous embedding. Moreover the embedding is compact if $q < \frac{n}{n-1}$. In particular

$$BH(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [1, \infty[, \quad \text{if } d = 2.$$

- ii. If $d = 2$
 - $BV(\Omega) \subset L^2(\Omega)$ with continuous embedding.
 - $BV(\Omega) \subset L^p(\Omega)$ with compact embedding, for every $p \in [1, 2)$.
- iii. If $d = 2$ and if Ω satisfies assumption (2.1) then $BH(\Omega) \subset C^0(\bar{\Omega})$.

So $BH(\Omega) \subset H^1(\Omega)$ with continuous embedding and $BH(\Omega) \subset W^{1,1}(\Omega)$ with compact embedding. Let us define the space $BV_0(\Omega)$ as the space of functions of bounded variation that vanish on the boundary $\partial\Omega$ of Ω . More precisely as Ω is bounded and $\partial\Omega$ is Lipschitz, functions of $BV(\Omega)$ have a trace of class L^1 on $\partial\Omega$ [3, 4, 31], and the trace mapping $T : BV(\Omega) \rightarrow L^1(\partial\Omega)$ is linear, continuous from $BV(\Omega)$ equipped with the intermediate convergence to $L^1(\partial\Omega)$ endowed with the strong topology ([4] Theorem 10.2.2 p 386). The space $BV_0(\Omega)$ is then defined as the kernel of T . It is a Banach space, endowed with the induced norm:

$$BV_0(\Omega) := \{u \in BV(\Omega) \mid u|_{\partial\Omega} = 0\}.$$

In addition, if $u \in BH(\Omega)$, $\nabla u \in BV(\Omega)^n$ (as a consequence of the definition of $BH(\Omega)$) and we may define the normal derivative trace operator $\nu : BH(\Omega) \rightarrow L^1(\partial\Omega)$ (with $\nu(u) = \nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial n}$). This operator is linear, continuous from $BH(\Omega)$ equipped with the strong convergence in $W^{1,1}$ and the convergence of TV^2 (intermediate convergence in BH) to $L^1(\partial\Omega)$ endowed with the strong topology. So we may define similarly

$$BH_0(\Omega) := \{u \in BH(\Omega) \mid \frac{\partial u}{\partial x_i} = 0 \text{ on } \partial\Omega, \quad i = 1, \dots, d\}.$$

We set also

$$BV_m(\Omega) := \{u \in BV(\Omega) \mid \int_{\Omega} u(x) dx = 0 \quad i = 1, \dots, n\},$$

and

$$BH_m(\Omega) := \{u \in BH(\Omega) \mid \int_{\Omega} \frac{\partial u}{\partial x_i} dx = 0 \ i = 1, \dots, d\} .$$

The Ostrograski formula gives

$$\int_{\Omega} \frac{\partial u}{\partial x_i} dx = - \int_{\partial\Omega} u_i n_i ,$$

where u_i is the i^{th} partial function with respect to the i^{th} coordinate and $n = (n_1, \dots, n_d)$ is the outer normal vector. In particular, if $u = 0$ on $\partial\Omega$, then $u \in BH_m(\Omega)$. At last we shall use the following result of [8]:

Lemma 1 *Let $\Omega \subset \mathbb{R}^n$ be an open Lipschitz bounded set. There exist generic constants only depending on Ω , $C_1, C_2 > 0$ such that*

$$\forall u \in BV_m(\Omega) \quad \|u\|_{L^1(\Omega)} \leq C_1 TV(u), \quad (2.5)$$

$$\forall u \in BH_0(\Omega) \cup BH_m(\Omega) \quad TV(u) \leq C_2 TV^2(u) \quad (2.6)$$

2.2 Comparison with BGV²

As already mentionned, a different approach of second-order total variation spaces has been set in [13]. The main difference lies in the choice of the test functions for the weak variational formulation. The authors define the *Total Generalized Variation* $TGV^2(u)$ as the supremum of the duality product between u and symmetric tests functions that are bounded together with their derivative. First, we note that we may define $TV^2(u)$ in a equivalent way as following: for any $\xi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{d \times d})$ recall that

$$\forall i, \xi_i = (\xi_{i,1}, \xi_{i,2}, \dots, \xi_{i,d}) \in \mathbb{R}^d, \quad \text{div}(\xi_i) = \sum_{j=1}^d \frac{\partial \xi_{i,j}}{\partial x_j}$$

and define as in [13]

$$\text{div}^2 \xi := \sum_{i,j=1}^d \frac{\partial^2 \xi_{i,j}}{\partial x_i \partial x_j} .$$

Let us call : $\mathcal{B} := \left\{ \xi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{d \times d}), \|\xi\|_{\infty} \leq 1 \right\}$. Then,

$$\forall u \in W^{1,1}(\Omega) \quad TV^2(u) := \sup \left\{ \int_{\Omega} u \text{div}^2 \xi dx, \xi \in \mathcal{B} \right\}, \quad (2.7)$$

Indeed, an integration by parts gives

$$\int_{\Omega} u \text{div}^2 \xi dx = - \int_{\Omega} (\nabla u, \text{div} \xi)_{\mathbb{R}^d} dx .$$

Let be $\alpha = (\alpha_0, \alpha_1) > 0$, we call

$$TGV_{\alpha}^2(u) = \sup \left\{ \int_{\Omega} u \text{div}^2 \xi dx, \xi \in \mathcal{B}_{\alpha} \right\},$$

where $\mathcal{B}_{\alpha} := \left\{ \xi \in \mathcal{K}, \xi_{ij} = \xi_{ji} \ \forall i, j, \|\xi\|_{\infty} \leq \alpha_0, \|\text{div} \xi\|_{\infty} \leq \alpha_1 \right\}$. We may define ([13])

$$BGV_{\alpha}^2(\Omega) = \left\{ u \in L^1(\Omega), TGV_{\alpha}^2(u) < +\infty \right\}. \quad (2.8)$$

Proposition 1 Let be $\alpha = (\alpha_0, \alpha_1) > 0$. For every function u in $W^{1,1}(\Omega)$ we get

$$TGV_\alpha^2(u) \leq \alpha_0 TV^2(u) .$$

Therefore

$$\forall \alpha > 0 \quad BH(\Omega) \subset BGV_\alpha^2(\Omega)$$

with continuous embedding.

Proof. As $\mathcal{B}_\alpha \subset \alpha_0 \mathcal{B}$ the first relation is obvious. Moreover if $u \in BH(\Omega)$, then $u \in W^{1,1}(\Omega)$ and $TGV_\alpha^2(u) < +\infty$. In addition

$$\|u\|_{BVG_\alpha^2} = \|u\|_{L^1} + TGV_\alpha^2(u) \leq \|u\|_{W^{1,1}} + \alpha_0 TV^2(u) \leq \max(1, \alpha_0) \|u\|_{BH} ,$$

which gives the continuity of the embedding mapping. \square

Corollary 1 For any $u \in BH(\Omega)$, $TV^2(u) = 0$ if and only if u is a polynomial function of order 1.

Proof. For any $u \in BH(\Omega)$, $TV^2(u) = 0 \implies TGV_\alpha^2(u) = 0$. Then we use Proposition 3.3 of [13]. \square

The main difference between the two approaches concerns the functions regularity. The $BGV^2(\Omega)$ functions do not necessarily belong to $L^1(\Omega)$. In particular, the indicator function of smooth open sets belong to $BGV^2(\Omega)$ and not to $BH(\Omega)$. On the other hand, we cannot have Sobolev-type embeddings for $BGV^2(\Omega)$.

3 A second-order variational model for image decomposition

3.1 Presentation of the model

We have already presented this model in [10] so that we do no detail so much. However we provide here an existence result that was expected but only proved in the finite dimensional case and give a inf-convolution formulation. We now assume that u_d belongs to $L^2(\Omega)$ and that the image we want to recover can be decomposed as $u_d = w + u + v$ where u , v and w are functions that characterize different parts of u_d . Components belong to different functional spaces: v is the (smooth) second order part and belongs to $BH(\Omega)$, u is a $BV(\Omega)$ component and $w \in L^2(\Omega)$ is the remainder term. We consider the following cost functional defined on $BV(\Omega) \times BH(\Omega)$:

$$\mathcal{F}_{\lambda,\mu}(u, v) = \frac{1}{2} \|u_d - u - v\|_{L^2(\Omega)}^2 + \lambda TV(u) + \mu TV^2(v), \quad (3.9)$$

where $\lambda, \mu > 0$. We are looking for a solution to the optimization problem

$$\inf \{ \mathcal{F}_{\lambda,\mu}(u, v) \mid (u, v) \in BV(\Omega) \times BH_0(\Omega) \} \quad (\mathcal{P}_{\lambda,\mu})$$

Remark 1 We decide to look for the minima of $\mathcal{F}_{\lambda,\mu}$ on $BV(\Omega) \times BH_0(\Omega)$ and not $BV(\Omega) \times BH(\Omega)$ to get an existence result. This will cause troubles to set the dual problem because of the computation of Legendre-Fenchel conjugate functions. Nevertheless, the constraint $v \in BH_0(\Omega)$ (that is $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$) is a usual one in image processing and the difficulty will be overcome in the discrete setting.

We expect v to be the smooth *colored* part of the image, u to be a $BV(\Omega) \setminus BH(\Omega)$ function which derivative is a measure supported by the contours and $w := u_d - u - v \in L^2$ is the noise and/or small textures (we shall detail this point later). Problem $\mathcal{P}_{\lambda,\mu}$ can be (formally) viewed as a minimization problem where the regularization term is an inf-convolution. Recall that the inf-convolution ([4] p 324) is defined as

$$(f \# g)(v) = \inf \{ f(u) + g(v - u), u \in V \},$$

where $f, g : V \rightarrow \mathbb{R} \cup \{+\infty\}$. If we set $V = L^2(\Omega)$,

$$\Phi_\lambda^1(u) = \begin{cases} \lambda TV(u) & \text{if } u \in BV(\Omega) \\ +\infty & \text{else.} \end{cases} \quad \text{and} \quad \Phi_\mu^2(v) = \begin{cases} \mu TV^2(v) & \text{if } v \in BH_0(\Omega) \\ +\infty & \text{else.} \end{cases} \quad (3.10)$$

then

$$(\Phi_\lambda^1 \# \Phi_\mu^2)(\varpi) = \inf_{u \in BV(\Omega), v \in BH_0(\Omega), u+v=\varpi} \lambda TV(u) + \mu TV^2(v),$$

and problem $\mathcal{P}_{\lambda,\mu}$ may be written as

$$\inf_{\varpi \in L^2(\Omega)} \frac{1}{2} \|u_d - \varpi\|_{L^2(\Omega)}^2 + (\Phi_\lambda^1 \# \Phi_\mu^2)(\varpi).$$

This formulation is to compare to the one by Bredies and al. [13,14,12]

$$\inf_{\varpi \in L^2(\Omega)} \frac{1}{2} \|u_d - \varpi\|_{L^2(\Omega)}^2 + TGV_\alpha^2(\varpi),$$

which seems to be more efficient for denoising. However, we are interested in the decomposition components u and v . First, we give an existence result for problem $(\mathcal{P}_{\lambda,\mu})$.

Theorem 3 (Existence) *The problem $(\mathcal{P}_{\lambda,\mu})$ has at least an optimal solution (u^*, v^*) in $BV(\Omega) \times BH_0(\Omega)$.*

Proof. We first prove that the auxiliary problem

$$\inf \{ \mathcal{F}_{\lambda,\mu}(u, v) \mid (u, v) \in BV_m(\Omega) \times BH_0(\Omega) \} \quad (3.11)$$

has an optimal solution. Let $(u_n, v_n) \in BV_m(\Omega) \times BH_0(\Omega)$ be a minimizing sequence, i.e.

$$\lim_{n \rightarrow +\infty} \mathcal{F}_{\lambda,\mu}(u_n, v_n) = \inf \{ \mathcal{F}_{\lambda,\mu}(u, v) \mid (u, v) \in BV_m(\Omega) \times BH_0(\Omega) \} < +\infty.$$

Therefore

- $TV^2(v_n)$ is bounded and with lemma 1, $\|\nabla v_n\|_{L^1}$ is bounded as well.
- $TV(u_n)$ is bounded. Using once again lemma 1 this yields that u_n is bounded in $L^1(\Omega)$. Therefore the sequence u_n is bounded in $BV(\Omega)$.
- As $u_n + v_n$ is L^2 -bounded, it is L^1 -bounded as well so that v_n is L^1 bounded. As $\|\nabla v_n\|_{L^1}$ and $TV^2(v_n)$ are bounded this means that the sequence v_n is bounded in $BH(\Omega)$.

With the compactness result of Theorem 2, this yields that $(v_n)_{n \in \mathbb{N}}$ strongly converges (up to a subsequence) in $W^{1,1}(\Omega)$ to $v^* \in BH(\Omega)$. Moreover, $v^* \in BH_0(\Omega)$ because the normal derivative operator is continuous (as mentioned before). Similarly $(u_n)_{n \in \mathbb{N}}$ strongly converges (up to a subsequence) in $L^1(\Omega)$ to $u^* \in BV_m(\Omega)$. Moreover $u_n + v_n$ weakly converges to $u^* + v^*$ in $L^2(\Omega)$. With theorem 1 we get

$$TV(u^*) \leq \liminf_{n \rightarrow +\infty} TV(u_n), \quad TV^2(v^*) \leq \liminf_{n \rightarrow +\infty} TV^2(v_n).$$

So

$$\mathcal{F}_{\lambda,\mu}(u^*, v^*) \leq \liminf_{n \rightarrow +\infty} \mathcal{F}_{\lambda,\mu}(u_n, v_n) = \min_{(u,v) \in BV_m(\Omega) \times BH_0(\Omega)} \mathcal{F}_{\lambda,\mu}(u, v),$$

and (u^*, v^*) is a solution to (3.11).

For every $(u, v) \in BV(\Omega) \times BH_0(\Omega)$, we have $(u - \bar{u}, v + \bar{u}) \in BV_m(\Omega) \times BH_0(\Omega)$ where $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u$ is the mean value of u . Moreover

$$\mathcal{F}_{\lambda,\mu}(u, v) = \mathcal{F}_{\lambda,\mu}(u - \bar{u}, v + \bar{u}) \geq \mathcal{F}_{\lambda,\mu}(u^*, v^*).$$

Therefore (u^*, v^*) is an optimal solution to $(\mathcal{P}_{\lambda,\mu})$. \square

Remark 2 *Uniqueness of the solution is challenging. We shall prove partial results in section 4.2.*

3.2 Optimality conditions

In what follows, we fix $\lambda > 0$ and $\mu > 0$ and set for any $u \in L^2(\Omega) : \mathcal{N}(u) = \frac{1}{2} \|u\|_2^2$.

Functionals Φ_{λ}^1 and Φ_{μ}^2 have been defined in (3.10).

It is easy to see that (\bar{u}, \bar{v}) is a solution to $(\mathcal{P}_{\lambda,\mu})$ if and only if

$$\bar{u} = \operatorname{argmin} \left\{ \frac{1}{2} \|u_d - \bar{v} - u\|^2 + \Phi_{\lambda}^1(u), \quad u \in L^2(\Omega) \right\}, \quad (3.12)$$

$$\bar{v} = \operatorname{argmin} \left\{ \frac{1}{2} \|u_d - \bar{u} - v\|^2 + \Phi_{\mu}^2(v), \quad v \in L^2(\Omega) \right\}. \quad (3.13)$$

and we may derive optimality conditions in a standard way :

Theorem 4 (\bar{u}, \bar{v}) is a solution to $(\mathcal{P}_{\lambda,\mu})$ if and only if

$$\bar{w} := u_d - \bar{u} - \bar{v} \in \partial\Phi_{\lambda}^1(\bar{u}) \cap \partial\Phi_{\mu}^2(\bar{v}). \quad (3.14)$$

The proof is obvious. Here $\partial f(u)$ stands for the subdifferential of f at u where $f : V \rightarrow \mathbb{R}$:

$$\partial f(u) = \{u^* \in V' \mid \forall v \in V \quad f(v) - f(u) \geq \langle u^*, v - u \rangle\},$$

and $\langle \cdot, \cdot \rangle$ is the duality product between V et V' .

3.3 Inf-convolution formulation

We have already noticed that the penalization term in $(\mathcal{P}_{\lambda,\mu})$ is an infimal convolution term. In addition, $(\mathcal{P}_{\lambda,\mu})$ can be viewed as successive inf-convolution processes.

Lemma 2 *The functionals $\mathcal{N}\#\Phi_\lambda^1$ and $\mathcal{N}\#\Phi_\mu^2$ are convex, continuous from $L^2(\Omega)$ to $L^2(\Omega)$.*

Proof. In the sequel we set $\Phi = \Phi_\lambda^1$ or Φ_μ^2 indifferently. As Φ and \mathcal{N} are convex so is $\mathcal{N}\#\Phi$ (see [23] for example). Let be $u \in L^2(\Omega)$:

$$(\mathcal{N}\#\Phi)(u) = \inf_{v \in L^2(\Omega)} \frac{1}{2} \|u - v\|_2^2 + \Phi(v) \leq \frac{1}{2} \|u\|_2^2 + \Phi(0) = \frac{1}{2} \|u\|_2^2 .$$

As $(\mathcal{N}\#\Phi)(0) = 0$ this gives the $\mathcal{N}\#\Phi$ continuity at 0 and its boundedness in a neighborhood of 0. As it is convex, it continuous on its whole domain $L^2(\Omega)$ (see [21] for example). \square

Note that problem (3.12) is equivalent to $\bar{u} \in \mathcal{N}\#\Phi_\lambda^1(u_d - \bar{v})$ and (3.13) is equivalent to $\bar{v} \in \mathcal{N}\#\Phi_\mu^2(u_d - \bar{u})$. In fact, problem $(\mathcal{P}_{\lambda,\mu})$ can be written as successive inf-convolution processes. More precisely we have

Theorem 5 *Let $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$ be a solution to $(\mathcal{P}_{\lambda,\mu})$ and $\bar{m} := \inf(\mathcal{P}_{\lambda,\mu})$. Then*

$$\begin{aligned} \bar{m} &= \mathcal{N}(u_d - \bar{u} - \bar{v}) + \Phi_\lambda^1(\bar{u}) + \Phi_\mu^2(\bar{v}) \\ &= (\mathcal{N}\#\Phi_\lambda^1)(u_d - \bar{v}) + \Phi_\mu^2(\bar{v}) = (\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u}) + \Phi_\lambda^1(\bar{u}) \\ &= (\Phi_\lambda^1\#\mathcal{N}\#\Phi_\mu^2)(u_d) = (\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d). \end{aligned}$$

Proof. Let $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$ be a solution to $(\mathcal{P}_{\lambda,\mu})$. Then, for every $(u, v) \in BV(\Omega) \times BH_0(\Omega)$, we get

$$\bar{m} = \mathcal{N}(u_d - \bar{u} - \bar{v}) + \Phi_\lambda^1(\bar{u}) + \Phi_\mu^2(\bar{v}) \leq \mathcal{N}(u_d - u - v) + \Phi_\lambda^1(\bar{u}) + \Phi_\mu^2(v) . \quad (3.15)$$

This gives, for every $v \in BH_0(\Omega)$

$$\bar{m} \leq \inf_{u \in L^2(\Omega)} \mathcal{N}(u_d - u - v) + \Phi_\lambda^1(\bar{u}) + \Phi_\mu^2(v) = (\mathcal{N}\#\Phi_\lambda^1)(u_d - v) + \Phi_\mu^2(v)$$

so that

$$\bar{m} \leq \inf_{v \in BH_0(\Omega)} (\mathcal{N}\#\Phi_\lambda^1)(u_d - v) + \Phi_\mu^2(v) \leq (\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d) .$$

Similarly

$$\bar{m} \leq (\Phi_\lambda^1\#\mathcal{N}\#\Phi_\mu^2)(u_d) .$$

Conversely, by definition of inf-convolution, we get for every $(u, v) \in BV(\Omega) \times BH_0(\Omega)$

$$(\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d) \leq (\mathcal{N}\#\Phi_\lambda^1)(u_d - v) + \Phi_\mu^2(v) \leq \mathcal{N}(u_d - v - u) + \Phi_\lambda^1(u) + \Phi_\mu^2(v) ,$$

so that $(\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d) \leq \bar{m}$.

We finally obtain $\bar{m} = (\Phi_\lambda^1\#\mathcal{N}\#\Phi_\mu^2)(u_d) = (\Phi_\mu^2\#\mathcal{N}\#\Phi_\lambda^1)(u_d)$. \square

3.4 Computing Fenchel conjugate function

We are going to write the dual problem of $(\mathcal{P}_{\lambda,\mu})$ and we need to compute the conjugate functions of Φ_λ^1 and Φ_μ^2 and $\tilde{f} : u \mapsto f(u_d + u)$. We recall that if $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$, the Legendre-Fenchel conjugate f^* is defined on V' as

$$\forall u^* \in V' \quad f^*(u^*) = \sup_{u \in V} \langle u^*, u \rangle - f(u) .$$

We obviously have

$$\forall \lambda > 0, \forall u^* \in V' \quad (\lambda f)^*(u^*) = \lambda f^*\left(\frac{u^*}{\lambda}\right) ,$$

and the following useful result:

Proposition 2 [4] *Let V be a normed space and $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ a closed, convex, proper function. then*

$$u^* \in \partial f(u) \iff u \in \partial f^*(u^*) \iff f(u) + f^*(u^*) = \langle u^*, u \rangle ,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality $V - V'$ product.

Lemma 3 *Let be $f : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ and \tilde{f} such that $\tilde{f}(u) = f(u_d + u)$. Then \tilde{f} conjugate function writes*

$$\forall u^* \in L^2(\Omega) \quad (\tilde{f})^*(u^*) = f^*(u^*) - (u^*, u_d)_2 ,$$

where $(\cdot, \cdot)_2$ denotes the $L^2(\Omega)$ inner product.

Proof. Let be $u^* \in L^2(\Omega)$. We have

$$\begin{aligned} (\tilde{f})^*(u^*) &= \sup_{u \in L^2(\Omega)} (u, u^*)_2 - f(u_d + u) = \sup_{w \in L^2(\Omega)} (w - u_d, u^*)_2 - f(w) \\ &= \sup_{w \in L^2(\Omega)} (w, u^*)_2 - f(w) - (u_d, u^*)_2 = f^*(u^*) - (u^*, u_d)_2 . \end{aligned}$$

□

In the sequel $\mathbf{1}_C$ denotes the indicator function of the set C :

$$\mathbf{1}_C(u) = \begin{cases} 0 & \text{if } u \in C \\ +\infty & \text{else.} \end{cases}$$

Lemma 4 *The conjugate function of Φ_λ^1 is $(\Phi_\lambda^1)^* = \lambda \mathbf{1}_{\lambda \mathcal{K}_1}$, where $\mathcal{K}_1 = \overline{\mathbf{K}_1}$ is the L^2 -closure of*

$$\mathbf{K}_1 := \left\{ \xi = \operatorname{div} \varphi \mid \varphi \in \mathcal{C}_c^1(\Omega), \|\varphi\|_\infty \leq 1 \right\} . \quad (3.16)$$

The conjugate function of Φ_μ^2 is $(\Phi_\mu^2)^ = \mu \mathbf{1}_{\mu \mathcal{K}_2}$, where $\mathcal{K}_2 \supset \overline{\mathbf{K}_2}$ and $\overline{\mathbf{K}_2}$ is the L^2 -closure of*

$$\mathbf{K}_2 := \left\{ \xi = \operatorname{div}^2 \psi \mid \psi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{d \times d}), \|\psi\|_\infty \leq 1 \right\} . \quad (3.17)$$

Proof. It is known that the conjugate TV^* of TV is the indicator function of $\overline{\mathcal{K}_1}$ (see [6,19] for example). As $\Phi_\lambda^1 = \lambda TV$ (or $+\infty$ outside $BV(\Omega)$) then $(\Phi_\lambda^1)^*(u^*) = \lambda TV^* \left(\frac{u^*}{\lambda} \right)$. This gives the result.

The result is not exactly the same since Φ_μ^2 is equal to μTV^2 on $BH_0(\Omega)$ and $+\infty$ outside (and in particular on $BH(\Omega) \setminus BH_0(\Omega)$). Therefore the conjugate of Φ_μ^2 is not the same as the conjugate of μTV^2 . We know that the conjugate function of TV^2 is $\mathbf{1}_{\overline{\mathcal{K}_2}}$ (see [11]); as Φ_1^2 is positively homogeneous ($\mu = 1$), it is the indicator function of a closed subset \mathcal{K}_2 of $L^2(\Omega)$. Moreover

$$\mathbf{1}_{\mathcal{K}_2}(v^*) = (\Phi_1^2)^*(v^*) = \sup_{v \in BH_0} \langle v^*, v \rangle - TV^2(v) \leq \sup_{v \in BH} \langle v^*, v \rangle - TV^2(v) = \mathbf{1}_{\overline{\mathcal{K}_2}}(v^*).$$

This implies that $\overline{\mathcal{K}_2} \subset \mathcal{K}_2$ but we cannot prove the converse inclusion (for example). We end the proof with the same argument as in the BV case. \square

Eventually it is easy to see that $\mathcal{N}^* = \mathcal{N}$.

3.5 Dual problem to $(\mathcal{P}_{\lambda,\mu})$

In the present subsection we use convex duality tools that we recall thereafter (see [4] for example).

Theorem 6 [[4] p 366] *Let V be a banach space, $f, g : V \rightarrow \mathbb{R} \cup \{+\infty\}$ lower semi-continuous convex functions and A a linear continuous operator from V to V . Assume there exists $u_o \in \text{dom } g$ and f continuous at Au_o . Then*

$$\inf_{u \in V} (f(Au) + g(u)) = \max_{u^* \in V^*} (-f^*(u^*) - g^*(-A^*u^*)).$$

Moreover, if \bar{u} is a solution to the primal problem and \bar{u}^* is a solution to the dual one then

$$\bar{u}^* \in \partial f(A\bar{u}) \text{ and } -A^*\bar{u}^* \in \partial g(\bar{u}),$$

where $\partial f(u)$ stands for the subdifferential of f at u .

Theorem 7 [[4] p 328] *Let V be a banach space and $f, g : V \rightarrow \mathbb{R} \cup \{+\infty\}$ proper functions. Then*

$$(f \# g)^* = f^* + g^*.$$

In addition if f and g satisfy the assumptions of Theorem 6, then

$$(f + g)^* = f^* \# g^*.$$

Now we may compute the dual problem to $(\mathcal{P}_{\lambda,\mu})$ and get the following

Theorem 8 *The dual problem to $(\mathcal{P}_{\lambda,\mu})$ writes*

$$\inf_{w \in \lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2} \frac{1}{2} \|u_d - w\|_2^2. \quad (3.18)$$

The unique solution w^* is the L^2 -projection of u_d on the closed convex set $\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2$:

$$w^* = \Pi_{\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2}(u_d).$$

Proof. Solving problem $(\mathcal{P}_{\lambda,\mu})$ is equivalent to solving

$$\inf_{u \in L^2(\Omega)} (\mathcal{N} \# \Phi_\mu^2)(u_d - u) + \Phi_\lambda^1(u) = \inf_{u \in L^2(\Omega)} (\widetilde{\mathcal{N} \# \Phi_\mu^2})(Au) + \Phi_\lambda^1(u)$$

with $Au = -u$. It clear that $A^* = A$. Moreover, Φ_λ^1 is lsc with respect to the L^1 -topology and thus for L^2 -topology since Ω is bounded. As $\mathcal{N} \# \Phi_\mu^2$, $\widetilde{\mathcal{N} \# \Phi_\mu^2}$, A and Φ_λ^1 fulfill assumptions of Theorem 6, the dual problem of $(\mathcal{P}_{\lambda,\mu})$ writes

$$(\mathcal{P}^*) \quad \max_{w \in L^2(\Omega)} -(\widetilde{\mathcal{N} \# \Phi_\mu^2})^*(w) - (\Phi_\lambda^1)^*(w),$$

where $(\widetilde{\mathcal{N} \# \Phi_\mu^2})(w) = (\mathcal{N} \# \Phi_\mu^2)(u_d + w)$. Using Lemma 3 and Theorem 7 it easy to see that

$$\begin{aligned} (\widetilde{\mathcal{N} \# \Phi_\mu^2})^*(w) &= -(u_d, w)_2 + (\mathcal{N} \# \Phi_\mu^2)^*(w) = -(u_d, w)_2 + \mathcal{N}^*(w) + (\Phi_\mu^2)^*(w) \\ &= -(u_d, w)_2 + \mathcal{N}(w) + (\Phi_\mu^2)^*(w). \end{aligned}$$

Therefore, (\mathcal{P}^*) writes

$$\begin{aligned} &\max_{w \in L^2(\Omega)} (u_d, w)_2 - \mathcal{N}(w) - (\Phi_\lambda^1)^*(w) - (\Phi_\mu^2)^*(w), \\ \Leftrightarrow & - \min_{w \in L^2(\Omega)} -(u_d, w)_2 + \mathcal{N}(w) + (\Phi_\lambda^1)^*(w) + (\Phi_\mu^2)^*(w). \end{aligned}$$

Finally, (\mathcal{P}^*) is equivalent to

$$\min_{w \in \lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2} \frac{1}{2} \|u_d - w\|_2^2, \quad (3.19)$$

The dual problem has obviously a unique solution w^* which is the L^2 projection of u_d on the closed convex set $\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2$. \square

Next we have a relation between the solutions to $(\mathcal{P}_{\lambda,\mu})$ and the (unique) solution of the dual problem.

Theorem 9 1. Let w^* be the (unique) solution to the dual problem $(\mathcal{P}_{\lambda,\mu})^*$:

$$w^* = \Pi_{\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2}(u_d).$$

Then there exists $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$ an optimal solution to $(\mathcal{P}_{\lambda,\mu})$ such that

$$w^* = u_d - \bar{u} - \bar{v} \text{ and } w^* \in \partial\Phi_\mu^2(\bar{v}) \cap \partial\Phi_\lambda^1(\bar{u}).$$

2. Conversely, if $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$ is any solution to $(\mathcal{P}_{\lambda,\mu})$ then

$$\bar{w} = u_d - \bar{u} - \bar{v} = \Pi_{\lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2}(u_d). \quad (3.20)$$

Proof. Let (\bar{u}, \bar{v}) be a solution to $(\mathcal{P}_{\lambda, \mu})$. A direct consequence of Theorem 6 is

$$w^* \in \partial\Phi_\lambda^1(\bar{u}) \quad \text{and} \quad w^* \in \partial(\widetilde{\mathcal{N}\#\Phi_\mu^2})(-\bar{u}).$$

A simple calculus shows that

$$\partial(\widetilde{\mathcal{N}\#\Phi_\mu^2})(-\bar{u}) = \partial(\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u})$$

so that

$$w^* \in \partial\Phi_\lambda^1(\bar{u}) \cap \partial(\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u}).$$

As

$$(\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u}) = \mathcal{N}(u_d - \bar{u} - \bar{v}) + \Phi_\mu^2(\bar{v}) = \operatorname{argmin}_{v \in L^2(\Omega)} \frac{1}{2} \|v + \bar{u} - u_d\|^2 + \Phi_\mu^2(v),$$

then

$$u_d - \bar{v} - \bar{u} \in \partial\Phi_\mu^2(\bar{v});$$

so, the inf-convolution is exact and we get ([23])

$$\partial(\mathcal{N}\#\Phi_\mu^2)(u_d - \bar{u}) = \bigcup_{v \in L^2(\Omega)} \partial\mathcal{N}(u_d - \bar{u} - v) \cap \partial\Phi_\mu^2(v).$$

As $\partial\mathcal{N}(u_d - \bar{u} - v) = \{u_d - \bar{u} - v\}$ this means that there exists $\bar{v} \in L^2(\Omega)$ such that

$$w^* = u_d - \bar{u} - \bar{v} \in \partial\Phi_\mu^2(\bar{v}).$$

So

$$w^* \in \partial\Phi_\mu^2(\bar{v}) \cap \partial\Phi_\lambda^1(\bar{u}),$$

with $\bar{v} = u_d - \bar{u} - w^*$. This prove that (\bar{u}, \bar{v}) is a solution to $(\mathcal{P}_{\lambda, \mu})$ as well: we use Theorem 4 with $\bar{w} = w^*$ to conclude.

Let us prove the converse property. Let $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$ be a solution to $(\mathcal{P}_{\lambda, \mu})$ and $\bar{w} = u_d - \bar{u} - \bar{v}$. Theorem 4 yields

$$\bar{w} \in \partial\Phi_\lambda^1(\bar{u}) \cap \partial\Phi_\mu^2(\bar{v}),$$

that is

$$\bar{u} \in \partial(\Phi_\lambda^1)^*(\bar{w}) \quad \text{and} \quad \bar{v} \in \partial(\Phi_\mu^2)^*(\bar{w}).$$

With the previous computations this gives

$$\bar{u} \in \partial\lambda_{\lambda\mathcal{K}_1}(\bar{w}) \quad \text{and} \quad \bar{v} \in \partial\lambda_{\mu\mathcal{K}_2}(\bar{w}).$$

Therefore

$$\forall w \in \lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2 \quad \langle \bar{u}, w - \bar{w} \rangle \leq 0 \quad \text{and} \quad \langle \bar{v}, w - \bar{w} \rangle \leq 0.$$

Adding the above inequalities gives

$$\forall w \in \lambda\mathcal{K}_1 \cap \mu\mathcal{K}_2 \quad \langle \bar{u} + \bar{v}, w - \bar{w} \rangle = \langle u_d - \bar{w}, w - \bar{w} \rangle \leq 0.$$

This is equivalent to (3.20).

Corollary 2 *If (\bar{u}, \bar{v}) is a solution to $(\mathcal{P}_{\lambda, \mu})$, then $\bar{w} = \bar{u} + \bar{v}$ is unique. In particular, there is a unique solution to $(\mathcal{P}_{\lambda, \mu})$ such that $\bar{u} = 0$ almost everywhere.*

Remark 3 *We cannot permute the role of Φ_λ^1 and Φ_μ^2 in the previous proof because Φ_μ^2 is not lower semi-continuous with respect to the L^2 topology. Indeed $L^2(\Omega)$ is not embedded in $W^{1,1}(\Omega)$.*

4 Solution properties ($d \leq 2$)

4.1 Structure of the solutions

Recall (see [5,24]) that the Meyer space $G(\Omega)$ is defined as

$$G(\Omega) := \{ f \in L^2(\Omega) \mid \exists \varphi = (\varphi_1, \varphi_2) \in L^\infty(\Omega, \mathbb{R}^2) \ f = \operatorname{div} \varphi \text{ and } \varphi \cdot n = 0 \text{ on } \partial\Omega \}$$

where n is the outer normal vector to $\partial\Omega$. The space G is endowed with a norm denoted by $\|\cdot\|_G$ and defined as

$$\|f\|_G = \inf\{ \|\sqrt{\varphi_1^2 + \varphi_2^2}\|_\infty \mid f = \operatorname{div} \varphi, \varphi \cdot n = 0 \text{ on } \partial\Omega \}.$$

We shall need the

Lemma 5 ([5], **Lemma 2.1**) *For every $u \in BV(\Omega)$ and $g \in G(\Omega)$ then*

$$\left| \int_\Omega u(x) g(x) dx \right| \leq TV(u) \|g\|_G.$$

We may now precise the structure of a generic solution.

Theorem 10 *Let us denote by $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$ a solution to problem $(\mathcal{P}_{\lambda, \mu})$ (for any fixed λ and μ) and set $\bar{w} = u_d - \bar{u} - \bar{v}$.*

- i. $\bar{w} = u_d - \bar{u} - \bar{v} \in G(\Omega)$*
- ii. If $d = 2$ and Ω satisfies assumption (2.1), \bar{v} is continuous on $\bar{\Omega}$.*
- iii. If $d = 2$, Ω satisfies assumption (2.1) and $u_d \in BV(\Omega) \cap L^\infty(\Omega)$, then the jump set of \bar{u} is included in the jump set of u_d .*

Proof. (i) This is a direct consequence of Theorem 9. Indeed $\bar{w} \in \lambda\mathcal{K}_1$. Therefore, there exists a sequence $\varphi_n \in C_c^1(\Omega, \mathbb{R}^2)$ with $\|\varphi_n\|_\infty \leq 1$ such that $w_n = \lambda \operatorname{div}(\varphi_n)$ L^2 -converges to \bar{w} . As $\|\varphi_n\|_\infty \leq 1$ on may extract a weak-star subsequence that converges to $\bar{\varphi}$ in $L^\infty(\Omega)$. Therefore $\bar{\varphi} \in L^\infty(\Omega)$ and $\bar{\varphi} \cdot n = 0$ on $\partial\Omega$. So, we get :

$$\forall u \in \mathcal{D}(\Omega) \quad (w_n, u)_{L^2} = \lambda \int_\Omega \operatorname{div} \varphi_n u = -\lambda \int_\Omega \varphi_n \nabla u \rightarrow -\lambda \int_\Omega \bar{\varphi} \nabla u.$$

As $(w_n, u)_{L^2} \rightarrow (\bar{w}, u)_{L^2}$ this gives

$$(\bar{w}, u) = -\lambda \langle \bar{\varphi}, \nabla u \rangle = \lambda \langle \operatorname{div} \bar{\varphi}, u \rangle,$$

in the distributional sense. Therefore $\bar{w} = \operatorname{div}(\lambda \bar{\varphi})$. Moreover, $\bar{\varphi} \cdot n = 0$ on $\partial\Omega$ since φ_n as compact support. This proves that $\bar{w} \in G(\Omega)$.

(ii) Assumption (2.1) yields that $\bar{v} \in BH(\Omega)$ is continuous (Theorem 2, (iii)).

(iii) With (ii), the jump discontinuity set of u_d is the same as the one of $u_d - \bar{v}$. Moreover \bar{u} is a solution to

$$\min_{u \in BV(\Omega)} \frac{1}{2} \|u_d - \bar{v} - u\|^2 + \lambda TV(u),$$

Therefore, following [[18], Theorem 3.3] we get the result.

Remark 4 The point (i) means that \bar{w} is an oscillating function: this is consistent with the fact that we expect \bar{w} to be the noise and/or micro-textures. The continuity of \bar{v} still hold if $d \geq 2$. Assumptions on Ω are slightly different (see [15, 20]).

Corollary 3 Let us denote by $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$ a solution to problem $(\mathcal{P}_{\lambda, \mu})$ (for any fixed λ and μ) and set $\bar{w} = u_d - \bar{u} - \bar{v}$. Then

$$\int_{\Omega} \bar{w}(x) dx = 0 .$$

Proof. This is a direct consequence of proposition 2.1 of [5].

The previous theorem deals with the case where $u_d \in BV(\Omega)$. This is not the case if u_d is noisy for example. In the case where $u_d \notin BV(\Omega)$ we have the following results due to W. Ring [28].

We first consider the 1D case where $\Omega = (a, b)$. Following Proposition 4 of [28], if we assume that

$$\forall \mathcal{U} \text{ open subset of } (a, b) \text{ with positive Lebesgue measure} \quad (\mathcal{H}_1)$$

$$u_d \text{ does not coincide on } \mathcal{U} \text{ with some function } u \in BV(a, b).$$

then $u_d - \bar{v}$ satisfies \mathcal{H}_1 and we get $D_a \bar{u} = 0$ where $D_a u$ is the absolutely continuous part of the measure Du . Let Γ be the support of the singular part of $D\bar{u}$. Therefore \bar{u} is piecewise constant on $(a, b) \setminus \Gamma$.

We have also a similar result for the 2D-case . Assume that

$$\forall \mathcal{U} \text{ open subset of } \Omega, u_d|_{\mathcal{U}} \text{ is not equal not a } BV(\Omega) \text{ function.} \quad (\mathcal{H}_2)$$

then $u_d - \bar{v}$ satisfies (\mathcal{H}_2) as well (since $\bar{v} \in W^{1,1}(\Omega)$). Following Proposition 6 of [28], there is no open subset ω of Ω on which both components $\frac{\partial \bar{u}}{\partial x_i}, i = 1, 2$ have constant, non-zero sign.

4.2 Uniqueness

The functional $\mathcal{F}_{\lambda, \mu}$ is convex but not strictly convex, because of the degenerating direction $u + v = 0$. It is obvious that if (u^*, v^*) is a solution then $(u^* + c, v^* - c)$, where c is constant, is a solution as well. Let us call

$$\mathbf{C}(\Omega) := \{(u, v) \in BV(\Omega) \times BH_0(\Omega) \mid \exists c \in \mathbb{R} \ u = c \text{ and } v = -c \text{ a.e on } \Omega \}. \quad (4.21)$$

The question of uniqueness reduces to uniqueness up to $\mathbf{C}(\Omega)$ functions. In other words, if (u_1, v_1) and (u_2, v_2) are two optimal solutions of $(\mathcal{P}_{\lambda, \mu})$ can we show that $u_2 = u_1 + c$ and $v_2 = v_1 - c$ where c is a constant function? It is still an open problem for the 2D case. We shall discuss this point more precisely in the numerical section. Nevertheless we may give partial results:

Proposition 3 Assume (u_1, v_1) and (u_2, v_2) are two optimal solutions of $(\mathcal{P}_{\lambda, \mu})$. Then there exists $\varphi \in BV(\Omega) \cap BH_0(\Omega)$ such that $u_2 = u_1 - \varphi$ and $v_2 = v_1 + \varphi$.

Proof. Set $u = u_2 - u_1 (\in BV(\Omega))$ and $v = v_2 - v_1 (\in BH_0(\Omega))$. As $u_d - u_1 - v_1 = u_d - u_2 - v_2$ (this is the unique solution of the dual problem), then $u + v = 0$. This yields that $u = -v \in BV(\Omega) \cap BH_0(\Omega)$ and we get the result.

Lemma 6 *The only solutions (\bar{u}, \bar{v}) to $(\mathcal{P}_{\lambda, \mu})$ that satisfies $\bar{u} + \bar{v} = 0$ are functions of $\mathbf{C}(\Omega)$.*

Proof. Assume that $\bar{u} + \bar{v} = 0$ then $\bar{u} \in BH_0(\Omega)$ and $\Phi^2(\bar{v}) = \Phi^2(-\bar{u}) = \Phi^2(\bar{u})$. As $\mathcal{F}_{\lambda, \mu}(\bar{u}, \bar{v}) \leq \mathcal{F}_{\lambda, \mu}(u, v)$, for every $(u, v) \in BV(\Omega) \times BH_0(\Omega)$ this yields

$$\|u_d\|_{L^2(\Omega)}^2 + 2\lambda TV(\bar{u}) + 2\mu TV^2(\bar{u}) \leq \|u_d - u - v\|_{L^2(\Omega)}^2 + 2\lambda TV(u) + 2\mu TV^2(v) .$$

Taking $u = v = 0$ gives

$$\|u_d\|_{L^2(\Omega)}^2 + 2\lambda TV(\bar{u}) + 2\mu TV^2(\bar{u}) \leq \|u_d\|_{L^2(\Omega)}^2 .$$

So we get $\lambda TV(\bar{u}) + \mu TV^2(\bar{u}) = 0$. This implies that $TV(\bar{u}) = 0$ and that \bar{u} is a constant function.

Theorem 11 *Let be (λ, μ) nonnegative real numbers such that $\lambda \geq \|u_d\|_G$ and $\mu \geq C_2\lambda$ where C_2 is the constant of Lemma 1. Then the $\mathbf{C}(\Omega)$ functions are the only solutions to $(\mathcal{P}_{\lambda, \mu})$.*

Proof. Let us assume that $\lambda \geq \|u_d\|_G$ and $\mu \geq C_2\lambda$ where C_2 is the constant of Lemma 1. Lemma 5 gives

$$\forall (u, v) \in BV(\Omega) \times BH_0(\Omega) \quad |(u_d, u + v)_2| \leq \lambda TV(u + v)$$

since $u_d \in L^2(\Omega)$ and $BH_0(\Omega) \subset BV(\Omega)$. Then

$$|(u_d, u + v)_2| \leq \lambda TV(u) + \lambda TV(v) .$$

Lemma 1 gives a constant C_2 (only depending on Ω) such that

$$\forall v \in BH_0(\Omega) \quad TV(v) \leq C_2 TV^2(v) ,$$

so that $\forall (u, v) \in BV(\Omega) \times BH_0(\Omega)$

$$|(u_d, u + v)_2| \leq \lambda TV(u) + C_2 \lambda TV^2(v) \leq \lambda TV(u) + \mu TV^2(v). \quad (4.22)$$

Finally, we get for every $(u, v) \in BV(\Omega) \times BH_0(\Omega)$

$$\frac{1}{2} \|u_d\|^2 = \frac{1}{2} \|u_d - u - v\|^2 - \frac{1}{2} \|u + v\|^2 + (u_d, u + v)_2 \leq \frac{1}{2} \|u_d - u - v\|^2 + \lambda TV(u) + \mu TV^2(v).$$

This means that $\mathcal{F}_{\lambda, \mu}(0, 0) \leq \mathcal{F}_{\lambda, \mu}(u, v)$: so $(0, 0)$ is a solution to $(\mathcal{P}_{\lambda, \mu})$. Let $(\bar{u}, \bar{v}) \in BV(\Omega) \times BH_0(\Omega)$ be another solution to $\mathcal{P}_{\lambda, \mu}$. With proposition 3, we get $\bar{u} + \bar{v} = 0$ and lemma 6 gives $(\bar{u}, \bar{v}) \in \mathbf{C}(\Omega)$. This ends the proof.

Remark 5 *The previous theorem tells that if $\frac{\mu}{\lambda}$ and λ are large enough then the set of solutions is $\mathbf{C}(\Omega)$. In addition, if we impose (for example) that $u \in G$ (that is u has a null mean value), then the unique solution is $(0, 0)$ since $\mathbf{C}(\Omega) \cap (G \times BH_0(\Omega)) = \{(0, 0)\}$.*

Eventually, we have a uniqueness result for the 1D case:

Theorem 12 Assume $n = 1$, $\Omega =]a, b[$ and that u_d satisfies assumption (\mathcal{H}_1) . Then, for every $\lambda > 0, \mu > 0$ problem $(\mathcal{P}_{\lambda, \mu})$ has a unique solution up to a $\mathbf{C}(\Omega)$ function. More precisely, if (u_1, v_1) and (u_2, v_2) are two optimal solutions of $(\mathcal{P}_{\lambda, \mu})$ then $\varphi := u_2 - u_1 = v_2 - v_1$ is a constant function. In particular, problem $(\mathcal{P}_{\lambda, \mu})$ has a unique solution (u^*, v^*) such that u^* has a null mean value.

Proof. Let (u_1, v_1) and (u_2, v_2) be two optimal solutions of $(\mathcal{P}_{\lambda, \mu})$. Then, with proposition 3, there exists $\varphi \in BV(\Omega) \cap BH_0(\Omega)$ such that $\varphi = u_2 - u_1 = v_2 - v_1$. If u_d satisfies (\mathcal{H}_1) then $u_d - v_i, i = 1, 2$ obviously satisfies this assumption as well. As $u_i, i = 1, 2$ is solution to the ROF problem

$$u_i = \operatorname{argmin} \left\{ \frac{1}{2} \|u_d - v_i - u\|^2 + \Phi_\lambda^1(u), u \in L^2(\Omega) \right\}, i = 1, 2.$$

then, u_1, u_2 and φ are piecewise constant on Ω . In addition $\varphi = v_2 - v_1 \in BH(\Omega) \subset W^{1,1}(\Omega)$. This implies that φ is continuous and proves that φ is constant.

5 Conclusion

The mathematical analysis of this problem has been completed by numerical experiments ([9]). The model is well adapted to texture extraction. In the case, where the data is noiseless and/or is not too much textured, the decomposition given par $\lambda \lesssim \mu$ and initialization $u_0 = v_0 = 0$, gives a cartoon part which is piecewise constant as expected. This means that $u = \sum_i u_i \mathbf{1}_{\Gamma_i}$ where $\bigcup_i \Gamma_i$ is the contour set. In this case, the remainder L^2 term is the texture and/or noise. The decomposition is robust with respect to quantification, sampling and is always the same for any $\mu \gg \lambda$, once λ has been chosen.

In the case where the image is highly textured the model provides a two-scale decomposition. The TV part represents the macro-texture and the L^2 part the micro-texture and/or noise. The scaling is tuned via the ratio $\rho = \frac{\lambda}{\mu}$.

From numerical computations, we infer that problem $(\mathcal{P}_{\lambda, \mu})$ has a unique solution (up to $\mathbf{C}(\Omega)$ functions) but the question is still open.

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