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► **To cite this version:**

Matthieu Astorg. The Teichmüller space of a rational map immerses into moduli space. *Advances in Mathematics*, 2017, 313, pp.991-1023. 10.1016/j.aim.2017.04.013 . hal-01002101v3

**HAL Id: hal-01002101**

**<https://hal.science/hal-01002101v3>**

Submitted on 1 Jul 2014

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# ON THE DYNAMICAL TEICHMÜLLER SPACE

MATTHIEU ASTORG

ABSTRACT. We prove that the Teichmüller space of a rational map immerses into the moduli space of rational maps of the same degree, answering a question of McMullen and Sullivan. This is achieved through a new description of the tangent and cotangent space of the dynamical Teichmüller space.

## 1. NOTATIONS

The following notations will be used throughout the article :

- $\mathcal{S}$  is a Riemann surface
- $\mathbb{P}^1$  is the Riemann sphere
- $\Omega$  is a hyperbolic open subset of  $\mathbb{P}^1$
- $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a rational map
- If  $\mathcal{S}$  is hyperbolic,  $\rho_{\mathcal{S}}$  is the hyperbolic metric on  $\mathcal{S}$

## 2. INTRODUCTION

Let us denote by  $\text{Rat}_d$  the space of rational fractions of degree  $d$ , and by  $\text{rat}_d$  its quotient under the action by conjugacy of the group of Möbius transformations. For  $f \in \text{Rat}_d$ , we will denote by  $\mathcal{O}(f)$  the orbit of  $f$  under the action of the group of Möbius transformations.

In order to study the geometry of the quasiconformal conjugacy class of  $f$  in  $\text{Rat}_d$  and  $\text{rat}_d$ , McMullen and Sullivan introduced in [MS98] the dynamical Teichmüller space of a rational map  $f$ , as a dynamical analogue of the Teichmüller theory of surfaces (see [GL00] and [Hub06] for an introduction to Teichmüller theory). McMullen and Sullivan constructed a natural complex structure on the Teichmüller space of a rational map  $f$  of degree  $d$ , making it into a complex manifold of dimension at most  $2d - 2$ . They also exhibited a holomorphic map of orbifolds  $\Psi : \text{Teich}(f) \rightarrow \text{rat}_d$  whose image is exactly the quasiconformal conjugacy class of  $f$  : thus one should think of the Teichmüller space of  $f$  as a complex manifold parametrizing the conjugacy class of  $f$ . In this context, a natural question arises concerning the parametrization  $\Psi$  : is it an immersion ? This question was asked by McMullen and Sullivan in their introductory paper. As it turns out, the answer is yes. Adam Epstein has an unpublished proof of this result; in [Mak10], Makienko also gives a proof in the same spirit. We present here a different approach, using more elementary tools : in particular, we won't need the explicit description of the Teichmüller space of  $f$  given in [EM88], and we will give a new method for constructing the complex structure on  $\text{Teich}(f)$  which does not rely on preexisting Teichmüller theory. A related question, also raised in [MS98], is to know whether or not the image of this map can accumulate on itself. In [Bra92], Branner showed that the answer is yes.

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*Key words and phrases.* Complex dynamics; Teichmüller theory; Quasiconformal analysis.  
This work was partially supported by the ANR Lambda project.

Denote by  $\text{bel}(f)$  the space of  $L^\infty$  Beltrami differentials invariant under  $f$ , and by  $\text{Bel}(f)$  its unit ball. We shall use the term "Beltrami forms" for elements of  $\text{Bel}(f)$ , and "Beltrami differentials" for elements of  $\text{bel}(f)$ , which we will think of as the tangent space to  $\text{Bel}(f)$ . Given a quasiconformal homeomorphism  $\phi$ , we will denote by  $K(\phi)$  its dilatation.

**Definition 2.1.**

- Denote by  $\text{QC}(f)$  the group of quasiconformal homeomorphisms commuting with  $f$ .
- Denote by  $\text{QC}_0(f)$  the normal subgroup of the elements  $\phi \in \text{QC}(f)$  such that there exists  $K > 1$  and an isotopy  $\phi_t \in \text{QC}(f)$  with  $\phi_0 = \text{Id}$ ,  $\phi_1 = \phi$  and for all  $t \in [0, 1]$ ,  $K(\phi_t) \leq K$ .
- The modular group of  $f$  is  $\text{Mod}(f) = \text{QC}(f)/\text{QC}_0(f)$ .
- The Teichmüller space of a rational map  $f$  (which we will denote by  $\text{Teich}(f)$ ) is  $\text{Bel}(f)$  quotiented by the right action of  $\text{QC}_0(f)$  by precomposition.

Let  $Z \subset \mathbb{P}^1$  be a set of cardinal 3. There is a holomorphic map  $\Psi^Z : \text{Bel}(f) \rightarrow \text{Rat}_d$  defined by  $\Psi(\mu) = \phi_\mu^Z \circ f \circ (\phi_\mu^Z)^{-1}$ , where  $\phi_\mu^Z$  is the unique solution of the Beltrami equation  $\bar{\partial}\phi_\mu^Z = \mu \circ \partial\phi_\mu^Z$  fixing  $Z$ . It descends to a holomorphic map of orbifolds  $\Psi : \text{Bel}(f) \rightarrow \text{rat}_d$  independant from the choice of  $Z$ , and to maps  $\Psi_T^Z : \text{Teich}(f) \rightarrow \text{Rat}_d$  and  $\Psi_T : \text{Teich}(f) \rightarrow \text{rat}_d$ .

The unit ball  $\text{Bel}(f)$  being an open subset of the Banach space  $L^\infty$ , it has a natural complex Banach manifold structure, and there exists at most one complex structure on  $\text{Teich}(f)$  making  $\pi : \text{Bel}(f) \rightarrow \text{Teich}(f)$  into a split submersion. Using the results of [EM88] on the equivalence between several notions of isotopies (isotopies relative to the ideal boundary, relative to the topological boundary, uniformly quasiconformal isotopies) McMullen and Sullivan constructed such a complex structure on  $\text{Teich}(f)$  and showed that  $\text{Teich}(f)$  is isomorphic to the cartesian product of a polydisk and of Teichmüller spaces of some finite type Riemann surfaces associated to the dynamics of  $f$ .

Once  $\text{Teich}(f)$  is endowed with its complex structure, one can verify that  $\Psi_T^Z$  and  $\Psi_T$  are holomorphic maps between complex manifolds and orbifolds respectively, and McMullen and Sullivan asked whether those maps are immersions. Since  $\text{rat}_d$  is not a manifold, we have to define what we mean by the statement that  $\Psi_T$  is an immersion.

**Definition 2.2.** We will say that  $\Psi_T$  is an immersion if the lift  $\Psi_T^Z$  is an immersion whose image is transverse to  $\mathcal{O}(f)$ . If this is true for one choice of normalization set  $Z$ , then it holds for all  $Z$ .

It turns out that  $\Psi_T$  is indeed an immersion, and Adam Epstein has an unpublished proof of this result. The idea of his proof is a dual approach using quadratic differentials. The key ingredients are the deformation spaces introduced in [Eps09] and a result of Bers concerning the density of rational quadratic differentials (cf [GL00], theorem 9 p.63).

The main result of this article is to give another proof of this result :

**Main Theorem.** The map  $\Psi_T : \text{Teich}(f) \rightarrow \text{rat}_d$  is an immersion.

Our proof uses a new and more elementary construction of the complex structure on  $\text{Teich}(f)$  (we will notably not use the results of [EM88]).

A key tool for this construction is the following analytical result on quasiconformal vector fields (see definition 3.3), which is interesting in its own right.

**Theorem A.** Let  $\Omega$  be a hyperbolic open subset of  $\mathbb{P}^1$  and  $\xi$  be a quasiconformal vector field on  $\Omega$ . The following properties are equivalent :

- i)* We have  $\rho_\Omega(\xi) \in L^\infty(\Omega)$ .

- ii) We have  $\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\bar{\partial}\xi\|_{L^\infty(\Omega)}$ .
- iii) There exists a quasiconformal extension  $\hat{\xi}$  of  $\xi$  on all of  $\mathbb{P}^1$  with  $\hat{\xi} = 0$  on  $\partial\Omega$ .
- iv) The extension  $\hat{\xi}$  defined by  $\hat{\xi}(z) = \xi(z)$  if  $z \in \Omega$  and 0 else is quasiconformal on  $\mathbb{P}^1$ , and  $\bar{\partial}\hat{\xi}(z) = 0$  for almost every  $z \notin \Omega$ .

In particular, we get a new characterization of infinitesimally trivial Teichmüller differentials on hyperbolic Riemann surfaces (see definition 3.13) :

**Corollary 1.** A Beltrami differential  $\mu$  on a hyperbolic Riemann surface  $\mathcal{S}$  is infinitesimally trivial if and only if it is of the form  $\mu = \bar{\partial}\xi$ , with  $\|\rho_{\mathcal{S}}(\xi)\|_{L^\infty(\mathcal{S})} \leq 4\|\bar{\partial}\xi\|_{L^\infty(\mathcal{S})}$ , where  $\rho_{\mathcal{S}}$  is the hyperbolic metric on  $\mathcal{S}$ .

We will also get a simplified proof of Bers' theorem on the density of rational quadratic differentials, which notably doesn't use Ahlfors' Mollifier :

**Corollary 2** (Bers' density theorem). Let  $K$  be a compact of  $\mathbb{P}^1$  containing at least 3 points, and let  $Z$  be a countable dense subset of  $K$ . The space of meromorphic quadratic differentials with simple poles in  $Z$  is dense (for the  $L^1$  norm) in the space of integrable quadratic differentials which are holomorphic outside of  $K$ .

The proof of the Main theorem will also yield the following description of the tangent and cotangent spaces to the Teichmüller space of  $f$  (here,  $\Lambda_f$  is the closure of the grand orbit of the critical points,  $Q(\Lambda_f)$  is the space of integrable quadratic differentials holomorphic outside  $\Lambda_f$ , and  $\nabla_f = \text{Id} - f_*$ ) :

**Corollary 3.** We have the following identification :

$$\begin{aligned} T_0\text{Teich}(f) &= \text{bel}(f)/\{\bar{\partial}\xi, \xi = f^*\xi\} \\ T_0^*\text{Teich}(f) &= Q(\Lambda_f)/\overline{\nabla_f Q(\Lambda_f)}. \end{aligned}$$

In section 3, we will be concerned only with non-dynamical, analytic results on quasiconformal vector fields. The main result of this section is theorem A. In section 4, we will apply theorem A to obtain the key fact that  $D\Psi^Z$  has constant rank. Lastly, we will prove the Main Theorem in section 5.

### 3. QUASICONFORMAL VECTOR FIELDS

**3.1. Generalities.** In this section, we introduce notations and recall important results on several mathematical objects involved in quasiconformal Teichmüller theory : Beltrami forms and differentials, quadratic differentials, and quasiconformal vector fields.

In all of the article,  $\mathcal{S}$  will denote a Riemann surface.

**Definition 3.1.** A quadratic differential on  $\mathcal{S}$  is a section of the vector bundle  $T^*\mathcal{S} \otimes T^*\mathcal{S}$  (symmetric tensor product).

**Definition 3.2.** If  $\mu$  is a section of  $\overline{\text{Hom}}(T\mathcal{S}, T\mathcal{S})$ , i.e. a section of the vector bundle of anti- $\mathbb{C}$ -linear endomorphisms of tangent planes, and  $z \in \mathcal{S}$ , then let  $|\mu|(z)$  denote the norm of the endomorphism  $\mu(z)$  of  $T_z\mathcal{S}$  :  $|\mu|$  is a well defined function on  $\mathcal{S}$ . If  $\mu$  is such a section verifying  $|\mu| \in L^\infty(\mathcal{S})$ ,  $\mu$  is called a Beltrami differential. If additionally  $\|\mu\|_{L^\infty(\mathcal{S})} < 1$ , we say that  $\mu$  is a Beltrami form.

**Definition 3.3.** Let  $\xi$  be a vector field on  $\mathcal{S}$ . We say that  $\xi$  is quasiconformal if  $\bar{\partial}\xi$  (in the sense of distribution theory) is a Beltrami differential.

More generally, it will be useful to define :

**Definition 3.4.** For  $(p, q) \in \mathbb{Z} \times \{0, 1\}$ , we define  $S_p^q$  as the space of sections of  $T^*\mathcal{S}^{\otimes p} \otimes \overline{T^*\mathcal{S}}^{\otimes q}$  if  $p \geq 0$ , and the space of sections of  $T\mathcal{S}^{\otimes p} \otimes \overline{T^*\mathcal{S}}^{\otimes q}$  if  $p < 0$  (all tensor products are symmetric).

Since  $E \otimes \overline{E^*}$  is canonically isomorphic to  $\overline{\text{Hom}}(E, E)$  for all complex vector space  $E$ , Beltrami differentials are exactly the  $\mu \in S_{-1}^1$  such that  $|\mu| \in L^\infty(\mathcal{S})$ . Similarly, quadratic differentials are the elements of  $S_2^0$ , vector fields are the elements of  $S_{-1}^0$ , generalized Beltrami differentials (introduced by Bers in [Ber67]) are the elements of  $S_{-k}^1$ ,  $k \in \mathbb{N}$ , and the differentials of order  $p$  are the elements of  $S_p^0$ ,  $p \in \mathbb{N}$ .

In local coordinates, elements of  $S_p^q$  are written  $u = u(z)dz^p d\overline{z}^q$ , and under a change of coordinates  $\phi(w) = z$ , we have  $u = u \circ \phi(w)\phi'(w)^p \overline{\phi'(w)}^q dw^p d\overline{w}^q$ .

**Definition 3.5.** Let  $u_i = v_i \otimes w_i \in S_{p_i}^{q_i}$ , with  $v_i \in (T^*\mathcal{S})^{\otimes p_i}$  and  $w_i \in (\overline{T^*\mathcal{S}})^{\otimes q_i}$ ,  $1 \leq i \leq 2$ . Assume that  $p_1 \leq p_2$ ,  $0 \leq p_1 + p_2 \leq 1$ , and  $0 \leq q_1 + q_2 \leq 1$ . We then define :

$$u_1 \cdot u_2 = u_1(u_2, \cdot) \wedge (v_1 \otimes v_2)$$

Thus  $u_1 \cdot u_2$  is an alternate  $(p_1 + p_2, q_1 + q_2)$  differential form on  $\mathcal{S}$ . By convention, we will set  $u_2 \cdot u_1 = u_1 \cdot u_2$ .

Note that in local coordinates, if  $u_i = u_i(z)dz^{p_i} d\overline{z}^{q_i}$  with  $1 \leq i \leq 2$  and  $p_1 + p_2 \in \{0, 1\}$  and  $q_1 + q_2 \in \{0, 1\}$ , then

$$u_1 \cdot u_2 = u_1(z)u_2(z)dz^{p_1+p_2} d\overline{z}^{q_1+q_2}.$$

The next definition is a particular case of the usual definition of  $\overline{\partial}u$ , where  $u$  is a section of a holomorphic vector bundle.

**Definition 3.6.** Let  $u \in S_p^0$  be of class  $C^1$ . We can write locally  $u = \phi v$ , where  $v$  is a holomorphic local section of  $(T^*\mathcal{S})^{\otimes p}$  and  $\phi : \mathcal{S} \rightarrow \mathbb{C}$  is  $C^1$ . We define  $\overline{\partial}u \in S_p^1$  by :

$$\overline{\partial}u = \overline{\partial}\phi \otimes v$$

This definition is independant of the choice of  $\phi$  and  $v$ .

Note that in local coordinates, if  $u = u(z)dz^p$ , then  $\overline{\partial}u(z) = \frac{\partial u}{\partial \overline{z}}(z)dz^p d\overline{z}$ . Also, it is not hard to see that  $\overline{\partial}(u \cdot v) = u \cdot \overline{\partial}v + \overline{\partial}u \cdot v$ .

There are three particular cases of the above definitions which are especially important and deserve to be explicitly worked out :

a) The case of  $q \cdot \mu$ , where  $q$  is a quadratic differential and  $\mu$  is a Beltrami differential :

Then  $q \cdot \mu$  is a (1,1) alternate form given by :

$$(q \cdot \mu)_z : T_z\mathcal{S} \times T_z\mathcal{S} \rightarrow \mathbb{C}$$

$$(u, v) \mapsto \frac{1}{2}(q_z(u, \mu_z(v)) - q_z(v, \mu_z(u)))$$

In local coordinates,  $q \cdot \mu = q(z)\mu(z)dz \wedge d\overline{z}$ .

b) The case of  $q \cdot \xi$ , where  $q$  is a quadratic differential and  $\xi$  is a vector field :

Then  $q \cdot \xi$  is a (1,0)-differential form, given by :

$$(q \cdot \xi)_z : T_z\mathcal{S} \rightarrow \mathbb{C}$$

$$u \mapsto q_z(u, \xi(z))$$

In local coordinates,  $q \cdot \xi = q(z)\xi(z)dz$ .

c) The case of  $\bar{\partial}q \cdot \xi$ , where  $q$  is a quadratic differential and  $\xi$  is a vector field :

We can write locally  $q = \phi q'$ , where  $\phi$  is a function and  $q'$  is a holomorphic quadratic differential. Then  $\bar{\partial}q = \bar{\partial}\phi \otimes q' \in S_2^1$ , and  $\bar{\partial}q \cdot \xi$  is a  $(1, 1)$  alternate differential form, given by :

$$\begin{aligned} (\bar{\partial}q \cdot \xi)_z : T_z\mathcal{S} \times T_z\mathcal{S} &\rightarrow \mathbb{C} \\ (u, v) &\mapsto \frac{1}{2}(\bar{\partial}\phi_z(u)q'_z(v, \xi(z)) - \bar{\partial}\phi_z(v)q'_z(u, \xi(z))) \end{aligned}$$

In local coordinates,  $\bar{\partial}q \cdot \mu = \frac{\partial q}{\partial \bar{z}}(z)\xi(z)dz \wedge d\bar{z}$ .

**Proposition 1** (Stokes' theorem for quasiconformal vector fields). Let  $U$  be an open subset of  $\mathbb{P}^1$  with piecewise  $C^1$  boundary, let  $q$  be a  $C^1$  quadratic differential continuous on  $\bar{U}$  and  $\xi$  a quasiconformal vector field on  $\mathbb{P}^1$ . Then

$$\int_U q \cdot \bar{\partial}\xi + \int_U \xi \cdot \bar{\partial}q = \int_{\partial U} q \cdot \xi$$

*Proof.* In the case where  $\xi$  is a  $C^1$  vector field, this is exactly the classical Stokes' theorem. We deduce the general case where  $\bar{\partial}\xi$  only exists in the sense of distribution with a density argument : let  $\xi$  be a quasiconformal vector field and  $\xi_n$  a sequence of vector fields which are  $C^1$  in the neighborhood of  $\bar{U}$  and converging uniformly to  $\xi$  on  $\bar{U}$  (such a sequence exists because  $\xi$  is continuous). Then  $\xi_n$  converges to  $\xi$  as a distribution on  $U$ , so  $\bar{\partial}\xi_n$  converges to  $\bar{\partial}\xi$  in the sense of distributions (by continuity of the  $\bar{\partial}$  operator for the topology of distributions). Since we know that  $\bar{\partial}\xi$  is in fact a  $L^\infty$  Beltrami differential, we deduce from this that for all test quadratic differential  $\phi$  (i.e. smooth and with compact support in  $U$ ), we have :

$$\lim_{n \rightarrow \infty} \int_U \phi \cdot \bar{\partial}\xi_n = \int_U \phi \cdot \bar{\partial}\xi$$

Since test quadratic differentials are dense for the  $L^1$  norm, this still holds for all quadratic differential  $\phi$  integrable on  $U$ , and in particular for  $q$ .

Therefore  $\lim_{n \rightarrow \infty} \int_U q \cdot \bar{\partial}\xi_n = \int_U q \cdot \bar{\partial}\xi$  and since  $\xi_n$  converges uniformly on  $\bar{U}$ , we also have

$$\lim_{n \rightarrow \infty} - \int_U \xi_n \cdot \bar{\partial}q + \int_{\partial U} q \cdot \xi_n = - \int_U \xi \cdot \bar{\partial}q + \int_{\partial U} q \cdot \xi.$$

□

**Definition 3.7.** For a rational map  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ , we will note  $\Delta_f = \text{Id} - f^*$  and  $\nabla_f = \text{Id} - f_*$ , where  $f^*$  et  $f_*$  are respectively the pullback by  $f$  on vector fields (and Beltrami differentials), and  $f_*$  is the pushforward by  $f$  on quadratic differentials, following the notations of [Eps09].

**Definition 3.8.** Let  $z_0 \in \mathcal{S}$  and  $\xi(z_0) \in T_{z_0}\mathcal{S}$ . If  $q$  is a meromorphic quadratic differential with a simple pole at  $z_0$ , we define the residue of  $q \cdot \xi$  at  $z_0$  as the residue of  $q \cdot \tilde{\xi}$  at  $z_0$ , where  $\tilde{\xi}$  is a vector field holomorphic in the neighborhood of  $z_0$  with  $\xi(z_0) = \tilde{\xi}(z_0)$ . This definition does not depend upon the choice of  $\tilde{\xi}$ .

**Proposition 2.** Let  $q$  be a meromorphic quadratic differential on an open domain  $\Omega$  with smooth boundary, relatively compact in a Riemann surface  $\mathcal{S}$ , with simple poles that are included in a finite set  $P$ . Let  $\xi$  be a quasiconformal vector field on  $\Omega$  extending

continuously to  $\bar{\Omega}$ . Then :

$$\int_{\Omega} q \cdot \bar{\partial}\xi = 2i\pi \sum_{z \in P} \text{Res}(q \cdot \xi, z) - \int_{\partial\Omega} q \cdot \xi$$

*Proof.* Let  $\Omega_\epsilon = \Omega - \cup_{z \in P} D(z, \epsilon)$  where  $D(z, \epsilon)$  is the closed disk of center  $z$  and radius  $\epsilon$  (for an arbitrary metric). Then, by Stokes' theorem,

$$\int_{\Omega_\epsilon} q \cdot \bar{\partial}\xi = - \int_{\partial\Omega_\epsilon} q \cdot \xi = - \int_{\partial\Omega} q \cdot \xi + \sum_{z \in P} \int_{\partial D(z, \epsilon)} q \cdot \xi$$

Let  $\xi_\epsilon$  be a quasiconformal vector field coinciding with  $\xi$  on  $P$  and on  $\Omega_\epsilon$  and holomorphic in the neighborhood of  $P$ . Then :

$$\int_{\Omega_\epsilon} q \cdot \bar{\partial}\xi_\epsilon = 2i\pi \sum_{z \in P} \text{Res}(q \cdot \xi, z) - \int_{\partial\Omega} q \cdot \xi$$

Since  $\int_{\Omega} q \cdot \bar{\partial}\xi - \int_{\Omega_\epsilon} q \cdot \bar{\partial}\xi_\epsilon = O(\epsilon)$ , the result follows by letting  $\epsilon$  tend to zero.  $\square$

Note that in the particular case  $\Omega = \Delta$  and  $q = \frac{dz^2}{z}$ , we get the usual Cauchy-Pompéiu formula.

### 3.2. Splitting and hyperbolic metric.

**Definition 3.9.** Let  $\mathcal{S}$  be a hyperbolic Riemann surface and  $\xi$  a vector field on  $\mathcal{S}$ . We say that  $\xi$  is hyperbolically bounded on  $\mathcal{S}$  if and only if  $\rho_{\mathcal{S}}(\xi) \in L^\infty(\mathcal{S})$ , where  $\rho_{\mathcal{S}}$  is the hyperbolic metric on  $\mathcal{S}$ .

**Theorem 3.10.** *Let  $\xi$  be a vector field hyperbolically bounded on an open hyperbolic subset  $\Omega$  of  $\mathbb{P}^1$ , quasiconformal on  $\Omega$  and identically vanishing outside  $\Omega$ . Then  $\xi$  is globally quasiconformal, and  $\bar{\partial}\xi = 0$  for almost every  $z \notin \Omega$ . Moreover,  $\|\rho(\xi)\|_{L^\infty(\Omega)} \leq 4\|\bar{\partial}\xi\|_{L^\infty(\Omega)}$ .*

*Proof.* The key point is the following lemma :

**Lemma 3.11.** *Let  $q$  be an integrable quadratic differential of class  $C^\infty$  on  $\Omega$ , and  $\xi$  a hyperbolically bounded quasiconformal vector field on  $\Omega$ . Assume that  $\xi \cdot \bar{\partial}q$  is integrable on  $\Omega$ . Then :*

$$\int_{\Omega} \bar{\partial}\xi \cdot q = - \int_{\Omega} \xi \cdot \bar{\partial}q$$

*Proof.* Let  $z_0 \in \Omega$  be an arbitrary base point, and let  $\delta(z) = d_\Omega(z, z_0)$ , where  $d_\Omega$  is the hyperbolic distance on  $\Omega$ . Let  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a smooth function such that  $\phi(x) = 1$  for  $x \in [0, 1]$  and  $\phi(x) = 0$  for  $x \geq 2$ . For all  $n \in \mathbb{N}$ , let us define  $\phi_n : \Omega \rightarrow \mathbb{R}^+$  by

$$\phi_n(z) = \phi\left(\frac{\delta(z)}{n}\right).$$

Set  $\mu = \bar{\partial}\xi$ .

Let  $q$  be a quadratic differential as in the statement of the lemma. Since  $\phi_n q$  is compactly supported in  $\Omega$ , we have :

$$\int_{\Omega} \mu \cdot (\phi_n q) = - \int_{\Omega} \xi \cdot \bar{\partial}(\phi_n q) = \int_{\Omega} \xi \cdot (\phi_n \bar{\partial}q) + \int_{\Omega} \xi \cdot (q \cdot \bar{\partial}\phi_n)$$

Moreover,  $\int_{\Omega} \xi \cdot (q \cdot \bar{\partial}\phi_n) = \int_{\Omega} q \cdot (\xi \cdot \bar{\partial}\phi_n)$ . Let us now evaluate the  $L^\infty$  norm of the Beltrami differential  $\bar{\partial}\phi_n \cdot \xi$ . Since  $\delta$  is a locally lipschitz function on  $\Omega$ , it has locally bounded distributional derivatives. We have :

$$\bar{\partial}\phi_n = \frac{1}{n} \phi'(\delta/n) \bar{\partial}\delta.$$

Let  $z \in \Omega$  and  $u \in T_z \mathbb{P}^1$ . We have  $\bar{\partial}\phi_n \cdot \xi(z) : u \mapsto \bar{\partial}\phi_n(u)\xi(z)$ , and the norm of this endomorphism for any hermitian metric is  $|\bar{\partial}\phi_n \cdot \xi|(z)$ . We can therefore work with the hyperbolic metric  $\rho_\Omega$ . Since  $\delta$  is 1-lipschitz for the hyperbolic metric in  $\Omega$ , the derivative  $\bar{\partial}\delta$  has hyperbolic norm less than one almost everywhere. We have :

$$\begin{aligned} \bar{\partial}\phi_n \cdot \xi(z; u) &= \frac{1}{n} \phi' \left( \frac{\delta(z)}{n} \right) \bar{\partial}\delta(z; u) \times \xi(z) \\ \rho_\Omega(\bar{\partial}\phi_n \cdot \xi(z; u)) &\leq \frac{\sup_{\mathbb{R}^+} |\phi'|}{n} \|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \rho_\Omega(u) \\ |\bar{\partial}\phi_n \cdot \xi(z)| &\leq \frac{\sup_{\mathbb{R}^+} |\phi'|}{n} \|\rho_\Omega(\xi)\|_{L^\infty(\Omega)}. \end{aligned}$$

Therefore :  $\|\bar{\partial}\phi_n \cdot \xi\|_{L^\infty} = O(1/n)$  and  $|\int_{\Omega} q \cdot (\xi \cdot \bar{\partial}\phi_n)| \leq \|q\|_{L^1} \|\bar{\partial}\phi_n \cdot \xi\|_{L^\infty} = O(1/n)$ .

We then have :

$$\int_{\Omega} \bar{\partial}\xi \cdot (\phi_n q) = \int_{\mathbb{P}^1} (\bar{\partial}\xi \cdot q) \phi_n = - \int_{\Omega} \phi_n (\xi \cdot \bar{\partial}q) + O(1/n)$$

so

$$\int_{\Omega} (\bar{\partial}\xi \cdot q) \phi_n = - \int_{\Omega} \phi_n (\xi \cdot \bar{\partial}q) + O(1/n)$$

Since we assumed that both  $\xi \cdot \bar{\partial}q$  and  $|q|$  are integrable, we can apply the dominated convergence theorem to get :

$$\int_{\Omega} \bar{\partial}\xi \cdot q = - \int_{\Omega} \xi \cdot \bar{\partial}q.$$

□

Let now  $q$  be a  $C^\infty$  quadratic differential on  $\mathbb{P}^1$  : its restriction to  $\Omega$  verifies the conditions of the lemma, therefore we have :

$$\int_{\mathbb{P}^1} \mu \cdot q = - \int_{\mathbb{P}^1} \xi \cdot \bar{\partial}q,$$

where  $\mu = \bar{\partial}\xi$  on  $\Omega$  and 0 elsewhere. This means precisely that  $\bar{\partial}\xi = \mu$  in the sense of distributions on  $\mathbb{P}^1$ , which proves the first assertion of the theorem.

Let us now prove the second assertion. Denote by  $\tilde{\xi} = p^* \xi|_{\Omega}$  where  $p : \Delta \rightarrow \Omega$  is a universal cover of  $\Omega$  mapping 0 to an arbitrary point  $z_0 \in \Omega$ . Proposition 2 applied to  $\tilde{\xi}$  and  $q = \frac{dz^2}{z}$  on  $\Delta$  yields :

$$\text{Res} \left( \frac{dz^2}{z} \cdot \tilde{\xi}(0), 0 \right) = dz(\tilde{\xi}(0)) = \frac{1}{2i\pi} \int_{\Delta_r} \bar{\partial}\tilde{\xi}(z) \cdot \frac{dz^2}{z} + \frac{1}{2i\pi} \int_{S_r} \tilde{\xi}(z) \cdot \frac{dz^2}{z}$$

where  $\Delta_r$  and  $S_r$  are respectively the disk of radius  $r$  and the circle of radius  $r$ . Since we assumed that  $\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} = \|\rho_\Delta(\tilde{\xi})\|_{L^\infty(\Delta)}$  is finite, the second term converges to 0 when  $r$  tends to 1. Therefore, by letting  $r$  converging to 1 :

$$dz(\tilde{\xi}(0)) = \frac{1}{2i\pi} \int_{\Delta} \bar{\partial}\tilde{\xi}(z) \cdot \frac{dz^2}{z}$$



and

$$\rho_{\Delta}(\tilde{\xi})(0) = 2|\tilde{\xi}(0)| \leq 2\frac{1}{2\pi}\left\|\frac{dz^2}{z}\right\|_{L^1(\Delta)}\|\bar{\partial}p^*\xi\|_{L^\infty}.$$

Since  $\left\|\frac{dz^2}{z}\right\|_{L^1(\Delta)} = 4\pi$  et  $\bar{\partial}p^*\xi = p^*\bar{\partial}\xi$ , we deduce

$$\rho_{\Delta}(\tilde{\xi})(0) = \rho_{\Omega}(\xi)(z_0) \leq 4\|\bar{\partial}\xi\|_{L^\infty(\Omega)}.$$

Since  $z_0$  is arbitrary, this concludes the proof of the second assertion.  $\square$

This last theorem states that if we have a hyperbolically bounded quasiconformal vector field on an open set  $\Omega$ , we can glue it together with the zero vector field outside  $\Omega$  and still get a globally quasiconformal vector field. The next proposition gives a little more than the converse. We will need the following lemma :

**Lemma 3.12.** *Let  $\Omega$  be a hyperbolic open subset of  $\mathbb{P}^1$ , and  $X$  a countable dense subset of  $\partial\Omega$ . Let  $(X_n)$  be an increasing sequence of finite subsets of  $X$  with  $\cup_n X_n = X$  and  $\text{card}X_n \geq 3$  for all  $n \in \mathbb{N}$ , and note  $\Omega_n = \mathbb{P}^1 - X_n$ . Then the hyperbolic metric  $\rho_{\Omega_n}$  of  $\Omega_n$  converges pointwise on  $\Omega$  to the hyperbolic metric  $\rho_{\Omega}$  of  $\Omega$ .*

The proof is not difficult and makes use of Montel's theorem and the Schwarz lemma applied to the inclusions  $\Omega \hookrightarrow \Omega_n$ .

**Proposition 3.** Let  $\xi$  be a quasiconformal vector field on  $\mathbb{P}^1$  vanishing on the boundary of a hyperbolic open subset  $\Omega$  of  $\mathbb{P}^1$ . Then :

$$\|\rho_{\Omega}(\xi)\|_{L^\infty(\Omega)} \leq 4\|\bar{\partial}\xi\|_{L^\infty(\Omega)}$$

*Proof.* Denote by  $K$  the boundary of  $\Omega$ . Let  $(X_n)_{n \in \mathbb{N}}$  be an increasing sequence of finite subsets of  $\partial\Omega$  whose union is dense in  $\partial\Omega$ , with  $\text{card}X_n \geq 3$ . Then by lemma 3.12, the hyperbolic metric  $\rho_{\Omega_n}$  of  $\Omega_n = \mathbb{P}^1 - X_n$  converges pointwise to the hyperbolic metric  $\rho_{\Omega}$  of  $\Omega$  on  $\Omega$ . By Theorem 3.10, it then suffices to show that for all  $n \in \mathbb{N}$ ,  $\|\rho_{\Omega_n}(\xi)\|_{L^\infty(\Omega)}$  is bounded.

Therefore it is enough to show the weaker property : for all  $n \in \mathbb{N}$ , there exists a constant  $C_n > 0$  such that  $\sup_{\Omega} \rho_{\Omega_n}(\xi) \leq C_n$ . Since  $\rho_{\Omega_n}(\xi)$  is a continuous function on  $\Omega_n = \mathbb{P}^1 - X_n$ , it is enough to show that  $\rho_{\Omega_n}(\xi)$  is bounded in the neighborhood of all  $z \in X_n$  (by a constant depending for now on  $n \in \mathbb{N}$ ). Let  $z_0 \in X_n$ , and  $r > 0$  such that the punctured disk  $U$  of center  $z_0$  and radius  $r$  is included in  $\Omega_n$ . Then by the Schwarz lemma, the hyperbolic metric of  $\Omega_n$  is smaller than that of  $U$ , so we have for all  $z \in U$  :

$$\rho_{\Omega_n}(\xi)(z) \leq \rho_U(\xi)(z) \leq C'_n |\xi(z)| (|z - z_0| \log |z - z_0|^{-1})^{-1}.$$

The second inequality is a classical estimate of the hyperbolic metric of the punctured disk in the neighborhood of  $z_0$  (see for example [GL00] or [Hub06]). The constant  $C'_n$  still depends a priori on  $r$  and therefore on  $n$ . Furthermore,  $\xi$  has a continuity modulus on  $-\epsilon \log \epsilon$  by virtue of quasiconformality (cf [GL00], theorem 7 p. 56), so there exists a constant  $C > 0$  (depending only on  $\xi$  and on the choice of coordinates) such that in the coordinates  $z$  :

$$|\xi(z)| = |\xi(z) - \xi(z_0)| \leq C|z - z_0| \log |z - z_0|^{-1}.$$

We therefore have, for all  $z \in D_r(z_0)$  :

$$\rho_{\Omega_n}(\xi)(z) \leq C_n.$$

The Theorem 3.10 applied to  $\xi$  on  $\Omega_n$  then allows us to get a uniform bound with respect to  $n$  :

$$\|\rho_{\Omega_n}(\xi)\|_{L^\infty(\Omega_n)} \leq 4\|\bar{\partial}\xi\|_{L^\infty(\Omega_n)} \leq 4\|\bar{\partial}\xi\|_{L^\infty(\mathbb{P}^1)}.$$

By passing to the limit, we get :

$$\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\bar{\partial}\xi\|_{L^\infty(\mathbb{P}^1)},$$

and a second application of the same theorem finally yields :

$$\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\bar{\partial}\xi\|_{L^\infty(\Omega)}.$$

□

By combining the results of Theorem 3.10 and proposition 3, we get :

**Theorem A.** Let  $\Omega$  be a hyperbolic open subset of  $\mathbb{P}^1$  and  $\xi$  be a quasiconformal vector field on  $\Omega$ . The following properties are equivalent :

- i) We have  $\rho_\Omega(\xi) \in L^\infty(\Omega)$ .
- ii) We have  $\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\bar{\partial}\xi\|_{L^\infty(\Omega)}$ .
- iii) There exists a quasiconformal extension  $\hat{\xi}$  of  $\xi$  on all of  $\mathbb{P}^1$  with  $\hat{\xi} = 0$  on  $\partial\Omega$ .
- iv) The extension  $\hat{\xi}$  defined by  $\hat{\xi}(z) = \xi(z)$  if  $z \in \Omega$  and 0 else is quasiconformal on  $\mathbb{P}^1$ , and  $\bar{\partial}\hat{\xi}(z) = 0$  for almost every  $z \notin \Omega$ .

**Corollary 4.** Let  $\Omega$  be a hyperbolic open subset of  $\mathbb{P}^1$  and  $\xi$  be a quasiconformal vector field vanishing on  $\mathbb{P}^1 - \Omega$ . Let  $\Omega = \bigsqcup_i \Omega_i$  a countable partition of  $\Omega$  into open sets  $\Omega_i$ . Then

$$\xi = \sum_i \xi_i$$

where  $\xi_i$  is a quasiconformal vector field coinciding with  $\xi$  on  $\Omega_i$  and vanishing outside  $\Omega_i$ .

*Proof.* By item iv) of theorem A, the vector fields  $\xi_i$  are quasiconformal. □

Recall the following notion, which is of importance in Teichmüller theory :

**Definition 3.13.** A Beltrami differential  $\mu$  on a Riemann surface  $\mathcal{S}$  is infinitesimally trivial if  $\int_{\mathcal{S}} q \cdot \mu = 0$  for all quadratic differential  $q$  holomorphic on  $\mathcal{S}$ .

The terminology comes from the fact that the tangent space to the base point  $T_0\text{Teich}(\mathcal{S})$  identifies canonically to the quotient of the space of Beltrami differentials on  $\mathcal{S}$  by the space of infinitesimally trivial Beltrami differentials (see [GL00] or [Hub06]).

The next result is a theorem due to Bers. Its proof classically involves a delicate mollifier introduced by Ahlfors, the so-called Ahlfors Mollifier, see [GL00], theorem 9 p. 63. The mollifier  $\phi_n$  of the proof of theorem 3.10 replaces the Ahlfors Mollifier and yields a simplified proof.

**Corollary 5** (Bers density theorem). Let  $K$  be a compact of  $\mathbb{P}^1$  containing at least 3 points, and  $A$  a countable dense subset of  $K$ . The space of meromorphic quadratic differentials with simple poles in  $A$  is dense (for the  $L^1$  topology) in the space of integrable quadratic differentials on  $\mathbb{P}^1$  which are holomorphic outside of  $K$ .

*Proof.* It is enough to show that any continuous linear form on the space of integrable quadratic differentials holomorphic outside  $\mathbb{P}^1$  vanishing against all meromorphic quadratic differentials with only simple poles in  $A$  must be trivial. By the Hahn-Banach theorem, any such linear form may be represented by a  $L^\infty$  Beltrami differential on  $\mathbb{P}^1$ . Let  $\mu$  be such a Beltrami differential and  $\xi$  a quasiconformal vector field such that  $\mu = \bar{\partial}\xi$ , and assume that

$$\int_{\mathbb{P}^1} q \cdot \bar{\partial}\xi = 0$$

for all meromorphic integrable quadratic differential  $q$  with simple poles in  $A$ . Let  $Z \subset A$  a set of cardinal 3 : by adding to  $\xi$  a holomorphic vector field, we lose no generality by assuming that  $\xi$  vanishes on  $Z$ . Then by proposition 2 applied to  $\Omega = \mathbb{P}^1$  and  $q$  a quadratic differential with simple poles precisely in  $Z$  and at  $z \in A \setminus Z$ , one sees that  $\xi$  must vanish at  $z$ . By continuity,  $\xi$  vanishes on all of  $K$ . So by theorem A,  $\xi$  is hyperbolically bounded on  $\Omega$ . Let  $q$  be an integrable quadratic differential that is holomorphic on  $\Omega$ . In particular,  $q$  is  $C^\infty$  and integrable on  $\Omega$ , and  $\bar{\partial}q$  vanishes on  $\Omega$ . Lemma 3.11 yields :

$$\int_{\Omega} q \cdot \bar{\partial}\xi = - \int_{\Omega} \bar{\partial}q \cdot \xi = 0.$$

Moreover, by theorem A, we have  $\bar{\partial}\xi = 0$  almost everywhere on  $K$ , so :

$$\int_K q \cdot \bar{\partial}\xi = 0,$$

which ends the proof.  $\square$

**Corollary 6.** Let  $\Omega$  be a hyperbolic open subset of  $\mathbb{P}^1$ , and  $\mu$  be a Beltrami differential on  $\Omega$ . Then  $\mu$  is infinitesimally trivial if and only if there exists a hyperbolically bounded quasiconformal vector field  $\xi$  on  $\Omega$  such that  $\mu = \bar{\partial}\xi$ .

*Proof.* We just proved that a Beltrami differential  $\mu$  is infinitesimally trivial on  $\Omega$  if and only if there exists a quasiconformal vector field  $\xi$  on  $\mathbb{P}^1$  such that  $\mu = \bar{\partial}\xi$  on  $\Omega$  and  $\xi = 0$  on  $\mathbb{P}^1 - \Omega$ . By theorem A, this property is equivalent to being hyperbolically bounded in  $\Omega$ .  $\square$

#### 4. DYNAMICAL TEICHMÜLLER SPACE

**4.1. The differential of  $\Psi$ .** If  $\lambda \mapsto f_\lambda$  is a holomorphic curve in  $\text{Rat}_d$  passing through  $f_0 = f$ , then  $\dot{f} = \frac{df_\lambda}{d\lambda}|_{\lambda=0}$  is a section of the bundle  $f^*T\mathbb{P}^1$ , and  $Df^{-1} \circ \dot{f}$  is a meromorphic vector field on  $\mathbb{P}^1$ , whose poles are included in  $\text{Crit}(f)$  and of multiplicity at most that of the critical points of  $f$ . Denoting by  $T(f)$  the complex vector space of such vector fields, we obtain a canonical identification between  $T_f\text{Rat}_d$  and  $T(f)$ . In the rest of this article, we will implicitly identify  $T_f\text{Rat}_d$  with  $T(f)$ .

Denote as well by  $\text{aut}(\mathbb{P}^1)$  the space of holomorphic vector fields on  $\mathbb{P}^1$  and by  $\mathcal{O}(f)$  the orbit of  $f$  by conjugacy via Möbius transformation. By [BE09], proposition 1,  $\mathcal{O}(f)$  is a complex submanifold of  $\text{Rat}_d$  of dimension 3, and  $T_f\mathcal{O}(f) = \Delta_f\text{aut}(\mathbb{P}^1) \subset T(f)$ .

**Proposition 4.** Let  $\xi$  be a quasiconformal vector field on  $\mathbb{P}^1$  such that  $\bar{\partial}\xi \in \text{bel}(f)$ . Then  $\Delta_f\xi \in T(f)$ . Moreover, if we assume that  $\xi$  vanishes on a set  $Z$  of cardinal 3, then :

$$D\Psi^Z(0) \cdot \bar{\partial}\xi = -\Delta_f\xi.$$

*Proof.* An easy calculation shows that for almost every  $z \notin \text{Crit}(f)$ ,  $\bar{\partial}f^*\xi = f^*\bar{\partial}\xi$ . Therefore by Weyl's lemma,  $\Delta_f\xi = \xi - f^*\xi$  is holomorphic on  $\mathbb{P}^1 - \text{Crit}(f)$ . Since  $\xi$  is continuous, we have  $\Delta_f\xi = O(1/f')$  in the neighborhood of  $\text{Crit}(f)$ , so  $\Delta_f\xi$  has at every critical point  $c$  of  $f$  a pole of at most the multiplicity of  $c$  as a critical point of  $f$  ; so  $\Delta_f\xi \in T(f)$ .

Moreover, if  $\mu_\lambda \in \text{bel}(f)$  is a holomorphic curve passing through 0, with  $\mu_\lambda = \lambda\bar{\partial}\xi + o(\lambda)$ , then we have :

$$\phi_{\mu_\lambda}^Z = \text{Id} + \lambda\xi + o(\lambda)$$

where  $\phi_{\mu_\lambda}^Z$  is the unique quasiconformal homeomorphism associated to  $\mu_\lambda$  fixing  $Z$  (see [GL00] or [Hub06]). If we differentiate with respect to  $\lambda$  the equality

$$\phi_{\mu_\lambda}^Z \circ f = f_\lambda \circ \phi_{\mu_\lambda}^Z,$$

we get :

$$\xi \circ f = \dot{f} + Df(\xi),$$

où  $\dot{f} = \frac{df_\lambda}{d\lambda}|_{\lambda=0}$ . This can be rewritten as :

$$\eta := Df^{-1}(\dot{f}) = -\Delta_f \xi.$$

□

With an abuse of notations, we will note  $D\Psi(0) : \text{bel}(f) \rightarrow T(f)/T_f\mathcal{O}(f)$  the quotient of the linear application  $D\Psi^Z(0) : \text{bel}(f) \rightarrow T(f)$ . This application does not depend on the choice of  $Z$ .

**Definition 4.1.** Let  $f$  be a rational map. We will note  $\Lambda_f$  the closure of the grand critical orbit of  $f$ , and  $\Omega_f = \mathbb{P}^1 - \Lambda_f$ .

**Proposition 5.** Let  $\xi$  be a quasiconformal vector field on  $\mathbb{P}^1$  such that  $\bar{\partial}\xi \in \text{bel}(f)$ . The following properties are equivalent :

- i)  $\bar{\partial}\xi \in \ker D\Psi(0)$
- ii)  $\Delta_f \xi \in \Delta_{f\text{aut}}(\mathbb{P}^1)$
- iii) There exists  $h \in \text{aut}(\mathbb{P}^1)$  such that  $\xi - h$  vanishes on  $\text{Crit}(f)$  with at least the multiplicity of each critical point of  $f$
- iv) There exists  $h \in \text{aut}(\mathbb{P}^1)$  such that  $\xi - h$  vanishes on  $\Lambda_f$

*Proof.* The first two items are equivalent by [BE09], proposition 1.

ii)  $\Rightarrow$  iii) : if  $\Delta_f \xi = \Delta_f h$ ,  $h \in \text{aut}(\mathbb{P}^1)$ , then  $\xi - h$  is a continuous  $f$ -invariant vector field. Hence  $\xi - h$  must vanish on  $\text{Crit}(f)$  with at least the multiplicity of the critical points of  $f$ .

iii)  $\Rightarrow$  ii) : If  $\xi - h$  vanishes on  $\text{Crit}(f)$  with at least the multiplicity of the critical points of  $f$ , then  $f^*(\xi - h)$  is well-defined and continuous at  $\text{Crit}(f)$ . By the above proposition,  $\Delta_f(\xi - h) \in T(f)$ , so  $\Delta_f(\xi - h) = 0$ .

iv)  $\Rightarrow$  ii) : If  $\xi - h$  vanishes on  $\Lambda_f$ , then since  $\Lambda_f$  is invariant  $\Delta_f(\xi - h)$  vanishes as well on  $\Lambda_f$ . Therefore  $\Delta_f(\xi - h)$  is a meromorphic vector field (by the above proposition) vanishing on  $\Lambda_f$  which is not discrete, so  $\Delta_f(\xi - h) = 0$  by the isolated zeros principle.

ii)  $\Rightarrow$  iv) : If  $\Delta_f(\xi - h) = 0$ , then we saw that  $\xi - h$  must vanish on  $\text{Crit}(f)$  (item iii)). Therefore  $(f^k)^*(\xi - h)(c) = (\xi - h)(c) = 0$  for all  $k \geq 0$ . Moreover, if  $f^p(z) = c \in \text{Crit}(f)$ , then  $(f^p)^*(\xi - h)(z) = 0 = (\xi - h)(z)$ . So  $(\xi - h)$  vanishes on the grand critical orbit of  $f$ , hence on  $\Lambda_f$  by continuity. □

Note that if we normalize  $\xi$  by imposing the condition that it vanishes on a set  $Z$  invariant by  $f$  of cardinal 3, then proposition 5 remains true by replacing  $h$  by 0 in items ii), iii) and iv), and  $D\Psi(0)$  by  $D\Psi^Z(0)$  in item i).

We will also need to know the differential  $\Psi^Z$  in an arbitrary point of  $\text{Bel}(f)$ . Recall the following fact of Teichmüller theory (see [Hub06]) :

**Definition 4.2.** Let  $\psi$  be a quasiconformal homeomorphism of  $\mathbb{P}^1$ . For all Beltrami form  $\mu$ , note  $\psi^*\mu$  the Beltrami form corresponding to  $\phi_\mu \circ \psi$ , where  $\phi_\mu$  is a quasiconformal homeomorphism associated to  $\mu$ .

We will also note  $\psi_* = (\psi^{-1})^*$ .

**Proposition 6.** For all quasiconformal homeomorphism  $\psi$ , the map  $\psi^*$  is biholomorphic.

We shall need to consider here maps  $\Psi_f^Z : \text{Bel}(f) \rightarrow \text{Rat}_d$  and  $\Psi_g^Z : \text{Bel}(g) \rightarrow \text{Rat}_d$  associated to different rational maps  $f$  and  $g$ . In the rest of the article, there will be no ambiguity and we will just use the notation  $\Psi^Z$ .

**Proposition 7.** Let  $\mu \in \text{Bel}(f)$  and  $\psi$  the unique corresponding quasiconformal homeomorphism fixing  $Z$ . Let  $g = \psi \circ f \circ \psi^{-1}$ . Then

$$D\Psi_f^Z(\mu) = D\Psi_g^Z(0) \circ D\psi_*(\mu)$$

In particular,  $\text{rg}D\Psi_f^Z(\mu) = \text{rg}D\Psi_g^Z(0)$ .

*Proof.* Remark that for all  $\phi_0$  and  $\phi_\lambda$  associated to elements  $\mu_\lambda$  and  $\mu_0$  of  $\text{bel}(f)$  :

$$\phi_\lambda \circ f \circ \phi_\lambda^{-1} = (\phi_\lambda \circ \phi_0^{-1}) \circ \phi_0 \circ f \circ \phi_0^{-1} \circ (\phi_\lambda \circ \phi_0^{-1})^{-1}$$

which may be rewritten as :

$$\Psi_f^Z(\mu_\lambda) = \Psi_g^Z(\phi_*\mu_\lambda)$$

if we assume additionally that  $\phi_\lambda$  and  $\phi_0$  fix  $Z$ . Then we only need to take a curve  $\mu_\lambda$  in  $\text{bel}(f)$ , and to differentiate at  $\lambda = 0$ .  $\square$

**4.2. Constant rank theorem in Banach spaces.** Recall the following version of the constant rank theorem in infinite dimension :

**Theorem 4.3** (Constant rank theorem). *Let  $\Psi : U \rightarrow F$  be an analytic map, where  $U$  is an open subset of a complex Banach space  $E$  and  $F$  is a complex finite-dimensional vector space. Assume that  $\text{rg}D\Psi = r$  is constant on  $U$ . Then for every  $x_0 \in U$ , there exists a germ of analytic diffeomorphism  $\chi : (F, f(x_0)) \rightarrow (F, f(x_0))$  and a germ of analytic diffeomorphism  $\phi : \text{Im}D\Psi(x_0) \oplus \ker D\Psi(x_0) \rightarrow E$  such that for all  $(u, v) \in \text{Im}D\Psi(x_0) \oplus \ker D\Psi(x_0)$  in the neighborhood of  $\phi^{-1}(x_0)$ ,*

$$\chi \circ \Psi \circ \phi(u, v) = u.$$

**Corollary 7.** Let  $\Psi : E \rightarrow F$  verifying the requirements of the above theorem. Then for all  $z_0 \in \Psi(E)$ , the level set  $M = \Psi^{-1}(z_0)$  is a Banach submanifold of  $E$ , of codimension  $r$  and whose tangent space at  $x_0 \in \Psi^{-1}(z_0)$  is  $T_{x_0}M = \ker D\Psi(x_0)$ .

*Proof.* With the notations of the constant rank theorem, we have  $\Psi(x) = z_0$  if and only if  $\chi \circ \Psi(u, v) = \chi(z_0) = u$ , where  $(u, v) = \psi^{-1}(x)$ , which is equivalent to  $\psi(\chi(z_0), v) = x$ . Since  $\psi$  is a (germ of) diffeomorphism, this gives a local chart at  $x_0$  for  $M$ , which is therefore a Banach submanifold modeled on  $\ker D\Psi(x_0)$ .  $\square$

**4.3. Counting dimensions.** The goal of this section is to show that the differential of  $\Psi^Z : \text{Bel}(f) \rightarrow \text{Rat}_d$  has constant rank.

**Definition 4.4.** We say that a critical point is acyclic if it is not preperiodic. We say that two acyclic critical points lie in the same foliated acyclic critical class if the closure of their grand orbits are the same.

The key point to apply the constant rank theorem is the following count of dimension :

**Theorem 4.5.** *Let  $f$  be a rational map of degree  $d \geq 2$ . Then*

$$\text{rg}D\Psi(0) = n_f + n_H + n_J - n_p$$

where  $n_H$  is the number of Herman rings of  $f$ ,  $n_J$  is the number of ergodic line fields of  $f$ ,  $n_f$  is the number of foliated acyclic critical classes lying in the Fatou set, and  $n_p$  is the number of parabolic cycles.

**Definition 4.6.** Let  $f : \mathcal{S} \rightarrow \mathcal{S}$  be a holomorphic function. Denote by  $M_f(\mathcal{S})$  the space of Beltrami forms that are invariant by  $f$ , and by  $N_f(\mathcal{S})$  the subspace of  $M_f(\mathcal{S})$  of Beltrami differentials of the form  $\bar{\partial}\xi$ , where  $\xi$  is a hyperbolically bounded quasiconformal vector field on  $\mathcal{S}$ .

**Theorem 4.7.** Let  $f$  be a rational map, and  $\Omega$  a hyperbolic open subset of  $\mathbb{P}^1$  completely invariant under  $f$ . Let  $\Omega = \bigsqcup_i \Omega_i$  be a partition of  $\Omega$  into open subsets  $\Omega_i$  completely invariant under  $f$ . Then

$$M_f(\Omega)/N_f(\Omega) \simeq \bigoplus_i M(\Omega_i)/N(\Omega_i)$$

*Proof.* Clearly  $M_f(\Omega) = \bigoplus_i M(\Omega_i)$ .

Let  $\bar{\partial}\xi \in N_f(\Omega)$ . By corollary 4, we have :

$$\xi = \sum_i \xi_i$$

where  $\xi_i$  is a quasiconformal vector field coinciding with  $\xi$  on  $\Omega_i$ , and such that  $\xi_i = 0$  outside of  $\Omega_i$ . This shows that  $N_f(\Omega) = \bigoplus_i N_f(\Omega_i)$ .

Hence  $M_f(\Omega)/N_f(\Omega) = \bigoplus_i M_f(\Omega_i)/N_f(\Omega_i)$ .  $\square$

Lastly, we will need the classification of Fatou components, which is a corollary of Sullivan's no wandering domain theorem. Note that McMullen (see [McM14]) has given a direct and purely infinitesimal proof of Sullivan's theorem, which does notably not rely on the theory of dynamical Teichmüller spaces. His proof is based on quasiconformal vector fields and is in the same spirit as the methods used here.

We will also need the following lemmas :

**Definition 4.8.** Let  $M(\mathcal{S})$  be the set of Beltrami differentials on the Riemann surface  $\mathcal{S}$  and  $N(\mathcal{S})$  be the subspace of Beltrami differentials on  $\mathcal{S}$  that are of the form  $\bar{\partial}\xi$ , where  $\xi$  is a hyperbolically bounded quasiconformal vector field on  $\mathcal{S}$ .

**Lemma 4.9.** Suppose  $\Omega$  is the grand orbit of a component of  $\Omega_f$  such that  $\Omega/f$  is a hyperbolic Riemann surface. Then the projection  $\pi_1 : \Omega \rightarrow \Omega/f$  induces an identification :

$$M_f(\Omega)/N_f(\Omega) \simeq M(\Omega/f)/N(\Omega/f)$$

*Proof.* It is clear that  $M_f(\Omega) \simeq M(\Omega/f)$ , and that any element of  $N_f(\Omega)$  passes to the quotient to an element of  $N(\Omega/f)$ . Let  $\mu = \bar{\partial}\xi \in N_f(\Omega)$ . Since the map  $\pi_1 : \Omega \rightarrow \Omega/f$  is a covering between hyperbolic Riemann surfaces, it is a local isometry for the hyperbolic metrics, and therefore  $\xi_1 = \pi_1^*\xi$  is a hyperbolically bounded quasiconformal vector field, which is invariant by  $f$  by construction. By theorem A,  $\hat{\xi}_1$  extended by 0 outside  $\Omega$  is still quasiconformal (and invariant). So  $\mu = \bar{\partial}\hat{\xi}_1 \in N_f(\Omega)$ . This proves that  $N_f(\Omega) \simeq N(\Omega/f)$ .  $\square$

**Lemma 4.10.** Let  $\Omega \subset \Omega_f$  be an open set completely invariant under  $f$  such that all connected component of  $\Omega$  is preperiodic to a component  $U$  of period  $p \in \mathbb{N}^*$ . Then the restriction to  $U$  induces an isomorphism  $M_f(\Omega) \rightarrow M_{f^p}(U)$ , mapping  $N_f(\Omega)$  onto  $N_{f^p}(U)$ . In particular,

$$M_f(\Omega)/N_f(\Omega) \simeq M_{f^p}(U)/N_{f^p}(U)$$

*Proof.* Every Beltrami differential  $\mu \in M_f(\Omega)$  is invariant under  $f$ , hence under  $f^p$ . Conversely, if  $\mu$  is a Beltrami differential on  $U$  invariant under  $f^p$ , then  $\mu$  extends to a Beltrami differential  $\tilde{\mu}$  invariant on  $\Omega_i$  in the following way : if  $V$  is a component of  $\Omega_i$ ,

then there exists  $k \in \mathbb{N}$  (defined up to a multiple of  $p$ ) such that  $f|_V^k : V \rightarrow U$ . We then set  $\tilde{\mu}|_V = (f^k)^*\mu$ , and this definition is valid if  $V$  belongs to the same cycle as  $U$  since  $\mu = (f^p)^*\mu$ .

This identification maps  $N_f(\Omega)$  onto  $N_{f^p}(U)$  since if  $\mu = \bar{\partial}\xi \in N_f(\Omega)$ , then  $\hat{\xi}(z) = \xi(z)$  if  $z \in U$  and 0 else is such that  $\bar{\partial}\hat{\xi} = \mu|_U$  by theorem A, and therefore  $\mu|_U \in N_{f^p}(U)$ .  $\square$

**Lemma 4.11.** *Let  $\mu$  be a Beltrami differential invariant under a holomorphic function  $g$ . In both of the following cases :  $g(z) = e^{2i\pi\alpha}z$ ,  $\alpha \notin \mathbb{Q}$ , and  $g(z) = z^d$ ,  $d \geq 2$ ,  $\mu$  is then invariant under all rotations, and we have in local coordinates :*

$$\mu(re^{it}) = c(r)e^{2it}\frac{\bar{d}z}{dz}$$

*Proof.* The proof is a modification of the usual proof of the ergodicity of rotations of irrational angles.

Let us start with the case of a rotation of irrational angle  $g(z) = e^{2i\pi\alpha}z$ . Let  $\mu$  be a Beltrami differential invariant by  $g$ . We have, in local coordinates :

$$\mu(z) = g^*\mu(z) = e^{-4i\pi\alpha}\mu(e^{2i\pi\alpha}z)$$

By expanding into Fourier series on the circles  $|z| = r$ , we obtain that  $\mu$  must be of the form

$$\mu(re^{it}) = c(r)e^{2it}\frac{\bar{d}z}{dz}$$

where  $c$  is a  $L^\infty$  function. In particular,  $\mu$  is invariant by rotations, and one easily verifies that all rotation-invariant Beltrami differential must be of this form.

If we now assume that  $g(z) = z^2$ ,  $d \geq 2$ , and that  $\mu$  is invariant by  $g$ , then  $\mu$  is invariant by all branches of  $g^{-n} \circ g^n$ , hence by all rotations of angles  $\frac{2k\pi}{d^n}$ , for all  $k \in \mathbb{N}$  :

$$\mu(z) = e^{-4ik\pi/d^n}\mu(e^{2ik\pi/d^n}z).$$

Similarly, by expanding into Fourier series on the circles centered on 0, we obtain :

$$\mu(re^{it}) = c(r)e^{2it}\frac{\bar{d}z}{dz}.$$

$\square$

**Lemma 4.12.** *Let  $\Omega$  be a rotation invariant planar open set. Let  $M(\Omega)$  be the space of rotation-invariant Beltrami differentials on  $\Omega$ , and  $N(\Omega)$  the subspace of  $M(\Omega)$  of elements of the form  $\bar{\partial}\xi$ , where  $\xi$  is a hyperbolically bounded quasiconformal vector field on  $\partial\Omega$ .*

- i) If  $\Omega$  is the unit disk,  $\dim M(\Omega)/N(\Omega) = 0$ .*
- ii) If  $\Omega$  is a ring of finite modulus, then  $\dim M(\Omega)/N(\Omega) = 1$ .*

*Proof.* Consider a vector field  $\xi$  of the form

$$\xi(re^{it}) = h(r)re^{it}\frac{d}{dz}$$

where  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a lipschitz function. One can easily verify that

$$\bar{\partial}\xi(re^{it}) = rh'(r)e^{2it}\frac{\bar{d}z}{dz}.$$

Therefore if  $\mu$  is a rotation-invariant Beltrami differential, hence of the form  $\mu(re^{it}) = c(r)e^{2it}\frac{\bar{d}z}{dz}$ , and if we denote by  $h$  the unique primitive of  $r \mapsto c(r)/r$  vanishing at  $r = 1$  and  $\xi(re^{it}) = rh(r)e^{it}\frac{d}{dz}$ , we have  $\bar{\partial}\xi = \mu$  in the sense of distributions, and  $\xi$  is a quasiconformal vector field on all of  $\mathbb{P}^1$  vanishing on the unit disk.

Therefore  $M(\Delta) = N(\Delta)$ . If now  $\Omega$  denotes a straight ring  $\Omega = \{r_0 < |z| < 1\}$ , the map

$$\mu = c(r)e^{2it}\frac{\overline{dz}}{dz} \mapsto h(r_0) = \int_1^{r_0} \frac{c(u)}{u} du$$

is a linear form on  $M(\Omega)$  whose kernel is exactly  $N(\Omega)$ . This linear form is not trivial, since if we take  $\mu = re^{2it}\frac{\overline{dz}}{dz}$ , then  $h(r_0) = r_0 - 1 \neq 0$ . Therefore  $\dim M(\Omega)/N(\Omega) = 1$ .  $\square$

We can now prove theorem 4.5.

*Proof of theorem 4.5.* Denote by  $\mathcal{F}$  the Fatou set of  $f$ , and  $\mathcal{J}$  its Julia set. We will also denote by  $\text{Fix}_{\mathcal{J}}$  the space of invariant line fields. Since  $\ker D\Psi(0) = N_f(\Omega_f) = N_f(\mathcal{F})$  by proposition 5, we have :

$$\text{bel}(f)/\ker D\Psi(0) = (\text{Fix}_{\mathcal{J}} \oplus M_f(\mathcal{F}))/N_f(\mathcal{F})$$

If  $c$  is a critical point of  $f$ , then the closure of its grand orbit is equal to the union of the Julia set  $\mathcal{J}$  and of a countable set of points and smooth circles (if the orbit of  $c$  is captured by a superattracting cycle, or a cycle of Siegel disks or Herman rings). Therefore  $\Lambda_f$  coincides with  $\mathcal{J}$  up to a set of Lebesgues measure zero. Hence  $M_f(\mathcal{F}) = M_f(\Omega_f)$ . We deduce from this observation that :

$$\text{Fix}(f)/\ker D\Psi(0) = \text{Fix}_{\mathcal{J}} \oplus M_f(\Omega_f)/N_f(\Omega_f)$$

Consider the equivalence relationship on the set of connected components of  $\Omega_f$  which identifies two components if and only if they have the same grand orbit, and let  $\Omega_i$  be the union of the elements of a class  $i$  of this equivalence relationship. The  $\Omega_i$  form a partition of  $\Omega_f$  into completely invariant open subsets. By theorem 4.7, we have :

$$\text{rg} D\Psi(0) = \dim \text{Fix}_{\mathcal{J}} + \sum_i \dim M_f(\Omega_i)/N_f(\Omega_i).$$

Each component  $\Omega_i$  is mapped by  $f^n$  for  $n$  large enough into a periodic Fatou component  $U$ .<sup>1</sup> Let us now compute  $\dim M_f(\Omega_i)/N_f(\Omega_i)$  depending on the nature of the periodic Fatou component  $U$  it meets. There are five cases to consider. Denote by  $n_i$  the number of foliated acyclic critical classes meeting the grand orbit of  $U$

a) The case of an attracting cycle

If  $U$  is a component of an attractive basin and  $\Omega_i$  meets  $U$ , then  $\Omega_i$  is the grand orbit of  $U$  with the countable set of the critical orbits captured by this cycle (and the cycle itself) removed. So every component of  $\Omega_i$  is preperiodic to  $U - \Lambda_f$ . Thus  $f|_{\Omega_i} : \Omega_i \rightarrow \Omega_i$  acts discretely, and  $X_i = \Omega_i/f$  is a Riemann surface. In a linearizing coordinate for  $f^k$  on the immediate basin of attraction (where  $k \in \mathbb{N}^*$  is the period of the cycle and  $\rho$  is its multiplier), note  $A = \{|\rho| \leq z < 1\}$ . It is a fundamental domain for the action of  $f$  on the cycle of Fatou components  $V$  containing  $U$ , and  $A - \Lambda_f$  is a fundamental domain for the action of  $f$  on  $\Omega_i$ . Therefore  $X_i$  is the torus  $X = A/f$  with a finite number  $n_i$  of points removed, where  $n_i$  is the number of points of the post-critical set meeting  $A$ , i.e. the number of foliated acyclic critical classes meeting  $V$ .

By lemma 4.9,  $\dim M_f(\Omega_i)/N_f(\Omega_i) = \dim M(X_i)/N(X_i)$ . Since  $X_i$  is a finitely punctured torus, any hyperbolically bounded quasiconformal vector field on  $X_i$  extends to a quasiconformal vector field on the torus vanishing on the marked points. Then the quotient  $M(X_i)/N(X_i)$  is exactly the tangent space to the Teichmüller space of  $X_i$ , which

<sup>1</sup>However, in the case of a superattracting cycle, the components  $\Omega_i$  need not be themselves preperiodic : if there is a critical orbit in a superattracting basin, one gets components  $\Omega_i$  which are annuli delimited by equipotentials that accumulate on the superattracting cycle.



has dimension equal to the number  $n_i$  of marked points (see for example [Hub06]).

b) The case of a parabolic cycle

If  $U$  is a parabolic cycle and  $\Omega_i$  meets  $U$ , then  $\Omega_i$  is the grand orbit of  $U$  minus the grand orbit of the critical points captured by  $U$ . In particular, all component of  $\Omega_i$  is iterated after a finite number of steps into  $U$  with at most a countable set of points removed, and is preperiodic. Moreover,  $f|_{\Omega_i} : \Omega_i \rightarrow \Omega_i$  acts discretely, so  $X_i = \Omega_i/f$  is a Riemann surface isomorphic to  $X = U/f^p$  minus the grand orbit of critical points captured by  $U$ , where  $p$  is the period of the parabolic cycle associated to  $U$ .

Via a Fatou coordinate, the action of  $f^p$  on  $U$  is conjugated to that of  $z \mapsto z+1$  on an upper half-plane, so  $X$  is isomorphic to a cylinder and  $X_i$  is isomorphic to a cylinder with  $n_i$  points removed, those points corresponding to the  $n_i$  grand critical orbits captured by  $U$ . So  $X$  is isomorphic to the Riemann sphere with two points  $a_1$  and  $a_2$  removed, and  $X_i$  is isomorphic to the Riemann sphere with  $n_i + 2$  points  $a_1, \dots, a_{n_i+2}$  removed, where the  $a_j$ ,  $j \geq 2$  correspond to the grand critical orbit meeting  $U$ .

By lemma 4.9,  $\dim M_f(\Omega_i)/N_f(\Omega_i) = \dim M(X_i)/N(X_i)$ . Since  $X_i$  is a finitely punctured sphere, any hyperbolically bounded quasiconformal vector field on  $X_i$  extends to a quasiconformal vector field on the torus vanishing on the marked points. Then the quotient  $M(X_i)/N(X_i)$  is exactly the tangent space to the Teichmüller space of  $X_i$ , which has dimension equal to the number  $n_i + 2 - 3 = n_i - 1$ , where  $n_i$  is the number of critical grand orbits meeting  $\Omega_i$  (see for example [Hub06]).

c) The case of a Siegel disk

If  $U$  is a Siegel disk, then the intersection of  $\Lambda_f$  and the cycle of Fatou components containing  $U$  consists in a finite union of  $n_i$  smooth circles, where  $n_i$  is the number of foliated acyclic critical classes captured by the cycle of Siegel disks (it may be that  $n_i = 0$ ). Therefore all components of  $\Omega_i$  are preperiodic and are iterated in a finite number of steps to a periodic ring  $A_i$  included in  $U$  or a topological disk strictly included in  $U$  (if  $n_i \neq 0$ ), or in all of the periodic Siegel disk if  $n_i = 0$ . In both cases, denote by  $V$  the periodic component of  $\Omega_i$  to which is iterated a given component of  $\Omega_i$ .

By lemma 4.10, the space  $M_f(\Omega_i)$  identifies to the space  $M_{f^p}(V)$  of Beltrami differentials on  $V$  that are invariant by  $f^p|_V$ , where  $p$  is the period of the cycle associated to  $U$ , and similarly  $N_f(\Omega_i)$  identifies to  $N_{f^p}(V)$ . A linearizing coordinate  $\phi$  for  $f^p$  conjugates  $f^p : V \rightarrow V$  to  $g(z) = e^{2i\pi\alpha}z$  on either the unit disk or an annulus  $A(R)$ , where  $\alpha$  is an irrational rotation number. Therefore, by lemmas 4.11 and 4.12,  $\dim M_f(\Omega_i)/N_f(\Omega_i) = 1$  if  $n_i \neq 0$  and 0 else. We then obtain  $\sum_{j \in J} \dim M_f(\Omega_j)/N_f(\Omega_j) = n_i$ .

d) The case of a Herman ring

This case is very similar to the case of a Siegel disk :  $\Omega_i$  still consists in the grand critical orbit of a periodic annulus. The only difference is that even if there are no critical orbit lying in the Herman ring, the components of  $\Omega_i$  are still preperiodic to a ring and not a disk, and therefore  $\dim M_f(\Omega_i)/N_f(\Omega_i) = 1$ . We deduce :  $\sum_{j \in J} \dim M_f(\Omega_j)/N_f(\Omega_j) = n_i + 1$  where  $n_i$  is the number of foliated acyclic critical classes captured by  $U$ .

e) The case of a superattracting cycle

If  $U$  is a component of a superattracting cycle, then  $\Lambda_f \cap U$  is a countable union of equipotentials (which are smooth circles) and the superattracting cycle itself.

Assume first that there are no critical orbits captured by the superattracting cycle. Then there is a unique  $\Omega_i$  intersecting  $U$ , and it is the whole grand orbit of  $U$ . By

lemma 4.10,  $M_f(\Omega_i)/N_f(\Omega_i) \simeq M_{f^p}(U)/N_{f^p}(U)$ , where  $p$  is the period of  $U$ . Through a Böttcher coordinate,  $f|_U : U \rightarrow U$  is conjugated to  $g(z) = z^k$ ,  $k \geq 2$ . By lemma 4.11, every Beltrami differential  $\mu \in M_{f^p}(U)$  is invariant by rotation. By lemma 4.12, we deduce that if there are no critical orbits meeting  $U$ , then  $\dim M_{f^p}(U)/N_{f^p}(U) = 0$ .

Assume now that  $n_i > 0$ , where  $n_i$  is the number of foliated acyclic critical classes meeting  $U$ . Let us denote by  $r_j$ ,  $j \leq n_i$ , the radii in Böttcher coordinates of the circles corresponding to foliated acyclic critical classes in  $U$ . Note  $A(r, r')$  the annulus  $\{r' < |z| < r\}$ . Let  $\Omega_j \subset \Omega_f$  meeting  $U$ . Then for every component  $V$  of  $\Omega_j$ , there exists a unique branch of  $f^{-k} \circ f^l$  mapping  $V$  into the annulus  $A(r_{j-1}, r_j)$  (with the convention  $r_{-1} = 1$ ). By lemma 4.11,  $M_f(\Omega_j)$  identifies to  $M(A(r_{j-1}, r_j))$ , and  $N_f(\Omega_j)$  to  $N(A(r_{j-1}, r_j))$ . We deduce from this that  $\dim M_f(\Omega_i)/N_f(\Omega_i) = 1$ . Therefore  $\sum_{j \in J} \dim M_f(\Omega_j)/N_f(\Omega_j) = n_i$ .

Summing things up, each Fatou component  $U$  contributes  $n_i$  to the dimension, where  $n_i$  is the number of foliated acyclic critical classes meeting  $U$ , except for Herman rings which contribute  $n_i + 1$  and the parabolic basins which contribute  $n_i - 1$ .

Moreover, ergodic line fields form a basis of the vector space  $\text{Fix}_J$  of invariant line fields, therefore  $\dim \text{Fix}_J = n_J$ . Thus we have :

$$\text{rg} D\Psi(0) = n_H + n_J + n_f - n_p.$$

□

## 5. PROOF OF THE MAIN THEOREM

The first application of theorem 4.5 is that  $\Psi^Z$  has constant rank :

**Corollary 8.** Let  $f$  be a rational map,  $Z$  be an invariant set of cardinal 3 and  $\mu \in \text{Bel}(f)$ . Then  $\text{rg} D\Psi^Z(0) = \text{rg} D\Psi^Z(\mu)$ .

*Proof.* It is clear that  $n_f$ ,  $n_p$  and  $n_H$  are invariant under quasiconformal conjugacy. The number  $n_J$  is invariant as well since a quasiconformal homeomorphism preserve sets of Lebesgues measure zero (see [GL00]). Therefore if  $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a quasiconformal conjugacy between  $f$  and another rational map  $g$ , then  $\phi^*$  maps invariant line fields for  $f$  to invariant line fields for  $g$ . Lemma 7 concludes the proof. □

**Corollary 9.** The group  $\text{QC}(f)$  is a Banach submanifold of  $\text{Bel}(f)$ , of tangent space to the identity equal to the space  $N_f(\Omega_f)$  of Beltrami differentials of the form  $\overline{\partial}\xi$ , where  $\xi$  is a quasiconformal vector field invariant by  $f$ .

*Proof.* The space of quasiconformal homeomorphisms commuting with  $f$  is exactly the fiber  $\Psi^{-1}(f)$ . But by the above corollary,  $\Psi^Z$  has constant finite rank on  $\text{Bel}(f)$ , therefore by the constant rank theorem,  $(\Psi^Z)^{-1}(f)$  is a Banach submanifold of finite codimension, whose tangent space to the identity is  $\ker D\Psi(0) = N_f(\Omega_f)$ . Moreover,  $N_f(\Omega_f)$  is also the space of Beltrami differentials of the form  $\overline{\partial}\xi$ , where  $\xi$  is a quasiconformal vector field invariant by  $f$  by proposition 5. □

Note that in particular,  $\text{QC}(f)$  is locally connected at the identity, and therefore on a neighborhood of the identity, any element of  $\text{QC}(f)$  belongs also to  $\text{QC}_0(f)$ .

**Corollary 10.** There exists a unique structure of complex manifold on  $\text{Teich}(f)$  making the projection  $\pi : \text{Bel}(f) \rightarrow \text{Teich}(f)$  holomorphic. For this complex structure,  $\pi$  is a split submersion.

*Proof.* Let  $\mu \in \text{Bel}(f)$ . By the constant rank theorem 8, there exists germs of biholomorphisms  $\phi : (\text{Im}D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu), 0) \rightarrow (\text{Bel}(f), \mu)$  et  $\chi : (\text{Rat}_d, g) \rightarrow (\text{Rat}_d, g)$  such that  $\chi \circ \Psi^Z \circ \phi(u, v) = u$  for all  $(u, v) \in \text{Im}D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu)$ , where  $g = \phi_\mu \circ f \circ \phi_\mu^{-1}$ .

In particular,  $\Psi^Z \circ \phi(u_1, v_1) = \Psi^Z \circ \phi(u_2, v_2)$  if and only if  $u_1 = u_2$ ; moreover, if we note  $\mu_i = \phi(u_i, v_i)$ ,  $1 \leq i \leq 2$ , then  $\Psi^Z(\mu_1) = \Psi^Z(\mu_2)$  if and only if  $\phi_1^Z \circ (\phi_2^Z)^{-1} \in \text{QC}(f)$  where  $\phi_i^Z$  is the quasiconformal homeomorphism corresponding to  $\mu_i$  and fixing  $Z$ .

We claim that  $\pi(\mu_1) = \pi(\mu_2)$  if and only if  $u_1 = u_2$ . Indeed, if  $\pi(\mu_1) = \pi(\mu_2)$ , then  $\phi_1^Z \circ (\phi_2^Z)^{-1} \in \text{QC}_0(f)$  and in particular  $\phi_1^Z \circ (\phi_2^Z)^{-1} \in \text{QC}(f)$ , so  $u_1 = u_2$ . If now we assume that  $u_1 = u_2$ , then  $\psi := \phi_1^Z \circ (\phi_2^Z)^{-1} \in \text{QC}(f)$ , and we have to prove that in fact  $\psi \in \text{QC}_0(f)$ . Let  $\phi_i^Z(t)$  be the quasiconformal homeomorphisms corresponding to  $\mu_i(t) = \phi(tu_i, tv_i)$ ,  $1 \leq i \leq 2$  and  $t \in [0, 1]$ , and  $\psi_t = \phi_1^Z(t) \circ (\phi_2^Z(t))^{-1}$ . Since for all  $t \in [0, 1]$ ,  $\mu_i(t) = \phi^{-1}(tu_i, tv_i)$ , we have  $\psi_t \in \text{QC}(f)$ , and  $\psi_0 = \text{Id}$ . The maps  $t \mapsto \mu_i(t)$  are analytic, so by the parametric Ahlfors-Bers theorem so are the maps  $t \mapsto \phi_i^Z(t)$ . Therefore, for all  $z \in \mathbb{P}^1$ , the map  $t \mapsto \psi_t(z) = \phi_1^Z(t) \circ (\phi_2^Z(t))^{-1}(z)$  is continuous and  $\psi_t$  is an isotopy to the identity through elements of  $\text{QC}(f)$ . Moreover, since the  $\phi_i^Z(t)$  have uniformly bounded dilatation, so does  $\psi_t$ .<sup>2</sup> So  $\psi = \psi_1 \in \text{QC}_0(f)$ , which proves the claim.

Therefore the map  $\tilde{\phi} : \text{Im}D\Psi^Z(0) \rightarrow \text{Rat}_d$  defined by  $\tilde{\phi}(u) = \pi \circ \phi(u, 0)$ , where  $\pi : \text{Bel}(f) \rightarrow \text{Teich}(f)$  is the projection, is a germ of homeomorphism and makes the following diagram commute :

$$\begin{array}{ccc} \text{Im}D\Psi^Z(0) \oplus \ker D\Psi^Z(0) & \xrightarrow{\phi} & \text{Bel}(f) \\ \downarrow \pi_1 & & \downarrow \pi \\ \text{Im}D\Psi^Z(0) & \xrightarrow{\tilde{\phi}} & \text{Teich}(f) \end{array}$$

The map  $\pi_1 : \text{Im}D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu) \rightarrow \text{Im}D\Psi^Z(\mu)$  being the projection onto the first factor.

We can now define local sections of  $\pi$  by transporting local holomorphic sections of  $\pi_1$  through the  $\phi$  coordinates.

Let us prove that these local sections of  $\pi$  can be glued together compatibly to define a complex atlas on  $\text{Teich}(f)$ . Let  $h_1, h_2$  be two such local sections of  $\pi$  defined in a neighborhood of  $[\mu] \in \text{Teich}(f)$  : we must prove that  $h_2 \circ h_1^{-1} = h_2 \circ \pi : \text{Bel}(f) \rightarrow \text{Bel}(f)$  is holomorphic. Let  $g_1$  and  $g_2$  be the corresponding right inverses of  $\pi_1 : \text{Im}D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu) \rightarrow \text{Im}D\Psi^Z(\mu)$ .

The following diagram commutes :

$$\begin{array}{ccc} \text{Bel}(f) & \xrightarrow{h_2 \circ \pi} & \text{Bel}(f) \\ \swarrow \pi & & \swarrow \pi \\ & \text{Teich}(f) & \end{array}$$

*(Note: The diagram above is a simplified representation of the commutative diagram in the image. The full diagram shows two Bel(f) nodes at the top, connected by a horizontal arrow h2 ∘ π. Below them is a Teich(f) node. Two diagonal arrows, π, point from each Bel(f) node down to Teich(f). Two other diagonal arrows, h1 and h2, point from Teich(f) up to the left and right Bel(f) nodes respectively.)*

and therefore this diagram commutes as well :

<sup>2</sup>Note however that the Beltrami coefficient of  $\psi_t$  needs not a priori depend continuously on  $t$ .

$$\begin{array}{ccccc}
 \text{Bel}(f) & \xrightarrow{h_2 \circ \pi} & \text{Bel}(f) & & \\
 \downarrow \phi^{-1} & \swarrow \pi & \searrow h_2 & & \uparrow \phi \\
 \text{Im}D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu) & \xrightarrow{h_1} & \text{Teich}(f) & \xrightarrow{\pi} & \text{Im}D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu) \\
 \swarrow \pi_1 & & \downarrow \tilde{\phi} & & \searrow g_2 \\
 & \xrightarrow{g_1} & \text{Im}D\Psi^Z(\mu) & \xrightarrow{\pi_1} & 
 \end{array}$$

A diagramm chase then shows that  $h_2 \circ \pi$  is holomorphic, which proves the existence of a complex structure on  $\text{Teich}(f)$  meeting the requirements.

Unicity comes from the fact that for any complex structure making the projection  $\pi : \text{Bel}(f) \rightarrow \text{Teich}(f)$  into a holomorphic map, the local holomorphic sections of  $\pi$  still define an atlas.

Lastly, the fact that  $\pi$  admits local holomorphic sections is precisely equivalent to  $\pi$  being a split submersion.  $\square$

We can finally prove the main theorem :

**Main Theorem.** The map  $\Psi_T^Z : \text{Teich}(f) \rightarrow \text{Rat}_d$  is an immersion, whose image is transverse to  $\mathcal{O}(f)$ .

*Proof.* By corollary 8 it is enough to show that it is an immersion at 0. By definition, we have  $\Psi^Z = \Psi_T^Z \circ \pi$ , and therefore

$$D\Psi^Z(0) = D\Psi_T^Z([0]) \circ D\pi(0)$$

Injectivity of  $D\Psi_T^Z([0])$  is then equivalent to the property  $\ker D\Psi^Z(0) = \ker D\pi(0)$ .

By proposition 5,  $\ker D\Psi^Z(0) = N_f(\Omega_f)$ , and by corollary 9,  $\ker D\pi(0) = N_f(\Omega_f)$ , which concludes the proof.  $\square$

**Definition 5.1.** If  $A \subset \mathbb{P}^1$  is closed, we note  $Q(A)$  the Banach space of integrable quadratic differentials on  $\mathbb{P}^1$  and holomorphic on  $\mathbb{P}^1 - A$ , equipped with the  $L^1$  norm.

**Corollary 11.** We have the following identification :

$$\begin{aligned}
 T_0\text{Teich}(f) &= \text{bel}(f)/\{\bar{\partial}\xi, \xi = f^*\xi\} \\
 T_0^*\text{Teich}(f) &= Q(\Lambda_f)/\overline{\nabla_f Q(\Lambda_f)}.
 \end{aligned}$$

*Proof.* The first statement is a direct consequence of corollary 9.

Since  $\text{Teich}(f)$  is a finite-dimensional manifold, it is enough to prove that

$$\left( Q(\Lambda_f)/\overline{\nabla_f Q(\Lambda_f)} \right)^*$$

identifies to  $T_0\text{Teich}(f)$ .

By the Hahn-Banach theorem, every linear form on  $Q(\Lambda_f)$  may be represented by a  $L^\infty$  Beltrami differential. Moreover, if  $q \in \overline{\nabla_f Q(\Lambda_f)}$ , then  $\int_{\mathbb{P}^1} q \cdot \mu = 0$  for all Beltrami differential  $\mu$  invariant under  $f$ , and  $\int_{\mathbb{P}^1} q \cdot \mu = 0$  for all quadratic differential  $q \in Q(\Lambda_f)$  and all infinitesimally trivial Beltrami differential on  $\Lambda_f$ , namely any Beltrami differential of the form  $\mu = \bar{\partial}\xi$ , where  $\xi$  is a quasiconformal vector field on  $\mathbb{P}^1$  vanishing on  $\Lambda_f$  (see [GL00]). Therefore by theorem A, every continuous linear form on  $Q(\Lambda_f)/\overline{\nabla_f Q(\Lambda_f)}$  (for the quotient norm corresponding to the  $L^1$  norm) may be represented by an element of  $T_0\text{Teich}(f)$ , with the dual norm coinciding with the quotient  $L^\infty$  norm (it is the Teichmüller metric of  $\text{Teich}(\Lambda_f)$ , see [GL00], [Hub06]).

This representation is unique, since if  $\mu$  is a  $L^\infty$  Beltrami differential annihilating all of  $Q(\Lambda_f)$ , then  $\mu \in N_f(\Lambda_f)$  by theorem A.  $\square$

Note that we obtain that  $Q(\Lambda_f)/\overline{\nabla_f Q(\Lambda_f)}$  has finite dimension which is less than  $2d - 2$ .

## REFERENCES

- [BE09] Xavier Buff and Adam L. Epstein. Bifurcation measure and postcritically finite rational maps. *Complex dynamics: families and friends; edited by Dierk Schleicher*, pages 491–512, 2009.
- [Ber67] Lipman Bers. Inequalities for finitely generated kleinian groups. *Journal d'Analyse Mathématique*, 18(1):23–41, 1967.
- [Bra92] Bodil Branner. *Cubic polynomials: turning around the connectedness locus*. Matematisk Inst., Danmarks tekniske Højskole, 1992.
- [EM88] Clifford J. Earle and Curtis T. McMullen. Quasiconformal isotopies. *Holomorphic functions and moduli*, 1:143–154, 1988.
- [Eps09] Adam L. Epstein. Transversality in holomorphic dynamics. <http://homepages.warwick.ac.uk/~mases/Transversality.pdf>, 2009.
- [GL00] Frederick P. Gardiner and Nikola Lakic. *Quasiconformal Teichmüller Theory*, volume 76. AMS Bookstore, 2000.
- [Hub06] John H. Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics*, volume 1. Matrix Pr, 2006.
- [Mak10] Peter M. Makienko. Remarks on the dynamic of the ruelle operator and invariant differentials. *Dal nevost. Mat. Zh.*, pages 180–205, 2010.
- [McM14] Curtis T. McMullen. Riemann surfaces, dynamics and geometry. <http://math.harvard.edu/~ctm/home/text/class/harvard/275/rs/rs.pdf>, 2014.
- [MS98] Curtis T. McMullen and Dennis P. Sullivan. Quasiconformal homeomorphisms and dynamics iii. the teichmüller space of a holomorphic dynamical system. *Advances in Mathematics*, 135(2):351–395, 1998.

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