



**HAL**  
open science

# Orthogonal polynomials with respect of a class of Fisher-Hartwig symbols.

Philippe Rambour

► **To cite this version:**

Philippe Rambour. Orthogonal polynomials with respect of a class of Fisher-Hartwig symbols.. 2014.  
hal-01002639v2

**HAL Id: hal-01002639**

**<https://hal.science/hal-01002639v2>**

Preprint submitted on 24 Jun 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Orthogonal polynomials with respect of a class of Fisher-Hartwig symbols.

Philippe Rambour\*

## Abstract

**Orthogonal polynomials with respect of a class of Fisher-Hartwig symbols.**

In this paper we give an asymptotic of the coefficients of the orthogonal polynomials on the unit circle, with respect of a weight of type  $f : \theta \mapsto \prod_{1 \leq j \leq M} |1 - e^{i(\theta_j - \theta)}|^{2\alpha_j} c$  with  $\theta_j \in ] - \pi, \pi]$ ,  $-\frac{1}{2} < \alpha_j < \frac{1}{2}$  and  $c$  a sufficiently smooth function.

**Mathematical Subject Classification (2000)**

**Primary** 15B05, 33C45; **Secondary** 33D45, 42C05, 42C10.

**Keywords:** Orthogonal polynomials, Fisher-Hartwig symbols, Gegenbauer polynomials on the unit circle, inverse of Toeplitz matrices.

## 1 Introduction

The study of the orthogonal polynomials on the unit circle is an old and difficult problem (see [27], [28] or [29]). Here we are interested in the asymptotic of the coefficient of the orthogonal polynomial with respect of an Fisher-Hartwig symbol. A Fisher-Hartwig symbol is a function  $\psi$  defined on the unit circle by  $\psi : e^{i\theta} \mapsto \prod_{1 \leq j \leq M} |e^{i(\theta - \theta_j)} - 1|^{2\alpha_j} e^{i\beta_j(\theta - \theta_j - \pi)} c(e^{i\theta})$  with  $0 < \theta, \theta_j < 2\pi$ ,  $-\frac{1}{2} < \Re(\alpha_j)$  and for all  $j, 1 \leq j \leq M$  and where the function  $c$  is assumed sufficiently smooth, continuous, non zero, and have winding number zero (see [1]). Here we consider the class of symbols  $f : e^{i\theta} \mapsto \prod_{1 \leq j \leq M} |e^{i(\theta - \theta_j)} - 1|^{2\alpha_j} c(e^{i\theta})$  with  $-\frac{1}{2} < \alpha_j < \frac{1}{2}$  and  $c$

a regular function sufficiently smooth. It is said that a function  $k$  is a regular function on the unit circle  $\mathbb{T}$  when  $k(\theta) > 0$  for all  $\theta \in \mathbb{T}$  and  $k \in L^1(\mathbb{T})$ . In [19] Martinez-Finkelstein, Mac-Laughin and Saff give the asymptotic behaviour of this polynomials. If  $M = 2$ ,  $\alpha_1 = \alpha_2$  and  $\theta_1 = -\theta_2$ ,  $\theta_1 \neq 0$  we can remark that these polynomials are Gegenbauer polynomials ([3, 2, 7]) (see Corollary 1). The main tool to compute this is the study of the Toeplitz matrix with symbol  $f$ . Given a function  $h$  in  $L^1(\mathbb{T})$  we denote by  $T_N(h)$  the Toeplitz matrix of order  $N$  with symbol  $h$  the  $(N + 1) \times (N + 1)$  matrix defined by

$$(T_N(h))_{i+1, j+1} = \hat{h}(j - i) \quad \forall i, j \quad 0 \leq i, j \leq N$$

where  $\hat{m}(s)$  is the Fourier coefficient of order  $s$  of the function  $m$  (see, for instance [4] and [5]). There is a close connection between Toeplitz matrices and orthogonal polynomials on

---

\*Université de Paris Sud, Bâtiment 425; F-91405 Orsay Cedex; tel : 01 69 15 57 28 ; fax 01 69 15 60 19  
e-mail : philippe.rambour@math.u-psud.fr

the complex unit circle. Indeed the coefficients of the orthogonal polynomial of degree  $N$  with respect of  $h$  are also the coefficients of the last column of  $T_N^{-1}(h)$  except for a normalisation (see [17]). Here we give an asymptotic expansion of the entries  $(T_N(f_\alpha))_{k+1,1}^{-1}$  (Theorem 2). Using the symmetries of the Toeplitz matrix  $T_N(f_\alpha)$ , we deduce from this last result an asymptotic of  $(T_N(f_\alpha))_{N-k+1,N+1}^{-1}$ .

The proof of our main Theorem often refers to results of [26]. In this last work we have treated the case of the symbols  $h_\alpha$  defined by  $\theta \mapsto (1 - \cos\theta)^\alpha c$  which  $-\frac{1}{2} < \alpha \leq \frac{1}{2}$  and the same hypothesis on  $c$  as on  $c_1$ . We have stated the following Theorem which is an important tool in the demonstration of Theorem 2.

**Theorem 1 ([26])** *If  $-\frac{1}{2} < \alpha \leq \frac{1}{2}$ ,  $\alpha \neq 0$  we have for  $c \in A(\mathbb{T}, \frac{3}{2})$  and  $0 < x < 1$*

$$c(1) (T_N(h_\alpha))_{[Nx]+1,1}^{-1} = N^{\alpha-1} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} (1-x)^\alpha + o(N^{\alpha-1}).$$

*uniformly in  $x$  for  $x \in [\delta_1, \delta_2]$  with  $0 < \delta_1 < \delta_2 < 1$ ,*

with  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ , and the definition

**Définition 1** *For all positive real  $\tau$  we denote by  $A(\mathbb{T}, \tau)$  the set*

$$A(\mathbb{T}, \tau) = \{h \in L^2(\mathbb{T}) \mid \sum_{s \in \mathbb{Z}} |s^\tau \hat{h}(s)| < \infty\}$$

This theorem has also been proved for  $\alpha \in \mathbb{N}^*$  in [24] and for  $\alpha \in ]\frac{1}{2}, +\infty[ \setminus \mathbb{N}^*$  in [23].

The results of this paper are of interest in the study of the random matrices (see [10], [9]) and in the analysis of time series. Indeed it is known that the  $n$ -th covariance matrix of a time series is a positive Toeplitz matrix. If  $\phi$  is the symbol of this Toeplitz matrix,  $\phi$  is called the spectral density of the time series. The time series with spectral density is the function  $f : \theta \mapsto |e^{i\theta} - e^{i\theta_0}|^{2\alpha} |e^{i\theta} - e^{-i\theta_0}|^{2\alpha} c$  with  $\theta_0 \in ]0, \pi[$  are also called GARMA processes. More-

over the time series with spectral density is the function  $f : \theta \mapsto \prod_{j=1}^k |e^{i\theta} - e^{i\theta_j}|^{2\alpha_j} |e^{i\theta} - e^{-i\theta_j}|^{2\alpha_j} c$

with  $\theta_0 \in ]0, \pi[$  are  $k$ -factors GARMA processes [12, 11]. For more on this processes we refer the reader to [3, 2, 7], and to [8, 13, 14, 3, 6, 16, 18] for Toeplitz matrices in times series.

On the other hand a random matrice is characterized by the distribution of its eigenvalues. For the case of random unitary matrices an important case is the Dyson generalized circular unitary ensemble the density of the vector  $(\theta_1, \theta_2, \dots, \theta_N)$  of eigenvalue angles is given for a  $N \times N$  matrix is ([22], [21], [20], [30])

$$P_N(\theta_1, \theta_2, \dots, \theta_N) = \prod_{1 \leq j \leq N} f(\theta_j) \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2,$$

where  $f$  is generally a regular function (see [15]), but it also can be a Fisher-Hartwing symbol. For the Dyson generalized circular ensemble the correlation function is written by means of the Christofel-Darboux kernel  $K_N$  ( see [29]) associated to the orthogonal polynomials with

respect of the weight  $f$ .

Lastly it is important to observe that Theorem 2 provides the entries and the trace of the matrix  $T_N^{-1}(f)$  with  $f = \prod_{1 \leq j \leq M} |1 - e^{i(\theta - \theta_j)}|^{2\alpha_j} c$  (see [26], [25]).

Now we have to precise the deep link between the orthogonal polynomials and the inverse of the Toeplitz matrices.

Let  $T_n(f)$  a Toeplitz matrix with symbol  $f$  and  $(\Phi_n)_{n \in \mathbb{N}}$  the orthogonal polynomials with respect to  $f$  ([17]). To have the polynomial used for the prediction theory we put

$$\Phi_n^*(z) = \sum_{k=0}^n \frac{(T_n(f))_{k+1, N+1}^{-1}}{(T_n(f))_{N+1, N+1}^{-1}} z^k, \quad |z| = 1. \quad (1)$$

We define the polynomial  $\Phi_n^*$  (see [27]) as

$$\Phi_n^*(z) = z^n \bar{\Phi}_n\left(\frac{1}{z}\right), \quad (2)$$

that implies, with the symmetry of the Toeplitz matrix

$$\Phi_n^*(z) = \sum_{k=0}^n \frac{(T_n(f))_{k+1, 1}^{-1}}{(T_n(f))_{1, 1}^{-1}} z^k, \quad |z| = 1. \quad (3)$$

The polynomials  $\tilde{\Phi}_n = \Phi_n^* \sqrt{(T_n(f))_{1, 1}^{-1}}$  are often called predictor polynomials. As we can see in the previous formula their coefficients are, up to a normalisation, the entries of the first column of  $T_n(f)^{-1}$ .

## 2 Main results

### 2.1 Main notations

In all the paper we consider the symbol defined by  $f : \theta \mapsto \prod_{1 \leq j \leq M} |1 - e^{i(\theta - \theta_j)}|^{2\alpha_j} c$  where  $c =$

$\left|\frac{P}{Q}\right|^2$  with  $P, Q \in \mathbb{R}[X]$ , without zeros on the united circle,  $-\frac{1}{2} < \alpha_j < \frac{1}{2}$  and  $0 \leq \theta_{j'} \neq \theta_j < 2\pi$ .

We consider also the function  $\tilde{f} : \theta \mapsto \prod_{1 \leq j \leq M} |1 - e^{i(\theta - \theta_j)}|^{2\alpha_j}$ . We have  $c = c_1 \bar{c}_1$  with  $c_1 = \frac{P}{Q}$ .

Obviously  $c_1 \in H^{2+}(\mathbb{T})$  since  $H^{2+}(\mathbb{T}) = \{h \in L^2(\mathbb{T}) | u < 0 \implies \hat{h}(u) = 0\}$ . If  $\chi$  is the function  $\theta \mapsto e^{i\theta}$  and if  $\chi_j = e^{i\theta_j}$  for all  $j, 1 \leq j \leq M$  we put  $g = \prod_{j=1}^M (1 - \bar{\chi}_j \chi)^{\alpha_j} c_1$  and

$\tilde{g} = \prod_{j=1}^M (1 - \bar{\chi}_j \chi)^{\alpha_j}$ . Clearly  $g, \tilde{g} \in H^{2+}(\mathbb{T})$  and  $f = g\bar{g}, \tilde{f} = \tilde{g}\bar{\tilde{g}}$ . Then we denote by  $\beta_k$  the

Fourier coefficient of order  $k$   $g^{-1}$  and by  $\tilde{\beta}_k$  the one of  $\tilde{g}^{-1}$ . Without loss of generality we assume  $\beta_0 = 1$ . Lastly for all real  $\alpha$  in  $]-\frac{1}{2}, \frac{1}{2}[$  we put  $\beta_u^{(\alpha)} = \widehat{(1 - \chi)^{-\alpha}}$ .

## 2.2 Orthogonal polynomials

**Theorem 2** Assume that for all  $j \in \{1, \dots, M\}$  we have  $\theta_j \in ]0, 2\pi[$ ,  $\theta_j \neq \theta_{j'}$  if  $j \neq j'$  and  $-\frac{1}{2} < \alpha_M \leq \dots \leq \alpha_j \leq \dots \alpha_1 < \frac{1}{2}$ . Let  $m$ ,  $1 \leq m \leq M$ , such that  $\alpha_j = \alpha_1$  for all  $j$ ,  $1 \leq j \leq m$ . Then for all integer  $k$ ,  $\frac{k}{N} \rightarrow x$ ,  $0 < x < 1$ , we have the asymptotic

$$\begin{aligned} & \left( T_N^{-1} \left( \prod_{1 \leq j \leq M} |\chi \overline{\chi_j} - 1|^{2\alpha_j c} \right) \right)_{k+1,1} = \\ & = \frac{k^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \left(1 - \frac{k}{N}\right)^{\alpha_1} \sum_{j=1}^m K_j \overline{\chi_j^k} c_1^{-1}(\chi_j) + o(k^{\alpha_1 - 1}) \end{aligned}$$

uniformly in  $k$  for  $x \in [\delta_0, \delta_1]$ ,  $0 < \delta_0 < \delta_1 < 1$ , and with  $K_j = \prod_{h=1}^M (1 - \overline{\chi_h} \chi_j)^{-\alpha_h}$ .

Then the following statement is an obvious consequence of Theorems 2.

**Corollary 1** Let  $\chi_0$  be  $e^{i\theta_0}$  with  $\theta_0 \in ]0, +\pi[$ . With the same hypotheses as in Theorem 2 we have

$$\begin{aligned} & \left( T_N^{-1} (|\chi \overline{\chi_0} - 1|^{2\alpha} |\chi \chi_0 - 1|^{2\alpha} c) \right)_{k+1,1} = \\ & = \frac{K_{\alpha, \theta_0, c_1}}{\Gamma(\alpha)} \cos(k\theta_0 + \omega_{\alpha, \theta_0}) k^{\alpha-1} \left(1 - \frac{k}{N}\right)^\alpha + o(k^{\alpha-1}) \end{aligned}$$

uniformly in  $k$  for  $x \in [\delta_0, \delta_1]$ ,  $0 < \delta_0 < \delta_1 < 1$ , and with

$$\omega_{\alpha, \theta_0} = \alpha \frac{\pi}{2} - \alpha \theta_0 - \arg(c_1(\chi_0)), \quad K_{\alpha, \theta_0, c_1} = 2^{-\alpha+1} (\sin \theta_0)^{-\alpha} \sqrt{c_1^{-1}(\chi_0)}.$$

We can also point out the asymptotic of the coefficients of order  $k$  of the predictor polynomial when  $\frac{k}{N} \rightarrow 0$ .

**Corollary 2** With the same hypotheses as in Theorem 2 we have, if  $\frac{k}{N} \rightarrow 0$  when  $N$  goes to the infinity

$$\left( T_N^{-1} \left( \prod_{1 \leq j \leq M} |\chi \overline{\chi_j} - 1|^{2\alpha_j c} \right) \right)_{k+1,1} = \beta_k + O\left(\frac{1}{N}\right).$$

## 3 Inversion formula

### 3.1 Definitions and notations

Let  $H^{2+}(\mathbb{T})$  and  $H^{2-}(\mathbb{T})$  the two subspaces of  $L^2(\mathbb{T})$  defined by  $H^{2+}(\mathbb{T}) = \{h \in L^2(\mathbb{T}) | u < 0 \implies \hat{h}(u) = 0\}$  and  $H^{2-}(\mathbb{T}) = \{h \in L^2(\mathbb{T}) | u \geq 0 \implies \hat{h}(u) = 0\}$ . We denote by  $\pi_+$  the orthogonal projector on  $H^{2+}(\mathbb{T})$  and  $\pi_-$  the orthogonal projector on  $H^{2-}(\mathbb{T})$ . It is known (see [14]) that if  $f \geq 0$  and  $\ln f \in L^1(\mathbb{T})$  we have  $f = g\bar{g}$  with  $g \in H^{2+}(\mathbb{T})$ . Put  $\Phi_N = \frac{g}{\bar{g}} \chi^{N+1}$ . Let  $H_{\Phi_N}$  and  $H_{\Phi_N}^*$  be the two Hankel operators defined respectively on  $H^{2+}$  and  $H^{2-}$  by

$$H_{\Phi_N} : H^{2+}(\mathbb{T}) \rightarrow H^{2-}(\mathbb{T}), \quad H_{\Phi_N}(\psi) = \pi_-(\Phi_N \psi),$$

and

$$H_{\Phi_N}^* : H^{2-}(\mathbb{T}) \rightarrow H^{2+}(\mathbb{T}), \quad H_{\Phi_N}^*(\psi) = \pi_+(\bar{\Phi}_N \psi).$$

### 3.2 A generalised inversion formula

We have stated in [26] for a precise class of non regular functions which contains  $\prod_{1 \leq j \leq M} |\chi \bar{\chi}_j - 1|^{2\alpha_j} c$  the following lemma (see the appendix of [26] for the demonstration),

**Lemma 1** *Let  $f$  be an almost everywhere positive function on the torus  $\mathbb{T}$  with  $\ln f$ ,  $f$ , and  $\frac{1}{f}$  are in  $\mathbb{L}^1(\mathbb{T})$ . Then  $f = g\bar{g}$  with  $g \in H^{2+}(\mathbb{T})$ . For all trigonometric polynomials  $P$  of degree at most  $N$ , we define  $G_{N,f}(P)$  by*

$$G_{N,f}(P) = \frac{1}{g} \pi_+ \left( \frac{P}{\bar{g}} \right) - \frac{1}{g} \pi_+ \left( \Phi_N \sum_{s=0}^{\infty} (H_{\Phi_N}^* H_{\Phi_N})^s \pi_+ \bar{\Phi}_N \pi_+ \left( \frac{P}{\bar{g}} \right) \right).$$

For all  $P$  we have

- The serie  $\sum_{s=0}^{\infty} (H_{\Phi_N}^* H_{\Phi_N})^s \pi_+ \bar{\Phi}_N \pi_+ \left( \frac{P}{\bar{g}} \right)$  converges in  $L^2(\mathbb{T})$ .
- $\det(T_N(f)) \neq 0$  and

$$(T_N(f))^{-1}(P) = G_{N,f}(P).$$

An obvious corollary of Lemma 1 is

**Corollary 3** *With the hypotheses of Lemma 1 we have*

$$(T_N(f))_{l+1,k+1}^{-1} = \left\langle \pi_+ \left( \frac{\chi^k}{\bar{g}} \right) \middle| \left( \frac{\chi^l}{\bar{g}} \right) \right\rangle - \left\langle \sum_{s=0}^{\infty} (H_{\Phi_N}^* H_{\Phi_N})^s \pi_+ \bar{\Phi}_N \pi_+ \left( \frac{\chi^k}{\bar{g}} \right) \middle| \bar{\Phi}_N \left( \frac{\chi^l}{\bar{g}} \right) \right\rangle.$$

Lastly if  $\gamma_u = \widehat{\frac{g}{f}}(u)$  we obtain as in [26] the formal result

$$\begin{aligned} (H_{\Phi_N}^* H_{\Phi_N})^m \pi_+ \bar{\Phi}_N \pi_+ \left( \frac{\chi^k}{\bar{g}} \right) &= \sum_{u=0}^k \overline{\beta_{u,\theta_0,c_1}^{(\alpha)}} \sum_{n_0=0}^{\infty} \left( \sum_{n_1=1}^{\infty} \bar{\gamma}_{-(N+1+n_1+n_0),\alpha,\theta_0} \right. \\ &\quad \sum_{n_2=0}^{\infty} \gamma_{-(N+1+n_1+n_2),\alpha,\theta_0} \cdots \sum_{n_{2m-1}=1}^{\infty} \bar{\gamma}_{-(N+1+n_{2m-1}+n_{2m-2}),\alpha,\theta_0} \\ &\quad \left. \sum_{n_{2m}=0}^{\infty} \gamma_{-(N+1+n_{2m-1}+n_{2m}),\alpha,\theta_0} \bar{\gamma}_{-(u-(N+1+n_{2m}),\alpha,\theta_0)} \right) \chi^{n_0} \end{aligned}$$

### 3.3 Application to the orthogonal polynomials

With the corollary 3 and the hypothesis on  $\beta_0$  the equality in the corollary 3 becomes, for  $l = 1$ ,

$$(T_N(f))_{1,k+1}^{-1} = \beta_k - \sum_{u=0}^k \beta_{k-u} H_N(u) \quad (4)$$

with

$$H_N(u) = \sum_{m=0}^{+\infty} \left( \sum_{n_0=0}^{\infty} \gamma_{N+1+n_0, \alpha, \theta_0} \left( \sum_{n_1=0}^{\infty} \bar{\gamma}_{-(N+1+n_1+n_0), \alpha, \theta_0} \right. \right. \\ \left. \sum_{n_2=0}^{\infty} \gamma_{-(N+1+n_1+n_2), \alpha, \theta_0} \cdots \sum_{n_{2m-1}=0}^{\infty} \bar{\gamma}_{-(N+1+n_{2m-1}+n_{2m-2}), \alpha, \theta_0} \right. \\ \left. \left. \sum_{n_{2m}=0}^{\infty} \gamma_{-(N+1+n_{2m-1}+n_{2m}), \alpha, \theta_0} \bar{\gamma}_{(u-(N+1+n_{2m}), \alpha, \theta_0)} \right) \right)$$

The remainder of the paper is devoted to the computation of the coefficients  $\beta_k = \widehat{g^{-1}}(k)$ ,  $\gamma_k = \widehat{\frac{g}{g}}$  and  $H_N(u)$  which appears in the inversion formula. For each step we obtain the corresponding terms for the symbol  $2^\alpha(1 - \cos \theta)c$  multiplied by a trigonometric coefficient (see [26]). That provides the expected link with the formulas in Theorem 2.

## 4 Demonstration of Theorem 2

### 4.1 Asymptotic of $\beta_k$

**Property 1** *With the hypothesis of Theorem 2 we have, for sufficiently large  $k$ ,*

$$\beta_k = \frac{k^{\alpha_1-1}}{\Gamma(\alpha_1)} \sum_{j=1}^m K_j \bar{\chi}_j^k c_1^{-1}(\chi_j) + o(k^{\tau_1-1})$$

*uniformly in  $k$ , with  $K_j = \prod_{h=1, h \neq j}^M (1 - \bar{\chi}_h \chi_j)^{-\alpha_h}$ , and  $\tau_1 = \alpha_1$  if  $\alpha_1 > 0$  and  $\tau_1 \leq \alpha_1 - \frac{1}{2}$  else.*

First we have to prove the lemma

**Lemma 2** *With the hypothesis of Theorem 2 we have, for a sufficiently large  $k$ .*

$$\tilde{\beta}_k = \frac{k^{\alpha_1-1}}{\Gamma(\alpha_1)} \sum_{j=1}^m K_j \bar{\chi}_j^k + o(k^{\alpha_1-1})$$

*uniformly in  $k$ , and with  $\tau_1$  as in Property 1*

**Remark 1** *In these two last statements “uniformly in  $k$ ” means*

$$\forall \epsilon > 0, \exists k_\epsilon \in \mathbb{N} \quad \text{such that : } \forall k, k \geq k_\epsilon$$

$$\left| \beta_k - \frac{k^{\alpha_1-1}}{\Gamma(\alpha_1)} \sum_{j=1}^m K_j \bar{\chi}_j^k c_1^{-1}(\chi_j) \right| < \epsilon k^{\tau_1-1}$$

*and*

$$\left| \tilde{\beta}_k - \frac{k^{\alpha_1-1}}{\Gamma(\alpha_1)} \sum_{j=1}^m K_j \bar{\chi}_j^k \right| < \epsilon k^{\tau_1-1}.$$

Proof of Lemma 2: Put  $g_M = \prod_{h=1}^M (1 - \overline{\chi_h \chi})^{-\alpha_h}$  and  $g_{M+1} = (1 - \overline{\chi_{M+1} \chi})^{-\alpha_{M+1}}$ . Assume

$\widehat{g}^{-1}(k) = \frac{k^{\alpha_1-1}}{\Gamma(\alpha_1)} \sum_{j=1}^m K_j \overline{\chi_j}^k$ . Put  $k_0 = k^\gamma$  and  $k_1 = k^{\gamma_1}$  with  $0 < \gamma, \gamma_1 < 1$  and for  $u > k_0, (k-k_1)$

we have

$$(1 - \widehat{\chi})^{-\alpha_{M+1}}(u) = \frac{u^{\alpha_{M+1}-1}}{\Gamma(\alpha_{M+1})} + O(k^{\alpha_{M+1}-2}) \quad (5)$$

uniformly in  $u$  (see [31]). Writting for  $k \geq k_0, \tilde{\beta}_k = S_1 + S_2 + S_3$  with

$$S_1 = \sum_{u=0}^{k_0} \widehat{g_M^{-1}}(u) \widehat{g_{M+1}^{-1}}(k-u), \quad S_2 = \sum_{u=k_0+1}^{k-k_1-1} \widehat{g_M^{-1}}(u) \widehat{g_{M+1}^{-1}}(k-u)$$

and  $S_3 = \sum_{u=k-k_1}^k \widehat{g_M^{-1}}(u) \widehat{g_{M+1}^{-1}}(k-u)$ . The first sum is also

$$\begin{aligned} S_1 &= \sum_{u=0}^{k_0} \widehat{g_M^{-1}}(u) \left( \widehat{g_{M+1}^{-1}}(k-u) - \overline{\chi_{M+1}}^{k-u} \frac{(k-u)^{\alpha_{M+1}-1}}{\Gamma(\alpha_{M+1})} \right) \\ &\quad + \sum_{u=0}^{k_0} \left( \overline{\chi_{M+1}}^u \frac{(k-u)^{\alpha_{M+1}-1}}{\Gamma(\alpha_{M+1})} \right) \overline{\chi_{M+1}}^k \end{aligned}$$

We observe that

$$\begin{aligned} &\sum_{u=0}^{k_0} \widehat{g_M^{-1}}(u) \left( \widehat{g_{M+1}^{-1}}(k-u) - \overline{\chi_{M+1}}^{k-u} \frac{(k-u)^{\alpha_{M+1}-1}}{\Gamma(\alpha_{M+1})} \right) \\ &= \sum_{u=0}^{k_0} O((k-u)^{\alpha_{M+1}-2}) = O((k-u)^{\alpha_{M+1}-1} - k^{\alpha_{M+1}-1}) \end{aligned}$$

Since  $0 \leq \alpha_1 - \alpha_{M+1} + \frac{1}{2}$  we may assume  $\gamma < \alpha_1 - \alpha_{M+1} + \frac{1}{2}$  and we get

$$\left| \sum_{u=0}^{k_0} \widehat{g_M^{-1}}(u) \left( \widehat{g_{M+1}^{-1}}(k-u) - \overline{\chi_{M+1}}^{k-u} \frac{(k-u)^{\alpha_{M+1}-1}}{\Gamma(\alpha_{M+1})} \right) \right| = o(k^{\tau_1-1}). \quad (6)$$

It turns out that

$$\begin{aligned} S_1 &= \sum_{u=0}^{k_0} \widehat{g_M^{-1}}(u) \overline{\chi_{M+1}}^{k-u} \left( \frac{(k-u)^{\alpha_{M+1}-1} - k^{\alpha_{M+1}-1}}{\Gamma(\alpha_{M+1})} \right) \\ &\quad + \sum_{u=0}^{k_0} \widehat{g_M^{-1}}(u) \overline{\chi_{M+1}}^{k-u} \frac{k^{\alpha_{M+1}-1}}{\Gamma(\alpha_{M+1})} + o(k^{\tau_1-1}) \end{aligned}$$

with, for  $\gamma < \frac{\alpha_1 - \alpha_{M+1} + 1}{2}$

$$\begin{aligned} \left| \sum_{u=0}^{k_0} \widehat{g_M^{-1}}(u) \left( \overline{\chi_{M+1}}^{k-u} \frac{(k-u)^{\alpha_{M+1}-1} - k^{\alpha_{M+1}-1}}{\Gamma(\alpha_{M+1})} \right) \right| &= O(k^{\alpha_{M+1}-2}) \sum_{u=0}^{k_0} u \\ &= O(k^{\alpha_{M+1}-2+\gamma} - 2) = o(k^{\tau_1-1}). \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{u=0}^{k_0} \widehat{g_M^{-1}}(u) \chi_{M+1}^u &= \sum_{u=0}^{+\infty} \widehat{g_M^{-1}}(u) \chi_{M+1}^u \\ &\quad - \sum_{u=k_0+1}^{\infty} \widehat{g_M^{-1}}(u) \chi_{M+1}^u. \end{aligned}$$

Using the appendix we get  $\left| \sum_{u=k_0+1}^{\infty} \widehat{g_M^{-1}}(u) \chi_{M+1}^u \right| = O(k_0^{\alpha_1-1})$ , and  $k^{\alpha_{M+1}-1} k_0^{\alpha_1-1} = o(k^{\tau_1-1})$  since  $\alpha_{M+1} - 1 + \gamma(\alpha_1 - 1) < \alpha_1 - \frac{3}{2}$ . Hence

$$S_1 = \frac{k^{\alpha_{M+1}-1}}{\Gamma(\alpha_{M+1})} \overline{\chi_{M+1}}^k \left( \prod_{j=1}^M (1 - \chi_{M+1} \overline{\chi_j})^{-\alpha_j} \right) + o(k^{\tau_1-1})$$

uniformly in  $k$ . Identically we get

$$S_3 = \frac{k^{\alpha_1-1}}{\Gamma(\alpha_1)} \sum_{j=0}^m \overline{\chi_j}^k \left( \prod_{h=1, h \neq j}^{M+1} (1 - \chi_j \overline{\chi_h})^{-\alpha_h} \right) + o(k^{\tau_1-1}),$$

uniformly in  $k$ . Finally we can remark that the appendix provides

$S_2 = O(\max(k_0^{\alpha_1-1} k^{\alpha_{M+1}-1}, k_1^{\alpha_{M+1}-1} k^{\alpha_1-1}) = o(k^{\tau_1-1})$  uniformly in  $k$ . We have obtain

1. for  $\alpha_{M+1} < \alpha_1$ ,

$$\beta_k = \frac{k^{\alpha_1}}{\Gamma(\alpha_1)} \sum_{j=0}^m \overline{\chi_j}^k \left( \prod_{h=1, h \neq j}^{M+1} (1 - \chi_j \overline{\chi_h})^{-\alpha_h} \right) + o(k^{\alpha_1-1}),$$

2. for  $\alpha_{M+1} = \alpha_1$

$$\begin{aligned} \beta_k &= \frac{k^{\alpha_1}}{\Gamma(\alpha_1)} \left( \sum_{j=0}^m \overline{\chi_j}^k \left( \prod_{h=1, h \neq j}^{M+1} (1 - \chi_j \overline{\chi_h})^{-\alpha_h} \right) \right. \\ &\quad \left. + \overline{\chi_{M+1}}^k \left( \prod_{j=1}^M (1 - \chi_{M+1} \overline{\chi_j})^{-\alpha_j} \right) \right) + o(k^{\alpha_1-1}). \end{aligned}$$

that ends the proof of the lemma.  $\square$

To ends the proof of the property we need to obtain  $\beta_k$  from  $\tilde{\beta}_k$  for a sufficiently large  $k$ . We can remark that a similar case has been treated in [23] for the function  $((1 - \chi)^\alpha c_1)^{-1}$ . Here we develop the same idea than in this last paper. Let  $c_m$  the coefficient of Fourier of order  $m$  of the function  $c_1^{-1}$ . The hypotheses on  $c_1$  imply that  $c_1^{-1}$  is in  $A(\mathbb{T}, p) = \{h \in$

$L^2(\mathbb{T})|\sum_{u \in \mathbb{Z}} u^p |\widehat{h}(u)| < \infty\}$  for all positive integer  $p$  ( because  $c_1^{-1} \in C^\infty(\mathbb{T})$  and for all positive integer  $|\widehat{h^{(p)}}| = \frac{1}{p}|\widehat{h}|$  ). We have,  $\beta_k = \sum_{s=0}^k \tilde{\beta}_s c_{k-s}$ . For  $0 < \nu < 1$  we can write

$$\sum_{s=0}^k \tilde{\beta}_s c_{k-s} = \sum_{s=0}^{k-k^\nu} \tilde{\beta}_s c_{k-s} + \sum_{s=k-k^\nu+1}^k \tilde{\beta}_s c_{k-s}.$$

Lemma 2 provides, with the same notations,

$$\sum_{s=k-k^\nu+1}^k \tilde{\beta}_s c_{k-s} = \sum_{j=0}^m K_j \sum_{s=k-k^\nu+1}^k \frac{s^{\alpha_1}}{\Gamma(\alpha_1)} \bar{\chi}_j^s c_{k-s} + R$$

with  $|R| = o(m^{\tau_1-1}) \sum_{s=k-k^\nu+1}^k |c_{k-s}|$ . Since  $\sum_{s \in \mathbb{Z}} |c_s| < \infty$ , we have

$$\sum_{s=k-k^\nu+1}^k \tilde{\beta}_s c_{k-s} = \sum_{j=0}^m K_j \sum_{s=k-k^\nu+1}^k \frac{s^{\alpha_1-1}}{\Gamma(\alpha_1)} \bar{\chi}_j^s c_{k-s} + o(m^{\tau_1-1}).$$

We have

$$\left| \sum_{s=k-k^\nu}^k (s^{\alpha_1-1} - k^{\alpha_1-1}) c_{k-s} \right| \leq (1 - \alpha) O(k^{\nu+\alpha_1-2}) \sum_{s=k-k^\nu+1}^m |c_{k-s}|. \quad (7)$$

and the convergence of  $(c_s)$  implies

$$\begin{aligned} & \sum_{s=k-k^\nu}^k \frac{s^{\alpha_1-1} - k^{\alpha_1-1} + k^{\alpha_1-1}}{\Gamma(\alpha_1)} \bar{\chi}_j^s c_{k-s} \\ &= \frac{k^{\alpha_1-1}}{\Gamma(\alpha_1)} \sum_{s=k-k^\nu}^k \bar{\chi}_j^s c_{k-s} + O(k^{\alpha_1-2+\nu}) \\ & \frac{k^{\alpha_1-1}}{\Gamma(\alpha_1)} \bar{\chi}_j^k \left( \sum_{v=0}^{\infty} \bar{\chi}_j^v c_v - \sum_{v=k^\nu+1}^{\infty} \bar{\chi}_j^v c_v \right) \end{aligned}$$

For all positive integer  $p$  the function  $c_1 \in A(p, \mathbb{T})$ . Hence one can prove first

$$\left| \sum_{v=k^\nu+1}^{\infty} e^{+iv\theta} c_v \right| \leq (k^{-p\nu}) \sum_{s \in \mathbb{Z}} |c_s| \quad (8)$$

and secondly

$$\sum_{s=k-k^\nu}^k \bar{\chi}_j^s c_{k-s} = \bar{\chi}_j^k c_1^{-1}(\bar{\chi}_j) + O(k^{-p\nu}).$$

On the other hand we have (always because  $c_1^{-1}$  in  $A(\mathbb{T}, p)$ )

$$\left| \sum_{s=0}^{k-k^\nu} \tilde{\beta}_s c_{k-s} \right| \leq \frac{1}{k^{p\nu}} \sum_{v \in \mathbb{Z}} v^p |c_v| \max_{s \in \mathbb{N}} (|\tilde{\beta}_s|). \quad (9)$$

For a good choice of  $p$  and  $\nu$  we obtain the expected formula for  $\beta_k$ . The uniformity is provided by Lemma 2 and the equation (7), (8) and (9).

## 4.2 Estimation of the Fourier coefficients of $\frac{g}{g}$ .

Let  $\gamma_k$  be  $\widehat{\frac{g}{g}}(k)$  and  $\tilde{\gamma}_k$  be  $\widehat{\frac{\tilde{g}}{g}}(k)$ .

**Property 2** *With the hypothesis of Theorem 2 we have, for all integer  $k \geq 0$  sufficiently large*

$$\gamma_{-k} = \frac{1}{k} \sum_{j=1}^M \frac{\sin(\pi\alpha_j)}{\pi} H_j \frac{c_1(\chi_j)}{c_1(\bar{\chi}_j)} \bar{\chi}_j^k + o(k^{\min(\alpha_1-1, -1)})$$

$$\text{uniformly in } k \text{ and with } H_j = \prod_{j=1, h \neq j}^M \left( \frac{\bar{\chi}_h \chi_j - 1}{\chi_h \bar{\chi}_j - 1} \right)^{\alpha_j}.$$

First we have to prove the lemma

**Lemma 3** *With the hypothesis of Theorem 2 we have, for all integer  $k \geq 0$  sufficiently large*

$$\tilde{\gamma}_{-k} = \frac{1}{k} \sum_{j=1}^M \frac{\sin(\pi\alpha_j)}{\pi} H_j \bar{\chi}_j^k + o(k^{\min(\alpha_1-1, -1)})$$

*uniformly in  $k$ .*

**Proof of Lemma 3:** In all this proof we denote respectively by  $\gamma_{1,k}, \gamma_{2,k}$  the Fourier coefficient of order  $k$  of  $\prod_{j=1}^{M-1} \left( \frac{\bar{\chi}_h \chi - 1}{\chi_h \bar{\chi} - 1} \right)^{\alpha_j}$  and  $\left( \frac{\chi \bar{\chi}_M - 1}{\bar{\chi} \chi_M - 1} \right)^{\alpha_M}$ . Clearly  $\gamma_{2,k} = (\bar{\chi}_M)^k \frac{\sin \pi \alpha_M}{\pi} \frac{1}{k + \alpha_M} =$

$$(\bar{\chi}_M)^k \gamma_{3,k}. \text{ Assume } k \geq 0 \text{ and } \gamma_{1,k} = \frac{1}{k} \sum_{j=1}^M \frac{\sin(\pi\alpha_j)}{\pi} H'_j \bar{\chi}_j^k + o\left(\frac{1}{k}\right) \text{ with } H'_j = \prod_{j=1, h \neq j}^{M-1} \left( \frac{\bar{\chi}_h \chi_j - 1}{\chi_h \bar{\chi}_j - 1} \right)^{\alpha_j}.$$

Assume also  $k \geq 0$ . We have  $\gamma_{-k} = \sum_{v+u=-k} \gamma_{1,u} \gamma_{2,v}$ . For  $k_0 = k^\tau$ ,  $0 < \tau < 1$  we can split this sum into

$$\begin{aligned} & \sum_{u < -k - k_0} \gamma_{1,u} \gamma_{2,-k-u} + \sum_{u = -k - k_0}^{-k+k_0} \gamma_{1,u} \gamma_{2,-k-u} + \sum_{u = -k+k_0+1}^{-k_0-1} \gamma_{1,u} \gamma_{2,-k-u} \\ & + \sum_{u = -k_0}^{k_0} \gamma_{1,u} \gamma_{2,k-u} + \sum_{u > k_0} \gamma_{1,u} \gamma_{2,-k-u}. \end{aligned}$$

Write

$$\sum_{u = -k_0}^{k_0} \gamma_{1,u} \gamma_{2,-k-u} = \sum_{u = -k_0}^{k_0} \gamma_{1,u} (\bar{\chi}_M)^{k+u} (\gamma_{3,-k-u} - \gamma_{3,-k} + \gamma_{3,-k}).$$

Since

$$\sum_{u = -k_0}^{k_0} \gamma_{1,u} (\bar{\chi}_M)^{k+u} (\gamma_{3,-k-u} - \gamma_{3,-k}) = \frac{\sin(\pi\alpha)}{\pi} \sum_{u = -k_0}^{k_0} \gamma_{1,u} (\bar{\chi}_M)^{k+u} \frac{-u}{(k+u+\alpha)(k+\alpha)} \quad (10)$$

it follows that (always with the appendix)

$$\begin{aligned}
\sum_{u=-k_0}^{k_0} \gamma_{1,u} \gamma_{2,-k-u} &= \gamma_{3,-k} \sum_{u=-k_0}^{k_0} \gamma_{1,u} (\bar{\chi}_M)^{-k-u} + O(k_0 k^{-2}) \\
&= \gamma_{3,-k} (\chi_M)^k \sum_{|u| \geq k_0} \gamma_{1u} \chi_M^u + O(k_0 k^{-2}) \\
&= \gamma_{3,-k} (\chi_M)^k \prod_{j=1}^{M-1} \left( \frac{\bar{\chi}_h \chi_M - 1}{\chi_h \bar{\chi}_M - 1} \right)^{\alpha_j} + O((k_0 k)^{-1}) + O(k_0 k^{-2}) \\
&= \gamma_{3,-k} (\chi_M)^k \prod_{j=1}^{M-1} \left( \frac{\bar{\chi}_h \chi_M - 1}{\chi_h \bar{\chi}_M - 1} \right)^{\alpha_j} + O(k^{\tau-2}).
\end{aligned}$$

In the same way we have

$$\sum_{u=-k-k_0}^{-k+k_0} \gamma_{1,u} \gamma_{2,k-u} = \sum_{j=1}^M \frac{\sin \pi \alpha_j}{\pi} H_j' \bar{\chi}_j^{-k} \left( \frac{\bar{\chi}_M \chi_j - 1}{\chi_M \bar{\chi}_j - 1} \right)^{\alpha_M} O(k^{\tau-2}).$$

Now using the appendix it is easy to see that

$$\sum_{u < -k-k_0} \gamma_{1,u} \gamma_{2,-k-u} \leq M_1 (k_0 k)^{-1} \tag{11}$$

$$\sum_{u > k_0} \gamma_{1,u} \gamma_{2,-k-u} \leq M_2 (k_0 k)^{-1} \tag{12}$$

with  $M_1$  and  $M_2$  no depending from  $k$ . For the sum  $S = \sum_{u=-k+k_0+1}^{-k_0-1} \gamma_{1,u} \gamma_{2,-k-u}$  we remark, using an Abel summation, that

$$|S| \leq M_3 (k_0 k)^{-1} + \sum_{u=-k+k_0+1}^{-k_0-1} \left| \frac{1}{(u+\alpha)(k-u+\alpha)} - \frac{1}{(u+1+\alpha)(k-u-1+\alpha)} \right|$$

with  $M_3$  no depending from  $k$ . Consequently

$$|S| \leq M_3 (k_0 k)^{-1} + \sum_{u=-k+k_0+1}^{-k_0-1} \frac{k-2u}{(k-u)^2 u^2}. \tag{13}$$

Then Euler and Mac-Laurin formula provides the upper bound

$$|S| \leq O((k_0 k)^{-1}) + \int_{-k+k_0+1}^{-k_0-1} \frac{k-2u}{(k-u)^2 u^2} du.$$

Since

$$\int_{-k+k_0+1}^{-k_0-1} \frac{k-2u}{(k-u)^2 u^2} du \leq \frac{3k}{(k+k_0)^2} \int_{-k+k_0+1}^{-k_0-1} \frac{1}{u^2} du$$

we get finally

$$\sum_{u < -k+k_0+1}^{-k_0-1} \gamma_{1,u} \gamma_{2,k-u} = O((k_0 k)^{-1})$$

and

$$\tilde{\gamma}_{-k} = \frac{1}{k} \sum_{j=1}^M \frac{\sin(\pi \alpha_j)}{\pi} H_j \overline{\chi_j^k} + O((k_0 k)^{-1}) + O(k^{\alpha-2}).$$

Then with a good choice of  $\tau$  we obtain the expected formula. The uniformity is a direct consequence of the equations (10), (11), (12), (13).  $\square$

The rest of the proof of Lemma 3 can be treated as the end of the proof of property 1.

### 4.3 Expression of $(T_N^{-1}(f))_{k+1,1}$ .

First we have to prove the next lemma

**Lemma 4** For  $\alpha \in ]-\frac{1}{2}, \frac{1}{2}[$  we have a function  $F_{N,\alpha} \in C^1[0, \delta]$  for all  $\delta \in ]0, 1[$ , satisfying the properties

i)

$$\forall z \in [0, \delta[ \quad |F_{N,\alpha}(z)| \leq K_0(1 + |\ln(1 - z + \frac{1+\alpha}{N})|)$$

where  $K_0$  is a constant no depending from  $N$ .

ii)  $F_N$  and  $F'_N$  have a modulus of continuity no depending from  $N$ .

iii) with the notations of Theorem 2 we have

$$\begin{aligned} (T_N^{-1}(f))_{k+1,1} &= \\ &= \beta_k - \frac{1}{N} \sum_{u=0}^k \beta_{k-u} \left( \sum_{j=1}^M F_{N,\alpha_j} \left( \frac{u}{N} \right) \overline{\chi_j^u} \right) + R_{N,\alpha_1} \end{aligned}$$

uniformly in  $k$ ,  $0 \leq k \leq N$ , with

$$R_{N,\alpha_1} = o \left( N^{-1} \sum_{u=0}^k \beta_{k-u} \left( \sum_{j=1}^M F_{N,\alpha_j} \left( \frac{u}{N} \right) \overline{\chi_j^u} \right) \right) \quad \text{if } \alpha > 0$$

and

$$R_{N,\alpha_1} = o \left( N^{\alpha_1-1} \sum_{u=0}^k \beta_{k-u} \left( \sum_{j=1}^M F_{N,\alpha_j} \left( \frac{u}{N} \right) \overline{\chi_j^u} \right) \right) \quad \text{if } \alpha < 0$$

**Remark 2 (Proof of the corollary 2)** for  $\frac{k}{N} \rightarrow 0$  Lemma 4 and the continuity of the function  $F_\alpha$  provide

$$(T_N^{-1}(f))_{k,1} = \beta_k + \frac{1}{N} \sum_{u=0}^k \beta_{k-u} \left( \sum_{j=0}^M F_{N,\alpha_j}(0) \overline{\chi_j^u} \right) (1 + o(1)).$$

Since  $F_{N,\alpha}(0) = \alpha^2 + o(1)$  (see [26]) the hypothesis  $\beta_0 = 1$  and the formula (4) imply the corollary.

**Proof of the lemma 4:** As for [26] and using the inversion formula and Corollary 3 we have to consider the sums

$$H_{p,N}(u) = \left( \sum_{n_0=0}^{\infty} \gamma_{-(N+1+n_0)} \sum_{n_1=0}^{\infty} \overline{\gamma_{-(N+1+n_1+n_0)}} \sum_{n_2=0}^{\infty} \gamma_{-(N+1+n_1+n_2)} \times \cdots \right. \\ \left. \times \sum_{n_{2m-1}=0}^{\infty} \overline{\gamma_{-(N+1+n_{2p-2}+n_{2p-1})}} \sum_{n_{2p}=0}^{\infty} \gamma_{-(N+1+n_{2m-1}+n_{2m})} \overline{\gamma_{u-(N+1+n_{2p})}} \right).$$

If

$$S_{2p} = \sum_{n_{2p}=0}^{\infty} \gamma_{-(N+1+n_{2p-1}+n_{2p})} \overline{\gamma_{u-(N+1+n_{2p})}}$$

we can write, following the previous Lemma,  $S_{2p} = S_{2p,0} + S_{2p,1} + R_{2p,\alpha_1}$  with

$$S_{2p,0} = \sum_{n_{2p}=0}^{\infty} \left( \sum_{j=0}^M \left( \frac{\sin \pi \alpha_j}{\pi} \right)^2 \chi_j^{n_{2p-1}+u} \right. \\ \left. \frac{1}{N+1+n_{2p-1}+n_{2p}+\alpha_j} \frac{1}{N+1+n_{2p}-u+\alpha_j} \right)$$

$$S_{2p,1} = \sum_{n_{2p}=0}^{\infty} \left( \sum_{jj'=0j \neq j'}^M H_j \overline{H(j')} \frac{\sin \pi \alpha_j}{\pi} \frac{\sin \pi \alpha_{j'}}{\pi} \frac{c_1(\chi^j)}{c_1(\chi^j)} \frac{c_1(\chi^{j'})}{c_1(\chi^{j'})} \right. \\ \left. \overline{\chi_j^{N+1+n_{2p}+n_{2p-1}}} \chi_{j'}^{N+1+n_{2p}-u} \frac{1}{N+1+n_{2p-1}+n_{2p}+\alpha_j} \frac{1}{N+1+n_{2p}-u+\alpha_{j'}} \right)$$

Let us study the order of  $S_{2p,1}$ . To do this we have to evaluate the order of the expression

$$\sum_{j=0}^H \chi_0^j \frac{1}{N+1+n_{2m-1}+j+\alpha} \frac{1}{N+1+j-u+\alpha}$$

where  $H$  goes to the infinity and  $N = o(H)$ . As for the previous proofs it is clear that this sum is bounded by

$$\sum_{j=0}^M \left| \frac{1}{N+2+n_{2p-1}+j} \frac{1}{N+2+j-u} - \frac{1}{N+1+n_{2p-1}+j} \frac{1}{N+1+j-u} \right|$$

Obviously

$$\left| \frac{1}{N+2+n_{2p-1}+j} \frac{1}{N+2+j-u} - \frac{1}{N+1+n_{2p-1}+j} \frac{1}{N+1+j-u} \right| \\ \leq \left| \frac{2N+2+2j+n_{2p-1}-u}{(N+1+n_{2p-1}+j)^2(N+1+j-u)^2} \right|$$

and

$$\begin{aligned}
& \left| \frac{2N + 2 + 2j + n_{2p-1} - u}{(N + 1 + n_{2p-1} + j)^2 (N + 1 + j - u)^2} \right| \\
&= \left| \frac{1}{N + 1 + j + n_{2p-1}} + \frac{1}{N + 1 + j - u} \right| \frac{1}{(N + 1 + j + n_{2p-1})(N + 1 + j - u)} \\
&\leq \frac{1}{N} \frac{1}{(N + 1 + j + n_{2p-1})(N + 1 + j - u)}.
\end{aligned}$$

In the other hand we have, for  $\alpha_1 \in ]0, \frac{1}{2}[$

$$R_{2p, \alpha_1} = o \left( \sum_{j=0}^{\infty} \frac{1}{N + 1 + n_{2p-1} + n_{2p}} \frac{1}{N + 1 + n_{2p} - u} \right)$$

and for  $\alpha_1 \in ]-\frac{1}{2}, 0[$ .

$$R_{2p, \alpha_1} = o \left( N^{\alpha_1} \sum_{j=0}^{\infty} \frac{1}{N + 1 + n_{2p-1} + n_{2p}} \frac{1}{N + 1 + n_{2p} - u} \right).$$

Hence we can write

$$S_{2p} = S'_{2p} \left( \sum_{j=0}^M \frac{\sin \pi \alpha_j}{\pi} \frac{1}{\chi_j^{n_{2p-1} + u}} + r_m \right),$$

with

$$S'_{2p} = \sum_{j=0}^{+\infty} \frac{1}{N + 1 + n_{2m-1} + n_{2m}} \frac{1}{N + 1 + n_{2m} - u}.$$

and

$$\begin{cases} r_{m, \alpha_1} = o(1) & \text{if } \alpha \in ]0, \frac{1}{2}[ \\ r_{m, \alpha_1} = o(N^{\alpha_1}) & \text{if } \alpha \in ]-\frac{1}{2}, 0[. \end{cases}$$

For  $z \in [0, 1]$  we define  $F_{p, N}(z)$  by

$$\begin{aligned}
F_{p, N}(z) &= \sum_{n_0=0}^{\infty} \frac{1}{N + 1 + n_0} \sum_{n_1=0}^{\infty} \frac{1}{N + 1 + w_1 + w_0} \times \dots \\
&\times \sum_{n_{2p-1}=0}^{\infty} \frac{1}{N + 1 + n_{2p-2} + n_{2p-1}} \\
&\times \sum_{n_{2p}=0}^{\infty} \frac{1}{N + 1 + n_{2p-1} + n_{2p}} \frac{1}{1 + \frac{1}{N} + \frac{n_{2p}}{N} - z}.
\end{aligned}$$

Repeating the same idea as previously for the sums on  $n_{2m-1}, \dots, n_0$  we finally obtain

$$H_{p, N}(u) = \frac{1}{N} \left( \sum_{j=0}^M \left( \frac{\sin(\pi \alpha_j)}{\pi} \right)^{2p+2} \frac{1}{\chi_j^u} \right) F_{m, N}\left(\frac{u}{N}\right) + R_{N, \alpha_1}.$$

with  $R_{N,\alpha_1}$  as announced previously.

For all  $\alpha \in ]-\frac{1}{2}, \frac{1}{2}[$  we established in [26] the continuity of the function  $F_{p,N}$  and the uniform convergence in  $[0, 1]$  of the sequence  $\sum_{p=0}^{\infty} \left(\frac{\sin(\pi\alpha)}{\pi}\right)^{2p} F_{p,N}(z)$ . For  $\alpha \in ]-\frac{1}{2}, \frac{1}{2}[$  let us denote

by  $F_{N,\alpha}(z)$  the sum  $\sum_{m=0}^{+\infty} \left(\frac{\sin \pi\alpha}{\pi}\right)^{2m} F_{m,N}(z)$ . The function  $F_{N,\alpha}$  is defined, continuous and derivable on  $[0, 1[$  (see [26] Lemma 4). Moreover for all  $z \in [0, \delta]$ ,  $0 < \delta < 1$  we have the upper bounds

$$\frac{1}{1 + \frac{1}{N} + \frac{n_{2p}}{N} - z} \leq \frac{1}{1 + \frac{1}{N} - \delta}.$$

Hence

$$\left(\frac{1 + \frac{1}{N} - \delta}{1 + \frac{1}{N} + \frac{n_{2p}}{N} - z}\right)^2 \leq \frac{1 + \frac{1}{N} - \delta}{1 + \frac{1}{N} + \frac{n_{2p}}{N} - z}$$

and

$$\left(\frac{1}{1 + \frac{1}{N} + \frac{n_{2p}}{N} - z}\right)^2 \leq \frac{1}{1 + \frac{1}{N} - \delta} \frac{1}{1 + \frac{1}{N} + \frac{n_{2p}}{N} - z}.$$

These last inequalities and the proof of Lemma 4 in [26] provide that  $F_{N,\alpha}$  is in  $C^1[0, 1[$ .

Always in [26] we have obtained that, for all  $z$  in  $[0, 1]$ ,

$$\left|F_{N,\alpha}(z)\right| \leq K_0 \left(1 + \left|\ln\left(1 - z + \frac{1+\alpha}{N}\right)\right|\right) \quad (14)$$

where  $K_0$  is a constant no depending from  $N$ .

Now we have to prove the point ii) of the statement. For  $z, z' \in [0, \delta]$

$$\begin{aligned} & \left| \frac{z - z'}{(1 + \frac{1+\alpha}{N} + \frac{n_{2m}}{N} - z)(1 + \frac{1+\alpha}{N} + \frac{n_{2m}}{N} - z')} \right| \\ & \leq \frac{|z - z'|}{1 - \delta} \frac{1}{1 + \frac{1+\alpha}{N} + \frac{n_{2m}}{N} - \delta} \end{aligned}$$

that implies, with (14)

$$|F_{N,\alpha}(z) - F_{N,\alpha}(z')| \leq |z - z'| \frac{K_0 \left(1 + \left|\ln\left(1 - \delta + \frac{1+\alpha}{N}\right)\right|\right)}{1 - \delta}. \quad (15)$$

In the same way we have

$$\begin{aligned} & |z - z'| \left| \frac{\left((1 + \frac{1+\alpha}{N} + \frac{n_{2m}}{N} - z) + (1 + \frac{1+\alpha}{N} + \frac{n_{2m}}{N} - z')\right)}{(1 + \frac{1+\alpha}{N} + \frac{n_{2m}}{N} - z)^2 (1 + \frac{1+\alpha}{N} + \frac{n_{2m}}{N} - z')^2} \right| \\ & \leq 2|z - z'| \frac{1}{(1 - \delta)^2} \frac{1}{1 + \frac{1+\alpha}{N} + \frac{n_{2m}}{N} - \delta} \end{aligned}$$

and always with the inequality (14)

$$|F'_{N,\alpha}(z) - F'_{N,\alpha}(z')| \leq 2|z - z'| \frac{K_0 \left(1 + \left|\ln\left(1 - \delta + \frac{1+\alpha}{N}\right)\right|\right)}{(1 - \delta)^2}. \quad (16)$$

Using (15) and (16) we get the point *ii*).

To achieve the proof we have to remark that the uniformity in  $k$  in the point *iii*) is a direct consequence of Property 2.  $\square$

We have now to state the following lemma.

**Lemma 5** For  $\frac{k}{N} \rightarrow x$ ,  $0 < x < 1$  we have, with the notations of Theorem 2,

$$\sum_{u=0}^k \beta_{k-u} \left( \sum_{j=1}^M F_{\alpha_j, N} \left( \frac{u}{N} \right) \overline{\chi_j^u} \right) = \left( \sum_{j=1}^m \overline{\chi_j^u} c_1^{-1}(\chi_j) K_j \right) \sum_{u=0}^k \beta_{k-u}^{(\alpha_1)} F_{N, \alpha_1} \left( \frac{u}{N} \right) + o(k^{\alpha_1-1}),$$

uniformly in  $k$  for  $x$  in all compact of  $]0, 1[$  and for  $K_j$  as in Property 1.

**Remark 3** This Lemma and Lemma 4 imply the equality

$$T_N^{-1}(f)_{k+1,1} = \left( \sum_{j=1}^m \overline{\chi_j^u} c_1^{-1}(\chi_j) K_j \right) T_N^{-1}(|1 - \chi|^{2\alpha_1})_{k+1,1} + o(k^{\alpha_1-1})$$

with (see [26] Lemma 3)

$$T_N^{-1}(|1 - \chi|^{2\alpha_1})_{k+1,1} = \left( \beta_k^{(\alpha)} - \frac{1}{N} \sum_{u=0}^k \beta_{k-u}^{(\alpha_1)} F_{N, \alpha_1} \left( \frac{u}{N} \right) \right).$$

**Proof of lemma 5:** With our notation assume  $x \in [0, \delta]$ ,  $0 < \delta < 1$ . Put  $k_0 = N^\gamma$  with  $\gamma \in ]\max(\frac{\alpha_1}{\tau_1}, \frac{-\alpha_1}{1-\alpha_1}), 1[$  if  $\alpha_1 < 0$ , and  $\gamma \in ]0, 1[$  if  $\alpha_1 > 0$ . For all integer  $h$ ,  $0 \leq h \leq M$

we can split the sum  $\sum_{u=0}^k \beta_{k-u} F_{N, \alpha_h} \left( \frac{u}{N} \right) \overline{\chi_h^u}$  into  $S = \sum_{u=k-k_0}^k \beta_{k-u} F_{N, \alpha_h} \left( \frac{u}{N} \right) \overline{\chi_h^u}$  and  $S' =$

$\sum_{u=0}^{k-k_0} \beta_{k-u} F_{N, \alpha_h} \left( \frac{u}{N} \right) \overline{\chi_h^u}$ . First we assume that  $0 \leq h \leq m$ . Then  $\alpha_1 = \alpha_h$  and Property 1 and the assumption on  $\tau_1$  show that

$$\begin{aligned} S' &= \sum_{u=0}^{k-k_0} \left( \sum_{j=1}^m K_j \overline{\chi_j^{k-u}} c_1^{-1}(\chi_j) \right) \frac{(k-u)^{\alpha_1-1}}{\Gamma(\alpha_1)} F_{N, \alpha_h} \left( \frac{u}{N} \right) \overline{\chi_h^u} \\ &= K_h \overline{\chi_h^k} c_1^{-1}(\chi_h) \sum_{u=0}^{k-k_0} \frac{(k-u)^{\alpha_1-1}}{\Gamma(\alpha_1)} F_{N, \alpha_1} \left( \frac{u}{N} \right) \\ &\quad + \left( \sum_{j=1, j \neq h}^m K_j \overline{\chi_j^k} c_1^{-1}(\chi_j) \right) \sum_{u=0}^{k-k_0} \frac{(k-u)^{\alpha_1-1}}{\Gamma(\alpha_1)} F_{N, \alpha_1} \left( \frac{u}{N} \right) (\overline{\chi_h} \chi_j)^u + o(k^{\alpha_1}) \end{aligned}$$

uniformly in  $k$  (5). Then an Abel summation provides that the quantity

$$\left| \sum_{u=0}^{k-k_0} (k-u)^{\alpha_1-1} F_{N, \alpha_1} \left( \frac{u}{N} \right) (\overline{\chi_h} \chi_j)^u \right|$$

is bounded by

$M_1 k_0^{\alpha-1} + \sum_{u=0}^{k-k_0} \left| (k-u-1)^{\alpha-1} F_{N,\alpha_1} \left( \frac{u+1}{N} \right) - (k-u)^{\alpha-1} F_{N,\alpha_1} \left( \frac{u}{N} \right) \right|$  with  $M_1$  no depending from  $k$ . Moreover

$$\begin{aligned} & \sum_{u=0}^{k-k_0} \left| (k-u-1)^{\alpha_1-1} F_{N,\alpha_1} \left( \frac{u+1}{N} \right) - (k-u)^{\alpha_1-1} F_{N,\alpha_1} \left( \frac{u}{N} \right) \right| \\ & \leq \sum_{u=0}^{k-k_0} |(k-u-1)^{\alpha_1-1} - (k-u)^{\alpha_1-1}| |F_{N,\alpha_1} \left( \frac{u}{N} \right)| \\ & + \sum_{u=0}^{k-k_0} |F_{N,\alpha_1} \left( \frac{u+1}{N} \right) - F_{N,\alpha_1} \left( \frac{u}{N} \right)| |(k-u-1)^{\alpha_1-1}| \end{aligned}$$

From the inequality (14) (we have assumed  $0 < \frac{k}{N} < \delta$ ) we infer

$$\sum_{u=0}^{k-k_0} |(k-u-1)^{\alpha_1-1} - (k-u)^{\alpha_1-1}| |F_{N,\alpha_1}(u)| \leq M_2 \sum_{w=k_0}^k v^{\alpha_1-2}$$

with  $M_2$  no depending from  $k$ . We finally get

$$\begin{aligned} \sum_{u=0}^{k-k_0} |(k-u-1)^{\alpha_1-1} - (k-u)^{\alpha_1-1}| |F_{N,\alpha_1}(u)| &= O \left( \sum_{w=k_0}^k v^{\alpha_1-2} \right) \\ &= O \left( k_0^{\alpha_1-1} \right) = o(k^{\alpha_1}) \end{aligned}$$

Identically Lemma 4 and the main value theorem provide (since  $F_{N,\alpha} \in C^1[0, \delta]$ ,  $\forall \delta \in ]0, 1[$ ).

$$\sum_{u=0}^{k-k_0} |F_{N,\alpha_1} \left( \frac{u+1}{N} \right) - F_{N,\alpha_1} \left( \frac{u}{N} \right)| |(k-u-1)^{\alpha_1-1}| \leq M_3 \frac{k_1^\alpha}{N} = o(k^{\alpha_1}).$$

always with  $M_3$  no depending from  $N$ . By definition of  $k_0$  and with Property 1 we have easily the existence of a constant  $M_4$ , always no depending from  $k$ , satisfying for  $\alpha_1 > 0$

$$\sum_{u=k-k_0}^k \beta_{k-u}^{(\alpha_1)} F_{N,\alpha_1} \left( \frac{u}{N} \right) \leq M_4 k_0^{\alpha_1} = o(k^{\alpha_1}).$$

Consequently for  $\alpha_1 > 0$  and  $0 \leq h \leq m$

$$\begin{aligned} & \sum_{u=0}^k \beta_{k-u} F_{N,\alpha_h} \left( \frac{u}{N} \right) \overline{\chi}_h^u \\ &= K_h \overline{\chi}_h^k c_1^{-1}(\chi_h) \sum_{u=0}^{k-k_0} \frac{(k-u)^{\alpha_1-1}}{\Gamma(\alpha_1)} F_{N,\alpha_1} \left( \frac{u}{N} \right) + o(k^{\alpha-1}) \\ &= K_h \overline{\chi}_h^k c_1^{-1}(\chi_h) \sum_{u=0}^{k-} \beta_{k-u}^{(\alpha_1)} F_{N,\alpha_1} \left( \frac{u}{N} \right) o(k^{\alpha_1}) \end{aligned}$$

uniformly in  $k$  with the definition of the constants  $M_i$ ,  $1 \leq i \leq 4$ . For  $h > m$  we obtain identically that

$$\sum_{u=0}^{k-k_0} \beta_{k-u} F_{N,\alpha_h} \left( \frac{u}{N} \right) \overline{\chi}_h^u = o(k^{\alpha_1}).$$

and we get the Lemma for  $\alpha_1 > 0$ .

Hence we assume in the rest of the demonstration that  $\alpha_1 \in ]-\frac{1}{2}, 0[$ . Recall that now  $\gamma \in ]\max(\frac{\alpha_1}{\beta}, \frac{-\alpha_1}{1-\alpha_1}), 1[$ .

We have to evaluate the sum  $\sum_{u=k-k_0}^k \beta_{k-u} F_{N,\alpha_h}(\frac{u}{N}) \overline{\chi h}^u$ .  $F_{N,\alpha_h} \in C^1[0, \delta]$  implies, for  $\frac{k-k_0}{N} \leq \frac{u}{N} \leq \frac{k}{N} \leq \delta < 1$ ,

$$F_{N,\alpha_h}(\frac{u}{N}) - F_{N,\alpha_h}(\frac{k}{N}) + F_{N,\alpha_h}(\frac{k}{N}) = F_{N,\alpha_h}(\frac{k}{N}) + O(\frac{k_0}{N}) = F_{N,\alpha_h}(\frac{k}{N}) + o(k^{\alpha_1})$$

uniformly in  $k$  (see once a more the definition of  $\gamma$  and  $\tau_1$ ).

Hence we can write, uniformly in  $k$ ,

$$\begin{aligned} \sum_{u=k-k_0}^k \beta_{k-u} F_{N,\alpha_h}(\frac{u}{N}) \overline{\chi h}^u &= \overline{\chi h}^k \sum_{u=k-k_0}^k \beta_{k-u} F_{N,\alpha_h}(\frac{k}{N}) \chi_h^{k-u} + o(k^{\alpha_1}) \\ &= -\overline{\chi h}^k F_{N,\alpha_h}(\frac{k}{N}) \sum_{v=k_0+1}^{+\infty} \beta_v \chi_h^v + o(k^{\alpha_1}). \end{aligned}$$

If  $0 \leq h \leq m$  we get

$$\sum_{v=k_0+1}^{+\infty} \beta_v \chi_h^v = \sum_{v=k_0+1}^{+\infty} \left( \sum_{j=1}^m K_j c_1^{-1}(\chi_j) \overline{\chi_j}^v \right) \frac{v^{\alpha_1-1}}{\Gamma(\alpha_1)} \chi_h^v + o(k_0^{\tau_1}),$$

that is also, with the definition  $k_0 = k^\gamma$ ,  $\gamma \in ]\max(\frac{\alpha_1}{\tau_1}, \frac{-\alpha_1}{1-\alpha_1}), 1[$ ,

$$\sum_{v=k_0+1}^{+\infty} \beta_v \chi_h^v = \sum_{v=k_0+1}^{+\infty} \left( \sum_{j=1}^m K_j c_1^{-1}(\chi_j) \overline{\chi_j}^v \right) \frac{v^{\alpha_1-1}}{\Gamma(\alpha_1)} \chi_h^v + o(k^{\alpha_1}).$$

We have

$$\sum_{v=k_0+1}^{+\infty} \left( \sum_{j=1}^m K_j c_1^{-1}(\chi_j) \overline{\chi_j}^v \right) \frac{v^{\alpha_1-1}}{\Gamma(\alpha_1)} \chi_h^v = K_h c_1^{-1}(\chi_h) \sum_{v=k_0+1}^{+\infty} \frac{v^{\alpha_1-1}}{\Gamma(\alpha_1)} + R$$

An Abdel summation provides  $|R| \leq M_5 k_0^{\alpha_1-1} = o(k^{\alpha_1})$  uniformly in  $k$ .

Hence we have

$$\sum_{u=k-k_0}^k \beta_{k-u} F_{N,\alpha_h}(\frac{u}{N}) \overline{\chi h}^u = -K_h c_1^{-1}(\chi_h) \overline{\chi h}^k F_{\alpha_1}(\frac{k}{N}) \sum_{v=k_0+1}^{+\infty} \frac{v^{\alpha_1-1}}{\Gamma(\alpha_1)} + o(k^{\alpha_1})$$

that is also

$$\begin{aligned} \sum_{u=k-k_0}^k \beta_{k-u} F_{N,\alpha_h}(\frac{u}{N}) \overline{\chi h}^u &= K_h c_1^{-1}(\chi_h) \overline{\chi h}^k F_{\alpha_1}(\frac{k}{N}) \sum_{u=k-k_0}^k \frac{\beta_{k-u}^{(\alpha_1)}}{\Gamma(\alpha_1)} + o(k^{\alpha_1}) \\ &= K_h c_1^{-1}(\chi_h) \overline{\chi h}^k \sum_{u=k-k_0}^k \frac{\beta_{k-u}^{(\alpha_1)}}{\Gamma(\alpha_1)} F_{\alpha_1}(\frac{u}{N}) + o(k^{\alpha_1}) \end{aligned}$$

uniformly in  $k$ . Since we have seen that the sum

$$\sum_{u=0}^{k-k_0} \beta_{k-u} F_{N, \alpha_h} \left( \frac{u}{N} \right) \overline{\chi}_h^u$$

is equal to

$$\overline{\chi}_h^k c_1^{-1}(\chi_h) \sum_{u=0}^{k-k_0} \frac{\beta_{k-u}^{(\alpha_1)}}{\Gamma(\alpha_1)} F_{N, \alpha_1} \left( \frac{u}{N} \right) + o(k^{\alpha_1})$$

we can also conclude, as for  $\alpha_1 > 0$ , that for  $1 \leq h \leq m$

$$\sum_{u=0}^k \beta_k F_{N, \alpha_h} \left( \frac{u}{N} \right) \overline{\chi}_h^u = K_h c_1^{-1}(\chi_h) \overline{\chi}_h^k \sum_{v=0}^k \frac{\beta_{k-v}^{(\alpha_1)}}{\Gamma(\alpha_1)} F_{\alpha_1} \left( \frac{v}{N} \right) + o(k^{\alpha_1}).$$

Identically if  $h > m$  we obtain  $\left| \sum_{u=0}^k \beta_{k-u} F_{N, \alpha_h} \left( \frac{u}{N} \right) \overline{\chi}_h^u \right| = o(k^{\alpha_1})$  uniformly in  $k$ . The uniformity is clearly provided by the uniformity in Lemma 4 and by the previous remarks. This last remark is sufficient to prove Lemma 5.  $\square$

Then Theorem 2 is a direct consequence of the inversion formula and of Lemma 5.

## 5 Appendix

### 5.1 Estimation of a trigonometric sum

**Lemma 6** *Let  $M_0, M_1$  two integers with  $0 < M_0 < M_1$ ,  $\chi \neq 1$  and  $f$  a function in  $\mathcal{C}^1 ]M_0, M_1[$  with for all  $t \in ]M_0, M_1[$   $f(t) = O(t^\beta)$  and  $f'(t) = O(t^{\beta-1})$ . Then*

$$\left| \sum_{u=M_0}^{M_1} f(u) \chi^u \right| = \begin{cases} O(M_1^\beta) & \text{if } \beta > 0 \\ O(M_0^\beta) & \text{if } \beta < 0. \end{cases}$$

Proof: With an Abel summation we obtain, if  $\sigma_u = 1 + \dots + \chi^u$ ,

$$\sum_{u=M_0}^{M_1} f(u) \chi^u = \sum_{u=M_0}^{M_1-1} (f(u+1) - f(u)) \sigma_u + f(M_1) \sigma_{M_1} + f(M_0) \sigma_{M_0-1}$$

and

$$\begin{aligned} \sum_{u=M_0}^{M_1-1} (f(u+1) - f(u)) \sigma_u &= (f(M_0) + f(M_1)) \left( \frac{1}{1-\chi} \right) - \sum_{u=M_0}^{M_1-1} (f(u+1) - f(u)) \frac{\chi^{u+1}}{1-\chi} \\ &= \sum_{u=M_0}^{M_1-1} f'(c_u) \frac{\chi^{u+1}}{1-\chi} + (f(M_0) + f(M_1)) \left( \frac{1}{1-\chi} \right) \end{aligned}$$

with  $c_u \in ]u, u+1[$ . We have

$$\left| \sum_{u=M_0}^{M_1-1} f'(c_u) \frac{\chi^u}{1-\chi} \right| \leq O \left( \sum_{u=M_0}^{M_1-1} u^{\beta-1} \right)$$

hence

$$\left| \sum_{u=M_0}^{M_1} f(u)\chi^u \right| = \begin{cases} O(M_1^\beta) & \text{if } \beta > 0 \\ O(M_0^\beta) & \text{if } \beta < 0. \end{cases}$$

□

## References

- [1] E. L. Basor. Toeplitz determinants, Fisher-Hartwig symbols and random matrices. In lecture note series London mathematical society, editor, *Recent perspectives in random matrix theory and number theory*, volume 322, pages 309–336, 2005.
- [2] P. Beaumont and R. Ramachandran. Robust estimation of GARMA model parameters with an application to cointegration among interest rates of industrialized country. *Computational economics*, 17:179–201, 2001.
- [3] J. Beran. *Statistics for long memory process*. Chapman and Hall, 1994.
- [4] A. Böttcher and B. Silbermann. Toeplitz matrices and determinants with Fisher-Hartwig symbols. *J. Funct. Anal.*, 63:178–214, 1985.
- [5] A. Böttcher and B. Silbermann. Toeplitz operators and determinants generated by symbols with one Fisher-Hartwig singularity. *Math. Nachr.*, 127:95–124, 1986.
- [6] P. J. Brockwell and R. A. Davis. *Times series: theory and methods*. Springer Verlag, 1986.
- [7] Q.C. Cheng, H. L. Gray, and W. A. Wayne. A k-factor GARMA long-memory model. *Journal of time series analysis*, 19(4):485–504, 1998.
- [8] R. Dahlhaus. Efficient parameter estimation for self-similar processes. *Ann. Statist.*, 17:1749–1766, 1989.
- [9] P. A. Deift, K. T. R. McLaughlin, T. Kriecherbauer, S Venakides, and X. Zhou. A riemann-Hilbert approach to asymptotic questions for orthogonal polynomials. *J. Approx. Theory*, 95:388–475, 1998.
- [10] P.A. Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*. AMS, New York, 1998.
- [11] Abdou Kâ Diongue and D. Guéguan. Estimating parameters for k-factor GIGARCH process. *C.R.A.S, Serie I*, 339,:435,440, 2004.
- [12] Abdou Kâ Diongue and D. Guéguan. Estimation of k-factor GIGARCH process : a Monte Carlo Study. *Communications in Statistic-Simulations and computations*, 37:2037,2049, 2009.
- [13] P. Doukhan, G. Oppenheim, and M. S. Taqqu. *Theory and applications of long-range dependence*, volume 54. Birkhäuser, Boston, 2003.

- [14] U. Grenander and G. Szegő. *Toeplitz forms and their applications*. Chelsea, New York, 2nd ed. edition, 1984.
- [15] Kurt Johansson. On random matrices from the compact classical groups. *Annals of Mathematics.*, 145:519–545, 1997.
- [16] A.P. Kirman and G. Teyssiere. *Long memory in economic*. Mathematical Review, 2007.
- [17] H.J. Landau. Maximum entropy and the moment problem. *Bulletin (New Series) of the american mathematical society*, 16(1):47–77, 1987.
- [18] Y. Lu and C. M. Hurvich. On the complexity of the preconditioned conjugate gradient algorithm for solving Toeplitz systems with a Fisher-Hartwig singularity. *SIAM J. Matrix Anal. Appl.*, 27:638–653, 2005.
- [19] A. Martinez-Finkelshtein, K. T. R McLaughlin, and E. B. Saff. Asymptotics of orthogonal polynomials with respect to an analytic weight with algebraic singularities on the circle. *Internat. Math. Research Notices*, 2:423–434, 1993.
- [20] Taro Nagao. Universal Correlations Near a Singularity of Random Matrix Spectrum. *Journal of the Physical Society of Japan.*, 64:3675–3681, 1995.
- [21] Taro Nagao and Miki Wadati. An Integration Method on Generalized Circular Ensembles. *Journal of the Physical Society of Japan.*, 61:1903–1909, 1992.
- [22] Taro Nagao and Miki Wadati. Eigenvalue distribution of random matrices at the spectrum edge. *Journal of the Physical Society of Japan.*, 62:3845–3856, 1993.
- [23] P. Rambour and A. Seghier. Inversion des matrices de Toeplitz dont le symbole admet un zéro d'ordre rationnel positif, valeur propre minimale. *Annales de la Faculté des Sciences de Toulouse*, XXI, n° 1:173–2011, 2012.
- [24] P. Rambour and A. Seghier. Formulas for the inverses of Toeplitz matrices with polynomially singular symbols. *Integr. equ. oper. theory*, 50:83–114, 2004.
- [25] P. Rambour and A. Seghier. Théorèmes de trace de type Szegő dans le cas singulier. *Bull. des Sci. Math.*, 129:149–174, 2005.
- [26] P. Rambour and A. Seghier. Inverse asymptotique des matrices de Toeplitz de symbole  $(1 - \cos \theta)^\alpha f_1$ ,  $\frac{-1}{2} < \alpha \leq \frac{1}{2}$ , et noyaux intégraux. *Bull. des Sci. Math.*, 134:155–188, 2008.
- [27] B. Simon. *Orthogonal polynomials on the unit circle, Part 1: classical theory*, volume 54. American Mathematical Society, 2005.
- [28] B. Simon. *Orthogonal polynomials on the unit circle, Part 2: spectral theory*, volume 54. American Mathematical Society, 2005.
- [29] G. Szegő. *Orthogonal polynomials*. American Mathematical Society, colloquium publication, Providence, Rhode Island, 3rd edition, 1967.
- [30] C.A. Tracy and H. Widom. Correlation functions, cluster functions and spacing distribution for random matrices. *J. Stat. Phys.*, 92:809–835, 1999.

[31] A. Zygmund. *Trigonometric series*, volume 1. Cambridge University Press,, 1968.