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# ORDER-STATISTICS MINIMUM ERROR RATE DETECTOR

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## ABSTRACT

A new methodology for random signal detection, in presence of closely spaced interfering signals, is introduced, namely the *Order-Statistics Minimum Error Rate* (OS-MER) detector. The latter is based on the minimization of the error probability instead of minimizing only the miss probability for a *Constant False Alarm Rate* (CFAR). Results show that the OS-MER detector is well adapted to this specific problem and overcomes, in particular, the CFAR-based one.

**Keywords:** Signal Detection, Order-Statistics, Incoherent Integration, Constant False Alarm Rate, Minimum Error Rate.

## 1. INTRODUCTION

Detection theory is a statistical tool used for making a decision about the presence or the absence of a signal of interest. This tool has been extensively studied and largely used in many fields, including: Radar, Sonar, communication systems, speech, image processing, biomedicine, control, seismology, etc. [1].

Initially developed for the adaptive radar detection, the CFAR-based detection technique has been widely used to resolve detection problems for other applications: CDMA multi-user detection and pseudonoise code acquisition, OFDM signal detection, acquisition of weak GPS signals, spectrum sensing in cognitive radio, mobile localization, UWB localization for through the wall imaging, distributed detection in sensor networks, adaptive subspace detection, failure detection in dynamic systems, target recognition for automotive applications [1, 2, 3, 4, 5]. More applications can be found in [6].

For a fixed and completely controlled false alarm rate, the CFAR-based detectors try to do their best for maximizing the detection probability under a constant false alarm probability constraint. By doing that, our intention is focused at first on keeping the first type of detection errors under control while the second type of detection errors, is minimized at the best [2, 4, 5].

This way of thinking is well adapted for radar or sonar detection applications where a false alarm, in general, costs more than a miss event. Nevertheless, in other types of detection

applications, this is not true, and both error types have to be taken into account when optimizing the detection process. For example, for time-based localization techniques, when detecting the first peak of the correlation function between the received signal and the transmitted pilot sequence, a false alarm or a miss of detection are equally harmful and both lead to errors of localization.

In this paper, we introduce a new class of adaptive detectors based on the same scheme and sufficient statistic of the CFAR detector. In contrast to the above mentioned approaches, this new methodology, namely the *Minimum Error Rate* MER, is based on the minimization of the error probability. The main difference between the MER-based detectors class and the CFAR-based one is in the thresholding process by means of which the detection threshold is obtained.

The proposed method takes into account the two types of errors, false alarm and a non-detection, and takes advantage from the prior knowledge about the presence or absence of a signal of interest.

The MER-based thresholding method is combined here with Order-Statistics method for the estimation of the background noise power. The resulted detector is developed for the detection of incoherently integrated signals in presence of interfering signals.

This methodology has been used for the time-based mobile localization in the context of multipaths communication systems. Two MER variants have been used for the detection of the first arriving signal; the *Cell-Averaging Minimum Error Rate* (CA-MER) detector if only one finger is present within the reference window or if the power of the first peak is high enough to shadow the other peaks [8], and the *Order-Statistics Minimum Error Rate* (OS-MER) detector if several closely spaced fingers are present within the reference window [9]. In [8] and [9], the optimization of the probability of error has been done numerically. Herein, the  $\gamma$  that minimizes the error probability is given in a closed-form.

## 2. DETECTION SCHEME AND STATISTICAL ASSUMPTIONS

### 2.1. Detection Scheme

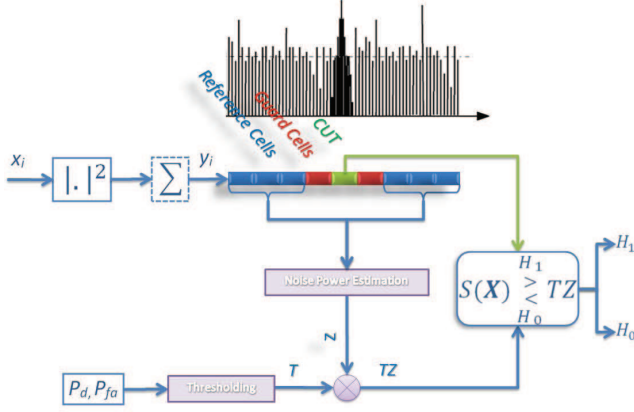


Fig. 1. Adaptive Detection Scheme.

In a typical detection problem we are interested in determining whether a received signal  $x(t)$  contains a reflected signal  $s(t)$  embedded in random noise  $w(t)$  or, on the contrary,  $x(t)$  is just a confusing manifestation of the background noise.

$$x(t) = s(t) + w(t) \quad (1)$$

where  $s(t)$  and  $w(t)$  are Gaussian complex-valued random signals such that  $s(t) \sim \mathcal{N}(0, 2\sigma_s^2)$  and  $w(t) \sim \mathcal{N}(0, 2\sigma_w^2)$ . Thus,  $x \sim \mathcal{N}(0, \sigma_x^2)$  where  $\sigma_x^2 = 2\sigma_w^2(1 + S)$ ,  $S = \sigma_s^2/\sigma_w^2$  being the signal-to-noise ratio.

Assuming that we have  $n$  measurements (samples)  $x_i, i = 1, \dots, n$  of the signal under investigation defined in (1) such that  $x_1, x_2, \dots, x_n \text{ iid} \sim \mathcal{N}(0, \sigma_x^2)$ . We wish to detect the presence of the signal considering the  $n$  samples stacked in the vector  $\mathbf{x} = [x_1, x_2, \dots, x_n]$ .

For more than forty years, adaptive detection has been used successfully to resolve an important number of detection problems in several fields [6]. An adaptive detector is an implementation of the following hypothesis test [3, 4]:

$$\mathcal{T} : \mathcal{S}(\mathbf{x}) \geq \gamma = TZ \quad (2)$$

The test statistic  $\mathcal{S}(\mathbf{x})$  is compared with the adaptive threshold  $\gamma$  which is the product of the noise power estimate  $Z$  and the scaling factor  $T$  obtained from a given combination of false alarm and detection probabilities. The process by means of which these two parameters are obtained defines the adaptive detector scheme.

In a typical adaptive detection scheme (see Fig.1), the in-phase and quadrature components of the received signal  $x_i$  are processed by a square-law detector and, eventually, an integrator. The resulting signals are set serially in a shift register.

The square-law detector and the integrator are the implementation of the sufficient<sup>1</sup> statistic  $\mathcal{S}(\mathbf{x})$ .

The noise power is estimated using the measurements within the reference window composed of  $m$  Reference Cells. The Guard Cells, immediate neighbours of the Cell Under Investigation (CUI), are excluded from the estimation process to avoid an eventual spillover from the CUI. The scaling factor  $T$  is obtained via an optimization process where the criterion is function of the probabilities of the two erroneous decisions, i.e. miss probability  $P_m$  and false alarm probability  $P_{fa}$ .

### 2.2. Statistical Assumptions

Knowing that the signal of interest and the noise are zero-mean complex-valued Gaussian-distributed random processes and that the signal samples  $s_i$  are i.i.d and independent from noise samples  $w_i$ , the square law detector outputs are exponentially distributed.

In practice, the detection process is often implemented after non-coherent integrator, the content of each cell within the analyzing window is the sum of  $n$  squared observations. Thus, the CUI content  $\mathcal{S}$  is a Chi-square random variable with  $2n$  degrees of freedom:

$$f_{\mathcal{S}|H_j}(s|H_j) = \frac{s^{n-1} e^{-s/\xi}}{(\xi)^n \Gamma(n)}; s > 0 \quad (3)$$

and thus for the two following situations:

$$\xi = \begin{cases} 2\sigma_w^2 & H_0 : \text{ noise alone} \\ 2\sigma_w^2(1 + S) & H_1 : \text{ noise + signal} \end{cases} \quad (4)$$

Also the reference cell contents  $Y_i, i = 1, 2, \dots, m$  are Chi-square random variables with  $2n$  degrees of freedom ( $Y_i \text{ iid} \sim \xi\chi_{2n}^2, i = 1, 2, \dots, m$ ). Thus, their pdf is given by:

$$f_{Y|H'_j}(y|H'_j) = \frac{y^{n-1} e^{-y/\xi}}{(\xi)^n \Gamma(n)}; y > 0 \quad (5)$$

and thus for the two following situations:

$$\xi = \begin{cases} 2\sigma_w^2 & : \text{ noise alone} \\ 2\sigma_w^2(1 + I) & : \text{ noise + interfering signal} \end{cases} \quad (6)$$

where  $I = \sigma_r^2/\sigma_w^2$  is the interference-to noise ratio ( $\sigma_r^2$  being the variance of the interfering signal).

## 3. ORDER-STATISTICS

The reference cell contents are sorted in increasing order:  $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(k)} \leq \dots \leq Y_{(m)}$ . Then, the noise power is adaptively estimated as the content of reference cell having the  $k$ -th rank [11]:

$$Z = Y_{(k)} \quad (7)$$

<sup>1</sup>This is the case if the considered signals are gaussian distributed.

Under the assumptions of homogeneous background environment, the statistical properties of the  $k^{\text{th}}$ -rank sample  $y^{(k)}$  are given by [11]:

$$\begin{cases} f_Z(z) = k C_k^m [1 - F_Y(y)]^{m-k} [F_Y(y)]^{k-1} f_Y(y) \\ F_Z(z) = \sum_{j=k}^m C_j^m [F_Y(y)]^j [1 - F_Y(y)]^{m-j} \end{cases} \quad (8)$$

where  $f_Z(z)$  and  $F_Z(z)$  represent the probability density function and the probability distribution function, respectively, of the random variable  $Z$ .

However, if  $r$  iid interfering signals appear in the reference window, the  $m$  reference cell outputs still statistically independent but are no longer identically distributed. The noise power estimate  $Z$  is chosen as the content of reference cell having the  $k$ -th order picked up from the more general set:

$$\{U_i\}_{i=1,2,\dots,r} \cup \{Y_i\}_{i=1,2,\dots,m-r} \quad (9)$$

where  $Y_i$  and  $U_i$  are Chi-square distributed random variables with  $2n$  degrees of freedom:

$$\begin{cases} Y_i & \text{iid} \sim 2\sigma_w^2 \chi_{2n}^2, & i = 1, 2, \dots, m-r \\ U_i & \text{iid} \sim 2\sigma_w^2 (1 + I_r) \chi_{2n}^2, & i = 1, 2, \dots, r \end{cases} \quad (10)$$

Thus, the probability distribution function of  $Z$  is given by:

$$F_Z(z) = \sum_{i=k}^m \sum_{j=\min(0,i-r)}^{\max(i,(m-r))} C_j^{m-r} C_{i-j}^r [F_Y(y)]^j [F_U(u)]^{i-j} [1 - F_Y(y)]^{m-r-j} [1 - F_U(u)]^{r-i+j} \quad (11)$$

### 3.1. Performance Analysis

Aiming at evaluating the performance of the OS-MER detector, the analytical formulas of the false alarm and detection probabilities have to be developed. Under the hypothesis of homogeneous background noise, these two entities defined as:

$$\begin{cases} P_{fa} = P(\mathcal{S} > TZ | H_0) \\ P_d = P(\mathcal{S} > TZ | H_1) \end{cases} \quad (12)$$

Replacing (3), (4), (6) and (8) in (12), the detection probability is given by<sup>2</sup>:

$$P_d = \frac{k C_k^m}{\Gamma(n)} \sum_{i=0}^{k-1} C_i^{k-1} (-1)^{k-i-1} \sum_{j_0, \dots, j_{n-1}}^{m-1-i} \frac{C_{j_0, \dots, j_{n-1}}^{m-1-i}}{\prod_{q=0}^{n-1} (q!)^{j_q}} \sum_{l=0}^{n-1} \left( \frac{T}{1+S} \right)^l \frac{\Gamma\left(\sum_{q=0}^{n-1} q j_q + n + l\right)}{l! \left(m - i + \frac{T}{1+S}\right)^{\sum_{q=0}^{n-1} q j_q + n + l}} \quad (13)$$

<sup>2</sup>Due to space limitation, all proofs are omitted.

where  $C_{j_0, \dots, j_{n-1}}^{m-1-i}$  are the multinomial coefficients and  $\Gamma(\cdot)$  is the gamma function. The probability of false alarm  $P_{fa}$ , in this case, is got by setting  $S = 0$  in (13).

In the aim of quantifying the degradation in detection performance of the OS-MER detectors caused by the presence of interfering signals, we can use the closed-form of the detection and false alarm probabilities defined, in this case, as follows:

$$\begin{cases} P_{fa} = P(\mathcal{S} > TZ | H_0') \\ P_d = P(\mathcal{S} > TZ | H_1') \end{cases} \quad (14)$$

Substituting (3), (4), (6) and (11) into (14), the detection probability is given by:

$$P_d = \frac{\left(\frac{T}{1+S}\right)^n}{\Gamma(n)} \sum_{i=k}^m \sum_{j=\max(0,i-r)}^{\min(i,m-r)} C_j^{m-r} C_{i-j}^r \sum_{p=0}^j \sum_{q=0}^{i-j} C_p^j C_q^{i-j} (-1)^{i-p-q} \sum_{i_1, \dots, i_{n-1}}^{m-r-p} \sum_{j_1, \dots, j_{n-1}}^{r-m} \frac{C_{i_0, \dots, i_{n-1}}^{m-r-p}}{\prod_{p=0}^{n-1} (p!)^{i_p}} \frac{C_{j_0, \dots, j_{n-1}}^{r-m}}{\prod_{q=0}^{n-1} (q!)^{j_q}} \left(\frac{1}{1+I}\right)^{\sum_{q=0}^{n-1} q j_q} \frac{\Gamma\left(\sum_{p=0}^{n-1} p i_p \sum_{q=0}^{n-1} q j_q + n\right)}{\left(m - r - p + \frac{T}{1+S} + \frac{r-q}{1+I}\right)^{\sum_{p=0}^{n-1} p i_p \sum_{q=0}^{n-1} q j_q + n}} \quad (15)$$

The probability of false alarm  $P_{fa}$ , in this case, is got by setting  $S = 0$  in (15). Notice that, in absence of interfering signals within the reference window ( $r = 0$  or  $I = 0$ ), (15) is identical to (13).

### 4. MINIMUM ERROR RATE THRESHOLDING

In order to control the probability of false alarm of CFAR-based detector, the scaling factor  $T$  is fixed so that the latter takes a low constant value.

Herein, we introduce a new method that combines the Order-Statistics technique for the noise power estimation with the MER criterion for optimum selection of the scaling factor. The probability that a binary hypothesis test of two simple hypothesis generates an error is given by:

$$P_e = \lambda P_{fa} + (1 - \lambda) P_m \quad (16)$$

where  $\lambda = P(H_0)$  is the prior about the absence the signal to be detected and  $P_m = 1 - P_d$  is the probability of miss. The MER-based thresholding consists in finding the optimal scaling factor  $T$  that minimizes the detection error probability. As can be seen in equations (13) (15), the complex formulas of  $P_d$  and  $P_{fa}$  makes the minimization of (16) a complex and numerically sensitive optimization problem.

As we can see, (16) is a simple version of the *Bayes Risk*. It can be shown that the derivation for the MER-based detector

can be seen as a special case of the more general Bayesian detector given by the *Conditional Likelihood Ratio Test* (CLRT) [1]:

$$\Lambda(\mathbf{x}) = \frac{P(\mathbf{x}|H_1)}{P(\mathbf{x}|H_0)} \geq \eta = \frac{P_{H_0}}{P_{H_1}} \quad (17)$$

which is equivalent in our context to the following test:

$$\mathcal{S}(\mathbf{x}) \geq \gamma = T\sigma_w^2 \quad (18)$$

with  $\mathcal{S}(\mathbf{x})$  is the resulting sufficient statistic and  $T$  is the scaling factor that minimises the probability of error in (16). It can be shown that the scaling factor is given as follows:

$$T = \log \left[ \left( (1+S)\eta^n \right)^{\frac{1+S}{S}} \right] \quad (19)$$

It is clear that the threshold  $\gamma$  is a function of the noise power  $\sigma_w^2$ . If the latter is unknown, it should be adaptively estimated and injected in the thresholding process. The resulting test is referred to as *Generalized Conditional Likelihood Ratio Test* (GCLRT), given by:

$$\mathcal{S}(\mathbf{x}) \geq \gamma = TZ \quad (20)$$

Notice that the probability of error remains invariant if the threshold is considered as the product of a constant scaling factor  $T$  and the estimate of the noise power  $Z$  or as the product of a random scaling factor  $T' = T \frac{Z}{\sigma_w^2}$  and the true noise power:

$$\gamma = TZ = T' \sigma_w^2 \quad (21)$$

Thus, in this case, the scaling factor that minimizes (16) is a random variable. If  $Z$  is a consistent estimator of the noise power, then the scaling factor  $T$  in (19) will be a good approximation of  $T'$ . Reference [7] explores and illustrates the relationships between the estimation and the detection theories for a binary composite hypothesis test.

For the OS-based detector,  $T'$  is a parametric function of the SNR depending on three parameters; the reference cell number  $m$ , the prior about the absence of a signal within the analyzing window  $\lambda$ , and the order of the noise sample  $k$ , chosen as an estimate of the noise power.

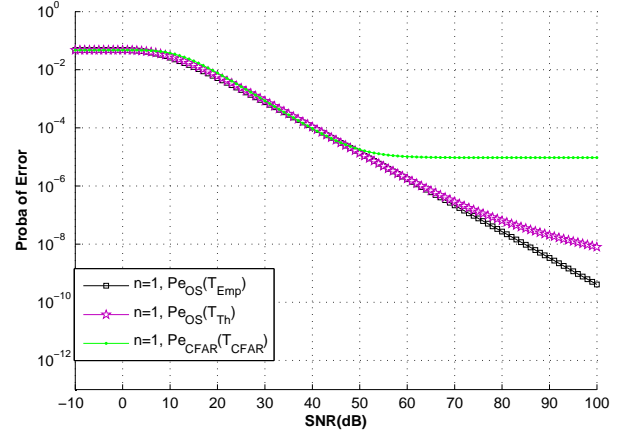
## 5. NUMERICAL RESULTS

To assess the performance of the OS-MER thresholding method and to compare its behavior w.r.t. OS-CFAR detector, different scenarios have been simulated. In the following simulations, the size of the reference window  $m$  is set to 20, the rank of the reference cell chosen as estimate of the noise power  $k$  is set to  $3/4m$ , the number of interfering signals  $r$  is set to 0, and the pre-fixed probability of false alarm of the OS-CFAR detector is set to  $10^{-5}$ .

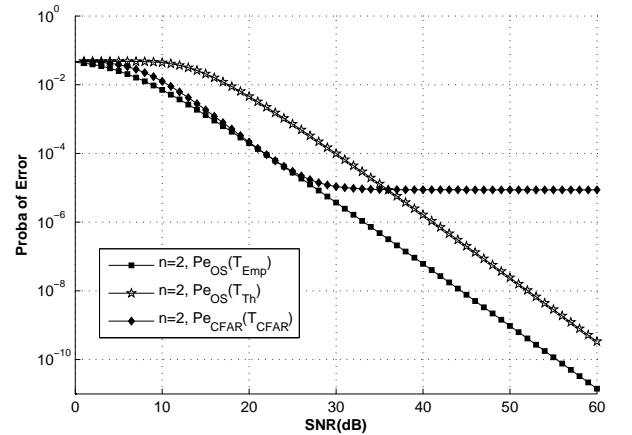
Fig.2 depicts the behavior of the probability of error vs. SNR. One can see that the error probability of the OS-MER is

less than that of the OS-CFAR. The gap between the two curves becomes more significant for high SNR values (SNR > 50 dB). The reason of this is that the error probability of the OS-CFAR detector is dominated by the prior fixed probability of false alarm for high SNR values.

The same results can be obtained for relatively low SNR if incoherent pulse integration process is used which is advantageous for the OS-MER detector. Fig.3 and Fig.5 show that we obtain similar results at 30dB for  $n = 2$  and at 0dB for  $n = 3$ .

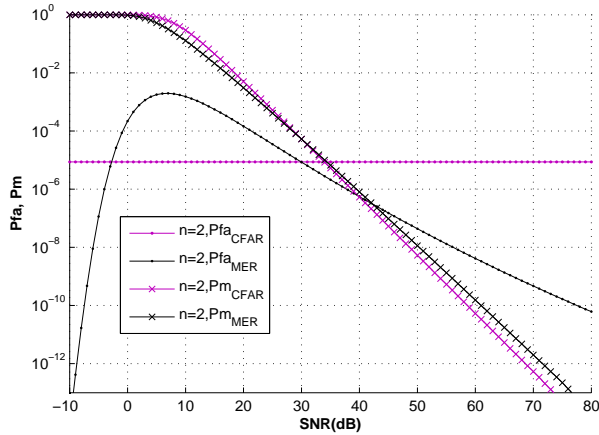


**Fig. 2.** Probability of error vs. SNR for OS-MER and OS-CFAR detectors ( $n = 1$ ).

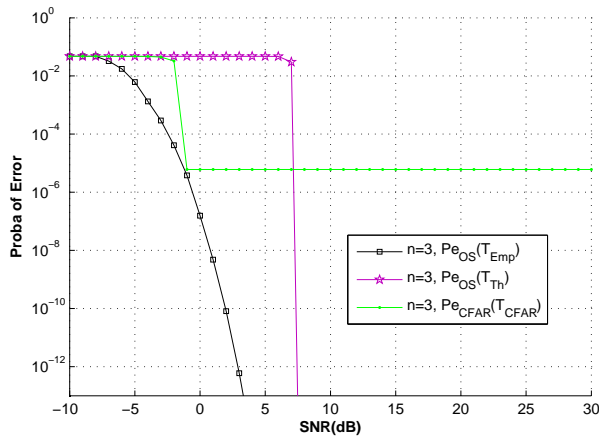


**Fig. 3.** Probability of error vs. SNR for OS-MER and OS-CFAR detectors ( $n = 2$ ).

In Fig.4 we show that for high SNR (SNR > 50 dB) a small loss in the detection probability of the OS-MER detector w.r.t the OS-CFAR is noticed. However, the false alarm rate of the latter is thousands times higher than the false alarm rate of the OS-MER.



**Fig. 4.** Probability of false alarm and probability of miss vs. SNR for OS-MER and OS-CFAR detectors ( $n = 2$ ).



**Fig. 5.** Probability of error vs. SNR for OS-MER and OS-CFAR detectors ( $n = 3$ ).

The set of curves presented in Fig.2, Fig.3 and Fig.5 show also the effect of approximating the scaling factor by its theoretical formulas presented in (19). For  $n = 1$  no difference between the probability of error computed using the theoretical scaling factor, referred to as  $T_{Th}$ , and the probability of error computed using the empirical scaling factor, referred to as  $T_{Emp}$ , which is obtained by minimizing (16) numerically.

Fig.3 and Fig.5 show that the gap between the two curves increases as  $n$  increases.

## 6. CONCLUSION

In this paper, the Order-Statistics based detector that minimizes the error probability has been introduced.

The proposed method takes into account the prior about the presence or absence of a signal of interest rather than con-

sidering only the false alarm rate as a criterion to be maintained constant.

The presented results clearly demonstrate the efficiency of the proposed method, i.e. the OS-MER, as compared to OS-CFAR detector, while presenting a small detection loss. The MER-based thresholding can be used for any symmetrical detection problem in which a false alarm and a non-detection event have the same cost. Perspective works include the investigation of the detector robustness w.r.t. erroneous prior information and the OS-MER performance in presence of interfering signals (considered in section 3.1 but not investigated in our simulation results).

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