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# REAL SUBMANIFOLDS OF MAXIMUM COMPLEX TANGENT SPACE AT A CR SINGULAR POINT

XIANGHONG GONG AND LAURENT STOLOVITCH

Abstract. We study a germ of real analytic $n$-dimensional submanifold of $\mathbf{C}^n$ that has a complex tangent space of maximal dimension at a CR singularity. Under the condition that its complexification admits the maximum number of deck transformations, we study its transformation to a normal form under the action of local (possibly formal) biholomorphisms at the singularity. We first conjugate formally its associated reversible map $\sigma$ to suitable normal forms and show that all these normal forms can be divergent. If the singularity is *abelian*, we show, under some assumptions on the linear part of $\sigma$ at the singularity, that the real submanifold is holomorphically equivalent to an analytic normal form. We also show that if a real submanifold is formally equivalent to a quadric, it is actually holomorphically equivalent to it, if a small divisors condition is satisfied. Finally, we prove that, in general, there exists a complex submanifold of positive dimension in $\mathbf{C}^n$ that intersects a real submanifold along two totally and real analytic submanifolds that intersect transversally at a CR singularity of the *complex type*.

## Contents



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## 1. Introduction and main results

1.1. **Introduction.** We are concerned with the local holomorphic invariants of a real analytic submanifold $M$ in $\mathbf{C}^n$. The tangent space of $M$ at a point $x$ contains a maximal complex subspace of dimension $d_x$. When this dimension is constant, $M$ is called a Cauchy-Riemann (CR) submanifold. The CR submanifolds have been extensively studied since E. Cartan. For the analytic real hypersurfaces in $\mathbf{C}^n$ with a non-degenerate Levi-form, the normal form problem has a complete theory achieved through the works of E. Cartan [Car32], [Car33], Tanaka [Tan62], and Chern-Moser [CM74]. In another direction, the relations between formal and holomorphic equivalences of real analytic hypersurfaces have been investigated by Baouendi-Ebenfelt-Rothschild [BER97], [BER00], Baouendi-Mir-Rothschild [BMR02], Juhlin-Lamel [JL13], where positive results were obtained. In a recent preprint, Kossovskiy and Shafikov [KS13] showed that there are real analytic real hypersurfaces which are formally but not holomorphically equivalent.

We say that a point $x_0$ in a real submanifold $M$ in $\mathbf{C}^n$ is a complex tangent, or a CR singularity, if the complex tangent spaces $T_xM \cap J_xT_xM$ do not have a constant dimension in any neighborhood of $x_0$. A real submanifold with a CR singularity must have codimension at least 2. The study of real submanifolds with CR singularities was initiated by E. Bishop in his pioneering work [Bis65], when the complex tangent space of $M$ at a CR singularity is minimal, that is exactly one-dimensional. The very elementary models of this kind of manifolds are classified as certain quadrics which depend on one non-negative number, the Bishop invariant. They are the Bishop quadrics, given by

$$Q \subset \mathbf{C}^2 \colon z_2 = |z_1|^2 + \gamma(z_1^2 + \overline{z}_1^2), \quad 0 \leq \gamma < \infty.$$

The origin is a complex tangent which is said to be *elliptic* if $0 \leq \gamma < 1/2$, *parabolic* if $\gamma = 1/2$, or *hyperbolic* if $\gamma > 1/2$.

In [MW83], Moser and Webster studied the normal form problem of a real analytic surface $M$ in $\mathbf{C}^2$ which is the higher order perturbation of $Q$. They showed that when $0 < \gamma < 1/2$, $M$ is holomorphically equivalent to a normal form which is an algebraic

surface that depends only on $\gamma$ and two discrete invariants. They also constructed a formal normal form of $M$ when the origin is a non-exceptional hyperbolic complex tangent point; although the normal form is still convergent, they showed that the normalizing transformation is divergent in general for the hyperbolic case. We mention that the Moser-Webster normal form theory, as in Bishop's work, actually deals with an $n$-dimensional real submanifold $M$ in $\mathbf{C}^n$, of which the complex tangent space has (minimum) dimension 1 at a CR singularity.

The main purpose of this work is to investigate an $n$-dimensional real analytic submanifold $M$ in $\mathbf{C}^n$ of which the complex tangent space has the *largest* possible dimension at a given CR singularity. The dimension must be $p = n/2$. Therefore, $n = 2p$ is even. We are interested in the geometry, the analytic classification, and the normal form problem of such real analytic manifolds.

In suitable holomorphic coordinates, a $2p$-dimensional real analytic submanifold $M$ in $\mathbf{C}^{2p}$ that has a complex tangent space of maximum dimension at the origin is given by

$$M\colon z_{p+j} = E_j(z', \overline{z}'), \quad 1 \le j \le p, \tag{1.1}$$

where $z' = (z_1, \dots, z_p)$ and

$$E_j(z', \overline{z'}) = h_j(z', \overline{z}') + q_j(\overline{z}') + O(|(z', \overline{z}')|^3).$$

Moreover, each $h_j(z', \overline{z}')$ is a homogeneous quadratic polynomial in $z', \overline{z}'$ without holomorphic or anti-holomorphic terms, and each $q_j(\overline{z}')$ is a homogeneous quadratic polynomial in $\overline{z}'$. One of our goals is to seek suitable normal forms of perturbations of quadrics at the CR singularity (the origin).

The study of these kind of real submanifolds, with $p > 1$, was initiated in [Sto07] by the second-named author.

1.2. **Basic invariants.** To study $M$, we consider its complexification in $\mathbf{C}^{2p} \times \mathbf{C}^{2p}$ defined by

$$\mathcal{M}\colon \begin{cases} z_{p+i} = E_i(z', w'), & i = 1, \dots, p, \\ w_{p+i} = \bar{E}_i(w', z'), & i = 1, \dots, p. \end{cases}$$

It is a complex submanifold of complex dimension $2p$ with coordinates $(z', w') \in \mathbf{C}^{2p}$. Let $\pi_1, \pi_2$ be the restrictions of the projections $(z, w) \to z$ and $(z, w) \to w$ to $\mathcal{M}$, respectively. Note that $\pi_2 = \rho_0 \pi_1 \rho_0$, where $\rho_0$ is the restriction to $\mathcal{M}$ of the anti-holomorphic involution $(z, w) \to (\overline{w}, \overline{z})$.

Our basic assumption is the following condition.

**Condition B.** $q(z') = (q_1(z'), \dots, q_p(z'))$ satisfies $q^{-1}(0) = \{0\}$.

Let us first describe the significance of condition B. When $p = 1$ this corresponds to the case that the Bishop invariant $\gamma$ of $M$ at the origin does not vanish. When $\gamma = 0$, Moser [Mos85] obtained a formal normal form that is still subject to further formal changes of coordinates. In [HY09a], Huang and Yin obtained a formal normal form with a complete set of formal holomorphic invariants of $M$ when $\gamma = 0$. They used their formal normal form to show that two such real analytic surfaces are holomorphically equivalent if and only if they have the same formal normal form. The formal normal forms for codimension two real submanifolds in $\mathbf{C}^n$ have been further studied by Huang-Yin [HY12] and

Burcea [Bur13]. Note that by a rapid iteration method, Coffman [Cof06] showed that any $m$ dimensional real analytic submanifold in $\mathbf{C}^n$ of one-dimensional complex tangent space at a CR singularity satisfying certain non-degeneracy conditions is locally holomorphically equivalent to a unique algebraic submanifold, provided $2(n+1)/3 \leq m < n$.

When $M$ is a *quadric*, i.e. each $E_j$ in (1.1) is a quadratic polynomial, our basic condition B is equivalent to $\pi_1$ being a $2^p$-to-1 branched covering. Since $\pi_2 = \rho_0\pi_1\rho_0$, then $\pi_2$ is also a $2^p$-to-1 branched covering. We will see that the CR singularities of the real submanifolds are closely connected with these branched coverings and their deck transformations.

We now introduce our main results. Some of them are analogous to the Moser-Wester theory. We will underline major differences which arise with $p > 1$.

1.2.1. *Branched covering and deck transformations.* In section 2, we study the existence of deck transformations for $\pi_1$. We will show that they must be involutions and they commute pairwise. We show that they form a group of order $2^k$ for some $0 \leq k \leq p$. This is a major difference between the real submanifolds with one dimensional complex tangent space at a CR singularity and the ones with maximum complex tangent space, when $p > 1$. Indeed, we recall that in the Moser-Webster theory, the branched covering $\pi_1$ is 2-to-1 and consequently the group of deck transformations of $\pi_1$ has order 2. The group is then generated by a unique involution $\tau_1$.

In this paper, we will focus on the case where the group of deck transformations of $\pi_1$ has the maximum order $2^p$. Thus, we will impose the following condition.

**Condition D.** *$M$ satisfies condition B and the branched covering $\pi_1$ of $\mathcal{M}$ admits the maximum $2^p$ deck transformations.*

Condition D gives rise to two families of commuting involutions $\{\tau_{i1}, \ldots, \tau_{i2^p}\}$ intertwined by the anti-holomorphic involution $\rho_0 \colon (z', w') \to (\overline{w}', \overline{z}')$ such that $\tau_{2j} = \rho_0\tau_{1j}\rho_0$ ($1 \leq j \leq 2^p$) are deck transformations of $\pi_2$. We will call $\{\tau_{11}, \ldots, \tau_{12^p}, \rho_0\}$ the set of *Moser-Webster involutions*. We will show that there is a unique set of $p$ generators for the deck transformations of $\pi_1$, denoted by $\tau_{11}, \ldots, \tau_{1p}$, which are characterized by the property that each $\tau_{1j}$ fixes a hypersurface in $\mathcal{M}$ pointwise. Then

$$\tau_1 = \tau_{11} \circ \cdots \circ \tau_{1p}$$

is the unique deck transformation of which the set of fixed-points has the smallest dimension $p$. Let $\tau_2 = \rho_0\tau_1\rho_0$ and

$$\sigma = \tau_1\tau_2.$$

Then $\sigma$ is *reversible* by $\tau_j$ and $\rho_0$, i.e. $\sigma^{-1} = \tau_j\sigma\tau_j^{-1}$ and $\sigma^{-1} = \rho_0\sigma\rho_0$.

As in the Moser-Webster theory, the existence of such $2^p$ deck transformations allows us to transfer the normal form problem for the real submanifolds into the normal form problem for the sets of involutions $\{\tau_{11}, \ldots, \tau_{1p}, \rho_0\}$.

In this paper we will make the following assumption.

**Condition E.** *$M$ satisfies condition D and $M$ has distinct eigenvalues, while the latter means that $\sigma$ has $2p$ distinct eigenvalues.*

Note that the condition excludes the higher dimensional analogous complex tangency of *parabolic* type, i.e. of $\gamma = 1/2$. The normal form problem for the parabolic complex tangents has been studied by Webster [Web92], and in [Gon96] where the normalization

is divergent in general. In [AG09], Ahern and Gong constructed a moduli space for real analytic submanifolds that are formally equivalent to the Bishop quadric with $\gamma = 1/2$.

We now introduce our main results.

Our first step is to normalize $\{\tau_1, \tau_2, \rho_0\}$. When $p = 1$, this normalization is the main step in order to obtain the Moser-Webster normal form; in fact a simple further normalization allows Moser and Webster to achieve a convergent normal form under a suitable non-resonance condition even for the non-exceptional hyperbolic complex tangent.

When $p > 1$, we need to carry out a further normalization for $\{\tau_{11}, \dots, \tau_{1p}, \rho_0\}$; this is our second step. Here the normalization has a large degree of freedom as shown by our formal and convergence results.

In sections 2 through 7, we will study the formal normal forms and the relations on the convergence of normalizations in these two steps. Let us first describe main results in these sections.

1.2.2. *Normal forms of quadrics with the maximum number of deck transformations.* The basic model for quadric manifolds with such a CR singularity is a product of 3 types of quadrics defined by

$$\begin{aligned} Q_{\gamma_e} \subset \mathbf{C}^2 \colon z_2 &= (z_1 + 2\gamma_e \overline{z}_1)^2; \\ Q_{\gamma_h} \subset \mathbf{C}^2 \colon z_2 &= (z_1 + 2\gamma_h \overline{z}_1)^2; \\ Q_{\gamma_s} \subset \mathbf{C}^4 \colon z_3 = (z_1 + 2\gamma_s \overline{z}_2)^2, &\quad z_4 = (z_2 + 2(1 - \overline{\gamma}_s)\overline{z}_1)^2. \end{aligned}$$

Here

$$0 < \gamma_e < 1/2, \quad 1/2 < \gamma_h < \infty, \quad \operatorname{Re} \gamma_s > 1/2, \quad \operatorname{Im} \gamma_s > 0. \tag{1.2}$$

Note that $Q_{\gamma_e}, Q_{\gamma_h}$ are elliptic and hyperbolic Bishop quadrics, respectively. Realizing a type of pairs of involutions introduced in [Sto07], we will say that the complex tangent of $Q_{\gamma_s}$ at the origin is *complex*. We emphasize that this last type of quadric is new as we will show that $Q_{\gamma_s}$ is not holomorphically equivalent to a product of two Bishop surfaces. A real submanifold of dimension $n$ in $\mathbf{C}^n$ with $n = 2p$ that is a product of the above quadrics will be called a *product of quadrics*, or a *product quadric*.

In section 3, we study all quadrics which admit the maximum number of deck transformations. For such quadrics, all deck transformations are linear. Under condition E, we will first normalize $\sigma, \tau_1, \tau_2$ and $\rho_0$ into $\hat{S}, \hat{T}_1, \hat{T}_2$ and $\rho$ where

$$\begin{aligned} \hat{T}_1 \colon &\quad \xi'_j = \lambda_j^{-1} \eta_j, \qquad \eta'_j = \lambda_j \xi_j, \\ \hat{T}_2 \colon &\quad \xi'_j = \lambda_j \eta_j, \qquad \eta'_j = \lambda_j^{-1} \xi_j, \\ \hat{S} \colon &\quad \xi'_j = \mu_j \xi_j, \qquad \eta'_j = \mu_j^{-1} \eta_j \end{aligned}$$

with

$$\lambda_e > 1, \quad |\lambda_h| = 1, \quad |\lambda_s| > 1, \quad \lambda_{s+s_*} = \overline{\lambda}_s^{-1}, \quad \mu_j = \lambda_j^2.$$

Here $1 \le j \le p$. Throughout the paper, the indices $e, h, s$ have the ranges: $1 \le e \le e_*$, $e_* < h \le e_* + h_*$, $e_* + h_* < s \le p - s_*$. Thus $e_* + h_* + 2s_* = p$. We will call $e_*, h_*, s_*$ the numbers of *elliptic, hyperbolic* and *complex* components of a product quadric, respectively. As in the Moser-Webster theory, at the complex tangent (the origin) an *elliptic* component of a product quadric corresponds a *hyperbolic* component of $\hat{S}$, while a *hyperbolic* component

of the quadric corresponds an *elliptic* component of $\hat{S}$. One could identify a *complex* component of the quadric with a *hyperbolic* (instead of complex) component of $\hat{S}$; however, each type of complex tangents exhibits striking differences in the formal normal forms, the convergence of normalizations, and the existence of attached complex submanifolds, as illustrated by the results in this section.

For the above normal form of $\hat{T}_1, \hat{T}_2$ and $\hat{S}$, we always normalize the anti-holomorphic involution $\rho_0$ as

$$\rho\colon \begin{cases} \xi'_e &= \overline{\eta}_e, & \eta'_e &= \overline{\xi}_e, \\ \xi'_h &= \overline{\xi}_h, & \eta'_h &= \overline{\eta}_h, \\ \xi'_s &= \overline{\xi}_{s+s_*}, & \eta'_s &= \overline{\eta}_{s+s_*}, \\ \xi'_{s+s_*} &= \overline{\xi}_s, & \eta'_{s+s_*} &= \overline{\eta}_s. \end{cases} \tag{1.3}$$

With the above normal forms $\hat{T}_1, \hat{T}_2, \hat{S}, \rho$ with $\hat{S} = \hat{T}_1\hat{T}_2$, we will then normalize the $\tau_{11}, \ldots, \tau_{1p}$ under linear transformations that commute with $\hat{T}_1, \hat{T}_2$, and $\rho$, i.e. the linear transformations belonging to the *centralizer* of $\hat{T}_1, \hat{T}_2$ and $\rho$. This is a subtle step. Instead of normalizing the involutions directly, we will use the pairwise commutativity of $\tau_{11}, \ldots, \tau_{1p}$ to associate to these $p$ involutions a non-singular $p \times p$ matrix $\mathbf{B}$. The normalization of $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$ is then identified with the normalization of the matrices $\mathbf{B}$ under a suitable equivalence relation. The latter is easy to solve. Our normal form of $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$ is then constructed from the normal forms of $T_1, T_2, \rho$, and the matrix $\mathbf{B}$. Following Moser-Webster [MW83], we will construct the normal form of the quadrics from the normal form of involutions.

**Theorem 1.1.** *Let $M$ be a quadratic submanifold defined by*

$$z_{p+j} = h_j(z', \overline{z}') + q_j(\overline{z}'), \quad 1 \le j \le p.$$

*Suppose that $M$ satisfies condition $E$, i.e. the branched covering of $\pi_1$ of complexification $\mathcal{M}$ has $2^p$ deck transformations and $M$ has $2p$ distinct eigenvalues. Then $M$ is holomorphically equivalent to*

$$Q_{\mathbf{B},\boldsymbol{\gamma}}\colon z_{p+j} = L_j^2(z', \overline{z}'), \quad 1 \le j \le p$$

*where $(L_1(z', \overline{z}'), \ldots, L_p(z', \overline{z}'))^t = \mathbf{B}(z' - 2\boldsymbol{\gamma}\overline{z}')$, $\mathbf{B} \in GL_p(\mathbf{C})$ and*

$$\boldsymbol{\gamma} := \begin{pmatrix} \boldsymbol{\gamma}_{e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\gamma}_{h_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\gamma}_{s_*} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} - \overline{\boldsymbol{\gamma}}_{s_*} & \mathbf{0} \end{pmatrix}.$$

*Here $p = e_* + h_* + 2s_*$, $\mathbf{I}_{s_*}$ denotes the $s_* \times s_*$ identity matrix, and*

$$\boldsymbol{\gamma}_{e_*} = \operatorname{diag}(\gamma_1, \ldots, \gamma_{e_*}), \quad \boldsymbol{\gamma}_{h_*} = \operatorname{diag}(\gamma_{e_*+1}, \ldots, \gamma_{e_*+h_*}),$$
$$\boldsymbol{\gamma}_{s_*} = \operatorname{diag}(\gamma_{e_*+h_*+1}, \ldots, \gamma_{p-s_*})$$

*with $\gamma_e, \gamma_h$, and $\gamma_s$ satisfying (1.2). Moreover, $\mathbf{B}$ is uniquely determined by an equivalence relation $\mathbf{B} \sim \mathbf{CBR}$ for suitable non-singular matrices $\mathbf{C}, \mathbf{R}$ which have exactly $p$ non-zero entries.*

See Theorem 3.7 for detail of the equivalence relation. The scheme of finding quadratic normal forms turns out to be useful. It will be applied to the study of normal forms of the general real submanifolds.

### 1.3. **Formal normalization and divergence of normal forms.**

1.3.1. *Formal submanifolds, formal involutions, and formal centralizers.* In section 4, we show that the formally holomorphic classification of formal submanifolds with the maximum number of formal deck transformations and the formally holomorphic classification of suitable families of involutions $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$ are equivalent. This equivalence will be used to derive the formal normal forms of the submanifolds. As mentioned earlier, we will first normalize $\sigma = \tau_1\tau_2$ under general formal biholomorphic transformations. The normal forms of $\sigma$ turn out to be in the centralizer of $\hat{S}$, the normal form of the linear part of $\sigma$. The family is subject to a second step of normalization, under mappings which again turn out to be in the centralizer of $\hat{S}$. Thus, before we introduce normalization, we will first study various centralizers. We will discuss the centralizer of $\hat{S}$ as well as the centralizer of $\{\hat{T}_1, \hat{T}_2\}$ in section 4. The centralizer of $\{\hat{T}_{11}, \ldots, \hat{T}_{1p}, \rho\}$ is more complicated, which will be discussed in section 10.

1.3.2. *Normalization of $\sigma$.* As mentioned earlier, we will divide the normalization for the families of non-linear involutions into two steps. This division will serve two purposes: first, it helps us to find the formal normal forms of the family of involutions $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$; second, it helps us understand the convergence of normalization of the original normal form problem for the real submanifolds. For purpose of normalization, we will assume that $M$ is *non-resonant*, i.e. $\sigma$ is *non-resonant*, if its eigenvalues $\mu_1, \ldots, \mu_p, \mu_1^{-1}, \ldots, \mu_p^{-1}$ satisfy

$$\mu^Q \neq 1, \qquad \forall Q \in \mathbf{Z}^p, \quad |Q| \neq 0.$$

In section 5, we obtain the normalization of $\sigma$ by proving the following.

**Theorem 1.2.** *Let $\sigma$ be a holomorphic map with linear part $\hat{S}$. Assume that $\mu_1, \ldots, \mu_p$ are non-resonant. Suppose that $\sigma = \tau_1\tau_2$ where $\tau_1$ is a holomorphic involution, $\rho$ is an anti-holomorphic involution, and $\tau_2 = \rho\tau_1\rho$. Then there exists a formal map $\Psi$ such that $\rho := \Psi^{-1}\rho\Psi$ is given by* (1.3)*, $\sigma^* = \Psi^{-1}\sigma\Psi$ and $\tau_i^* = \Psi^{-1}\tau_i\Psi$ have the form*

$$\begin{aligned} \sigma^*\colon \xi_j' = M_j(\xi\eta)\xi_j, \quad \eta_j' = M_j^{-1}(\xi\eta)\eta_j, \quad 1 \leq j \leq p, \\ \tau_i^* = \Lambda_{ij}(\xi\eta)\eta_j, \quad \eta_j' = \Lambda_{ij}^{-1}(\xi\eta)\xi_j. \end{aligned} \tag{1.4}$$

*Here, $\xi\eta = (\xi_1\eta_1, \ldots, \xi_p\eta_p)$. Assume further that $\log M$ (see* (5.31) *for definition) is tangent to the identity. Under a further change of coordinates that preserves $\rho$, $\sigma^*$ and $\tau_i^*$ are transformed into*

$$\begin{aligned} \hat{\sigma}\colon \xi_j' = \hat{M}_j(\xi\eta)\xi_j, \quad \eta_j' = \hat{M}_j^{-1}(\xi\eta)\eta_j, \quad 1 \leq j \leq p, \\ \hat{\tau}_i = \hat{\Lambda}_{ij}(\xi\eta)\eta_j, \quad \eta_j' = \hat{\Lambda}_{ij}^{-1}(\xi\eta)\xi_j, \quad \hat{\Lambda}_{2j} = \hat{\Lambda}_{1j}^{-1}. \end{aligned} \tag{1.5}$$

*Here the $j$th component of $\log \hat{M}(\zeta) - I$ is independent of $\zeta_j$. Moreover, $\hat{M}$ is unique.*

**Remark 1.3.** The condition that $\log M$ is tangent to identity at the origin has to be understood as a non-degeneracy condition of which it is the simplest instance. When there is no ambiguity, "tangent to identity" stands for "tangent to identity at the origin".

We will conclude section 5 with an example showing that although $\sigma, \tau_1, \tau_2$ are both linear, $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$ are not necessary linear, provided $p > 1$.

Section 6 is devoted to the proof of the following divergence result.

**Theorem 1.4.** *There exists a non-resonant real analytic submanifold $M$ with pure elliptic complex tangent in $\mathbf{C}^6$ such that if its corresponding $\sigma$ is transformed into a map $\sigma^*$ which commutes with the linear part of $\sigma$ at the origin, then $\sigma^*$ must diverge.*

Note that the theorem says that all normal forms of $\sigma$ (by definition, they belong to the centralizer of its linear part, i.e. they are in the Poincare-Dulac normal forms) are divergent. It implies that any transformation for $M$ that transforms $\sigma$ into a Poincaré-Dulac normal form must diverge. This is in contrast with the Moser-Webster theory: For $p = 1$, a convergent normal form can always be achieved even if the associated transformation is divergent (in the case of hyperbolic complex tangent), and furthermore in case of $p = 1$ and elliptic complex tangent with a non-varnishing Bishop invariant, the normal form can be achieved by a convergent transformation. The divergent Birkhoff normal form for the classical Hamiltonian systems was obtained in [Gon12] by the first-named author. We refer to [SM71, Mos73] as general references concerning Hamiltonian and reversible dynamics.

1.3.3. *Normalization on the family $\{\tau_{ij}, \rho\}$.* In section 7, we will follow the scheme developed for the quadric normal forms in order to normalize $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$. Let $\hat{\sigma}$ be given by (1.5). We define

$$\hat{\tau}_{1j} \colon \xi_j' = \hat{\Lambda}_{1j}(\xi\eta)\eta_j, \quad \eta_j' = \hat{\Lambda}_{1j}^{-1}(\xi\eta)\xi_j, \quad \xi_k' = \xi_k, \quad \eta_k' = \eta_k,$$

where $k \neq j$, $\hat{\Lambda}_{1j}(0) = \lambda_j$, and $\hat{M}_j = \hat{\Lambda}_{1j}^2$. We have the following formal normal form.

**Theorem 1.5.** *Let $M$ be a real analytic submanifold that is a higher order perturbation of a non-resonant product quadric. Suppose that its associated $\sigma$ is formally equivalent to $\hat{\sigma}$ given by* (1.5)*. Suppose that the formal mapping $\log \hat{M}$ in Theorem* 1.2 *is tangent to the identity. Then the formal normal form of $M$ is completely determined by*

$$\hat{M}(\zeta), \quad \Phi.$$

*Here $\Phi$ is a formal invertible mapping in $\mathcal{C}^c(\hat{\tau}_{11}, \ldots, \hat{\tau}_{1p})$. Moreover, $\Phi$ is uniquely determined up to the equivalence relation $\Phi \sim R_\epsilon \Phi R_\epsilon^{-1}$ with $R_\epsilon \colon \xi_j = \epsilon_j \xi, \eta_j' = \epsilon_j \eta_j$ $(1 \leq j \leq p)$ and $R_\epsilon^2 = I$. Furthermore, if the normal form* (1.4) *of $\sigma$ can be achieved by a convergent transformation, so does the normal form of $M$.*

The set $\mathcal{C}^c(\hat{\tau}_{11}, \ldots, \hat{\tau}_{1p})$ is defined by Definition 7.2.

The second part of the paper is devoted to geometric properties of $M$ and in particular those obtained through convergent normalization, under additional assumptions on $M$.

We first turn to a holomorphic normalization of a real analytic submanifold $M$ with the so-called abelian CR singularity. This will be achieved by studying an integrability problem on a general family of commuting biholomorphisms described below. The holomorphic normalization will be used to construct the local hull of holomorphy of $M$. We will also study the rigidity problem of a quadric under higher order analytic perturbations, i.e. the

problem if such a perturbation remains holomorphically equivalent to the quadric if it is formally equivalent to the quadric. The rigidity problem is reduced to a theorem of holomorphic linearization of one or several commuting diffeomorphisms along a suitable ideal that was devised in [Sto13]. Finally, we will study the existence of holomorphic submanifolds attached to the real submanifold $M$. These are complex submanifolds of dimension $p$ intersecting $M$ along two totally real analytic submanifolds that intersect transversally at a CR singularity. Attaching complex submanifolds has less constraints than finding a convergent normalization. Therefore, our only assumption is that $M$ is non-resonant and admits the maximum number of deck transformations. A remarkable feature of attached complex submanifolds is that their existence depends only on the existence of suitable (convergent) invariant submanifolds of $\sigma$. When the real submanifold has a complex tangent of pure complex type, the existence is ensured under a mild non-resonance condition but without any further restriction such as small divisors condition on the eigenvalues of $M$.

## 1.4. **Abelian CR singularity and analytic hull of holomorphy.**

1.4.1. *Normal form of commuting biholomorphisms.* Let $\mathcal{F} = \{F_1, \dots, F_\ell\}$ be a finite family of germs of biholomorphisms of $\mathbf{C}^n$ fixing the origin. Let $D_m$ be the linear part of $F_m$ at the origin. We say that the family $\mathcal{F}$ is (formal) *completely integrable*, if there is a (formal) biholomorphic mapping $\Phi$ such that $\{\Phi^{-1} F_m \Phi \colon 1 \le m \le \ell\} = \{\hat{F}_m \colon 1 \le m \le \ell\}$ satisfies

- (*i*) $\hat{F}_m(z) = (\mu_{m1}(z)z_1, \dots, \mu_{mn}(z)z_n)$ and $\mu_{mj} \circ D_{m'} = \mu_{mj}$ for $1 \le m, m' \le \ell$ and $1 \le j \le n$. In particular, $\hat{F}_m$ commutes with $D_{m'}$ for all $1 \le m, m' \le \ell$.
- (*ii*) For each $j$ and each $Q \in \mathbf{N}^n$ with $|Q| > 1$, $\mu_m^Q(0) = \mu_{mj}(0)$ hold for all $m$ if and only if $\mu_m^Q(z) = \mu_{mj}(z)$ hold for all $m$.

Note that a necessary condition for $\mathcal{F}$ to be formally completely integrable is that $F_1, \dots, F_\ell$ commute pairwise. The main result of section 8 is the following.

**Theorem 1.6.** *Let $\mathcal{F}$ be a family of finitely many germs of biholomorphisms at the origin. Suppose that $\mathcal{F}$ is formally completely integrable. Then it is holomorphically completely integrable, provided the family of linear parts $\mathcal{D}$ of $\mathcal{F}$ is of the Poincaré type. In particular, $\mathcal{F}$ is holomorphically equivalent to a normal form in which each element commutes with $D_1, \dots, D_\ell$.*

The definition of Poincaré type is in Definition 8.9. Such results for commuting germs of vector fields were obtained in [Sto00, Sto05] under a collective Brjuno-type of small divisors condition. For a single germ of real analytic hyperbolic area-preserving mapping, the result was due to Moser [Mos56], and for a single germ of reversible hyperbolic holomorphic mapping $\sigma = \tau_1\tau_2$ of which $\tau_1$ fixes a hypersurface, this result was due to Moser-Webster [MW83]. Our proof is inspired by these proofs.

1.4.2. *Convergence of normalization for the abelian CR singularity.* In sections 7 we will obtain the convergence of normalization for an *abelian* CR singularity which we now define. We characterize the abelian CR singularity as follows. We first consider a product quadric $Q$ which satisfies condition E. So the branched covering $\pi_1$ for the complexification of $Q$ are generated by $p$ involutions of which each fixes a hypersurface pointwise. We denote them by $T_{11}, \dots, T_{1p}$. Let $T_{2j} = \rho T_{1j} \rho$. It turns out that each $T_{1j}$ commutes with all $T_{ik}$ except one,

$T_{2k_j}$ for some $1 \le k_j \le p$. When we formulate $S_j = T_{1j}T_{2k_j}$ for $1 \le j \le p$, then $S_1, \dots, S_p$ commute pairwise. Consider a general $M$ that is a third-order perturbation of product quadric $Q$ and satisfies condition E. We define $\sigma_j = \tau_{1j}\tau_{2k_j}$. In suitable coordinates, $T_{ij}$ (resp. $S_j$) is the linear part of $\tau_{ij}$ (resp. $\sigma_j$) at the origin. We say that the complex tangent of a third order perturbation $M$ of a product quadric at the origin is of *abelian type*, if $\sigma_1, \dots, \sigma_p$ commute pairwise. If each linear part $S_j$ of $\sigma_j$ has exactly two eigenvalues $\mu_j, \mu_j^{-1}$ that are different from 1, then $\mathcal{S} := \{S_1, \dots, S_p\}$ is of Poincaré type if and only if $|\mu_j| \neq 1$ for all $j$. As mentioned previously, Moser and Webster actually dealt with $n$-dimensional real submanifolds in $\mathbf{C}^n$ that have the minimal dimension of complex tangent subspace at a CR singular point. In their situation, there is only one possible composition, that is $\sigma = \tau_1\tau_2$. When the complex tangent has an elliptic but non-vanishing Bishop invariant, $\sigma$ has exactly two positive eigenvalues that are separated by 1, while the remaining eigenvalues are 1 with multiplicity $n-2$.

As an application of Theorem 1.6, we will prove the following convergent normalization.

**Theorem 1.7.** *Let $M$ be a germ of real analytic submanifold in $\mathbf{C}^{2p}$ at an abelian CR singularity. Suppose that $M$ has distinct eigenvalues and has no hyperbolic component of complex tangent. Then $M$ is holomorphically equivalent to*

$$\widehat{M}\colon z_{p+j} = \Lambda_{1j}(\zeta)\zeta_j, \quad 1 \le j \le p,$$

*where $\zeta = (\zeta_1, \dots, \zeta_p)$ are the convergent solutions to*

$$\begin{aligned}
\zeta_e &= A_e(\zeta) z_e \overline{z}_e - B_e(\zeta)(z_e^2 + \overline{z}_e^2),\\
\zeta_s &= A_s(\zeta) z_s \overline{z}_{s+s_*} - B_s(\zeta)(z_s^2 + \Lambda_{1s}^2(\zeta)\overline{z}_{s+s_*}^2),\\
\zeta_{s+s_*} &= A_{s+s_*}(\zeta)\overline{z}_s z_{s+s_*} - B_{s+s_*}(\zeta)(z_{s+s_*}^2 + \Lambda_{1(s+s_*)}^2(\zeta)\overline{z}_s^2)
\end{aligned}$$

*with*

$$\begin{aligned}
A_e(\zeta) &:= \frac{1+\Lambda_{1e}^2(\zeta)}{(1-\Lambda_{1e}^2(\zeta))^2}, \quad A_j(\zeta) := \Lambda_{1j}(\zeta)\frac{1+\Lambda_{1j}^2(\zeta)}{(1-\Lambda_{1j}^2(\zeta))^2},\ j = s, s+s_*,\\
B_j(\zeta) &:= \frac{\Lambda_{1j}(\zeta)}{(1-\Lambda_{1j}^2(\zeta))^2}, \quad j = e, s, s+s_*.
\end{aligned}$$

*Moreover, $\Lambda_{1j}(0) = \lambda_j$, and $\Lambda_1 = (\Lambda_{11}, \dots, \Lambda_{1p})$ commutes with the anti-holomorphic involution $\rho_z\colon \zeta_e \to \overline{\zeta}_e, \zeta_s \to \overline{\zeta}_{s+s_*}, \zeta_{s+s_*} \to \overline{\zeta}_s$.*

We will also present a more direct proof by using a convergence theorem of Moser and Webster [MW83] and some formal results from section 8.

In the above theorem $M_j = \Lambda_{1j}^2$, and they are obtained by Theorem 9.3 for the Jacobian matrix of $\log M$ to be arbitrary. When $2\log \operatorname{diag}(\Lambda_{11}, \dots \Lambda_{1p})$ is tangent to the identity, $M$ can be further uniquely normalized in suitable holomorphic coordinates to obtain a unique normal form for $M$; see Remark 9.4. When $p > 1$, the unique normal form shows that $M$ has infinitely many holomorphic invariants and $M$ is not biholomorphic to the product of Bishop surfaces in $\mathbf{C}^2$ even if the CR singularity has pure elliptic type. As an application of Theorem 1.7, we will prove the following.

**Corollary 1.8.** *Under the conditions in Theorem* 1.7*, the manifold $M$ can be holomorphically flattened. More precisely, in suitable holomorphic coordinates, $M$ is contained in the*

*linear subspace* $\mathbf{C}^p \times \mathbf{R}^p$ *defined by* $z_{p+e} = \overline{z}_{p+e}$ *and* $z_{p+s} = \overline{z}_{p+s+s_*}$ *where* $1 \leq e \leq e_*$ *and* $e_* < s \leq e_* + s_*$.

One of significances of the Bishop quadrics is that their higher order analytic perturbation at an elliptic complex tangent has a non-trivial hull of holomorphy. As another application of the above normal form, we will construct the local hull of holomorphy of $M$, that is the intersection of domains of holomorphy in $\mathbf{C}^n$ that contain $M$, via higher dimensional non-linear analytic polydiscs.

**Corollary 1.9.** *Let* $M$ *be a germ of real analytic submanifold at an abelian CR singularity. Suppose that* $M$ *has distinct eigenvalues and has only elliptic component of complex tangent. Then in suitable holomorphic coordinates,* $\mathcal{H}_{loc}(M)$*, the local hull of holomorphy of* $M$*, is filled by a real analytic family of analytic polydiscs of dimension* $p$*. Moreover,* $\mathcal{H}_{loc}(M)$ *is the transversal intersection of* $p$ *real analytic submanifolds* $\mathcal{H}_j(M)$ *with boundary and in suitable holomorphic coordinates all* $\mathcal{H}_j(M)$ *are contained in* $\mathbf{R}^p \times \mathbf{C}^p$.

For a precise statement of the corollary, see Theorem 9.6. The hulls of holomorphy for real submanifolds with a CR singularity have been studied extensively, starting with the work of Bishop. In the real analytic case with minimum complex tangent space at an elliptic complex tangent, we refer to Moser-Webster [MW83] for $\gamma > 0$, and Krantz-Huang [HK95] for $\gamma = 0$. For the smooth case, see Kenig-Webster [KW82, KW84], Huang [Hua98]. For global results on hull of holomorphy, we refer to [BG83, BK91].

1.5. **Rigidity of quadrics.** In Section 10, as an application of the theorem of linearization of holomorphic mappings on an ideal $\mathcal{I}$ [Sto13] (see Theorem 11.8 below), we will prove the following theorem, which corresponds to the case $\mathcal{I} = 0$ :

**Theorem 1.10.** *Let* $M$ *be a germ of real analytic submanifold at the origin of* $\mathbf{C}^n$*. Suppose that* $M$ *is formally equivalent a product quadric that has distinct eigenvalues. Suppose that each hyperbolic component has an eigenvalue* $\mu_h$ *which is either a root of unity or satisfies Brjuno small divisors condition. Then* $M$ *is holomorphically equivalent to the product quadric.*

Brjuno small divisors condition is defined by (11.32). When $p = 1$, this result is due to the first-named author under a stronger small divisor condition, namely Siegel's condition [Gon94]. In the case $p = 1$ with a vanishing Bishop invariant, such rigidity result was obtained by Moser [Mos85] and by Huang-Yin [HY09b] in a more general context.

1.6. **Attached complex submanifolds.** We now describe convergent results for attached complex submanifolds. The convergent results are for a general $M$, including the one of which the complex tangent might not be of abelian type.

We say that a formal complex submanifold $K$ is *attached* to $M$ if $K \cap M$ contains at least two germs of totally real and formal submanifolds $K_1, K_2$ that intersect transversally at a given CR singularity. In [Kli85], Klingenberg showed that when $M$ is non-resonant and $p = 1$, there is a unique formal holomorphic curve attached to $M$ with a hyperbolic complex tangent. He also proved the convergence of the attached formal holomorphic curve under a Siegel small divisors condition. When $p > 1$, we will show that generically there is no formal complex submanifold that can be attached to $M$ if $M$ does not admit the maximum

number of deck transformations or if the CR singularity has an elliptic component. When $p > 1$ and $M$ is a higher order perturbation of a product quadric of $Q_{\gamma_h}, Q_{\gamma_s}$, we will encounter various interesting situations.

Firstly, by adapting Klingenberg's proof for $p = 1$ and using a theorem of Pöschel [Pös86], we will prove the following.

**Proposition 1.11.** *Let $M$ be a third order perturbation of a product of quadrics of which each has complex type at the origin. Suppose that $M$ admits the maximum number of deck transformations and it is non-resonant. Then $M$ admits an attached complex submanifold.*

The proposition does not need any small divisors condition and a more detailed version in the presence of hyperbolic components is in Theorem 11.5. Furthermore, the non resonance condition is satisfied for $\gamma_1, \dots, \gamma_{s_*}$ outside the union of countable algebraic hypersurfaces.

Secondly, we will show that a non-resonant product quadric has a unique attached complex manifolds. However, under a perturbation of the quadric, the attached complex submanifold of the quadric can split into different attached formal submanifolds which may or may not be convergent. In fact, we will show that the coexistence of divergent and convergent attached complex submanifolds for a complex tangent of the complex type; see Proposition 11.6.

Finally, for the convergence of *all* attached formal complex submanifolds, we have the following.

**Theorem 1.12.** *Let $M$ be a third order perturbation of a product quadric. Suppose that $M$ admits the maximum number of deck transformations and is non resonant. Suppose that $M$ has no elliptic component and the eigenvalues of $\sigma$ satisfy a Bruno type condition, then all attached formal submanifolds are convergent.*

The above theorem for hyperbolic complex tangency was drafted in [Sto07]. For the Bruno type of condition in the theorem, see (11.32), which was introduced in [Sto13] for linearization on ideals.

1.6.1. *Notation.* We briefly introduce the notation used in the paper. The identity map is denoted by $I$. We denote by $LF$ the linear part at the origin of a mapping $F\colon \mathbf{C}^m \to \mathbf{C}^n$ with $F(0) = 0$. Let $F'(0)$ or $DF(0)$ denote the Jacobian matrix of the $F$ at the origin. Then $LF(z) = F'(0)z$. We also denote by $DF(z)$ or simply $DF$, the Jacobian matrix of $F$ at $z$, when there is no ambiguity. By an analytic (or holomorphic) function, we shall mean a *germ* of analytic function at a point (which will be defined by the context) otherwise stated. We shall denote by $\mathcal{O}_n$ (resp. $\widehat{\mathcal{O}}_n$, $\mathfrak{M}_n$, $\widehat{\mathfrak{M}}_n$) the space of germs of holomorphic functions of $\mathbf{C}^n$ at the origin (resp. of formal power series in $\mathbf{C}^n$, holomorphic germs, and formal germs vanishing at the origin).

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## 2. CR singularities and deck transformations

We will consider a real submanifold $M$ of $\mathbf{C}^n$. The simplest local holomorphic invariant of $M$ is the dimension of its complex tangent subspace $T^{(1,0)}_{x_0}M$ at a given point $x_0$. Here

$T^{(1,0)}_{x_0}M$ is the space of tangent vectors of $M$ at $x_0$ of the form $\sum_{j=1}^n a_j \frac{\partial}{\partial z_j}$. Let $M$ have dimension $n$. In this paper, we assume that $T^{(1,0)}_{x_0}M$ has the largest possible dimension $p = n/2$ at a given point $x_0$, or equivalently, that the complexified tangent space $T_{x_0}M \otimes \mathbf{C}$ is the direct sum of $T^{(1,0)}_{x_0}M$ and its complex conjugate. We study local invariants of $M$ under a local holomorphic change of coordinates fixing the $x_0$. In suitable holomorphic affine coordinates, we have $x_0 = 0$ and

$$M\colon z_{p+j} = E_j(z', \bar{z}'), \quad 1 \le j \le p. \tag{2.1}$$

Here we have set $z' = (z_1, \dots, z_p)$ and we will denote $z'' = (z_{p+1}, \dots, z_{2p})$. The 1-jet at the origin of the complex analytic functions $E_j$ vanishes; in other words, $E_j$ together with their first order derivatives vanish at 0. The tangent space $T_0M$ is then the $z'$-subspace. For the local theory, the only interesting case is when $M$ is not a complex submanifold, that is that $E(z', \overline{z}')$ is not holomorphic in $z'$, which we assume throughout the paper.

The main purpose of this section is to obtain some basic invariants and a relation between two families of involutions and the real analytic submanifolds which we want to normalize.

2.1. **CR singular set.** Let $M$ be given by (2.1). Then

$$X = \sum_{j=1}^{p} \left\{ a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial z_{p+j}} \right\}$$

is tangent to $M$ at $(z', z'')$ if and only if

$$b_k = \sum_{j=1}^{p} a_j \frac{\partial E_k(z', \overline{z}')}{\partial z_j}, \quad \sum_{j=1}^{p} a_j \frac{\partial \overline{E}_k(\overline{z}', z')}{\partial z_j} = 0, \quad 1 \le k \le p.$$

To consider the second set of equations, we introduce

$$C(z', \overline{z}') := \begin{vmatrix} \frac{\partial E_1}{\partial \overline{z}_1} & \cdots & \frac{\partial E_1}{\partial \overline{z}_p} \\ \vdots & \vdots & \vdots \\ \frac{\partial E_p}{\partial \overline{z}_1} & \cdots & \frac{\partial E_p}{\partial \overline{z}_p} \end{vmatrix}. \tag{2.2}$$

Note that $M$ is totally real at $(z', z'') \in M$ if and only if $C(z', \overline{z'}) \neq 0$. We will assume that $C(z', \overline{z}')$ is not identically zero in any neighborhood of the origin. Then the zero set of $C$ on $M$, denoted by $M_{CRsing}$, is called *CR singular set* of $M$, or the set of *complex tangents* of $M$. We assume that $M$ is real analytic. Then $M_{CRsing}$ is a possibly singular proper real analytic subset of $M$ that contains the origin.

2.2. **Existence of deck transformations and examples.** We first derive some quadratic invariants. Applying a quadratic change of holomorphic coordinates, we obtain

$$E_j(z', \overline{z}') = h_j(z', \overline{z}') + q_j(\overline{z}') + O(|(z', \overline{z}')|^3). \tag{2.3}$$

Here we have used the convention that if $x = (x_1, \dots, x_n)$, then $O(|x|^k)$ denotes a formal power series in $x$ without terms of order $< k$. A biholomorphic map $f$ that preserves the form of the above submanifolds $M$ and fixes the origin must preserve their complex tangent

spaces at the origin, i.e. $z'' = 0$. Thus if $\tilde{z}$ denote the old coordinates and $z$ denote the new coordinates then $f$ has the form

$$\tilde{z}' = \mathbf{A}z' + \mathbf{B}z'' + O(|z|^2), \quad \tilde{z}'' = \mathbf{U}z'' + O(|z|^2).$$

Here $\mathbf{A}$ and $\mathbf{U}$ are non-singular $p \times p$ complex matrices. Now $f(M)$ is given by

$$\mathbf{U}z'' = h(\mathbf{A}z', \overline{\mathbf{A}}\overline{z}') + q(\overline{\mathbf{A}}\overline{z}') + O(|z|^3).$$

We multiply the both sides by $\mathbf{U}^{-1}$ and solve for $z''$; the vectors of $p$ quadratic forms $\{h(\tilde{z}', \overline{\tilde{z}'}), q(\tilde{z}')\}$ are transformed into

$$\{\hat{h}(z', z'), \hat{q}(\overline{z}')\} = \{\mathbf{U}^{-1}h(\mathbf{A}z', \overline{\mathbf{A}}\overline{z}'), \mathbf{U}^{-1}q(\overline{\mathbf{A}}\overline{z}')\}. \tag{2.4}$$

This shows that if $M$ and $\hat{M}$ are holomorphically equivalent, their corresponding quadratic terms are equivalent via (2.4). Therefore, we obtain a holomorphic invariant

$$q_* = \dim_{\mathbf{C}}\{z' \colon q_1(z') = \cdots = q_p(z') = 0\}.$$

We remark that when $M, \hat{M}$ are quadratic (i.e. when their corresponding $E, \hat{E}$ are homogeneous quadratic polynomials), the equivalence relation (2.4) implies that $M, \hat{M}$ are linearly equivalent, Therefore, the above transformation of $h$ and $q$ via $\mathbf{A}$ and $\mathbf{U}$ determines the classifications of the quadrics under local biholomorphisms as wells as under linear biholomorphisms. We have shown that the two classifications for the quadrics are identical.

Recall that $M$ is real analytic. Let us complexify such a real submanifold $M$ by replacing $\bar{z}'$ by $w'$ to obtain a complex $n$-submanifold of $\mathbf{C}^{2n}$, defined by

$$\mathcal{M} \colon \begin{cases} z_{p+i} = E_i(z', w'), \\ w_{p+i} = \bar{E}_i(w', z'), \quad i = 1, \ldots, p. \end{cases}$$

We use $(z', w')$ as holomorphic coordinates of $\mathcal{M}$ and define the anti-holomorphic involution $\rho$ on it by

$$\rho(z', w') = (\bar{w}', \bar{z}'). \tag{2.5}$$

Occasionally we will also denote the above $\rho$ by $\rho_0$ for clarity. We will identify $M$ with a totally real and real analytic submanifold of $\mathcal{M}$ via embedding $z \to (z, \overline{z})$. We have $M = \mathcal{M} \cap \mathrm{Fix}(\rho)$ where $\mathrm{Fix}(\rho)$ denotes the set of fixed points of $\rho$. Let $\pi_1 \colon \mathcal{M} \mapsto \mathbf{C}^n$ be the restriction of the projection $(z, w) \to z$ and let $\pi_2$ be the restriction of $(z, w) \to w$. It is clear that $\pi_2 = \overline{\pi_1\rho}$ on $\mathcal{M}$. Throughout the paper, $\pi_1, \pi_2, \rho$ are restricted on $\mathcal{M}$ unless stated otherwise.

Our first basic assumption on $M$ is the following condition.

**Condition B.** $q_* = 0$.

Note that a necessary condition for $q_* = 0$ is that functions $q_1(z')$, $q_2(z'), \ldots, q_p(z')$ are linearly independent, since the intersection of $k$ germs of holomorphic hypersurfaces at 0 in $\mathbf{C}^p$ has dimension at least $p - k$. (See [Chi89], p. 35; [Gun90][Corollary 8, p. 81].)

When $\pi_1 \colon \mathcal{M} \to \mathbf{C}^{2p}$ is a branched covering, we define a *deck transformation* on $\mathcal{M}$ for $\pi_1$ to be a germ of biholomorphic mapping $F$ defined at $0 \in \mathcal{M}$ that satisfies $\pi_1 \circ F = \pi_1$.

In other words, $F(z', w') = (z', f(z', w'))$ and

$$E_i(z', w') = E_i(z', f(z', w')), \quad i = 1, \dots, p.$$

**Lemma 2.1.** *Suppose that $q_* = 0$. Then $M_{CRsing}$ is a proper real analytic subset of $M$ and $M$ is totally real away from $M_{CRsing}$, i.e. the CR dimension of $M$ is zero. Furthermore, $\pi_1$ is a $2^p$-to-1 branched covering. The group of deck transformations of $\pi_1$ consists of $2^\ell$ commuting involutions with $0 \le \ell \le p$.*

*Proof.* Since $q^{-1}(0) = \{0\}$, then $z' \to q(0, z')$ is a finite holomorphic map. Hence its Jacobian determinant is not identically zero. In particular, $C(z', \overline{z}')$, defined by (2.2), is not identically zero. This shows that $M$ has CR dimension 0.

Since $w' \to q(w')$ is a homogeneous quadratic mapping of the same space which vanishes only at the origin, then

$$|q(w')| \ge c|w'|^2.$$

We want to verify that $\pi_1$ is a $2^p$–to–1 branched covering. Let $\Delta_r = \{z \in \mathbf{C} \colon |z| < r\}$. We choose $C > 0$ such that $\pi_1(z, w) = (z', E(z', w'))$ defines a proper and onto mapping

$$\pi_1 \colon \mathcal{M}_1 := \mathcal{M} \cap ((\Delta_\delta^p \times \Delta_{\delta^2}^p) \times (\Delta_{C\delta}^p \times \Delta_{C\delta^2}^p)) \mapsto \Delta_\delta^p \times \Delta_{\delta^2}^p. \tag{2.6}$$

By Sard's theorem, the regular values of $\pi_1$ have the full measure. For each regular value $z$, $\pi_1^{-1}(z)$ has exactly $2^p$ distinct points (see [Chi89], p. 105 and p. 112). It is obvious that $\mathcal{M}_1$ is smooth and connected. We fix a fiber $F_z$ of $2^p$ points. Then the group of deck transformations of $\pi_1$ acts on $F_z$ in such a way that if a deck transformation fixes a point in $F_z$, then it must be the identity. Therefore, the number of deck transformations divides $2^p$ and each deck transformation has period $2^\ell$ with $0 \le \ell \le p$.

We first show that each deck transformation $f$ of $\pi_1$ is an involution. We know that $f$ is periodic and has the form

$$z' \to z', \quad w' \to \mathbf{A}w' + \mathbf{B}z' + O(2),$$

where $\mathbf{A}, \mathbf{B}$ are matrices. Assume that $f$ has period $m$. Then $\hat{f}(z', w') = (z', \mathbf{A}w' + \mathbf{B}z')$ satisfies $\hat{f}^m = I$ and $f$ is locally equivalent to $\hat{f}$; indeed $\hat{f} g f^{-1} = g$ for

$$g = \sum_{i=1}^{m} (\hat{f}^i)^{-1} \circ f^i.$$

Therefore, it suffices to show that $\hat{f}$ is an involution.

We have

$$\hat{f}^m(z', w') = (z', \mathbf{A}^m w' + (\mathbf{A}^{m-1} + \cdots + \mathbf{A} + \mathbf{I})\mathbf{B}z').$$

Since $f$ is a deck transformation, then $E(z', w')$ is invariant under $f$. Recall from (2.3) that $E(z', \overline{z}')$ starts with quadratic terms of the form $h(z', \overline{z}') + q(\overline{z}')$. Comparing quadratic terms in $E(z', w') = E \circ \hat{f}(z', w')$, we see that the linear map $\hat{f}$ has invariant functions

$$z'' = h(z', w') + q(w').$$

We know that $\mathbf{A}^m = \mathbf{I}$. By the Jordan normal form, we choose a linear transformation $\tilde{w}' = \mathbf{S}w'$ such that $\mathbf{SAS}^{-1}$ is the diagonal matrix $\operatorname{diag} \mathbf{a}$ with $\mathbf{a} = (a_1, \dots, a_p)$. In $(z', \tilde{w}')$ coordinates, the mapping $\hat{f}$ has the form $(z', \tilde{w}') \to (z', (\operatorname{diag} \mathbf{a})\tilde{w}' + \mathbf{SB}z')$. Now

$$\tilde{h}_j(z', \tilde{w}') + \tilde{q}_j(\tilde{w}') := h_j(z', \mathbf{S}^{-1}\tilde{w}) + q_j(\mathbf{S}^{-1}\tilde{w}')$$

are invariant under $\hat{f}$. Hence $\tilde{q}_j(\tilde{w}')$ are invariant under $\tilde{w}' \mapsto (\operatorname{diag} \mathbf{a})\tilde{w}'$. Since the common zero set of $q_1(w'), \dots, q_p(w')$ is the origin, then

$$V = \{\tilde{w}' \in \mathbf{C}^p \colon \tilde{q}(\tilde{w}') = 0\} = \{0\}.$$

We conclude that $\tilde{q}(\tilde{w}_1, 0, \dots, 0)$ is not identically zero; otherwise $V$ would contain the $\tilde{w}_1$-axis. Now $\tilde{q}((\operatorname{diag} \mathbf{a})\tilde{w}') = \tilde{q}(\tilde{w}')$, restricted to $\tilde{w}' = (\tilde{w}_1, 0, \dots, 0)$, implies that $a_1 = \pm 1$. By the same argument, we get $a_j = \pm 1$ for all $j$. This shows that $\mathbf{A}^2 = \mathbf{I}$. Let us combine it with

$$\mathbf{A}^m = \mathbf{I}, \quad (\mathbf{A}^{m-1} + \cdots + \mathbf{A} + \mathbf{I})\mathbf{B} = \mathbf{0}.$$

If $m = 1$, it is obvious that $\hat{f} = I$. If $m = 2^\ell > 1$, then $(\mathbf{A} + \mathbf{I})\mathbf{B} = \mathbf{0}$. Thus $\hat{f}^2(z', w') = (z', \mathbf{A}^2 w' + (\mathbf{A} + \mathbf{I})\mathbf{B}z') = (z', w')$. This shows that every deck transformation of $\pi_1$ is an involution.

For any two deck transformations $f$ and $g$, $fg$ is still a deck transformation. Hence $(fg)^2 = I$ implies that $fg = gf$. □

Next, we want to introduce types of complex tangents. When $p = 1$, the types give the classification for quadratic parts of the real submanifolds. For higher dimensions, the types serves a crude classification, but they are significant to characterize our results.

Let us first recall types of complex tangents for surfaces. The Moser-Webster theory deals with the case $p = 1$ for a real analytic surface

$$z_2 = |z_1|^2 + \gamma_1(z_1^2 + \overline{z}_1^2) + O(|z_1|^3).$$

Here $\gamma_1 \geq 0$ is the Bishop invariant of $M$. The complex tangent of $M$ is said to be *elliptic*, *parabolic*, or *hyperbolic* according to $0 \leq \gamma_1 < 1/2, \gamma_1 = 1/2$ or $\gamma_1 > 1/2$, respectively. One of most important properties of the Moser-Webster theory is the existence of the above mentioned deck transformations. When $\gamma_1 \neq 0$, there is a pair of Moser-Webster involutions $\tau_1, \tau_2$ with $\tau_2 = \rho\tau_1\rho$ such that $\tau_1$ generates the deck transformations of $\pi_1$. In fact, $\tau_1$ is the only possible non-trivial deck transformation of $\pi_1$. When $\gamma_1 \neq 1/2$, in suitable coordinates their composition $\sigma = \tau_1\tau_2$ is of the form

$$\tau \colon \xi' = \mu\xi + O(|(\xi, \eta)|^2), \quad \eta' = \mu^{-1}\eta + O(|(\xi, \eta)|^2).$$

Here $\rho(\xi, \eta) = (\overline{\eta}, \overline{\xi})$ when $0 < \gamma < 1/2$, and $\rho(\xi, \eta) = (\overline{\xi}, \overline{\eta})$ when $\gamma > 1/2$. When the complex tangent is elliptic, $\sigma$ is *hyperbolic* with $\mu > 1$; when the complex tangent is hyperbolic, then $\sigma$ is *elliptic* with $|\mu| = 1$. When the complex tangent is parabolic, the linear part of $\sigma$ is not diagonalizable and 1 is the eigenvalue. We also remark that the Moser-Webster theory deals with a more general case where $n$-dimensional submanifolds $M$ in $\mathbf{C}^n$ have the form

$$z_2 = |z_1|^2 + \gamma_1(z_1^2 + \overline{z}_1^2) + O(3), \quad y_j = O(2), \quad 2 \leq j \leq n$$

with the Bishop invariant $0 < \gamma_1 < \infty$. Here $n > 1$ is not necessarily even. The origin is then a complex tangent of $M$ of which the complex tangent space at the origin has the minimum dimension 1.

Our basic model is the product of the above-mentioned Bishop quadrics

$$Q_\gamma \colon z_{p+j} = |z_j|^2 + \gamma_j(z_j^2 + \overline{z}_j^2), \quad 1 \leq j \leq p.$$

Here $0 < \gamma_j < \infty$, $\gamma_j \neq 1/2$, $\gamma = (\gamma_1, \ldots, \gamma_p)$ and $Q_\gamma := Q_{\gamma_1} \times \cdots \times Q_{\gamma_p}$. We will see later that with $p \geq 2$, there is yet another simple model that is not in the product. This is the quadric in $\mathbf{C}^4$ defined by

$$Q_{\gamma_s} \colon z_3 = z_1\overline{z}_2 + \gamma_s\overline{z}_2^2 + (1-\gamma_s)z_1^2, \quad z_4 = \overline{z}_3. \tag{2.7}$$

Here $\gamma_s$ is a complex number. We will, however, exclude $\gamma_s = 0$ or equivalently $\gamma_s = 1$ by condition B. We also exclude $\gamma_s = 1/2$ by condition E. Note that $\gamma_s = 1/2$ does not correspond to a product Bishop quadrics either, by examining the CR singular sets. Under these mild non degeneracy conditions, we will show that $\gamma_s$ is an invariant when it is normalized to the range

$$\gamma_s \in (1/2, \infty) + i(0, \infty). \tag{2.8}$$

In this case, the complex tangent is said of *complex* type. Notice that $Q_{\gamma_j}$ is contained in a real hyperplane when $\gamma_j \geq 0$, while $Q_{\gamma_s}$ is contained in $\mathbf{C}^2 \times \mathbf{R}^2$.

We have introduced the types of the complex tangent at the origin. Of course a product of quadrics, or a product quadric, can exhibit a combination of the above basic 4 types. We will see soon that quadrics have other invariants when $p > 1$. Nevertheless, in our results, the above invariants that describe the types of the complex tangent will play a major role in the convergence or divergence of normalizations.

Before we proceed to discussing the deck transformations, we give some examples. The first example turns out to be a holomorphic equivalent form of a real submanifold that admits the maximum number of deck transformations and satisfies other mild conditions.

**Example 2.2.** Let $\mathbf{B} = (b_{jk})$ be a non-singular $p \times p$ matrix. Let $M$ be defined by

$$z_{p+j} = \left( \sum_k b_{jk}\overline{z}_k + R_j(z', \overline{z}') \right)^2, \quad 1 \leq j \leq p, \tag{2.9}$$

where each $R_j(0, \overline{z}')$ starts with terms of order at least 2. Then $M$ admits $2^p$ deck transformations for $\pi_1$. Indeed, let $\mathbf{E}_1, \ldots, \mathbf{E}_{2^p}$ be the set of diagonal $p \times p$ matrices with $\mathbf{E}_j^2 = \mathbf{I}$, and let $\mathbf{R}$ is the column vector $(R_1, \ldots, R_p)^t$. Any deck transformation $(z', w') \to (z', \tilde{w}')$ must satisfy

$$\mathbf{B}\tilde{w}' + \mathbf{R}(z', \tilde{w}') = \mathbf{E}_j(\mathbf{B}w' + \mathbf{R}(z', w')), \tag{2.10}$$

for some $\mathbf{E}_j$. Since $\mathbf{B}$ is invertible, it has a unique solution

$$\tilde{w} = \mathbf{B}^{-1}\mathbf{E}_j\mathbf{B}w' + O(|z'|) + O(|w'|^2).$$

Finally, $(z', w') \to (z', \tilde{w}')$ is an involution, as if $(z', w', \tilde{w}') = (z', w', f(z', w'))$ satisfy (2.10) if and only if $(z', f(z', w'), w')$, substituting for $(z', w', \tilde{w}')$ in (2.10), satisfy (2.10).

**Example 2.3.** Let $M$ be defined by

$$\begin{aligned}
z_{p+j} &= z_j\overline{z}_j + b_j\overline{z}_j^2 + E_j(z', \overline{z}_j), && 1 \leq j \leq p - 2s_*, \\
z_s &= \overline{z}_s z_{s+s_*} + b_{s+s_*}\overline{z}_s^2 + E_s(z', \overline{z}_s), \\
z_{s_*+s} &= \overline{z}_{s+s_*} z_s + b_s\overline{z}_{s+s_*}^2 + E_{s+s_*}(z', \overline{z}_{s+s_*}), && p - 2s_* \leq s \leq p - s_*.
\end{aligned}$$

Here $b_j \neq 0$ and $E_j = O(3)$ for $1 \leq j \leq p$. Then $M$ admits $2^p$ deck transformations for $\pi_1$.

We now present two examples to show that the deck transformations can be destroyed by perturbations when $p > 1$. This is the major difference between real submanifolds with $p > 1$ and the ones with $p = 1$.

The first example shows that a small perturbation can reduce the number of deck transformations to any number $2^\ell$.

**Example 2.4.** Let $M_{\gamma,\epsilon}$ be defined by

$$z_{p+j} = z_j\overline{z}_j + \gamma_j\overline{z}_j^2 + \epsilon_{j-1}\overline{z}_{j-1}^2, \quad 1 \le j \le p$$

with $z_0 = z_p$. Suppose that $\epsilon_j \neq 0$ and

$$\gamma_1 \cdots \gamma_p + (-1)^{p-1}\epsilon_0 \cdots \epsilon_{p-1} \neq 0. \tag{2.11}$$

We want to show that $M_{\gamma,\epsilon}$ admits the identity deck transformation only. Let $\tau(z', w') = (z', A(z', w'))$ be a deck transformation. Then

$$z_j A_j(z', w') + \gamma_j A_j^2(z', w') + \epsilon_{j-1}A_{j-1}^2(z', w') = z_j w_j + \gamma_j w_j^2 + \epsilon_{j-1}w_{j-1}^2. \tag{2.12}$$

Let $a_j(w') = A_j(0, w')$. Set $z' = 0$. By (2.11), we can solve for $a_j^2$ to get unique solutions

$$a_j^2(w') = w_j^2.$$

This shows that $a_j(w') = \pm w_j$. Since $\epsilon_{j-1} \neq 0$, setting $w_j = 0$ and comparing the coefficients of $z_i w_{j-1}$ in (2.12) yield $A_{j-1}(z', 0) = O(|z'|^2)$. Comparing the coefficients of $z_j w_j$ in (2.12), we conclude that $A_j(z', w') = w_j + O(|(z', w')|^2)$. This shows that $L\tau = I$. Since $\tau$ is periodic, then $\tau = I$.

The next example shows that the number of deck transformations can be reduced to any number $2^\ell$ by a higher order perturbation, too.

**Example 2.5.** Let $N_{\gamma,\epsilon}$ be a perturbation of $Q_\gamma$ defined by

$$z_{p+j} = z_j\overline{z}_j + \gamma_j\overline{z}_j^2 + \epsilon_{j-1}\overline{z}_{j-1}^3, \quad 1 \le j \le p.$$

Here $\epsilon_j \neq 0$ for all $j$. Let $\tau$ be a deck transformation of $N_{\gamma,\epsilon}$ for $\pi_1$. We know that $\tau$ has the form

$$z_j' = z_j, \quad w_j' = A_j(z', w') + B_j(z', w') + O(|(z', w')|^3).$$

Here $A_j$ are linear and $B_j$ are homogeneous quadratic polynomials. We then have

$$z_j A_j(z', w') + \gamma_j A_j^2(z', w') + A_{j-1}^2(z', w') = z_j w_j + \gamma_j w_j^2, \tag{2.13}$$

$$z_j B_j(z', w') + 2\gamma_j (A_j B_j)(z', w') + A_{j-1}^3(z', w') = w_{j-1}^3. \tag{2.14}$$

We know that $L\tau$ is a deck transformation for $Q_\gamma$. Thus $a_j(w') = A_j(0, w') = \pm w_j$. Set $z_j = 0$ in (2.13)-(2.14) to get $a_j(w')|\epsilon_{j-1}(w_{j-1}^3 - a_{j-1}^3(w'))$. Thus $a_{j-1}(w') = w_{j-1}$. Hence, the matrix of $L\tau$ is triangular and its diagonal entries are 1. Since $L\tau$ is periodic then $L\tau = I$. Since $\tau$ is periodic, then $\tau = I$.

Based the above two examples, we impose the second basic assumption.

**Condition D.** *$M$ satisfies condition B and the branched covering $\pi_1$ of $\mathcal{M}$ admits the maximum $2^p$ deck transformations.*

Let us first derive some significant properties for real submanifolds that satisfy conditions B and D.

2.3. **Real submanifolds and Moser-Webster involutions.** The main result of this subsection is to show the equivalence of classification of the real submanifolds with that of families of involutions $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$. The relation between two classifications plays an important role in the Moser-Webster theory for $p = 1$. This will be the base of our approach to the normal form problems.

Let $\mathcal{F}$ be a family of holomorphic maps in $\mathbf{C}^n$ with coordinates $z$. Let $L\mathcal{F}$ denote the set of linear maps $z \to f'(0)z$ with $f \in \mathcal{F}$. Let $\mathcal{O}_n^{\mathcal{F}}$ denote the set of germs of holomorphic functions $h$ at $0 \in \mathbf{C}^n$ so that $h \circ f = h$ for each $f \in \mathcal{F}$. Let $[\mathfrak{M}_n]_1^{L\mathcal{F}}$ be the subset of linear functions of $\mathfrak{M}_n^{L\mathcal{F}}$.

**Lemma 2.6.** *Let $G$ be an abelian group of holomorphic (resp. formal) involutions fixing $0 \in \mathbf{C}^n$. Then $G$ has $2^\ell$ elements which are simultaneously diagonalizable by a holomorphic (resp. formal) transformation. If $k = \dim_{\mathbf{C}}[\mathfrak{M}_n]_1^{LG}$ then $\ell \le n - k$. Assume furthermore that $\ell = n - k$ then, in suitable holomorphic $(z_1, \ldots, z_n)$ coordinates, the group $G$ is generated by $Z_{k+1}, \ldots, Z_n$ with*

$$Z_j \colon z_j' = -z_j, \quad z_i' = z_i, \quad i \ne j, \quad 1 \le i \le n. \tag{2.15}$$

*In the $z$ coordinates, the set of convergent (resp. formal) power series in $z_1, \ldots, z_k$, $z_{k+1}^2, \ldots, z_n^2$ is equal to $\mathcal{O}_n^G$ (resp. $\widehat{\mathcal{O}}_n^G$), and with $Z = Z_{n-k} \cdots Z_n$,*

$$[\mathfrak{M}_n]_1^G = [\mathfrak{M}_n]_1^Z, \quad \mathrm{Fix}(Z) = \bigcap_{j=k+1}^{n} \mathrm{Fix}(Z_j). \tag{2.16}$$

*Proof.* We first want to show that $G$ has $2^\ell$ elements. Suppose that it has more than one element and we have already found a subgroup of $G$ that has $2^i$ elements $f_1, \ldots, f_{2^i}$. Let $g$ be an element in $G$ that is different from the $2^i$ elements. Since $g$ is an involution and commutes with each $f_j$, then

$$f_1, \ldots, f_{2^i}, \quad gf_1, \ldots, gf_{2^i}$$

form a group of $2^{i+1}$ elements. We have proved that every finite subgroup of $G$ has exactly $2^\ell$ elements. Moreover, if $G$ is infinite then it contains a subgroup of $2^\ell$ elements for every $\ell \ge 0$. Let $\{f_1, \ldots, f_{2^\ell}\}$ be such a subgroup of $G$. It suffices to show that $\ell \le n-k$. We first linearize all $f_j$ simultaneously. We know that $Lf_1, \ldots, Lf_{2^\ell}$ commute pairwise. Note that $I + f_1'(0)^{-1} f_1$ linearizes $f_1$. Assume that $f_1$ is linear. Then $f_1 = Lf_1$ and $Lf_2$ commute, and $I + f_2'(0)^{-1} f_2$ commutes with $f_1$ and linearizes $f_2$. Thus $f_j$ can be simultaneously linearized by a holomorphic (resp. formal) change of coordinates. Without loss of generality, we may assume that each $f_j$ is linear. We want to diagonalize all $f_j$ simultaneously. Let $E_i^1$ and $E_i^{-1}$ be the eigenspaces of $f_i$ with eigenvalues 1 and $-1$, respectively. Since $f_i = f_j^{-1} f_i f_j$, each eigenspace of $f_i$ is invariant under $f_j$. Then we can decompose

$$\mathbf{C}^n = \bigoplus_{(i_1, \ldots, i_s)} E_1^{i_1} \cap \cdots \cap E_s^{i_s}. \tag{2.17}$$

Here $(i_1, \ldots, i_s)$ runs over $\{-1, 1\}^s$ with subspaces $E^{(i_1, \ldots, i_s)} := E_1^{i_1} \cap \cdots \cap E_s^{i_s} \ne \{0\}$. On each of these subspaces, $f_j = I$ or $-I$. We are ready to choose a new basis for $\mathbf{C}^n$ whose elements are in the subspaces. Under the new basis, all $f_j$ are diagonal.

Let us rewrite (2.17) as

$$\mathbf{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell.$$

Here $V_j = E^{I_j}$ and $I_1 = (1, \ldots, 1)$. Also, $I_j \neq (1, \ldots, 1)$ and $\dim V_j > 0$ for $j > 1$. We have $\dim_{\mathbf{C}} \mathrm{Fix}(G) = \dim_{\mathbf{C}} V_1 = \dim_{\mathbf{C}} [\mathfrak{M}_n]_1^{LG} = k$. Therefore, $\ell \leq n - \dim_{\mathbf{C}} V_1 \leq n - k$. We have proved that in suitable coordinates $G$ is generated by $Z_{k+1}, \ldots, Z_n$. The remaining assertions follow easily. □

We will need an elementary result about invariant functions.

**Lemma 2.7.** *Let $Z_{k+1}, \ldots, Z_n$ be defined by (2.15). Let $F = \{f_{k+1}, \ldots, f_n\}$ be a family of germs of holomorphic mappings at the origin $0 \in \mathbf{C}^n$. Suppose that the family $F$ is holomorphically equivalent to $\{Z_{k+1}, \ldots, Z_n\}$. Let $b_1(z), \ldots, b_n(z)$ be germs of holomorphic functions that are invariant under $F$. Suppose that for $1 \leq j \leq k$, $b_j(0) = 0$ and the linear part of $b_j$ at the origin is $\tilde{b}_j$. Suppose that for $i > k$, $b_i(z) = O(|z|^2)$ and the quadratic part of $b_i$ at the origin is $b_i^*$. Suppose that $\tilde{b}_1, \ldots, \tilde{b}_k$ are linear independent, and that $b_{k+1}^*, \ldots, b_n^*$ are linearly independent modulo $\tilde{b}_1, \ldots, \tilde{b}_k$, i.e.*

$$\sum c_i b_i^*(z) = \sum d_j(z) \tilde{b}_j(z) + O(|z|^3)$$

*holds for some constants $c_i$ and formal power series $d_j$, if and only if all $c_i$ are zero. Then invariant functions of $F$ are power series in $b_1, \ldots, b_n$. Furthermore, $F$ is uniquely determined by $b_1, \ldots, b_n$. The same conclusion holds if $F$ and $b_j$ are given by formal power series.*

*Proof.* Without loss of generality, we may assume that $F$ is $\{Z_{k+1}, \ldots, Z_n\}$. Hence, for all $j$, there is a formal power series $a_j$ such that $b_j(z) = a_j(z_1, \ldots, z_k, z_{k+1}^2, \ldots, z_n^2)$. Let us show that the map $w \to a(w) = (a_1(w), \ldots, a_n(w))$ is invertible.

By Lemma 2.6, $\tilde{b}_1(z), \ldots, \tilde{b}_k(z)$ are linear combinations of $z_1, \ldots, z_k$, and vice versa. By Lemma 2.6 again, $b_{k+1}^*, \ldots, b_n^*$ are linear combinations of $z_{k+1}^2, \ldots, z_n^2$ modulo $z_1, \ldots, z_k$. This shows that

$$b_i^*(z) = \sum_{j>k} c_{ij} z_j^2 + \sum_{\ell \leq k} d_{i\ell}(z) \tilde{b}_\ell(z), \quad i > k.$$

Since $b_{k+1}^*, \ldots, b_n^*$ are linearly independent modulo $\tilde{b}_1, \ldots, \tilde{b}_k$. Then $(c_{ij})$ is invertible; so is the linear part of $a$.

To show that $F$ is uniquely determined by its invariant functions, let $\tilde{F}$ be another such family that is equivalent to $\{Z_{k+1}, \ldots, Z_n\}$. Assume that $F$ and $\tilde{F}$ have the same invariant functions. Without loss of generality, assume that $\tilde{F}$ is $\{Z_{k+1}, \ldots, Z_n\}$. Then $z_1, \ldots, z_k$ are invariant by each $F_j$, i.e. the $i$th component of $F_j(z)$ is $z_i$ for $i \leq k$. Also $F_{j,\ell}^2(z) = z_\ell^2$ for $\ell > k$. We get $F_{j,\ell} = \pm z_\ell$. Since $z_\ell$ is not invariant by $\tilde{F}$, then it is not invariant by $F$ either. Then $F_{j_\ell,\ell}(z) = -z_\ell$ for some $\ell_j > k$. Since $F_{j_\ell}$ is equivalent to some $Z_i$, the set of fixed points of $F_{j_\ell}$ is a hypersurface. This shows that $F_{j_\ell} = Z_\ell$. So the family $F$ is $\{Z_{k+1}, \ldots, Z_n\}$. □

We now want to find a special set of generators for the deck transformations and its basic properties, which will be important to our study of the normal form problems.

**Lemma 2.8.** *Let $M$ be defined by (2.1) and (2.3) with $q_* = 0$. Suppose that $\mathcal{T}_i$, the group of deck transformations of $\pi_i\colon \mathcal{M} \to \mathbf{C}^p$, has exactly $2^p$ elements. Then the followings hold.*

*(i) $\mathcal{T}_1$ is generated by $p$ distinct involutions $\tau_{1j}$ such that $\mathrm{Fix}(\tau_{11})$, …, $\mathrm{Fix}(\tau_{1p})$ are hypersurfaces intersecting transversally at 0. And $\tau_1 = \tau_{11}\cdots\tau_{1p}$ is the unique deck transformation of which the set of fixed points has dimension $p$. Moreover, $\mathrm{Fix}(\tau_1) = \bigcap \mathrm{Fix}(\tau_{1j})$.*

*(ii) $\mathcal{O}_n^{\mathcal{T}_1}$ (resp. $\widehat{\mathcal{O}}_n^{\mathcal{T}_1}$) is precisely the set of convergent (resp. formal) power series in $z'$ and $E(z', w')$. $\mathcal{O}_n^{\mathcal{T}_2}$ (resp. $\widehat{\mathcal{O}}_n^{\mathcal{T}_2}$) is the set of convergent (resp. formal) power series in $w'$ and $\overline{E}(w', z')$. In particular, in $(z', w')$ coordinates of $\mathcal{M}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy*

$$[\mathfrak{M}_n]_1^{L\mathcal{T}_1} \cap [\mathfrak{M}_n]_1^{L\mathcal{T}_2} = \{0\}, \tag{2.18}$$
$$\dim \mathrm{Fix}(\tau_i) = p, \quad \mathrm{Fix}(\tau_1) \cap \mathrm{Fix}(\tau_2) = \{0\}.$$

*Here $[\mathfrak{M}_n]_1$ is the set of linear functions in $z', w'$ without constant terms.*

*Proof.* (i). Since $z_1, \ldots, z_p$ are invariant under deck transformations of $\pi_1$, we have $p' = \dim_{\mathbf{C}}[\mathcal{O}_n]_1^{L\mathcal{T}_1} \geq p$. By Lemma 2.6, $\pi_1$ has at most $2^{2p-p'}$ deck transformations. Therefore, $p' = p$. By Lemma 2.6 again, we may assume that in suitable $(\xi, \eta)$ coordinates, the deck transformations are generated by

$$Z_j\colon (\xi, \eta) \to (\xi, \eta_1, \ldots, \eta_{j-1}, -\eta_j, \eta_{j+1}, \ldots, \eta_p), \quad 1 \leq j \leq p. \tag{2.19}$$

It follows that $Z = Z_1 \cdots Z_p$ is the unique deck transformation of $\pi_1$, of which the set of fixed points has dimension $p$.

(ii). We have proved that in $(\xi, \eta)$ coordinates the deck transformations are generated by the above $Z_1, \ldots, Z_p$. Thus, the invariant holomorphic functions of $Z_1, \ldots, Z_p$ are precisely the holomorphic functions in $\xi_1, \ldots, \xi_p, \eta_1^2, \ldots, \eta_p^2$. Since $z_1, \ldots, z_p$ and $E_i(z', w')$ are invariant under deck transformations, then on $\mathcal{M}$

$$z' = f(\xi, \eta_1^2, \ldots, \eta_p^2), \quad E(z', w') = g(\xi, \eta_1^2, \ldots, \eta_p^2). \tag{2.20}$$

Since $(z', w')$ are local coordinates of $\mathcal{M}$, the differentials of $z_1, \ldots, z_p$ under any coordinate system of $\mathcal{M}$ are linearly independent. Computing the differentials of $z'$ in variables $\xi, \eta$ by using (2.20), we see that the mapping $\xi \to f(\xi, 0)$ is a local biholomorphism. Expressing both sides of the second identity in (2.20) as power series in $\xi, \eta$, we obtain

$$E(f(\xi, 0), w') = g(\xi, \eta_1^2, \ldots, \eta_p^2) + O(|(\xi, \eta)|^3).$$

We set $\xi = 0$, compute the left-hand side, and rewrite the identity as

$$g(0, \eta_1^2, \ldots, \eta_p^2) = q(w') + O(|(\xi, \eta)|^3). \tag{2.21}$$

As coordinate systems, $(z', w')$ and $(\xi, \eta)$ vanish at $0 \in \mathcal{M}$. We now use $(z', w') = O(|(\xi, \eta)|)$. By (2.20), $f(0) = g(0) = 0$ and $g(\xi, 0) = O(|\xi|^2)$. Let us verify that the linear parts of $g_1(0, \eta), \ldots, g_p(0, \eta)$ are linearly independent. Suppose that $\sum_{j=1}^p c_j g_j(0, \eta) = O(|\eta|^2)$. Replacing $\xi, \eta$ by $O(|(z', w')|)$ in (2.21) and setting $z' = 0$, we obtain

$$\sum_{j=1}^p c_j q_j(w') = O(|w'|^3), \quad i.e. \quad \sum_{j=1}^p c_j q_j(w') = 0.$$

As remarked after condition B was introduced, $q_* = 0$ implies that $q_1(w'), \ldots, q_p(w')$ are linearly independent. Thus all $c_j$ are 0. We have verified that $\xi \to f(\xi, 0)$ is biholomorphic near $\xi = 0$. Also $\eta \to g(0, \eta)$ is biholomorphic near $\eta = 0$ and $g(\xi, 0) = O(|\xi|^2)$. Therefore, $(\xi, \eta) \to (f, g)(\xi, \eta)$ is invertible near 0. By solving (2.20), the functions $\xi, \eta_1^2, \ldots, \eta_p^2$ are expressed as power series in $z'$ and $E(z', w')$.

It is clear that $z_1, \ldots, z_p$ are invariant under $\tau_{1j}$. From linearization of $\mathcal{T}_1$, we know that the space of invariant linear functions of $L\mathcal{T}_1$ is the same as the space of linear invariant functions of $L\tau_1$, which has dimension $p$. This shows that $z_1, \ldots, z_p$ span the space of linear invariant functions of $L\tau_1$. Also $w_1, \ldots, w_p$ span the space of linear invariant functions of $L\tau_2$. We obtain $[\mathfrak{M}_n]_1^{L\mathcal{T}_1} \cap [\mathfrak{M}_n]_1^{L\mathcal{T}_2} = \{0\}$. We have verified (2.18).

In view of the linearization of $\mathcal{T}_1$ in (i), we obtain $\dim \mathrm{Fix}(\tau_1) = \dim \mathrm{Fix}(\mathcal{T}_1) = p$. Moreover, $\mathrm{Fix}(\tau_i)$ is a smooth submanifold of which the tangent space at the origin is $\mathrm{Fix}(L\tau_i)$. We choose a basis $u_1, \ldots, u_p$ for $\mathrm{Fix}(L\tau_1)$. Let $v_1, \ldots, v_p$ be any $p$ vectors such that $u_1, \ldots, u_p, v_1, \ldots, v_p$ form a basis of $\mathbf{C}^n$. In new coordinates defined by $\sum \xi_i u_i + \eta_i v_i$, we know that linear invariant functions of $L\tau_1$ are spanned by $\xi_1, \ldots, \xi_p$. The linear invariant functions in $(\xi, \eta)$ that are invariant by $L\tau_2$ are spanned by $f_j(\xi, \eta) = \sum_k (a_{jk}\xi_k + b_{jk}\eta_k)$ for $1 \le j \le p$. Since $[\mathfrak{M}_n]^{L\tau_1} \cap [\mathfrak{M}_n]^{L\tau_2} = \{0\}$, then $\xi_1, \ldots, \xi_p, f_1, \ldots, f_p$ are linearly independent. Equivalently, $(b_{jk})$ is non-singular. Now $\mathrm{Fix}(L\tau_2)$ is spanned by vectors $\sum_k (a_{jk}u_k + b_{jk}v_k)$. This shows that $\mathrm{Fix}(L\tau_1) \cap \mathrm{Fix}(L\tau_2) = \{0\}$. Therefore, $\mathrm{Fix}(\tau_1)$ intersects $\mathrm{Fix}(\tau_2)$ transversally at the origin and the intersection must be the origin. □

We remark that the proof of the above lemma actually gives us a more general result.

**Corollary 2.9.** *Let $0 \le p \le n$. Let $\mathfrak{I}$ be a group of commuting holomorphic (formal) involutions on $\mathbf{C}^n$.*

(*i*) $\mathrm{Fix}(L\mathfrak{I}) = \{0\}$ *if and only if* $[\mathfrak{M}_n]_1^{L\mathfrak{I}}$ *has dimension* 0.

(*ii*) *Let $\widetilde{\mathfrak{I}}$ be another family of commuting holomorphic (resp. formal) involutions such that $[\mathfrak{M}_n]^{L\mathfrak{I}} \cap [\mathfrak{M}_n]^{L\widetilde{\mathfrak{I}}} = \{0\}$. Then $\mathrm{Fix}(L\mathfrak{I}) \cap \mathrm{Fix}(L\widetilde{\mathfrak{I}}) = \{0\}$. Moreover, $\mathrm{Fix}(\mathfrak{I}) \cap \mathrm{Fix}(\widetilde{\mathfrak{I}}) = \{0\}$ if $\mathfrak{I}$ and $\widetilde{\mathfrak{I}}$ consist of convergent involutions.*

In view of Lemma 2.8, we will refer to

$$\{\tau_{1j}, \tau_{2j}, \rho;\ 1 \le j \le p\}$$

as the Moser-Webster involutions, while the two groups of the $2^p$ involutions intertwined by $\rho$ will be called the extended family of Moser-Webster involutions. Recall that $\tau_{2j} = \rho\tau_{1j}\rho$. Let us denote

$$\mathcal{T}_1 := \{\tau_{11}, \ldots, \tau_{1p}\}, \quad \mathcal{T}_2 := \{\tau_{21}, \ldots, \tau_{2p}\}.$$

Thus the sets of involutions are uniquely determined by

$$\{\mathcal{T}_1, \rho\} = \{\tau_{11}, \ldots, \tau_{1p}, \rho\}.$$

The significance of the two sets of involutions is the following proposition that transforms the normalization of the real manifolds into that of two families of commuting involutions.

For clarity, recall the anti-holomorphic involution $\rho_0 \colon (z', w') \to (\overline{w'}, \overline{z'})$.

**Proposition 2.10.** *Let $M$ and $\widetilde{M}$ be two real analytic submanifolds of the form* (2.1) *and* (2.3) *that admit Moser-Webster involutions $\{\mathcal{T}_1, \rho_0\}$ and $\{\widetilde{\mathcal{T}}_1, \rho_0\}$, respectively. Then $M$ and*

*$\widetilde{M}$ are holomorphically equivalent if and only if $\{\mathcal{T}_1, \rho_0\}$ and $\{\widetilde{\mathcal{T}}_1, \rho_0\}$ are holomorphically equivalent, i.e. if there is a biholomorphic map $f$ commuting with $\rho_0$ such that $f\mathcal{T}_1 f^{-1} = \widetilde{\mathcal{T}}_1$, that is that $f\tau_{1j}f^{-1} = \tilde{\tau}_{1i_j}$ for $1 \le j \le p$. Here $\{i_1, \dots, i_p\} = \{1, \dots, p\}$.*

*Let $\mathcal{T}_1 = \{\tau_{11}, \dots, \tau_{1p}\}$ be a family of $p$ distinct commuting holomorphic involutions. Suppose that $\mathrm{Fix}(\tau_{11}), \dots, \mathrm{Fix}(\tau_{1p})$ are hypersurfaces intersecting transversely at the origin. Let $\rho$ be an anti-holomorphic involutions and let $\mathcal{T}_2$ be the family of involutions $\tau_{2j} = \rho\tau_{1j}\rho$ with $1 \le j \le p$. Suppose that*

$$[\mathfrak{M}_n]_1^{L\mathcal{T}_1} \cap [\mathfrak{M}_n]_1^{L\mathcal{T}_2} = \{0\}. \tag{2.22}$$

*There exists a real analytic real $n$-submanifold*

$$M \subset \mathbf{C}^{2p} \colon z_{p+j} = A_j^2(z', \bar{z}'), \quad 1 \le j \le p \tag{2.23}$$

*such that the set of Moser-Webster involutions $\{\widetilde{\mathcal{T}}_1, \rho_0\}$ of $M$ is holomorphically equivalent to $\{\mathcal{T}_1, \rho\}$.*

*Proof.* We recall from (2.6) the branched covering

$$\pi_1 \colon \mathcal{M}_1 := \mathcal{M} \cap ((\Delta_\delta^p \times \Delta_{\delta^2}^p) \times (\Delta_{C\delta}^p \times \Delta_{C\delta^2}^p)) \longrightarrow \Delta_\delta^p \times \Delta_{\delta^2}^p.$$

Here $C \ge 1$. Let $\pi_1$ be restricted to $\mathcal{M}_1$. Then $\pi_2 = \overline{\pi_1 \circ \rho}$ is defined on $\rho(\mathcal{M}_1)$. Note that

$$\pi_2 \colon \rho(\mathcal{M}_1) \longrightarrow \Delta_\delta^p \times \Delta_{\delta^2}^p.$$

We have $\pi_1^{-1}(z) \cap \mathrm{Fix}(\rho) = \{(z, \overline{z})\}$ for $z \in M$ and $\pi_1(\mathrm{Fix}(\rho)) = M$. Let $\mathcal{B}_0 \subset \Delta_\delta^p \times \Delta_{\delta^2}^p$ be the branched locus. Take $\mathcal{B} = \pi_1^{-1}(\mathcal{B}_0)$. We will denote by $\widetilde{\mathcal{M}}_1, \tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}_0$ the corresponding data for $\widetilde{M}$. Here $\widetilde{\mathcal{M}}_1$ is an analogous branched covering over $\pi_1(\widetilde{\mathcal{M}}_1)$. We assume that the latter contains $f(\pi_1(\mathcal{M}_1))$ if $\widetilde{M}$ is equivalent to $M$ via $f$.

Assume that $f$ is a biholomorphic map sending $M$ into $\widetilde{M}$. Let $f^c$ be the restriction of biholomorphic map $f^c(z, w) = (f(z), \overline{f}(w))$ to $\mathcal{M}$. Let $M$ be defined by $z'' = E(z', \overline{z}')$ and $\widetilde{M}$ be defined by $z'' = \tilde{E}(z', \overline{z}')$. By $f(M) \subset \widetilde{M}$, $f = (f', f'')$ satisfies

$$f''(z', E(z', \overline{z}')) = \tilde{E}(f'(z', E(z', \overline{z}')), \overline{f'}(\overline{z}', \overline{E}(\overline{z}', z'))).$$

Using the defining equations for $\mathcal{M}$, we get $f^c(\mathcal{M}) \subset \widetilde{\mathcal{M}}$ and $\rho f^c = f^c \rho$ on $\mathcal{M} \cap \rho(\mathcal{M})$. We will also assume that $f^c(\mathcal{M}_1)$ is contained in $\widetilde{\mathcal{M}}_1$. It is clear that $f^c$ sends a fiber $\pi_1^{-1}(z)$ onto the fiber $\pi_1^{-1}(f(z))$ for $z \in \Omega = \pi_1(\mathcal{M}_1) \setminus (\mathcal{B}_0 \cup f^{-1}(\tilde{B}_0))$, since the two fibers have the same number of points and $f$ is injective. Thus $f^c\tau_{1j} = \tilde{\tau}_{1i_j} f^c$ on $\pi_1^{-1}(\Omega)$. Here $i_j$ is of course locally determined on $\pi_1^{-1}(\Omega)$. Since $\mathcal{B}$ has positive codimension in $\mathcal{M}_1$ then $\mathcal{M}_1 \setminus \mathcal{B}$ is connected. Hence $i_j$ is well-defined on $\pi_1^{-1}(\Omega)$. Then $f^c\tau_{1j} = \tilde{\tau}_{1i_j} f^c$ on $\mathcal{M}_1 \setminus B$. This shows that $f^c$ conjugates simultaneously the deck transformations of $\mathcal{M}$ to the deck transformations of $\widetilde{\mathcal{M}}$ for $\pi_1$. The same conclusion holds for $\pi_2$.

Conversely, assume that there is a biholomorphic map $g \colon \mathcal{M} \to \widetilde{\mathcal{M}}$ such that $\rho g = g\rho$ and $g\tau_{1i} = \tilde{\tau}_{1j_i} g$. Since $\tau_{11}, \dots, \tau_{12^p}$ are distinct and $M_1 \setminus \mathcal{B}$ is connected, then $\bigcup_{j \ne i}\{x \in \mathcal{M}_1 \setminus \mathcal{B} \colon \tau_{1i}(x) = \tau_{1j}(x)\}$ is a complex subvariety of positive codimension in $\mathcal{M}_1 \setminus \mathcal{B}$. Its image under the proper projection $\pi_1$ is a subvariety of positive codimension in $\Delta_\delta^p \times \Delta_{\delta^2}^p \setminus \mathcal{B}_0$. This shows that the latter contains a non-empty open subset $\omega$ such that $\{\tau_{11}(x), \dots, \tau_{12^p}(x)\} = \pi_1^{-1}\pi_1(x)$ has $2^p$ distinct points for each $\pi_1(x) \in \omega$. Therefore, $\tau_{11}, \dots, \tau_{12^p}$ are all deck

transformations of $\pi_1$ over $\omega$. Hence they are all deck transformations of $\pi_1\colon \mathcal{M}_1 \setminus \mathcal{B} \to \Delta_\delta^p \times \Delta_{\delta^2}^p \setminus \mathcal{B}_0$, too. This shows that $\pi_1^{-1}(\pi_1(x)) = \{\tau_{1j}(x)\colon 1 \le j \le 2^p\}$ for $x \in \mathcal{M}_1 \setminus \mathcal{B}$. Now, $g$ sends $\tau_{1j}(x)$ to $\tilde{\tau}_{1i_j}(g(x))$ for each $j$. Hence $f(z) = \pi_1 g \pi_1^{-1}(z)$ is well-defined and holomorphic for $z \in \Delta_\delta^p \times \Delta_{\delta^2}^p \setminus \mathcal{B}_0$. By the Riemann extension for bounded holomorphic functions, $f$ extends to a holomorphic mapping, still denoted by $f$, which is defined near the origin. We know that $f$ is invertible and in fact the inverse can be obtained by extending the mapping $z \to \pi_1 g^{-1} \pi_1(z)$. If $z = (z', E(z', w')) \in M$, then $w' = \overline{z}'$ and $f(z) = \pi_1 g \pi_1^{-1}(z) = \pi_1 g(z, \overline{z})$ with $(z, \overline{z}) \in \mathrm{Fix}(\rho)$. Since $\rho g = g\rho$, then $g(z, \overline{z}) \in \mathrm{Fix}(\rho)$. Thus $f(z) = \pi_1 g(z, \overline{z}) \in \widetilde{M}$.

Assume that $\{\tau_{1j}\}$ and $\rho$ are germs of involutions defined at the origin of $\mathbf{C}^n$. Assume that they satisfy the conditions in the proposition. From Lemma 2.6 it follows that $\tau_{11}, \dots, \tau_{1p}$ generate a group of $2^p$ involutions, while the $p$ generators are the only elements of which each fixes a hypersurface pointwise. To realize them as deck transformations of the complexification of a real analytic submanifold, we apply Lemma 2.6 to find a coordinate map $(\xi, \eta) \to \phi(\xi, \eta) = (A, B)(\xi, \eta)$ such that invariant holomorphic functions of $\{\tau_{1j}\}$ are precisely holomorphic functions in

$$z' = (A_1(\xi, \eta), \dots, A_p(\xi, \eta)), \quad z'' = (B_1^2(\xi, \eta), \dots, B_p^2(\xi, \eta)).$$

Note that $B_j$ is skew-invariant under $\tau_{1j}$ and is invariant under $\tau_{1i}$ for $i \ne j$ and $A$ is invariant under all $\tau_{1j}$. Set

$$w_j' = \overline{A_j \circ \rho(\xi, \eta)}, \quad w_j'' = \overline{B_j^2 \circ \rho(\xi, \eta)}.$$

Since $\tau_{2j} = \rho \tau_{1j} \rho$, the holomorphic functions invariant under all $\tau_{2j}$ are precisely the holomorphic functions in the above $w', w''$. We now draw conclusions for the linear parts of invariant functions and involutions. Since $\phi$ is biholomorphic, then $LA_1, \dots, LA_p$ are linearly independent. They are also invariant under $L\tau_{1j}$. Since $\tau_{2j} = \rho\tau_{1j}\rho$, the $p$ functions $L\overline{A_i \circ \rho}$ are linearly independent and invariant under $L\tau_{2j}$. Thus

$$LA_1, \dots, LA_p, \ L\overline{A_1 \circ \rho}, \dots, L\overline{A_p \circ \rho}$$

are linearly independent, since $[\mathfrak{M}_n]_1^{L\mathcal{T}_1} \cap [\mathfrak{M}_n]_1^{L\mathcal{T}_2} = \{0\}$. This shows that the map $(\xi, \eta) \to (z', w') = (A(\xi, \eta), \overline{A \circ \rho(\xi, \eta)})$ has an inverse $(\xi, \eta) = \psi(z', w')$. Define

$$M\colon z'' = (B_1^2, \dots, B_p^2) \circ \psi(z', \overline{z}').$$

The complexification of $M$ is given by

$$\mathcal{M}\colon z'' = (B_1^2, \dots, B_p^2) \circ \psi(z', w'), \quad w'' = (\overline{B}_1^2, \dots, \overline{B}_p^2) \circ \overline{\psi}(w', z').$$

Note that $\phi \circ \psi(z', w') = (z', B \circ \psi(z', w'))$ is biholomorphic. In particular, we can write

$$B_j^2 \circ \psi(z', \overline{z}') = h_j(z', \overline{z}') + q_j(\overline{z}') + b_j(z') + O(|(z', \overline{z}')|^3).$$

Here $q_j(\overline{z}') = \tilde{q}_j^2(\overline{z}')$, and $\tilde{q}(w')$ is the linear part of $w' \to B \circ \psi(0, w')$. Therefore, $|q(w')| \ge c|w'|^2$ and $q_* = 0$. By Lemma 2.1, $\pi_1\colon \mathcal{M} \to \mathbf{C}^p$ is a $2^p$-to-1 branched covering defined near $0 \in \mathcal{M}$. Since $B^2$ is invariant by $\tau_{1j}$, then $z'' = B^2 \circ \psi(z', w')$ is invariant by $\psi^{-1}\tau_{1j}\psi(z', w')$. Also $A$ is invariant under $\tau_{1j}$. Then $z' = A \circ \psi(z', w')$ is invariant by $\psi^{-1}\tau_{1j}\psi(z', w')$. This show that $\{\psi^{-1}\tau_{1j}\psi\}$ has the same invariant functions as of the deck transformations of $\pi_1$.

By Lemma 2.7, $\{\psi^{-1}\tau_{1j}\psi\}$ agrees with the set of deck transformations of $\pi_1$. For $\rho_0(z', w') = (\overline{w'}, \overline{z'})$ we have $\rho_0\psi^{-1} = \psi^{-1}\rho$. This shows that $M$ is a realization for $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$. $\square$

**Remark 2.11.** (i) We choose the realization in such a way that $z_{p+j}$ are square functions. Of course, the choice is not unique. In fact, we can replace $z''$ by $f(z) = (f_{p+1}(z), \ldots, f_{2p}(z))$ as long as the mapping $z \to (z', f(z))$ is biholomorphic. However, this particular holomorphic equivalent form of $M$ will be crucial to study the asymptotic manifolds in section 11. In fact, by Example 2.4, (2.9) provides a general equation for $M$ to admit $2^p$ deck transformations. (ii) An interesting case is when $f(z)$ can be so chosen that $M$ is holomorphically flattened, i.e, $M$ is contained in $\operatorname{Im} z'' = 0$. In [MW83], such a choice is always possible at least at the formal level. We will discuss the holomorphic flatness in Theorem 9.5.

Next we want to compute the deck transformations for a product quadric. We will first recall the Moser-Webster involutions for elliptic and hyperbolic complex tangents. We will then compute the deck transformations for complex tangents of complex type.

Let us first recall involutions in [MW83] where the complex tangents are elliptic (with non-vanishing Bishop invariant) or hyperbolic. When $\gamma_1 \neq 0$, the non-trivial deck transformations of

$$z_2 = |z_1|^2 + \gamma_1(z_1^2 + \overline{z}_1^2)$$

for $\pi_1, \pi_2$ are $\tau_1, \tau_2$, respectively. They are

$$\tau_1 \colon z_1' = z_1, \quad w_1' = -w_1 - \gamma_1^{-1} z_1; \quad \tau_2 = \rho\tau_1\rho$$

with $\rho$ being defined by (2.5). Note that $\tau_1$ and $\tau_2$ do not commute and $\sigma = \tau_1\tau_2$ satisfies

$$\sigma^{-1} = \tau_i\sigma\tau_i = \rho\sigma\rho, \quad \tau_i^2 = I, \quad \rho^2 = I.$$

When the complex tangent is not parabolic, the eigenvalues of $\sigma$ are $\mu, \mu^{-1}$ with $\mu = \lambda^2$ and

$$\gamma\lambda^2 - \lambda + \gamma = 0.$$

For the elliptic complex tangent, we can choose a solution $\lambda > 1$, and in suitable coordinates we obtain

$$\begin{gathered} \tau_1 \colon \xi' = \lambda\eta + O(|(\xi,\eta)|^2), \quad \eta' = \lambda^{-1}\xi + O(|(\xi,\eta)|^2), \\ \tau_2 = \rho\tau_1\rho, \quad \rho(\xi,\eta) = (\overline{\eta}, \overline{\xi}), \\ \sigma \colon \xi' = \mu\xi + O(|(\xi,\eta)|^2), \quad \eta' = \mu^{-1}\eta + O(|(\xi,\eta)|^2), \quad \mu = \lambda^2. \end{gathered} \tag{2.24}$$

When the complex tangent is hyperbolic, i.e. $1/2 < \gamma < \infty$, $\tau_i$ and $\sigma$ still have the above form, while $|\mu| = 1 = |\lambda|$ and

$$\rho(\xi,\eta) = (\overline{\xi}, \overline{\eta}).$$

When the complex tangent is parabolic, i.e. $\gamma = 1/2$, the pair of involutions still exists. However, $L\sigma$ is not diagonalizable and 1 is its only eigenvalue.

For the complex type, new situations arise. Recall that such a quadric has the form

$$Q_{\gamma_s} \colon z_3 = z_1\overline{z}_2 + \gamma_s\overline{z}_2^2 + (1-\gamma_s)z_1^2, \quad z_4 = \overline{z}_3. \tag{2.25}$$

Here $\gamma_s$ is a complex number. Let us first check that such a quadric is not the product of two Bishop quadrics : Its CR singular set is defined by

$$(z_1 + 2\gamma_s\overline{z}_2)(z_2 + 2(1-\overline{\gamma}_s)\overline{z}_1) = 0.$$

It is the union of a complex line and a totally real plane, or two totally real planes. The CR singular set of a quadric defined by $z_3 = |z_1|^2 + \gamma_1(z_1^2 + \overline{z}_1^2)$ and $z_4 = |z_2|^2 + \gamma_2(z_2^2 + \overline{z}_2^2)$ is given by

$$(z_1 + 2\gamma_1\overline{z}_1)(z_2 + 2\gamma_2\overline{z}_2) = 0.$$

It is the union of two complex lines, or one complex line and a 3 dimensional plane.

By condition B, we know that $\gamma_s \neq 0, 1$. Let us compute the deck transformations of the complexification of (2.25). According to Lemma 2.8 (i), the deck transformations for $\pi_1$ are generated by two involutions

$$\tau_{11}\colon \begin{cases} z_1' = z_1, \\ z_2' = z_2, \\ w_1' = -w_1 - \gamma_s^{-1} z_2, \\ w_2' = w_2; \end{cases} \qquad \tau_{12}\colon \begin{cases} z_1' = z_1, \\ z_2' = z_2, \\ w_1' = w_1, \\ w_2' = -w_2 - (1 - \overline{\gamma}_s)^{-1} z_1. \end{cases}$$

We still have $\rho$ defined by (2.5). Let $\tau_{2j} = \rho\tau_{1j}\rho$. Then $\tau_{21}, \tau_{22}$ generate the deck transformations of $\pi_2$. Note that

$$\tau_{21}\colon \begin{cases} z_1' = -z_1 - \overline{\gamma}_s^{-1} w_2, \\ z_2' = z_2, \\ w_1' = w_1, \\ w_2' = w_2; \end{cases} \qquad \tau_{22}\colon \begin{cases} z_1' = z_1, \\ z_2' = -z_2 - (1 - \gamma_s)^{-1} w_1, \\ w_1' = w_1, \\ w_2' = w_2. \end{cases}$$

Recall that $\tau_i = \tau_{i1}\tau_{i2}$ is the unique deck transformation of $\pi_i$ that has the smallest dimension of the set of fixed-points among all deck transformations. They are

$$\tau_1\colon \begin{cases} z_1' = z_1, \\ z_2' = z_2, \\ w_1' = -w_1 - \gamma_s^{-1} z_2, \\ w_2' = -w_2 - (1 - \overline{\gamma}_s)^{-1} z_1; \end{cases} \qquad \tau_2\colon \begin{cases} z_1' = -z_1 - \overline{\gamma}_s^{-1} w_2, \\ z_2' = -z_2 - (1 - \gamma_s)^{-1} w_1, \\ w_1' = w_1, \\ w_2' = w_2. \end{cases}$$

And $\tau_1\tau_2$ is given by

$$\sigma_s\colon \begin{cases} z_1' = -z_1 - \overline{\gamma}_s^{-1} w_2, \\ z_2' = -z_2 - (1 - \gamma_s)^{-1} w_1, \\ w_1' = \gamma_s^{-1} z_2 + ((\gamma_s - \gamma_s^2)^{-1} - 1) w_1, \\ w_2' = (1 - \overline{\gamma}_s)^{-1} z_1 + ((\overline{\gamma}_s - \overline{\gamma}_s^2)^{-1} - 1) w_2. \end{cases}$$

In contrast to the elliptic and hyperbolic cases, $\tau_{11}$ and $\rho\tau_{11}\rho$ commute; in other words, $\tau_{11}\rho\tau_{11}\rho$ is actually an involution. And $\tau_{12}$ and $\rho\tau_{12}\rho$ commute, too. However, $\tau_{11}$ and $\tau_{22}$ do not commute, and $\tau_{12}, \tau_{21}$ do not commute either. Thus, we form compositions

$$\sigma_{s1} = \tau_{11}\tau_{22}, \quad \sigma_{s2} = \tau_{12}\tau_{21}, \quad \sigma_{s2}^{-1} = \rho\sigma_{s1}\rho.$$

By a simple computation, we have

$$\sigma_{s1}\colon \begin{cases} z_1' = z_1, \\ z_2' = -z_2 - (1-\gamma_s)^{-1} w_1, \\ w_1' = \gamma_s^{-1} z_2 + ((\gamma_s - \gamma_s^2)^{-1} - 1) w_1, \\ w_2' = w_2; \end{cases}$$

$$\sigma_{s2}\colon \begin{cases} z_1' = -z_1 - \overline{\gamma}_s^{-1} w_2, \\ z_2' = z_2, \\ w_1' = w_1, \\ w_2' = (1-\overline{\gamma}_s)^{-1} z_1 + ((\overline{\gamma}_s - \overline{\gamma}_s^2)^{-1} - 1) w_2. \end{cases}$$

We verify that

$$\sigma_{s1}\sigma_{s2} = \sigma_s = \tau_1\tau_2.$$

This allows us to compute the eigenvalues of $\sigma_{s1}\sigma_{s2}$ easily:

$$\mu_s, \quad \mu_s^{-1}, \quad \overline{\mu}_s^{-1}, \quad \overline{\mu}_s, \qquad \mu_s = (\gamma_s^{-1} - 1)^{-1}. \tag{2.26}$$

In fact we compute them by observing that the first two in (2.26) and 1 with multiplicity are eigenvalues of $\sigma_{s1}$, while the last two in (2.26) and 1 with multiplicity are eigenvalues of $\sigma_{s2}$. Therefore, for $\gamma_s \neq 1/2$, i.e. $\mu_s \neq 1$, we can find a linear transformation of the form

$$\psi\colon (z_1, w_2) \to (\xi_2, \eta_2) = \overline{\phi}(z_1, w_2), \quad (z_2, w_1) \to (\xi_1, \eta_1) = \phi(w_1, z_2)$$

such that $\sigma_{s1}, \sigma_{s2}, \sigma_s = \sigma_{s1}\sigma_{s2}$ are simultaneously diagonalized as

$$\begin{array}{rllll} \sigma_{s1}\colon & \xi_1' = \mu_s\xi_1, & \eta_1' = \mu_s^{-1}\eta_1, & \xi_2' = \xi_2, & \eta_2' = \eta_2, \\ \sigma_{s2}\colon & \xi_1' = \xi_1, & \eta_1' = \eta_1, & \xi_2' = \overline{\mu}_s^{-1}\xi_2, & \eta_2' = \overline{\mu}_s\eta_2, \\ \sigma_s\colon & \xi_1' = \mu_s\xi_1, & \eta_1' = \mu_s^{-1}\eta_1, & \xi_2' = \overline{\mu}_s^{-1}\xi_2, & \eta_2' = \overline{\mu}_s\eta_2. \end{array} \tag{2.27}$$

Under the transformation $\psi$, the involution $\rho$, defined by (2.5), takes the form

$$\rho(\xi_1, \xi_2, \eta_1, \eta_2) = (\overline{\xi}_2, \overline{\xi}_1, \overline{\eta}_2, \overline{\eta}_1). \tag{2.28}$$

Moreover, for $i, j = 1, 2$, we have

$$\tau_{ij}\colon \xi_j' = \lambda_j\eta_j, \quad \eta_j' = \lambda_j^{-1}\xi_j; \quad \xi_i' = \xi_i, \quad \eta_i' = \eta_i, \quad i \neq j; \qquad \lambda_1 = \lambda_s, \quad \lambda_2 = \overline{\lambda}_s^{-1}, \quad \mu_s = \lambda_s^2. \tag{2.29}$$

When $\gamma_s = 1/2$, the only eigenvalue of $\sigma_{s1}$ is 1. We can choose a suitable $\phi$ such that $\psi$ transforms $\sigma_{s1}, \sigma_{s2}, \sigma_s$ into

$$\begin{array}{rllll} \sigma_{s1}\colon & \xi_1' = \xi_1, & \eta_1' = \eta_1 + \xi_1, & \xi_2' = \xi_2, & \eta_2' = \eta_2 \\ \sigma_{s2}\colon & \xi_1' = \xi_1, & \eta_1' = \eta_1, & \xi_2' = \xi_2, & \eta_2' = -\xi_2 + \eta_2, \\ \sigma_s\colon & \xi_1' = \xi_1, & \eta_1' = \xi_1 + \eta_1, & \xi_2' = \xi_2, & \eta_2' = -\xi_2 + \eta_2. \end{array} \tag{2.30}$$

Note that eigenvalues formulae (2.26) and the Jordan normal form (2.30) tell us that $\tau_1$ and $\tau_2$ do not commute, while $\sigma_{s1}, \sigma_{s2}$ commute as mentioned earlier.

**Remark 2.12.** The mappings $\sigma_{s1}, \sigma_{s2}$ behave like a hyperbolic mapping when $|\mu_s| > 1$, an elliptic mapping when $|\mu_s| = 1$, or a parabolic mapping when $\mu_s = 1$. Recall that $\sigma$ has 4 distinct eigenvalues for the first case, 2 distinct eigenvalues with multiplicity for the second case, and only eigenvalue of 1 for the last case. The $\sigma$ is diagonalizable for the first two cases, but it has a Jordan bock with multiplicity for the last case. In this paper, we will only study the first case of complex type, i.e.

$$|\mu_s| > 1,$$

which follows from condition E.

For later purpose, we summarize some facts for complex type in the following.

**Proposition 2.13.** *Let $Q_{\gamma_s} \subset \mathbf{C}^4$ be the quadric defined by* (2.7) *and* (2.8)*. Then $\pi_1$ admits two deck transformations $\tau_{11}, \tau_{12}$ such that the set of fixed points of each $\tau_{1j}$ has dimension* 3*. Also, $\tau_{2j} = \rho\tau_{1j}\rho$ are the deck transformations of $\pi_2$ and*

$$\tau_{11}\tau_{21} = \tau_{21}\tau_{11}, \quad \tau_{12}\tau_{22} = \tau_{22}\tau_{12}.$$

*Let $\sigma_{s1} = \tau_{11}\tau_{22}$, $\sigma_{s2} = \tau_{12}\tau_{21}$, $\tau_i = \tau_{i1}\tau_{i2}$, and $\sigma_s = \tau_1\tau_2$. Then*

$$\sigma = \sigma_{s1}\sigma_{s2} = \sigma_{s2}\sigma_{s1}, \quad \sigma_{s2}^{-1} = \rho\sigma_{s1}\rho, \quad \sigma_s^{-1} = \rho\sigma_s\rho.$$

*In suitable coordinates $\sigma_{s1}, \sigma_{s2}, \sigma, \rho_s$ are given by* (2.27)-(2.28) *when $\gamma_s \neq 1/2$; when $\gamma_s = 1/2$, they are given by* (2.28) *and* (2.30)*. If*

$$\gamma_s \in \{z \in \mathbf{C} \colon \operatorname{Re} z > 1/2, \operatorname{Im} z > 0\},$$

*then $\sigma_s$ admits* 4 *distinct eigenvalues $(\gamma_s^{-1} - 1)^{-1}, \overline{\gamma}_s^{-1} - 1, \gamma_s^{-1} - 1$, and $(\overline{\gamma}_s^{-1} - 1)^{-1}$.*

The commutativity of $\sigma_h, \sigma_e, \sigma_{s1}, \sigma_{s2}$ will be important to understand the convergence of normalization for the abelian CR singularity to be introduced in section 9.

Let us summarize some facts in this section.

Let $\tau_i = \tau_{i1} \cdots \tau_{ip}$ for $i = 1, 2$. Note that they are intertwined by the anti-holomorphic involution via $\tau_2 = \rho\tau_1\rho$. Each $\tau_i$ is the unique deck transformation for $\pi_i$ whose set of fixed points has minimum dimension $p$. Then $\sigma = \tau_1\tau_2$ is *reversible* by $\tau_i$ and $\rho$ in the sense that

$$\tau_i\sigma\tau_i = \sigma^{-1}, \quad \rho\sigma\rho = \sigma^{-1}, \quad \tau_i^2 = I, \quad \rho^2 = I.$$

The reversible map $\sigma$ will play a central role to the study of the submanifolds $M$, as we will demonstrate this in the classification of quadratic manifolds. In particular, they carry some geometry and dynamics associated to the real manifolds; for instance the attached complex submanifolds are closely related the invariant submanifolds of $\sigma$, which is discussed in section 11. We will also call $\tau_{11}, \ldots, \tau_{1p}$ the generators of the deck transformations, which are unique as each $\operatorname{Fix}(\tau_{1j})$ has codimension 1.

For various reversible mappings and their relations with general mappings, the reader is referred to [OZ11] for recent results and references therein.

To derive our normal forms, we shall transform $\{\tau_1, \tau_2, \rho\}$ into a normal form first. We will further normalize $\{\tau_{1j}, \rho\}$ by using the group of biholomorphic maps that preserve the normal form of $\{\tau_1, \tau_2, \rho\}$, i.e. the centralizer of the normal form of $\{\tau_1, \tau_2, \rho\}$.

## 3. Quadrics with the maximum number of deck transformations

In section 2, we establish the basic relation between the classification of real manifolds and that of two families of involutions intertwined by an antiholomorphic involution; see Proposition 2.10. As a first application, we obtain in this section a normal form for two families of linear involutions and use it to construct the normal form for their associated quadrics. This section also serves an introduction to our approach to find the normal forms of the real submanifolds at least at the formal level.

3.1. **Normal form of two families of linear involutions.** To formulate our results, we first discuss the normal forms which we are seeking for the involutions. We are given two families of commuting linear involutions $\mathcal{T}_1 = \{T_{11}, \dots, T_{1p}\}$ and $\mathcal{T}_2 = \{T_{21}, \dots, T_{2p}\}$ with $T_{2j} = \rho T_{1j} \rho$. Here $\rho$ is a linear anti-holomorphic involution. We set

$$T_1 = T_{11} \cdots T_{1p}, \quad T_2 = \rho T_1 \rho.$$

Recall that our involutions satisfy the additional (2.16) and (2.22). Thus

$$\dim[\mathfrak{M}_n]_1^{\mathcal{T}_i} = p, \quad [\mathfrak{M}_n]_1^{\mathcal{T}_i} = [\mathfrak{M}_n]_1^{T_i}, \tag{3.1}$$

$$[\mathfrak{M}_n]_1^{T_1} \cap [\mathfrak{M}_n]_1^{T_2} = \{0\}. \tag{3.2}$$

Recall that $[\mathfrak{M}_n]_1$ denotes the linear functions without constant terms. We would like to find a change of coordinates $\varphi$ such that $\varphi^{-1} T_{1j} \varphi$ and $\varphi^{-1} \rho \varphi$ have a simpler form. We would like to show that two such families of involutions $\{\mathcal{T}_1, \rho\}$ and $\{\widetilde{\mathcal{T}}_1, \tilde{\rho}\}$ are holomorphically equivalent, if there are normal forms are equivalent under a much smaller set of changes of coordinates, or if they are identical in the ideal situation.

Next, we describe our plans to derive the normal forms for linear involutions. The scheme to derive the linear normal forms turns out to be essential to understand the derivation of normal forms for non-linear involutions and the perturbed quadrics. We define

$$S = T_1 T_2.$$

Besides conditions (3.1)-(3.2), we will soon impose condition E below that $S$ has $2p$ distinct eigenvalues.

We first use a linear map $\psi$ to diagonalize $S$ to its normal form

$$\hat{S}\colon \xi_j' = \mu_j \xi, \quad \eta_j' = \mu_j^{-1} \eta_j, \quad 1 \le j \le p.$$

The choice of $\psi$ is not unique. We further normalize $T_1, T_2, \rho$ under linear transformations commuting with $\hat{S}$, i.e. the invertible mappings in the *linear centralizer* of $\hat{S}$. We use a linear map that commutes with $\hat{S}$ to transform $\rho$ into a normal form too, which is still denoted by $\rho$. We then use a transformation $\psi_0$ in the linear centralizer of $\hat{S}$ and $\rho$ to normalize the $T_1, T_2$ into the normal form

$$\hat{T}_i\colon \xi_j' = \lambda_{ij} \eta_j, \quad \eta_j' = \lambda_{ij}^{-1} \xi_j, \quad 1 \le j \le p.$$

Here we require $\lambda_{2j} = \lambda_{1j}^{-1}$. Thus $\mu_j = \lambda_{1j}^2$ for $1 \le j \le p$, and $\lambda_{11}, \dots, \lambda_{1p}$ form a complete set of invariants of $T_1, T_2, \rho$, provided they are normalized into the regions

$$\lambda_{1e} > 1, \quad \operatorname{Im} \lambda_{1h} > 0, \quad \arg \lambda_{1s} \in (0, \pi/2), \quad |\lambda_s| > 1.$$

Next we normalize the family $\mathcal{T}_1$ of linear involutions under mappings in the linear centralizer of $\hat{T}_1, \rho$. Let us assume that $T_1, \rho$ are in the normal forms $\hat{T}_1, \rho$. To normalize the families $\{\mathcal{T}_1, \rho\}$, we use the crucial property that $T_{11}, \ldots, T_{1p}$ commute pairwise and each $T_{1j}$ fixes a hyperplane. This allows us to express the family of involutions via a single linear mapping $\phi_1$:

$$T_{1j} = \varphi_1 \phi_1 Z_j \phi_1^{-1} \varphi_1^{-1}.$$

Here the linear mapping $\varphi_1$ depends only on $\lambda_1, \ldots, \lambda_p$ and

$$Z_j \colon \xi' = \xi, \quad \eta_i' = \eta_i \ (i \neq j), \quad \eta_j' = -\eta_j.$$

Expressing $\phi_1$ in a non-singular $p \times p$ constant matrix $\mathbf{B}$, the normal form for $\{T_{11}, \ldots, T_{1p}, \rho\}$ consists of invariants $\lambda_1, \ldots, \lambda_p$ and a normal form of $\mathbf{B}$. After we obtain the normal form for $\mathbf{B}$, we will construct the normal form of the quadrics by using the realization procedure in the proof of Proposition 2.10.

We now carry out the details.

Let $T_1 = T_{11} \cdots T_{1p}$, $T_2 = \rho T_1 \rho$ and $S = T_1 T_2$. Since $T_i$ and $\rho$ are involutions, then $S$ is reversible with respect to $T_i$ and $\rho$, i.e.

$$S^{-1} = T_i^{-1} S T_i, \quad S^{-1} = \rho^{-1} S \rho, \quad T_i^2 = I, \quad \rho^2 = I.$$

Therefore, if $\kappa$ is an eigenvalue of $S$ with a (non-zero) eigenvector $u$, then

$$Su = \kappa u, \quad S(T_i u) = \kappa^{-1} T_i u, \quad S(\rho u) = \overline{\kappa}^{-1} \rho u, \quad S(\rho T_i u) = \overline{\kappa} \rho T_i u.$$

Following [MW83] and [St07], we will divide eigenvalues into 4 types: $\mu$ is *elliptic* if $\mu \neq \pm 1$ and $\mu$ is real, $\mu$ is *hyperbolic* if $|\mu| = 1$ and $\mu \neq 1$, $\mu$ is *parabolic* if $\mu = 1$, and $\mu$ is *complex* otherwise. The classification of $\sigma$ into the types corresponds to the classification of the types of complex tangents described in section 2; namely, an elliptic (resp. hyperbolic) complex tangent is tied to a hyperbolic (resp. elliptic) mapping $\sigma$. A complex tangent of parabolic (reps. complex) type is tied to a mapping of parabolic (resp. complex) type.

To classify the families of linear involutions, we need a mild assumption to exclude multiplicity in $\gamma_1, \ldots, \gamma_p$ and also parabolic complex tangent at the origin. We therefore impose the following condition on quadrics.

**Condition E.** The composition $S$ has $2p$ distinct eigenvalues.

**Lemma 3.1.** *Under conditions E and* (3.2)*, neither* 1 *nor* $-1$ *is an eigenvalue of* $S$*.*

*Proof.* Assume for the sake of contradiction that 1 is an eigenvalue. We have seen that eigenvalues arrive in pairs $\mu, \mu^{-1}$ if $\mu \neq \pm 1$. Since there are $n = 2p$ eigenvalues by condition E, both $-1$ and 1 are eigenvalues. Let $u, v$ be eigenvectors such that

$$\begin{aligned} Su = u, \quad T_1 u = \epsilon_1 u, \quad T_2 u = \epsilon_1 u, \quad \epsilon_1 = \pm 1; \\ Sv = -v, \quad T_1 v = \epsilon_2 v, \quad T_2 v = -\epsilon_2 v, \quad \epsilon_2 = \pm 1. \end{aligned}$$

Since $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) = \{0\}$, then $\epsilon_1 = -1$. Without loss of generality, we may assume that $\epsilon_2 = -1$. Let $V$ be the span of eigenvectors of $S$ with eigenvalues other than $\pm 1$. Thus $T_i$ preserves $V$, $\dim V = 2p - 2$, $\dim \mathrm{Fix}(T_1|_V) = p$, and $\dim \mathrm{Fix}(T_2|_V) = p - 1$. Since $p + (p-1) > 2p - 2$, then $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$ has dimension at least one, a contradiction. $\square$

We now assume conditions E and (3.1)-(3.2) for the rest of the section to derive a normal form for $T_{1j}$ and $\rho$.

We need to choose the eigenvectors of $S$ and their eigenvalues in such a way that $T_1, T_2$ and $\rho$ are in a normal form. We will first choose eigenvectors to put $\rho$ into a normal form. After normalizing $\rho$, we will then choose eigenvectors to normalize $T_1$ and $T_2$.

First, let us consider an elliptic eigenvalue $\mu_e$. Let $u$ be an eigenvector of $\mu_e$. Then $u$ and $v = \rho(u)$ satisfy

$$S(v) = \mu_e^{-1} v, \quad T_j(u) = \lambda_j^{-1} v, \quad \mu_e = \lambda_1 \lambda_2^{-1}. \tag{3.3}$$

Now $T_2(u) = \rho T_1 \rho(u)$ implies that

$$\lambda_2 = \overline{\lambda}_1^{-1}, \quad \mu_e = |\lambda_1|^2.$$

Replacing $(u, v)$ by $(cu, \overline{c}v)$, we may assume that $\lambda_1 > 0$ and $\lambda_2 = \lambda_1^{-1}$. Replacing $(u, v)$ by $(v, u)$ if necessary, we may further achieve

$$\rho(u) = v, \quad \lambda_1 = \lambda_e > 1, \quad \mu_e = \lambda_e^2 > 1.$$

We still have the freedom to replace $(u, v)$ by $(ru, rv)$ for $r \in \mathbf{R}^*$, while preserving the above conditions.

Next, let $\mu_h$ be a hyperbolic eigenvalue of $S$ and $S(u) = \mu_h u$. Then $u$ and $v = T_1(u)$ satisfy

$$\rho(u) = au, \quad \rho(v) = bv, \quad |a| = |b| = 1.$$

Replacing $(u, v)$ by $(cu, v)$, we may assume that $a = 1$. Now $T_2(v) = \rho T_1 \rho(v) = \overline{b}u$. To obtain $b = 1$, we replace $(u, v)$ by $(u, \sqrt{b}^{-1} v)$. This give us (3.3) with $|\lambda_j| = 1$. Replacing $(u, v)$ by $(v, u)$ if necessary, we may further achieve

$$\rho(u) = u, \quad \rho(v) = v, \quad \lambda_1 = \lambda_h, \quad \mu_h = \lambda_h^2, \quad \arg \lambda_h \in (0, \pi/2).$$

Again, we have the freedom to replace $(u, v)$ by $(ru, rv)$ for $r \in \mathbf{R}^*$, while preserving the above conditions.

Finally, we consider a complex eigenvalue $\mu_s$. Let $S(u) = \mu_s u$. Then $\tilde{u} = \rho(u)$ satisfies $S(\tilde{u}) = \overline{\mu}_s^{-1} \tilde{u}$. Let $u^* = T_1(u)$ and $\tilde{u}^* = \rho(u^*)$. Then $S(u^*) = \mu_s^{-1} u^*$ and $S(\tilde{u}^*) = \overline{\mu}_s \tilde{u}^*$. We change eigenvectors by

$$(u, \tilde{u}, u^*, \tilde{u}^*) \to (u, \tilde{u}, cu^*, \overline{c}\tilde{u}^*)$$

so that

$$\rho(u) = \tilde{u}, \quad \rho(u^*) = \tilde{u}^*,$$
$$T_j(u) = \lambda_j^{-1} u^*, \quad T_j(\tilde{u}) = \overline{\lambda}_j \tilde{u}^*, \quad \lambda_2 = \lambda_1^{-1}.$$

Note that $S(u) = \lambda_1^2 u$, $S(u^*) = \lambda_1^{-2} u^*$, $S(\tilde{u}) = \overline{\lambda}_1^{-2} \tilde{u}$, and $S(\tilde{u}^*) = \overline{\lambda}_1^2 \tilde{u}^*$. Replacing $(u, \tilde{u}, u^*, \tilde{u}^*)$ by $(u^*, \tilde{u}^*, u, \tilde{u})$ changes the argument and the modulus of $\lambda_1$ as $\lambda_1^{-1}$ becomes $\lambda_1$. Replacing them by $(\tilde{u}, u, \tilde{u}^*, u^*)$ changes only the modulus as $\lambda_1$ becomes $\overline{\lambda}_1^{-1}$ and then replacing them by $(u^*, \tilde{u}^*, -u, -\tilde{u})$ changes the sign of $\lambda_1$. Therefore, we may achieve

$$\mu_s = \lambda_s^2, \quad \lambda_1 = \lambda_s, \quad \arg \gamma_s \in (0, \pi/2), \quad |\lambda_s| > 1.$$

We still have the freedom to replace $(u, u^*, \tilde{u}, \tilde{u}^*)$ by $(cu, cu^*, \overline{c}\tilde{u}, \overline{c}\tilde{u}^*)$.

We summarize the above choice of eigenvectors and their corresponding coordinates. First, $S$ has distinct eigenvalues

$$\lambda_e^2 = \overline{\lambda}_e^2, \quad \lambda_e^{-2}; \qquad \lambda_h^2, \quad \overline{\lambda}_h^2 = \lambda_h^{-2}; \qquad \lambda_s^2, \quad \lambda_s^{-2}, \quad \overline{\lambda}_s^{-2}, \quad \overline{\lambda}_s^2.$$

Also, $S$ has linearly independent eigenvectors satisfying

$$\begin{gathered} Su_e = \lambda_e^2 u_e, \quad Su_e^* = \lambda_e^{-2} u_e^*, \\ Sv_h = \lambda_h^2 v_h, \quad Sv_h^* = \lambda_h^{-2} v_h^*, \\ Sw_s = \lambda_s^2 w_s, \quad Sw_s^* = \lambda_s^{-2} w_s^*, \quad S\tilde{w}_s = \overline{\lambda}_s^{-2} \tilde{w}_s, \quad S\tilde{w}_s^* = \overline{\lambda}_s^2 \tilde{w}_s^*. \end{gathered}$$

Furthermore, the $\rho$, $T_1$, and the chosen eigenvectors of $S$ satisfy

$$\begin{gathered} \rho u_e = u_e^*, \quad T_1 u_e = \lambda_e^{-1} u_e^*; \\ \rho v_h = v_h, \quad \rho v_h^* = v_h^*, \quad T_1 v_h = \lambda_h^{-1} v_h^*; \\ \rho w_s = \tilde{w}_s, \quad \rho w_s^* = \tilde{w}_s^*, \quad T_1 w_s = \lambda_s^{-1} w_s^*, \quad T_1 \tilde{w}_s = \overline{\lambda}_s \tilde{w}_s^*. \end{gathered}$$

For normalization, we collect elliptic eigenvalues $\mu_e$ and $\mu_e^{-1}$, hyperbolic eigenvalues $\mu_h$ and $\mu_h^{-1}$, and complex eigenvalues in $\mu_s, \mu_s^{-1}, \overline{\mu}_s^{-1}$ and $\overline{\mu}_s$. We put them in the order

$$\begin{gathered} \mu_e = \overline{\mu}_e, \quad \mu_{p+e} = \mu_e^{-1}, \\ \mu_h, \quad \mu_{p+h_*+h} = \overline{\mu}_h, \\ \mu_s, \quad \mu_{s+s_*} = \overline{\mu}_s^{-1}, \quad \mu_{p+s} = \mu_s^{-1}, \quad \mu_{p+s_*+s} = \overline{\mu}_s. \end{gathered}$$

Here and throughout the paper the ranges of subscripts $e, h, s$ are restricted to

$$1 \le e \le e_*, \quad e_* < h \le e_* + h_*, \quad e_* + h_* < s \le p - s_*.$$

Thus $e_* + h_* + 2s_* = p$. Using the new coordinates

$$\sum (\xi_e u_e + \eta_e u_e^*) + \sum (\xi_h v_h + \eta_h v_h^*) + \sum (\xi_s w_s + \xi_{s+s_*} \tilde{w}_s + \eta_s w_s^* + \eta_{s+s_*} \tilde{w}_s^*),$$

we have normalized $\sigma, T_1, T_2$ and $\rho$. In summary, we have the following normal form.

**Lemma 3.2.** *Let $T_1, T_2$ be linear holomorphic involutions on $\mathbf{C}^n$ that satisfy* (3.2). *Then $n = 2p$ and $\dim[\mathfrak{M}_n]_1^{T_i} = p$. Suppose that $T_2 = \rho_0 T_1 \rho_0$ for some anti-holomorphic linear involution $\rho_0$. Assume that $S = T_1 T_2$ has $n$ distinct eigenvalues. There exists a linear change of holomorphic coordinates that transforms $T_1, T_2, S, \rho_0$ simultaneously into the normal forms $\hat{T}_1, \hat{T}_2, \hat{S}, \rho$ :*

$$\hat{T}_1 \colon \xi_j' = \lambda_j \eta_j, \quad \eta_j' = \lambda_j^{-1} \xi, \quad 1 \le j \le p; \tag{3.4}$$

$$\hat{T}_2 \colon \xi_j' = \lambda_j^{-1} \eta_j, \quad \eta_j' = \lambda_j \xi_j, \quad 1 \le j \le p; \tag{3.5}$$

$$\hat{S} \colon \xi_j' = \mu_j \xi_j, \quad \eta_j' = \mu_j^{-1} \eta_j, \quad 1 \le j \le p; \tag{3.6}$$

$$\rho \colon \begin{cases} \xi_e' = \overline{\eta}_e, & \eta_e' = \overline{\xi}_e, \\ \xi_h' = \overline{\xi}_h, & \eta_h' = \overline{\eta}_h, \\ \xi_s' = \overline{\xi}_{s+s_*}, & \xi_{s+s_*}' = \overline{\xi}_s, \\ \eta_s' = \overline{\eta}_{s+s_*}, & \eta_{s+s_*}' = \overline{\eta}_s. \end{cases} \tag{3.7}$$

*Moreover, the eigenvalues $\mu_1, \ldots, \mu_p$ satisfy*

$$\mu_j = \lambda_j^2, \quad 1 \le j \le p; \tag{3.8}$$

$$\lambda_e > 1, \quad |\lambda_h| = 1, \quad |\lambda_s| > 1, \quad \lambda_{s+s_*} = \overline{\lambda}_s^{-1}; \tag{3.9}$$

$$\arg \lambda_h \in (0, \pi/2), \quad \arg \lambda_s \in (0, \pi/2); \tag{3.10}$$

$$\lambda_{e'} < \lambda_{e'+1}, \quad 0 < \arg \lambda_{h'} < \arg \lambda_{h'+1} < \pi/2; \tag{3.11}$$

$$\arg \lambda_{s'} < \arg \lambda_{s'+1}, \text{ or } \arg \lambda_{s'} = \arg \lambda_{s'+1} \text{ and } |\lambda_{s'}| < |\lambda_{s'+1}|. \tag{3.12}$$

*Here $1 \le e' < e_*$, $e_* < h' < e_* + h_*$, and $e_* + h_* < s' < p - s_*$. And $1 \le e \le e_*$, $e_* < h \le e_* + h_*$, and $e_* + h_* < s \le p - s_*$. If $\tilde{S}$ is also in the normal form (3.6) for possible different eigenvalues $\tilde{\mu}_1, \ldots, \tilde{\mu}_p$ satisfying (3.8)-(3.12), then $S$ and $\tilde{S}$ are equivalent if and only if their eigenvalues are identical.*

The above normal form of $\rho$ will be fixed for the rest of paper. Note that in case of non-linear involutions $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$ of which the linear part are given by $\{T_{11}, \ldots, T_{1p}, \rho\}$ we can always linearize $\rho$ first under a holomorphic map of which the linear part at the origin is described in above normalization for the linear part of $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$. Indeed, we may assume that the linear part of the latter family is already in the normal form. Then $\psi = \frac{1}{2}(I + (L\rho) \circ \rho)$ is tangent to the identity and $(L\rho) \circ \psi \circ \rho = \psi$, i.e. $\psi$ transforms $\rho$ into $L\rho$ while preserving the linear parts of $\tau_{11}, \ldots, \tau_{1p}$. Therefore in the non-linear case, we can assume that $\rho$ is given by the above normal form. The above lemma tells us the ranges of eigenvalues $\mu_e, \mu_h$ and $\mu_s$ that can be realized by quadrics that satisfy conditions E and (3.1)-(3.2).

Having normalized $T_1$ and $\rho$, we want to further normalize $\{T_{11}, \ldots, T_{1p}\}$ under linear maps that preserve the normal forms of $\hat{T}_1$ and $\rho$. We know that the composition of $T_{1j}$ is in the normal form, i.e.

$$T_{11} \cdots T_{1p} = \hat{T}_1 \tag{3.13}$$

is given in Lemma 3.2. We first need to find an expression for all $T_{1j}$ that commute pairwise and satisfy (3.13), by using invariant and skew-invariant functions of $\hat{T}_1$. Let

$$(\xi, \eta) = \varphi_1(z^+, z^-)$$

be defined by

$$z_e^+ = \xi_e + \lambda_e \eta_e, \qquad z_e^- = \eta_e - \lambda_e^{-1} \xi_e, \tag{3.14}$$

$$z_h^+ = \xi_h + \lambda_h \eta_h, \qquad z_h^- = \eta_h - \overline{\lambda}_h \xi_h, \tag{3.15}$$

$$z_s^+ = \xi_s + \lambda_s \eta_s, \qquad z_s^- = \eta_s - \lambda_s^{-1} \xi_s, \tag{3.16}$$

$$z_{s+s_*}^+ = \xi_{s+s_*} + \overline{\lambda}_s^{-1} \eta_{s+s_*}, \quad z_{s+s_*}^- = \eta_{s+s_*} - \overline{\lambda}_s \xi_{s+s_*}. \tag{3.17}$$

In $(z^+, z^-)$ coordinates, $\varphi_1^{-1} \hat{T}_1 \varphi_1$ becomes

$$Z \colon z^+ \to z^+, \quad z^- \to -z^-.$$

We decompose $Z = Z_1 \cdots Z_p$ by using

$$Z_j \colon (z^+, z^-) \to (z^+, z_1^-, \ldots, z_{j-1}^-, -z_j^-, z_{j+1}^-, \ldots, z_p^-).$$

To keep simple notation, let us use the same notions $x, y$ for a linear transformation $y = A(x)$ and its matrix representation:

$$A\colon x \to \mathbf{A}x.$$

The following lemma, which can be verified immediately, shows the advantages of coordinates $z^+, z^-$.

**Lemma 3.3.** *The linear centralizer of $Z$ is the set of mappings of the form*

$$\phi\colon (z^+, z^-) \to (\mathbf{A}z^+, \mathbf{B}z^-), \tag{3.18}$$

*where $\mathbf{A}, \mathbf{B}$ are constant and possibly singular matrices. Let $\nu$ be a permutation of $\{1, \dots, p\}$. Then $Z_j\phi = \phi Z_{\nu(j)}$ for all $j$ if and only if $\phi$ has the above form with $\mathbf{B} = \operatorname{diag}_\nu \mathbf{d}$. Here*

$$\operatorname{diag}_\nu(d_1, \dots, d_p) := (b_{ij})_{p\times p}, \quad b_{j\nu(j)} = d_j, \quad b_{jk} = 0 \text{ if } k \neq \nu(j). \tag{3.19}$$

*In particular, the linear centralizer of $\{Z_1, \dots, Z_p\}$ is the set of mappings (3.18) in which $\mathbf{B}$ are diagonal.*

To continue our normalization for the family $\{T_{1j}\}$, we note that $\varphi_1^{-1}T_{11}\varphi_1, \dots, \varphi_1^{-1}T_{1p}\varphi_1$ generate an abelian group of $2^p$ involutions and each of these $p$ generators fixes a hyperplane. By Lemma 2.6 there is a linear transformation $\phi_1$ such that

$$\phi_1^{-1}\varphi_1^{-1}T_{1j}\varphi_1\phi_1 = Z_j, \quad 1 \le j \le p.$$

Computing two compositions on both sides, we see that $\phi_1$ must be in the linear centralizer of $Z$. Thus, it is in the form (3.18). Of course, $\phi_1$ is not unique; $\tilde{\phi}_1$ is another such linear map for the same $T_{1j}$ if and only if $\tilde{\phi}_1 = \phi_1\psi_1$ with $\psi_1 \in \mathcal{C}(Z_1, \dots, Z_p)$. By (3.18), we may restrict ourselves to $\phi_1$ given by

$$\phi_1\colon (z^+, z^-) \to (z^+, \mathbf{B}z^-). \tag{3.20}$$

Then $\tilde{\phi}_1$ yields the same $T_{1j}$ if and only if its corresponding matrix $\widetilde{\mathbf{B}} = \mathbf{BD}$ for a diagonal matrix $\mathbf{D}$.

In the above we have expressed all $T_{11}, \dots, T_{1p}$ via equivalence classes of matrices. It will be convenient to restate them via matrices.

For simplicity, $T_i$ and $S$ denote $\hat{T}_i, \hat{S}$, respectively. In matrices, we write

$$T_1\colon \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \mathbf{T}_1 \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \rho\colon \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \boldsymbol{\rho} \begin{pmatrix} \overline{\xi} \\ \overline{\eta} \end{pmatrix}, \quad S\colon \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \mathbf{S} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Recall that the bold faced $\mathbf{A}$ represents a linear map $A$. Then

$$\mathbf{T_1} = \begin{pmatrix} \mathbf{0} & \mathbf{\Lambda}_1 \\ \mathbf{\Lambda}_1^{-1} & \mathbf{0} \end{pmatrix}_{2p\times 2p}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{\Lambda}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_1^{-2} \end{pmatrix}_{2p\times 2p}.$$

We will abbreviate

$$\boldsymbol{\xi}_{e_*} = (\xi_1, \dots, \xi_{e_*}), \quad \boldsymbol{\xi}_{h_*} = (\xi_{e_*+1}, \dots, \xi_{e_*+h_*}), \quad \boldsymbol{\xi}_{2s_*} = (\xi_{e_*+h_*+1}, \dots, \xi_p).$$

We use the same abbreviation for $\eta$. Then $(\boldsymbol{\xi}_{e_*}, \boldsymbol{\eta}_{e_*})$, $(\boldsymbol{\xi}_{h_*}, \boldsymbol{\eta}_{h_*})$, and $(\boldsymbol{\xi}_{2s_*}, \boldsymbol{\eta}_{2s_*})$ subspaces are invariant under $T_{1j}$, $T_1$, and $\rho$. We also denote by $T_1^{e_*}, T_1^{h_*}, T_1^{s_*}$ the restrictions of $T_1$ to

these subspaces. Define analogously for the restrictions of $\rho, S$ to these subspaces. Define diagonal matrices $\mathbf{\Lambda}_{1e_*}, \mathbf{\Lambda}_{1h_*}, \mathbf{\Lambda}_{1s_*}$, of size $e_* \times e_*, h_* \times h_*$ and $s_* \times s_*$ respectively, by

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \mathbf{\Lambda}_{1e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{1h_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\mathbf{\Lambda}}_{1s_*}^{-1} \end{pmatrix}, \quad \overline{\mathbf{\Lambda}_1} = \begin{pmatrix} \mathbf{\Lambda}_{1e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_{1h_*}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \overline{\Lambda}_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_{1s_*}^{-1} \end{pmatrix}.$$

Thus, we can express $T_1^{s_*}$ and $S^{s_*}$ in $(2s_*) \times (2s_*)$ matrices

$$\mathbf{T}_1^{s_*} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\mathbf{\Lambda}}_{1s_*}^{-1} \\ \mathbf{\Lambda}_{1s_*}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{\Lambda}}_{1s_*} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{S}^{s_*} = \begin{pmatrix} \mathbf{\Lambda}_{1s_*}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{\Lambda}}_{1s_*}^{-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}_{1s_*}^{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\mathbf{\Lambda}}_{1s_*}^2 \end{pmatrix}.$$

Let $\mathbf{I}_k$ denote the $k \times k$ identity matrix. With the abbreviation, we can express $\rho$ as

$$\boldsymbol{\rho}^{e_*} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{e_*} \\ \mathbf{I}_{e_*} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\rho}^{h_*} = \mathbf{I}_{2h_*},$$
$$\boldsymbol{\rho}^{s_*} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} \end{pmatrix}.$$

Note that $\rho$ is anti-holomorphic linear transformation. If $A$ is a complex linear transformation, in $(\xi, \eta)$ coordinates the matrix of $\rho A$ is $\boldsymbol{\rho}\overline{\mathbf{A}}$, i.e.

$$\rho A \colon \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \boldsymbol{\rho}\overline{\mathbf{A}} \begin{pmatrix} \overline{\xi} \\ \overline{\eta} \end{pmatrix}$$

with

$$\boldsymbol{\rho} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{h_*} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_{e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{h_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{s_*} & \mathbf{0} \end{pmatrix}.$$

For an invertible $p \times p$ matrix $\mathbf{A}$, let us define an $n \times n$ matrix $\mathbf{E}_\mathbf{A}$ by

$$\mathbf{E}_\mathbf{A} := \frac{1}{2} \begin{pmatrix} \mathbf{I}_p & -\mathbf{A} \\ \mathbf{A}^{-1} & \mathbf{I}_p \end{pmatrix}, \quad \mathbf{E}_\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{I}_p & \mathbf{A} \\ -\mathbf{A}^{-1} & \mathbf{I}_p \end{pmatrix}. \tag{3.21}$$

For a $p \times p$ matrix $\mathbf{B}$, we define

$$\mathbf{B}_* := \begin{pmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}. \tag{3.22}$$

Therefore, we can express

$$\mathbf{T}_{1j} = \mathbf{E}_{\mathbf{\Lambda}_1}\mathbf{B}_*\mathbf{Z}_j\mathbf{B}_*^{-1}\mathbf{E}_{\mathbf{\Lambda}_1}^{-1}, \quad \mathbf{T}_{2j} = \boldsymbol{\rho}\overline{\mathbf{T}_{1j}}\boldsymbol{\rho}, \tag{3.23}$$

$$\mathbf{Z}_j = \operatorname{diag}(1, \ldots, 1, -1, 1, \ldots, 1). \tag{3.24}$$

Here $-1$ is at the $(p+j)$-th place. Moreover, $\mathbf{B}$ is uniquely determined up to equivalence relation via diagonal matrices $\mathbf{D}$:

$$\mathbf{B} \sim \mathbf{BD}. \tag{3.25}$$

We have expressed all $\{T_{11}, \ldots, T_{1p}, \rho\}$ for which $\hat{T}_1 = T_{11}\cdots T_{1p}$ and $\rho$ are in the normal forms in Lemma 3.2 and we have found an equivalence relation to classify the involutions. Let us summarize the results in a lemma.

**Lemma 3.4.** *Let $\{T_{11}, \ldots, T_{1p}, \rho\}$ be the involutions of a quadric manifold $M$. Assume that $S = T_1\rho T_1\rho$ has distinct eigenvalues. Then in suitable linear $(\xi, \eta)$ coordinates, $T_{11}, \ldots, T_{1p}$ are given by* (3.23)*, while $T_{11}\cdots T_{1p} = \hat{T}_1$ and $\rho$ are given by* (3.4) *and* (3.7)*, respectively. Moreover, $\mathbf{B}$ in* (3.23) *is uniquely determined by the equivalence relation* (3.25) *for diagonal matrices $\mathbf{D}$.*

We remind the reader that we divide the classification for $\{T_{11}, \ldots, T_{1p}, \rho\}$ into two steps. We have obtained the classification for the composition $T_{11}\cdots T_{1p} = \hat{T}_1$ and $\rho$ in Lemma 3.2. Having found all $\{T_{11}, \ldots, T_{1p}, \rho\}$ and an equivalence relation, we are ready to reduce their classification to an equivalence problem that involves two dilatations and a coordinate permutation.

**Lemma 3.5.** *Let $\{T_{i1}, \ldots, T_{ip}, \rho\}$ be given by* (3.23)*. Suppose that $\hat{T}_1 = T_{11}\cdots T_{1p}$, $\rho$, $\hat{T}_2 = \rho\hat{T}_1\rho$, and $\hat{S} = \hat{T}_1\hat{T}_2$ have the form in Lemma* 3.2*. Suppose that $\hat{S}$ has distinct eigenvalues. Let $\{\hat{T}_{11}, \ldots, \hat{T}_{1p}, \rho\}$ be given by* (3.23) *where $\lambda_j$ are unchanged and $\mathbf{B}$ is replaced by $\hat{\mathbf{B}}$. Suppose that $R^{-1}T_{1j}R = \widehat{T}_{1\nu(j)}$ for all $j$ and $R\rho = \rho R$. Then the matrix of $R$ is $\mathbf{R} = \operatorname{diag}(\mathbf{a}, \mathbf{a})$ with $\mathbf{a} = (\mathbf{a}_{e_*}, \mathbf{a}_{h_*}, \mathbf{a}_{s_*}, \mathbf{a}'_{s_*})$, while $\mathbf{a}$ satisfies the reality condition*

$$\mathbf{a}_{e_*} \in (\mathbf{R}^*)^{e_*}, \quad \mathbf{a}_{h_*} \in (\mathbf{R}^*)^{h_*}, \quad \overline{\mathbf{a}_{s_*}} = \mathbf{a}'_{s_*} \in (\mathbf{C}^*)^{s_*}. \tag{3.26}$$

*Moreover, there exists $\mathbf{d} \in (\mathbf{C}^*)^p$ such that*

$$\hat{\mathbf{B}} = (\operatorname{diag}\mathbf{a})^{-1}\mathbf{B}(\operatorname{diag}_\nu \mathbf{d}), \quad \textit{i.e.}, \quad a_i^{-1}b_{i\nu^{-1}(j)}d_{\nu^{-1}(j)} = \hat{b}_{ij}, \quad 1 \le i, j \le p. \tag{3.27}$$

*Conversely, if $\mathbf{a}, \mathbf{d}$ satisfy* (3.26) *and* (3.27)*, then $R^{-1}T_{1j}R = \hat{T}_{1\nu(j)}$ and $R\rho = \rho R$.*

*Proof.* Suppose that $R^{-1}T_{1j}R = \widehat{T}_{1\nu(j)}$ and $R\rho = \rho R$. Then $R^{-1}\hat{T}_1R = \hat{T}_1$ and $R^{-1}\hat{S}R = \hat{S}$. The latter implies that the matrix of $R$ is diagonal. The former implies that

$$R\colon \xi'_j = a_j\xi_j, \quad \eta'_j = a_j\eta_j$$

with $a_j \in \mathbf{C}^*$. Now $R\rho = \rho R$ implies (3.26). We express $R^{-1}T_{1j}R = \widehat{T}_{1\nu(j)}$ via matrices:

$$\mathbf{E}_{\mathbf{\Lambda}_1}\widehat{\mathbf{B}}_*\mathbf{Z}_{\nu(j)}\widehat{\mathbf{B}}_*^{-1}\mathbf{E}_{\mathbf{\Lambda}_1}^{-1} = \mathbf{R}^{-1}\mathbf{E}_{\mathbf{\Lambda}_1}\mathbf{B}_*\mathbf{Z}_j\mathbf{B}_*^{-1}\mathbf{E}_{\mathbf{\Lambda}_1}^{-1}\mathbf{R}. \tag{3.28}$$

In view of formula (3.21), we see that $\mathbf{E}_{\mathbf{\Lambda}_1}$ commutes with $\mathbf{R} = \operatorname{diag}(\mathbf{a}, \mathbf{a})$. The above is equivalent to that $\boldsymbol{\psi} := \mathbf{B}_*^{-1}\mathbf{R}\widehat{\mathbf{B}}_*$ satisfies $\mathbf{Z}_{\nu(j)} = \boldsymbol{\psi}^{-1}\mathbf{Z}_j\boldsymbol{\psi}$. By Lemma 3.3 we obtain

$\boldsymbol{\psi} = \operatorname{diag}(\mathbf{A}, \operatorname{diag}_\nu \mathbf{d})$. This shows that

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \operatorname{diag}_\nu \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}^{-1} \begin{pmatrix} \operatorname{diag} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \operatorname{diag} \mathbf{a} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{B}} \end{pmatrix}.$$

The matrices on diagonal yield $\mathbf{A} = \operatorname{diag} \mathbf{a}$ and (3.27). The lemma is proved. □

Lemma 3.5 does not give us an explicit description of the normal form for the families of involutions $\{T_{11}, \dots, T_{1p}, \rho\}$. Nevertheless by the lemma, we can always choose a $\nu$ and $\operatorname{diag} \mathbf{d}$ such that the diagonal elements of $\tilde{\mathbf{B}}$, corresponding to $\{\tilde{T}_{1\nu(1)}, \dots, \tilde{T}_{1\nu(p)}, \rho\}$, are 1.

**Remark 3.6.** In what follows, we will fix a $\mathbf{B}$ and its associated $\{\mathcal{T}_1, \rho\}$ to further study our normal form problems.

3.2. **Normal form of the quadrics.** We now use the matrices $\mathbf{B}$ to express the normal form for the quadratic submanifolds. Here we follow the realization procedure in the proof of Proposition 2.10. We will use the coordinates $z^+, z^-$ again to express invariant functions of $T_{1j}$ and use them to construct the corresponding quadric. We will then pull back the quadric to the $(\xi, \eta)$ coordinates and then to the $z, \overline{z}$ coordinates to achieve the final normal form of the quadrics.

We return to the construction of invariant and skew-invariant functions $z^+, z^-$ in (3.14)-(3.17). when $\mathbf{B}$ is the identity matrix. For a general $\mathbf{B}$, we define $\Phi_1$ and the matrix $\mathbf{\Phi}_1^{-1}$ by

$$\Phi_1(Z^+, Z^-) = (\xi, \eta), \quad \mathbf{\Phi}_1^{-1} := \mathbf{B}_*^{-1} \mathbf{E}_{\mathbf{\Lambda}_1}^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{\Lambda}_1 \\ -\mathbf{B}^{-1} \mathbf{\Lambda}_1^{-1} & \mathbf{B}^{-1} \end{pmatrix}.$$

Note that $Z^+ = z^+$ and $\Phi_1^{-1} T_{1j} \Phi_1 = Z_j$. The $Z^+, Z_i^-$ with $i \neq j$ are invariant functions of $T_{1j}$, while $Z_j^-$ is a skew-invariant function of $T_{1j}$. They can be written as

$$Z^+ = \xi + \mathbf{\Lambda}_1 \eta, \quad Z^- = \mathbf{B}^{-1}(-\mathbf{\Lambda}_1^{-1} \xi + \eta).$$

Therefore, the invariant functions of $\mathcal{T}_1$ are generated by

$$Z_j^+ = \xi_j + \lambda_j \eta_j, \quad (Z_j^-)^2 = (\tilde{\mathbf{B}}_j(-\mathbf{\Lambda}_1^{-1} \xi + \eta))^2, \quad 1 \leq j \leq p.$$

Here $\tilde{\mathbf{B}}_j$ is the $j$th row of $\mathbf{B}^{-1}$. The invariant (holomorphic) functions of $\mathcal{T}_2$ are generated by

$$W_j^+ = \overline{Z_j^+ \circ \rho}, \quad (W_j^-)^2 = (\overline{Z_j^- \circ \rho})^2, \quad 1 \leq j \leq p. \tag{3.29}$$

Here $W_j^- = \overline{Z_j^- \circ \rho}$. We will soon verify that

$$m \colon (\xi, \eta) \to (z', w') = (Z^+(\xi, \eta), W^+(\xi, \eta))$$

is biholomorphic. A straightforward computation shows that $m \rho m^{-1}$ equals

$$\rho_0 \colon (z', w') \to (\overline{w'}, \overline{z'}).$$

We define

$$M \colon z''_{p+j} = (Z_j^- \circ m^{-1}(z', \overline{z'}))^2.$$

We want to find a simpler expression for $M$. We first separate $B$ from $Z^-$ by writing

$$\hat{\mathbf{Z}}^- := (-\mathbf{\Lambda}_1^{-1} \ \mathbf{I}), \quad \mathbf{Z}^- = \mathbf{B}^{-1} \hat{\mathbf{Z}}^-. \tag{3.30}$$

Note that $m$ does not depend on $\mathbf{B}$. To compute $\hat{Z}^- \circ m^{-1}$, we will use matrix expressions for $(\boldsymbol{\xi}_{e_*}, \boldsymbol{\eta}_{e_*})$, $(\boldsymbol{\xi}_{h_*}, \boldsymbol{\eta}_{h_*})$ and $(\boldsymbol{\xi}_{2s_*}, \boldsymbol{\eta}_{2s_*})$ subspaces. Let $m_{e_*}, m_{h_*}, m_{s_*}$ be the restrictions $m$ to these subspaces. In the matrix form, we have by (3.29)

$$\mathbf{W}^+ = \overline{\mathbf{Z}^+ \boldsymbol{\rho}}, \quad \mathbf{W}^- = \overline{\mathbf{Z}^- \boldsymbol{\rho}}.$$

Recall that $\boldsymbol{\Lambda}_1 = \operatorname{diag}(\boldsymbol{\Lambda}_{e_*}, \boldsymbol{\Lambda}_{h_*}, \boldsymbol{\Lambda}_{1s_*}, \overline{\boldsymbol{\Lambda}}_{1s_*}^{-1})$. Thus

$$\begin{aligned}
\mathbf{m}_{e_*} &= \begin{bmatrix} \mathbf{I} & \boldsymbol{\Lambda}_{1e_*} \\ \boldsymbol{\Lambda}_{1e_*} & \mathbf{I} \end{bmatrix}, \quad \mathbf{m}_{e_*}^{-1} = \begin{bmatrix} \mathbf{I} & -\boldsymbol{\Lambda}_{1e_*} \\ -\boldsymbol{\Lambda}_{1e_*} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{I} - \boldsymbol{\Lambda}_{1e_*}^2)^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \boldsymbol{\Lambda}_{1e_*}^2)^{-1} \end{bmatrix}, \\
\mathbf{m}_{h_*} &= \begin{bmatrix} \mathbf{I} & \boldsymbol{\Lambda}_{1h_*} \\ \mathbf{I} & \boldsymbol{\Lambda}_{1h_*}^{-1} \end{bmatrix}, \qquad \mathbf{m}_{h_*}^{-1} = \begin{bmatrix} \mathbf{I} & -\boldsymbol{\Lambda}_{1h_*}^2 \\ -\boldsymbol{\Lambda}_{1h_*} & \boldsymbol{\Lambda}_{1h_*} \end{bmatrix} \begin{bmatrix} (\mathbf{I} - \boldsymbol{\Lambda}_{1h_*}^2)^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \boldsymbol{\Lambda}_{1h_*}^2)^{-1} \end{bmatrix}, \\
\mathbf{m}_{s_*} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} & \boldsymbol{\Lambda}_{1s_*} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \overline{\boldsymbol{\Lambda}}_{1s_*}^{-1} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \overline{\boldsymbol{\Lambda}}_{1s_*} \\ \mathbf{I} & \mathbf{0} & \boldsymbol{\Lambda}_{1s_*}^{-1} & \mathbf{0} \end{bmatrix}, \\
\mathbf{m}_{s_*}^{-1} &= \begin{bmatrix} \boldsymbol{\Lambda}_{1s_*}^{-1} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\Lambda}_{1s_*} \\ \mathbf{0} & \overline{\boldsymbol{\Lambda}}_{1s_*} & -\overline{\boldsymbol{\Lambda}}_{1s_*}^{-1} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L}_{s_*} & \mathbf{0} \\ \mathbf{0} & -\overline{\mathbf{L}}_{s_*} \end{bmatrix}, \\
\mathbf{L}_{s_*} &= \begin{bmatrix} (\boldsymbol{\Lambda}_{1s_*}^{-1} - \boldsymbol{\Lambda}_{1s_*})^{-1} & \mathbf{0} \\ \mathbf{0} & (\overline{\boldsymbol{\Lambda}}_{1s_*} - \overline{\boldsymbol{\Lambda}}_{1s_*}^{-1})^{-1} \end{bmatrix}.
\end{aligned}$$

Note that $\mathbf{I} - \boldsymbol{\Lambda}_1^2$ is diagonal. Using (3.30) and the above formulae, the matrices of $\hat{Z}_{e_*}^{-1} \circ m^{-1}$, $\hat{Z}_{h_*}^- \circ m^{-1}$, and $\hat{Z}_{s_*}^{-1} \circ m^{-1}$ are respectively given by

$$\begin{aligned}
\hat{\mathbf{Z}}_{e_*}^- \mathbf{m}_{e_*}^{-1} &= \mathbf{L}_{e_*} \begin{bmatrix} \mathbf{I} & -2(\boldsymbol{\Lambda}_{1e_*} + \boldsymbol{\Lambda}_{1e_*}^{-1})^{-1} \end{bmatrix}, \\
\mathbf{L}_{e_*} &= (\mathbf{I} - \boldsymbol{\Lambda}_{1e_*}^2)^{-1}(-\boldsymbol{\Lambda}_{1e_*} - \boldsymbol{\Lambda}_{1e_*}^{-1}), \\
\hat{\mathbf{Z}}_{h_*}^- \mathbf{m}_{h_*}^{-1} &= \mathbf{L}_{h_*} \begin{bmatrix} \mathbf{I} & -2\boldsymbol{\Lambda}_{1h_*}(\boldsymbol{\Lambda}_{1h_*} + \boldsymbol{\Lambda}_{1h_*}^{-1})^{-1} \end{bmatrix}, \\
\mathbf{L}_{h_*} &= (\mathbf{I} - \boldsymbol{\Lambda}_{1h_*}^2)^{-1}(-\boldsymbol{\Lambda}_{1h_*} - \boldsymbol{\Lambda}_{1h_*}^{-1}), \\
\hat{\mathbf{Z}}_{s_*}^- \mathbf{m}_{s_*}^{-1} &= \begin{bmatrix} -\mathbf{I} - \boldsymbol{\Lambda}_{1s_*}^{-2} & \mathbf{0} & \mathbf{0} & 2\mathbf{I} \\ \mathbf{0} & -\mathbf{I} - \overline{\boldsymbol{\Lambda}}_{1s_*}^2 & 2\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L}_{s_*} & \mathbf{0} \\ \mathbf{0} & -\overline{\mathbf{L}}_{s_*} \end{bmatrix} \\
&= \tilde{\mathbf{L}}_{s_*} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & -2(\mathbf{I} + \boldsymbol{\Lambda}_{1s_*}^{-2})^{-1} \\ \mathbf{0} & \mathbf{I} & -2(\mathbf{I} + \overline{\boldsymbol{\Lambda}}_{1s_*}^2)^{-1} & \mathbf{0} \end{bmatrix}, \\
\tilde{\mathbf{L}}_{s_*} &= \begin{bmatrix} (\mathbf{I} + \boldsymbol{\Lambda}_{1s_*}^{-2})(\boldsymbol{\Lambda}_{1s_*} - \boldsymbol{\Lambda}_{1s_*}^{-1})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} + \overline{\boldsymbol{\Lambda}}_{1s_*}^2)(\overline{\boldsymbol{\Lambda}}_{1s_*}^{-1} - \overline{\boldsymbol{\Lambda}}_{1s_*})^{-1} \end{bmatrix}.
\end{aligned}$$

Combining the above identities, we obtain

$$\hat{\mathbf{Z}}^{-1}\mathbf{m}^{-1} = \operatorname{diag}(\mathbf{L}_{e_*}, \mathbf{L}_{h_*}, \tilde{\mathbf{L}}_{s_*}) \left( \mathbf{I}_p, -2 \operatorname{diag}\left( \boldsymbol{\Gamma}_{e_*}, \boldsymbol{\Lambda}_{1h_*} \boldsymbol{\Gamma}_{h_*}, \begin{bmatrix} 0 & \boldsymbol{\Gamma}_{s_*} \\ \tilde{\boldsymbol{\Gamma}}_{s_*} & 0 \end{bmatrix} \right) \right)$$

with $\tilde{\mathbf{\Gamma}}_{s_*} = \mathbf{I} - \overline{\mathbf{\Gamma}}_{1s_*}$ and

$$\mathbf{\Gamma}_{e_*} = (\mathbf{\Lambda}_{1e_*} + \mathbf{\Lambda}_{1e_*}^{-1})^{-1}, \quad \mathbf{\Gamma}_{h_*} = (\mathbf{\Lambda}_{1h_*} + \mathbf{\Lambda}_{1h_*}^{-1})^{-1}, \quad \mathbf{\Gamma}_{s_*} = (\mathbf{I} + \mathbf{\Lambda}_{1s_*}^{-2})^{-1}. \tag{3.31}$$

We define $\tilde{\mathbf{B}}_j$ to be the $j$-th row of

$$\tilde{\mathbf{B}} := \mathbf{B}^{-1} \operatorname{diag}(\mathbf{L}_{e_*}, \mathbf{L}_{h_*}, \tilde{\mathbf{L}}_{s_*}). \tag{3.32}$$

With $\mathbf{z}'_{s_*} = (z_{p-s_*+1}, \dots, z_p)$, the defining equations of $M$ are given by

$$z''_{p+j} = \Big\{ \tilde{\mathbf{B}}_j \operatorname{diag}(\mathbf{z}_{e_*} - 2\mathbf{\Gamma}_{e_*}\overline{\mathbf{z}}_{e_*}, \mathbf{z}_{h_*} - 2\mathbf{\Gamma}_{h_*}\mathbf{\Lambda}_{1h_*}\overline{\mathbf{z}}_{h_*}, \mathbf{z}_{s_*} - 2\mathbf{\Gamma}_{s_*}\overline{\mathbf{z}}'_{s_*}, \mathbf{z}'_{s_*} - 2(\mathbf{I} - \overline{\mathbf{\Gamma}_{s_*}})\overline{\mathbf{z}}_{s_*}) \Big\}^2.$$

Let us replace $z_j$ with $j \neq h$, $z_h$ by $iz_j$ and $i\sqrt{\lambda_h}^{-1} z_h$, respectively. We also multiply the $h$-th column of $\tilde{\mathbf{B}}$ by $-i\sqrt{\lambda_h}$ and its $j$-th column, $j \neq h$, by $-i$. In the new coordinates, $M$ is given by

$$z''_{p+j} = \Big\{ \hat{\mathbf{B}}_j \operatorname{diag}(\mathbf{z}_{e_*} + 2\mathbf{\Gamma}_{e_*}\overline{\mathbf{z}}_{e_*}, \mathbf{z}_{h_*} + 2\mathbf{\Gamma}_{h_*}\overline{\mathbf{z}}_{h_*}, \mathbf{z}_{s_*} + 2\mathbf{\Gamma}_{s_*}\overline{\mathbf{z}}'_{s_*}, \mathbf{z}'_{s_*} + 2(\mathbf{I} - \overline{\mathbf{\Gamma}}_{s_*})\overline{\mathbf{z}}_{s_*}) \Big\}^2.$$

Explicitly, we have

$$\begin{aligned} Q_{\mathbf{B},\gamma} \colon z_{p+j} = \Big( & \sum_{\ell=1}^{e_*+h_*} \hat{b}_{j\ell}(z_\ell + 2\gamma_\ell \overline{z}_\ell) \\ & + \sum_{s=e_*+h_*+1}^{p-s_*} \hat{b}_{js}(z_s + 2\gamma_s \overline{z}_{s+s_*}) + \hat{b}_{j(s+s_*)}(z_{s+s_*} + 2\gamma_{s+s_*}\overline{z}_s) \Big)^2 \end{aligned} \tag{3.33}$$

for $1 \leq j \leq p$. Here

$$\gamma_{s+s_*} = 1 - \overline{\gamma}_s. \tag{3.34}$$

By (3.32), we also obtain the following identity

$$\hat{\mathbf{B}} = -i\mathbf{B}^{-1} \operatorname{diag}(\mathbf{L}_{e_*}, \mathbf{L}_{h_*}, \tilde{\mathbf{L}}_{s_*}) \operatorname{diag}(\mathbf{I}_{e_*}, \mathbf{\Lambda}_{1h_*}^{1/2}, \mathbf{I}_{2s_*})$$

The equivalence relation (3.27) on the set of non-singular matrices $\mathbf{B}$ now takes the form

$$\hat{\tilde{\mathbf{B}}} = (\operatorname{diag}_\nu \mathbf{d})^{-1} \hat{\mathbf{B}} \operatorname{diag} \mathbf{a}, \tag{3.35}$$

where $\mathbf{a}$ satisfies (3.26) and $\operatorname{diag}_\nu \mathbf{d}$ is defined in (3.19).

Therefore, by Proposition 2.10 we obtain the following classification for the quadrics.

**Theorem 3.7.** *Let $M$ be a quadratic submanifold defined by* (2.1) *and* (2.3) *with $q_* = 0$. Assume that the branched covering $\pi_1$ has $2^p$ deck transformations. Let $T_1, T_2$ be the pair of Moser-Webster involutions of $M$. Suppose that $S = T_1T_2$ has $2p$ distinct eigenvalues. Then $M$ is holomorphically equivalent to* (3.33) *with $\hat{\mathbf{B}} \in GL(p, \mathbf{C})$ being uniquely determined by the equivalence relation* (3.35).

When $\hat{\mathbf{B}}$ is the identity, we obtain the product of 3 types of quadrics

$$\begin{gathered} \mathcal{Q}_{\gamma_e} \colon z_{p+e} = (z_e + 2\gamma_e \overline{z}_e)^2; \\ \mathcal{Q}_{\gamma_h} \colon z_{p+h} = (z_h + 2\gamma_h \overline{z}_h)^2; \\ \mathcal{Q}_{\gamma_s} \colon z_{p+s} = (z_s + 2\gamma_s \overline{z}_{s+s_*})^2, \quad z_{p+s+s_*} = (z_{s+s_*} + 2(1 - \overline{\gamma}_s)\overline{z}_s)^2 \end{gathered}$$

with

$$\gamma_e = \frac{1}{\lambda_e + \lambda_e^{-1}}, \quad \gamma_h = \frac{1}{\lambda_h + \overline{\lambda}_h}, \quad \gamma_s = \frac{1}{1+\lambda_s^{-2}}. \tag{3.36}$$

Note that $\arg \lambda_s \in (0, \pi/2)$ and $|\lambda_s| > 1$. Thus

$$0 < \gamma_e < 1/2, \quad \gamma_h > 1/2, \quad \gamma_s \in \{z \in \mathbf{C} \colon \operatorname{Re} z > 1/2, \operatorname{Im} z > 0\}.$$

We define the following invariants.

**Definition 3.8.** We call $\mathbf{\Gamma} = \operatorname{diag}(\mathbf{\Gamma}_{e_*}, \mathbf{\Gamma}_{h_*}, \mathbf{\Gamma}_{s_*}, \mathbf{I}_{s_*} - \bar{\mathbf{\Gamma}}_{s_*})$, given by formulae (3.31), the *Bishop invariants* of the quadrics. The equivalence classes $\hat{\mathbf{B}}$ of non-singular matrices $\mathbf{B}$ under the equivalence relation (3.27) are called the *extended Bishop invariants* for the quadrics.

Note that $\mathbf{\Gamma}_{e_*}$ has diagonal elements in $(0, 1/2)$, and $\mathbf{\Gamma}_{h_*}$ has diagonal elements in $(1/2, \infty)$, and $\mathbf{\Gamma}_{s_*}$ has diagonal elements in $(1/2, \infty) + i(0, \infty)$.

We remark that $Z_j^-$ is skew-invariant by $T_{1i}$ for $i \neq j$ and invariant by $\tau_{1j}$. Therefore, the square of a linear combination of $Z_1^-, \ldots, Z_p^-$ might not be invariant by all $T_{1j}$. This explains the presence of $\mathbf{B}$ as invariants in the normal form.

It is worthy stating the following normal form for two families of linear holomorphic involutions which may not satisfy the reality condition.

**Proposition 3.9.** *Let $\mathcal{T}_i = \{T_{i1}, \ldots, T_{ip}\}, i = 1, 2$ be two families of distinct and commuting linear holomorphic involutions on $\mathbf{C}^n$. Let $T_i = T_{i1} \cdots T_{ip}$. Suppose that for each $i$, $\operatorname{Fix}(T_{i1}), \ldots, \operatorname{Fix}(T_{ip})$ are hyperplanes intersecting transversally. Suppose that $T_1, T_2$ satisfy* (3.2) *and $S = T_1 T_2$ has $2p$ distinct eigenvalues. In suitable linear coordinates, the matrices of $T_i, S$ are*

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{0} & \mathbf{\Lambda}_i \\ \mathbf{\Lambda}_i^{-1} & \mathbf{0} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{\Lambda}_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_1^{-2} \end{pmatrix}$$

*with $\mathbf{\Lambda}_2 = \mathbf{\Lambda}_1^{-1}$ being diagonal matrix whose entries do not contain $\pm 1, \pm i$. The $\mathbf{\Lambda}_1^2$ is uniquely determined up to a permutation in diagonal entries. Moreover, the matrices of $T_{ij}$ are*

$$\mathbf{T}_{ij} = \mathbf{E}_{\mathbf{\Lambda}_i}(\mathbf{B}_i)_* \mathbf{Z}_j(\mathbf{B}_i)_*^{-1} \mathbf{E}_{\mathbf{\Lambda}_i}^{-1} \tag{3.37}$$

*for some non-singular complex matrices $\mathbf{B}_1, \mathbf{B}_2$ uniquely determined by the equivalence relation*

$$(\mathbf{B}_1, \mathbf{B}_2) \sim (\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2) := (\mathbf{R}^{-1}\mathbf{B}_1 \operatorname{diag}_{\nu_1} \mathbf{d}_1, \mathbf{R}^{-1}\mathbf{B}_2 \operatorname{diag}_{\nu_2} \mathbf{d}_2), \tag{3.38}$$

*where $\operatorname{diag}_{\nu_1} \mathbf{d}_1, \operatorname{diag}_{\nu_2} \mathbf{d}_2$ are defined as in* (3.19)*, and $\mathbf{R}$ is a non-singular diagonal complex matrix representing the linear transformation $\varphi$ such that*

$$\varphi^{-1} T_{ij} \varphi = \tilde{T}_{i\nu_i(j)}, \quad i = 1, 2, j = 1, \ldots, p.$$

*Here $\tilde{\mathcal{T}}_i$ is the family of the involutions associated to the matrices $\tilde{\mathbf{B}}_i$, and $\mathbf{E}_{\mathbf{\Lambda}_i}$ and $\mathbf{B}_*$ are defined by* (3.21) *and* (3.22)*.*

*Proof.* Let $\kappa$ be an eigenvalue of $S$ with (non-zero) eigenvector $u$. Since $T_iST_i = S^{-1}$. Then $S(T_i(u)) = \kappa^{-1}T_i(u)$. This shows that $\kappa^{-1}$ is also an eigenvalue of $S$. By Lemma 3.1, 1 and $-1$ are not eigenvalues of $S$. Thus, we can list the eigenvalues of $S$ as $\mu_1, \dots, \mu_p, \mu_1^{-1}, \dots, \mu_p^{-1}$. Let $u_j$ be an eigenvector of $S$ with eigenvalue $\mu_j$. Fix $\lambda_j$ such that $\lambda_j^2 = \mu_j$. Then $v_j := \lambda_j T_1(u_j)$ is an eigenvector of $S$ with eigenvalue $\mu_j^{-1}$. The $\sum \xi_j u_j + \eta_j v_j$ defines a coordinate system on $\mathbf{C}^n$ such that $T_i, S$ have the above matrices $\mathbf{\Lambda}_i$ and $\mathbf{S}$, respectively. By (3.20) and (3.23), $T_{ij}$ can be expressed in (3.37), where each $\mathbf{B}_i$ is uniquely determined up to $\mathbf{B}_i \operatorname{diag} \mathbf{d}_i$. Suppose that $\{\tilde{T}_{1j}\}, \{\tilde{T}_{2j}\}$ are another pair of families of linear involutions of which the corresponding matrices are $\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2$. If there is a linear change of coordinates $\varphi$ such that $\varphi^{-1}T_{ij}\varphi = \tilde{T}_{i\nu_i(j)}$, then in the matrix $\mathbf{R}$ of $\varphi$, we obtain (3.38); see a similar computation for (3.27) by using (3.28). Conversely, (3.28) implies that the corresponding pairs of families of involutions are equivalent. □

## 4. Formal deck transformations and centralizers

In section 2 we show the equivalence of the classification of real analytic submanifolds $M$ that admit the maximum number of deck transformations and the classification of the families of involutions $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ that satisfy some mild conditions (see Proposition 2.10). To classify the families of involutions and to find their normal forms, we will first study normal forms at the formal level. The main purpose of this section is to show that at the formal level, the classification of the formal submanifolds of the desired CR singularity and the classification of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ are equivalent under these mild conditions.

We will also study the centralizers of various linear maps to deal with resonance. This is relevant as the normal form of $\sigma$ will belong to the centralizer of its linear part and any further normalization will also be performed by transformations that are in the centralizer.

4.1. **Formal submanifolds and formal deck transformations.** We first need some notation. Let $I$ be an ideal of the ring $\mathbf{R}[[x]]$ of formal power series in $x = (x_1, \dots, x_N)$. Since $\mathbf{R}[[x]]$ is noetherian, then $I$ and its radical $\sqrt{I}$ are finitely generated. We say that $I$ defines a formal submanifold $M$ of dimension $N - k$ if $\sqrt{I}$ is generated by $r_1, \dots, r_k$ such that at the origin all $r_j$ vanish and $dr_1, \dots, dr_k$ are linearly independent. For such an $M$, let $I(M)$ denote $\sqrt{I}$ and let $T_0M$ be defined by $dr_1(0) = \cdots = dr_k(0) = 0$. If $F = (f_1, \dots, f_N)$ is a formal mapping with $f_j \in \mathbf{R}[[x]]$, we say that its set of (formal) fixed points is a submanifold if the ideal generated by $f_1(x) - x_1, \dots, f_N(x) - x_N$ defines a submanifold. Let $I, \tilde{I}$ be ideals of $\mathbf{R}[[x]], \mathbf{R}[[y]]$ and let $\sqrt{I}, \sqrt{\tilde{I}}$ define two formal submanifolds $M, \tilde{M}$, respectively. We say that a formal map $y = G(x)$ maps $M$ into $\widetilde{M}$ if $\tilde{I} \circ G \subset \sqrt{I}$. If $M, \tilde{M}$ are in the same space, we write $M \subset \tilde{M}$ if $\tilde{I} \subset \sqrt{I}$. We say that a formal map $F$ fixes $M$ pointwise if $I(M)$ contain each component of the mapping $F - \mathrm{I}$.

We now consider a formal $p$-submanifold in $\mathbf{C}^{2p}$ defined by

$$M\colon z_{p+j} = E_j(z', \bar{z}'), \quad 1 \le j \le p. \tag{4.1}$$

Here $E_j$ are formal power series in $z', \overline{z}'$. We assume that

$$E_j(z', \bar{z}') = h_j(z', \overline{z}') + q_j(\overline{z}') + O((|(z', \overline{z}')|^3) \tag{4.2}$$

and $h_j, q_j$ are homogeneous quadratic polynomials. The formal complexification of $M$ is defined by

$$\begin{cases} z_{p+i} = E_i(z', w'), & i = 1, \dots, p, \\ w_{p+i} = \bar{E}_i(w', z'), & i = 1, \dots, p. \end{cases}$$

We define a *formal deck transformation* of $\pi_1$ to be a formal biholomorphic map

$$\tau \colon (z', w') \to (z', f(z', w')), \quad \tau(0) = 0$$

such that $\pi_1 \tau = \pi_1$, i.e. $E \circ \tau = E$. Recall that condition B says that $q_* = \dim\{z' \in \mathbf{C}^n \colon q(z') = 0\}$ is zero, i.e. $q$ vanishes only at the origin in $\mathbf{C}^p$.

**Lemma 4.1.** *Let $M$ be a formal submanifold defined by* (4.1)-(4.2). *Suppose that $M$ satisfies condition* B. *Then formal deck transformations of $\pi_1$ are commutative involutions. Each formal deck transformation $\tau$ of $\pi_1 \colon \mathcal{M} \to \mathbf{C}^p$ is uniquely determined by its linear part $L\tau$ in the $(z', w')$ coordinates, while $L\tau$ is a deck transformation for the complexification for $\pi_1 \colon \mathcal{Q} \to \mathbf{C}^p$, where $\mathcal{Q}$ is the complexification of the quadratic part $Q$ of $M$. If $M$ is real analytic, all formal deck transformations of $\pi_1$ are convergent.*

*Proof.* Let us recall some results about the quadric $Q$. We already know that $q_* = 0$ implies that $\pi_1$ for the complexification of $Q$ is a branched covering. As used in the proof of Lemma 2.1, $\pi_1$ is an open mapping near the origin and its regular values are dense. In particular, we have

$$\det \partial_{w'} \{h(z', w') + q(w')\} \not\equiv 0. \tag{4.3}$$

Let $\tau$ be a formal deck transformation for $M$. To show that $\tau$ is an involution, we note that its linear part at the origin, $L\tau$, is a deck transformation of $Q$. Hence $L\tau$ is an involution. Replacing $\tau$ by the deck transformation $\tau^2$, we may assume that $\tau$ is tangent to the identity. Write

$$\tau(z', w') = (z', w' + u(z', w')).$$

We want to show that $u = 0$. Assume that $u(z', w') = O(|(z', w')|^k)$ and let $u_k$ be homogeneous and of degree $k$ such that $u(z', w') = u_k(z', w') + O(|(z', w')|^{k+1})$. We have

$$E(z', w' + u(z', w')) = E(z', w').$$

Comparing terms of order $k + 1$, we get

$$\partial_{w'} \{h(z', w') + q(w')\} u_k(z', w') = 0.$$

By (4.3), $u_k = 0$. This shows that each formal deck transformation $\tau$ of $\pi_1$ for $M$ is an involution. As mentioned above, $L\tau$ is a deck transformation of $\pi_1$ for $Q$. Also if $\tau, \tilde{\tau}$ are commuting formal involutions then $\tau^{-1}\tilde{\tau}$ is an involution and $\tau = \tilde{\tau}$ if and only if $L\tau = L\tilde{\tau}$.

Assume now that $M$ is real analytic. We want to show that each formal deck transformation $\tau$ is convergent. By a theorem of Artin [Art68], there is a convergent $\tilde{\tau}(z', w') = \tau(z', w') + O(|(z', w')|^2)$ such that $E \circ \tilde{\tau} = E$, i.e. $\tilde{\tau}$ is a deck transformation. Then $\tilde{\tau}^{-1}\tau$ is a deck transformation tangent to the identity. Since it is a formal involution by the above argument, then it must be identity. Therefore, $\tau = \tilde{\tau}$ converges. $\square$

Analogous to real analytic submanifolds, we say that a formal manifold defined by (4.1)-(4.2) satisfies condition D if its formal branched covering $\pi_1$ admits $2^p$ formal deck transformations.

Recall from section 2 that it is crucial to distinguish a special set of generators for the deck transformations in order to relate the classification of real analytic manifolds to the classification of certain $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$. The set of generators is uniquely determined by the dimension of fixed-point sets. We want to extend these results at the formal level.

**Proposition 4.2.** *Let $M, \tilde{M}$ be formal $p$-submanifolds in $\mathbf{C}^n$ of the form* (4.1)-(4.2)*. Suppose that $M, \tilde{M}$ satisfy condition* D*. Then the following hold* :

(*i*) *$M$ and $\tilde{M}$ are formally equivalent if and only if their associated families of involutions $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ and $\{\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}, \rho\}$ are formally equivalent.*

(*ii*) *Let $\mathcal{T}_1 = \{\tau_{11}, \dots, \tau_{1p}\}$ be a family of formal holomorphic involutions which commute pairwise. Suppose that the tangent spaces of $\mathrm{Fix}(\tau_{11}), \dots, \mathrm{Fix}(\tau_{1p})$ are hyperplanes intersecting transversally at the origin. Let $\rho$ be an anti-holomorphic formal involution and let $\mathcal{T}_2 = \{\tau_{21}, \dots, \tau_{2p}\}$ with $\tau_{2j} = \rho\tau_{1j}\rho$. Suppose that $\sigma = \tau_1\tau_2$ has distinct eigenvalues for $\tau_i = \tau_{i1} \cdots \tau_{ip}$, and*

$$[\mathfrak{M}_n]_1^{L\mathcal{T}_1} \cap [\mathfrak{M}_n]_1^{L\mathcal{T}_2} = \{0\}.$$

*There exists a formal submanifold defined by*

$$z'' = (B_1^2, \dots, B_p^2)(z', \overline{z}') \tag{4.4}$$

*for some formal power series $B_1, \dots, B_p$ such that $M$ satisfies condition* D*. The set of involutions $\{\tilde{\tau}_{11}, \dots, \tilde{\tau}_{1p}, \rho_0\}$ of $M$ is formally equivalent to $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$.*

*Proof.* (i) Let $M$ and $\tilde{M}$ be given by $z'' = E(z', \overline{z'})$ and $\tilde{z}'' = \tilde{E}(\tilde{z}', \overline{\tilde{z}'})$, respectively. Suppose that $f$ is a formal holomorphic transformation sending $M$ into $\tilde{M}$. We have

$$f''(z', E(z', w')) = \tilde{E}(f'(z', E(z', w')), \overline{f}'(w', \overline{E}(w', z'))). \tag{4.5}$$

Here $f = (f', f'')$. Recall that $\rho_0(z', w') = (\overline{w'}, \overline{z'})$. Define a formal mapping $(z', w') \to (\tilde{z}', \tilde{w}') = F(z', w')$ by

$$F(z', w') := (f'(z', E(z', w')), \overline{f'}(w', \overline{E}(w', z'))). \tag{4.6}$$

It is clear that $F\rho_0 = \rho_0 F$. By Lemma 2.7, we know that $\tilde{z}'$ and $\tilde{z}'' = \tilde{E}(\tilde{z}', \tilde{w}')$ generate invariant formal power series of $\{\tilde{\tau}_{1j}\}$. Thus, $\tilde{z}' \circ F(z', w') = f'(z', E(z', w'))$ and $\tilde{E} \circ F(z', w')$ are invariant by $F^{-1} \circ \tilde{\tau}_{1j} \circ F$. By (4.5) and the definition of $F$,

$$\tilde{E} \circ F(z', w') = f''(z', E(z', w')).$$

This shows that $f(z', E(z', w'))$ is invariant under $F^{-1} \circ \tilde{\tau}_{1j} \circ F$. Since $f$ is invertible, then $z'$ and $E(z', w')$ are invariant under $F^{-1} \circ \tilde{\tau}_{1j} \circ F$. Therefore, $\{\tau_{1j}\}$ and $\{F^{-1} \circ \tilde{\tau}_{1i} \circ F\}$ are the same by Lemma 2.7 as they have the same invariant functions.

Assume now that $\{\tau_{1j}\} = \{F^{-1} \circ \tilde{\tau}_{1i} \circ F\}$ for some formal biholomorphic map $F$ commuting with $\rho_0$. Recall that $\tilde{z}', \tilde{z}''$ are invariant by $\tilde{\tau}_{1j}$. Then $\tilde{z}' \circ F$ and $\tilde{E} \circ F$ are invariant

by $\{\tau_{1j}\}$. By Lemma 2.7, invariant power series of $\tau_{1j}$ are generated by $z', E(z', w')$. Thus we can write

$$\begin{aligned} \tilde{z}' \circ F(z', w') &= f'(z', E(z', w')), \\ \tilde{E} \circ F(z', w') &= f''(z', E(z', w')) \end{aligned} \tag{4.7}$$

for some formal power series map $f = (f', f'')$. Since $\rho_0 F = F\rho_0$, then by (4.6)

$$F(z', w') = (f'(z', 0), \overline{f}'(w', 0)) + O(|(z', w')|^2).$$

Since $F$ is (formal) biholomorphic then $z' \to f'(z', 0)$ is biholomorphic. Then

$$f''(0, E(0, w')) = \tilde{E}(0, \overline{f}'(w', 0)) + O(|w'|^3).$$

We have $E(0, w') = q(w') + O(|w'|^3)$ and $\tilde{E}(0, w') = \tilde{q}(w') + O(|w'|^3)$. Here $q(w'), \tilde{q}(w')$ are quadratic. By condition $q_* = 0$, we know that $\tilde{q}_1, \ldots, \tilde{q}_p$ and hence $\tilde{q}_1 \circ L, \ldots, \tilde{q}_p \circ L$ are linearly independent. Here $L$ is the linear part of the mapping $w' \to \overline{f}'(w', 0)$, which is invertible. This shows that the linear part of $w' \to f''(0, w')$ is biholomorphic. By (4.7), $f''(z', 0) = O(|z'|^2)$. Hence $f = (f', f'')$ is biholomorphic. By a simple computation, we have $f(M) = \tilde{M}$, i.e.

$$\tilde{E}(f'(z), \overline{f'(z)}) = f''(z)$$

for $z'' = E(z', \overline{z}')$.

(ii) Assume that $\{\tau_{1j}\}$ and $\rho$ are given in the $(\xi, \eta)$ space. We want to show that a formal holomorphic equivalence class of $\{\tau_{1j}, \rho\}$ can be realized by a formal submanifold satisfying condition D. The proof is almost identical to the realization proof of Proposition 2.10 and we will be brief. Using a formal, instead of convergent, change of coordinates, we know that invariant formal power series of $\{\tau_{1j}\}$ are generated by

$$z' = (A_1(\xi, \eta), \ldots, A_p(\xi, \eta)), \quad z'' = (B_1^2(\xi, \eta), \ldots, B_p^2(\xi, \eta)),$$

where $B_j$ is skew-invariant by $\tau_{1j}$, and $A, B_i$ are invariant under $\tau_{1j}$ for $i \neq j$. Moreover, $\phi(\xi, \eta) = (A, B)(\xi, \eta)$ is formal biholomorphic. Set

$$w_j' = \overline{A_j \circ \rho(\xi, \eta)}, \quad w_j'' = \overline{B_j^2 \circ \rho(\xi, \eta)}.$$

Then $(\xi, \eta) \to (A(\xi, \eta), \overline{A \circ \rho(\xi, \eta)})$ has an inverse $\psi$. Define

$$M\colon z'' = (B_1^2, \ldots, B_p^2) \circ \psi(z', \overline{z}').$$

The complexification of $M$ is given by

$$\mathcal{M}\colon z'' = (B_1^2, \ldots, B_p^2) \circ \psi(z', w'), \quad w'' = (\overline{B}_1^2, \ldots, \overline{B}_p^2) \circ \overline{\psi}(w', z').$$

Note that $\phi \circ \psi(z', w') = (z', B \circ \psi(z', w'))$. Since $\phi\psi$ is invertible, the linear part $D$ of $B \circ \psi$ satisfies

$$|D(0, w')| \geq |w'|/C.$$

This shows that $q_* = 0$. As in the proof of Proposition 2.10, we can verify that $M$ is a realization for $\{\tau_{1j}, \rho\}$. $\square$

4.2. **Centralizers and normalized transformations.** In this subsection, we describe several centralizers regarding $\hat{S}, \hat{T}_1$ and $\hat{\mathcal{T}}_1$. We will also describe the complement sets of the centralizers, i.e. the sets of mappings which satisfy suitable normalizing conditions. Roughly speaking, our normal forms are in the centralizers and coordinate transformations that achieve the normal forms are normalized, while an arbitrary formal transformation admits a unique decomposition of a mapping in a centralizer and a mapping in the complement of the centralizer. The description of the centralizer of $\{\mathcal{T}_1, \rho\}$ is more complicated and it will be given in section 11. We will also deal with the convergence for the decomposition.

Recall that

$$\hat{S}\colon \xi_j' = \mu_j \xi_j, \quad \eta_j' = \mu_j^{-1}\eta_j, \quad 1 \le j \le p, \tag{4.8}$$

$$\hat{T}_i\colon \xi_j' = \lambda_{ij}\eta_j, \quad \eta_j' = \lambda_{ij}^{-1}\xi, \quad 1 \le j \le p \tag{4.9}$$

with $\mu_j = \lambda_{1j}^2$ and $\lambda_{2j}^{-1} = \lambda_{1j} = \lambda_j$.

**Definition 4.3.** Let $\mathcal{F}$ be a family of formal mappings on $\mathbf{C}^n$ fixing the origin. Let $\mathcal{C}(\mathcal{F})$ be the *centralizer* of $\mathcal{F}$, i.e. the set of formal holomorphic mappings $g$ that fix the origin and commute with each element $f$ of $\mathcal{F}$, i.e., $f \circ g = g \circ f$.

Note that we do not require that elements in $\mathcal{C}(\mathcal{F})$ be invertible or convergent.

We first compute the centralizers.

**Lemma 4.4.** *Let $\hat{S}$ be given by (4.8) with $\mu_1, \dots, \mu_p$ being non-resonant. Then $\mathcal{C}(\hat{S})$ consists of mappings of the form*

$$\psi\colon \xi_j' = a_j(\xi\eta)\xi_j, \quad \eta_j' = b_j(\xi\eta)\eta_j, \quad 1 \le j \le p. \tag{4.10}$$

*Let $\tau_1, \tau_2$ be formal holomorphic involutions such that $\hat{S} = \tau_1\tau_2$. Then*

$$\tau_i\colon \xi_j' = \Lambda_{ij}(\xi\eta)\eta_j, \quad \eta_j' = \Lambda_{ij}^{-1}(\xi\eta)\xi_j, \quad 1 \le j \le p$$

*with $\Lambda_{1j}\Lambda_{2j}^{-1} = \mu_j$. The centralizer of $\{\hat{T}_1, \hat{T}_2\}$ consists of the above transformations satisfying*

$$b_j = a_j, \quad 1 \le j \le p. \tag{4.11}$$

*Proof.* Let $e_j = (0, \dots, 1, \dots, 0) \in \mathbf{N}^p$, where 1 is at the $j$th place. Let $\psi$ be given by

$$\xi_j' = \sum a_{j,PQ}\xi^P\eta^Q, \quad \eta_j' = \sum b_{j,PQ}\xi^P\eta^Q.$$

By the non-resonance condition, it is straightforward that if $\psi\hat{S} = \hat{S}\psi$, then $a_{j,PQ} = b_{j,QP} = 0$ if $P - Q \ne e_j$. Note that $\hat{S}^{-1} = T_0\hat{S}T_0$ for $T_0\colon (\xi, \eta) \to (\eta, \xi)$. Thus $\tau_1T_0$ commutes with $\hat{S}$. So $\tau_1T_0$ has the form (4.10) in which we rename $a_j, b_j$ by $\Lambda_{1j}, \tilde{\Lambda}_{1j}$, respectively. Now $\tau_1^2 = \mathrm{I}$ implies that

$$\Lambda_{1j}((\Lambda_{11}\tilde{\Lambda}_{11})(\zeta)\zeta_1, \dots, (\Lambda_{1p}\tilde{\Lambda}_{1p})(\zeta)\zeta_p)\tilde{\Lambda}_{1j}(\zeta) = 1, \quad 1 \le j \le p.$$

Then $\Lambda_{1j}(0)\tilde{\Lambda}_{1j}(0) = 1$. Applying induction on $d$, we verify that for all $j$

$$\Lambda_{1j}(\zeta)\tilde{\Lambda}_{1j}(\zeta) = 1 + O(|\zeta|^d), \quad d > 1.$$

Having found the formula for $\tau_1T_0$, we obtain the desired formula of $\tau_1$ via composition $(\tau_1T_0)T_0$. □

Let $\mathbf{D}_1 := \mathrm{diag}(\mu_{11}, \dots, \mu_{1n}), \dots, \mathbf{D}_\ell := \mathrm{diag}(\mu_{\ell 1}, \dots, \mu_{\ell n})$ be diagonal invertible matrices of $\mathbf{C}^n$. Let us set $D := \{\mathbf{D}_i z\}_{i=1,\dots \ell}$.

**Definition 4.5.** Let $F$ be a formal mapping of $\mathbf{C}^n$ that is tangent to the identity.

($i$) Let $n = 2p$. $F$ is *normalized* with respect to $\hat{S}$, if $F = (f, g)$ is tangent to the identity and $F$ contains no resonant terms, i.e.

$$f_{j,(A+e_j)A} = 0 = g_{j,A(A+e_j)}, \quad |A| > 1.$$

($ii$) Let $n = 2p$. $F$ is *normalized* with respect to $\{\hat{T}_1, \hat{T}_2\}$, if $F = (f, g)$ is tangent to the identity and

$$f_{j,(A+e_j)A} = -g_{j,A(A+e_j)}, \quad |A| > 1.$$

($iii$) $F$ is *normalized* with respect to $D$ if it does not have components along the centralizer of $D$, i.e. for each $Q$ with $|Q| \geq 2$,

$$f_{j,Q} = 0, \quad \text{if } \mu_i^Q = \mu_{ij} \text{ for all } i.$$

Let $\mathcal{C}^{\mathsf{c}}(\hat{S})$ (resp. $\mathcal{C}^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$, $\mathcal{C}^{\mathsf{c}}(D)$) denote the set of formal mappings normalized with respect to $\hat{S}$ (resp. $\{\hat{T}_1, \hat{T}_2\}$, the family $D$).

For convenience, we let $\mathcal{C}_2^{\mathsf{c}}(\hat{S})$ (resp. $\mathcal{C}_2^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$, $\mathcal{C}_2^{\mathsf{c}}(D)$) denote the set of formal mappings $F - \mathrm{I}$ with $F \in \mathcal{C}^{\mathsf{c}}(\hat{S})$ (resp. $\mathcal{C}^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$, $\mathcal{C}^{\mathsf{c}}(D)$).

**Remark 4.6.** Note that if $f \in \mathcal{C}^{\mathsf{c}}(\hat{S})$ (resp. $\mathcal{C}^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$), then $\rho f \rho$ is in $\mathcal{C}^{\mathsf{c}}(\hat{S})$ (resp. $\mathcal{C}^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$).

We now deal with the following decomposition problem: Let $\mathcal{C}$ be a set of analytic mappings. We shall decompose an arbitrary invertible mapping into the composition of an element of a centralizer of $\mathcal{C}$ and an element which is normalized with respect to $\mathcal{C}$. We shall also deal with the convergence issue. The following lemma, which deals with a general situation, will be used several times.

**Definition 4.7.** Let $\mathcal{A}$ be a group of permutations of $\{1, \dots, n\}$. Then $\mathcal{A}$ acts on the right (resp. on the left) on $\widehat{\mathcal{O}}_n^n$ by permutation of variables $z = (z_1, \dots, z_n)$ as follows: Let $F(z) = \sum_{|Q|>0} F_Q z^Q$ be a formal mapping from $\mathbf{C}^n$ to $\mathbf{C}^n$, and let $\nu, \mu \in \mathcal{A}$; set

$$\nu \circ F \circ \mu(z) := \sum_{Q \in \mathbf{N}^n} F_{\nu(i),\mu^{-1}(Q)} z^Q.$$

Define the components $(\mathcal{A}F)_i, (F\mathcal{A})_i$, and consequently $(\mathcal{A}F\mathcal{A})_i$ by

$$(\mathcal{A}F)_i(z) := \sum_{Q \in \mathbf{N}^n} \max_{\nu \in \mathcal{A}} |F_{\nu(i),Q}| z^Q,$$

$$(F\mathcal{A})_i(z) := \sum_{Q \in \mathbf{N}^n} \max_{\mu \in \mathcal{A}} |F_{i,\mu^{-1}(Q)}| z^Q,$$

$$(\mathcal{A}F\mathcal{A})_i(z) = \sum_{Q \in \mathbf{N}^n} \max_{(\nu,\mu) \in \mathcal{A}^2} |F_{\nu(i),\mu^{-1}(Q)}| z^Q.$$

We see that $F\mathcal{A}$ is the smallest (w.r.t. $\prec$) power series mapping that majorizes $F$ and is right-invariant under $\mathcal{A}$, while $\mathcal{A}F$ is the smallest power series mapping that majorizes $F$ and is left-invariant under $\mathcal{A}$. In particular, if $F, G$ are mappings without constant or linear terms, then

$$\mathcal{A}(F \circ (I+G))\mathcal{A} \prec (\mathcal{A}F\mathcal{A})(\mathcal{A}I\mathcal{A} + \mathcal{A}G\mathcal{A}), \tag{4.12}$$

where the last relation holds if the composition is well-defined.

To simply our notation, we will take $\mathcal{A}$ to be the full permutation group of $\{1, \ldots, n\}$. We will denote

$$F_{sym} = \mathcal{A}F\mathcal{A}.$$

**Lemma 4.8.** *Let $\hat{\mathcal{H}}$ be a real subspace of $(\widehat{\mathfrak{M}}_n^2)^n$. Let $\pi : (\widehat{\mathfrak{M}}_n^2)^n \to \hat{\mathcal{H}}$ be a* **R** *linear projection (i.e. $\pi^2 = \pi$) that preserves the degrees of the mappings and let $\hat{\mathcal{G}} := (\mathrm{I} - \pi)(\widehat{\mathfrak{M}}_n^2)^n$. Suppose that there is a positive constant $C$ such that*

$$\pi(E) \prec CE_{sym} \tag{4.13}$$

*for any $E \in (\widehat{\mathfrak{M}}_n^2)^n$. Let $F$ be a formal map tangent to the identity. There exists a unique decomposition*

$$F = HG^{-1} \tag{4.14}$$

*with $G - I \in \hat{\mathcal{G}}$ and $H - I \in \hat{\mathcal{H}}$. If $F$ is convergent, then $G$ and $H$ are also convergent.*

*Proof.* If $f$ is a formal mapping, we define the $k$-jet:

$$J^k f(z) = \sum_{|Q| \le k} f_Q z^Q.$$

Write $F = I + f$, $G = I + g$ and $H = I + h$. We need to solve $FG = H$, i.e to solve

$$h - g = f(I + g).$$

Since $f'(0) = 0$, then for any $k \ge 2$, the $k$-jet of $f(I+g)$ depends only on the $(k-1)$-jet of $g$. Since $\pi$ is linear and preserves degrees, (4.13) implies that $J^k$ commutes with $\pi$. Hence we can define, for all $k \ge 2$,

$$-J^k(g) := \pi\left(J^k(f(I+g)\right), \quad J^k(h) := (I-\pi)\left(J^k(f(I+g)\right).$$

This solves the formal decomposition uniquely. Assume that $F$ is a germ of holomorphic mapping. Hence, we have

$$g \prec C(f(I+g))_{sym} \prec Cf_{sym}(I_{sym} + g_{sym}).$$

Since $g_{sym}$ is the smallest left and right $\mathcal{A}$ invariant power series that dominates $g$, we have

$$g_{sym} \prec Cf_{sym}(I_{sym} + g_{sym}).$$

Therefore, $g_{sym}$ is dominated by the solution $u$ to

$$u = Cf_{sym}(I_{sym} + u), \quad u(0) = 0.$$

Notice that $u$ is real analytic near the origin by the implicit function theorem. So, $g_{sym}$ is convergent, and both $g$ and $h = g + f(I+g)$ are convergent in a neighborhood of the origin. $\square$

**Corollary 4.9.** *The previous decomposition* (4.14) *is valid with* $\hat{\mathcal{G}} := \mathcal{C}_2(\hat{S})$ *and* $\hat{\mathcal{H}} := \mathcal{C}_2^{\mathsf{c}}(\hat{S})$ *(resp.* $\hat{\mathcal{G}} := \mathcal{C}_2(\hat{T}_1, \hat{T}_2)$ *and* $\hat{\mathcal{H}} := \mathcal{C}_2^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$*;* $\hat{\mathcal{G}} := \mathcal{C}_2(D)$ *and* $\hat{\mathcal{H}} := \mathcal{C}_2^{\mathsf{c}}(D)$*).*

*Proof.* We apply the previous lemma by finding $\pi$. The first case is obvious since $K$ is in $\mathcal{C}_2(\hat{S})$ (resp. $\mathcal{C}_2^{\mathsf{c}}(\hat{S})$) if and only if $K_Q z^Q \in \mathcal{C}_2(\hat{S})$ (resp. $\mathcal{C}_2^{\mathsf{c}}(\hat{S})$) for all $Q$. So we take

$$(I-\pi)(K) = \sum_{j=1}^{n} \sum_{e_j z^Q \in \widehat{\mathcal{G}}} K_{j,Q} z^Q e_j.$$

Next, we consider the case where $\hat{\mathcal{G}} = \mathcal{C}_2(\hat{T}_1, \hat{T}_2)$ and $\hat{\mathcal{H}} = \mathcal{C}_2^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$. We need to find a projection such that $\hat{\mathcal{H}} = \pi(\widehat{\mathfrak{M}}_n^2)^n$ and $\hat{\mathcal{G}} = (\mathrm{I}-\pi)(\widehat{\mathfrak{M}}_n^2)^n$. Note that $g \in \mathcal{C}_2(\hat{T}_1, \hat{T}_2)$ and $h \in \mathcal{C}_2^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$ are determined by conditions

$$g_{j,(\gamma+e_j)\gamma} = g_{(j+p),\gamma(\gamma+e_j)}, \quad h_{j,(\gamma+e_j)\gamma} = -h_{(j+p),\gamma(\gamma+e_j)}, \quad 1 \le j \le p,$$
$$g_{j,PQ} = g_{(j+p),QP} = 0, \quad P - Q \neq e_j.$$

Thus, if $h - g = K$, we determine $g$ uniquely by combining the above identities with

$$g_{j,(\gamma+e_j)\gamma} = \frac{-1}{2}\left\{K_{j,(\gamma+e_j)\gamma} + K_{(j+p),\gamma(\gamma+e_j)}\right\},$$
$$h_{j,(\gamma+e_j)\gamma} = \frac{1}{2}\left\{K_{j,(\gamma+e_j)\gamma} - K_{(j+p),\gamma(\gamma+e_j)}\right\}$$

for $1 \le j \le p$. For the remaining coefficients of $h$, set $h_{i,PQ} = K_{i,PQ}$. Therefore, $\pi(K) := h \prec K_{sym}$ and the proof is complete. □

**Remark 4.10.** Let $\mathcal{A}, \mathcal{B}$ be two subgroups of permutations. Instead of using the full permutations group, we could have used $G_{sym} := \mathcal{A}G\mathcal{B}$. We have

$$G \prec \mathcal{A}G\mathcal{B} \prec C\mathcal{A}(F \circ (I+G))\mathcal{B} \prec (\mathcal{A}F\mathcal{A})(\mathcal{A}I\mathcal{B} + \mathcal{A}G\mathcal{B}).$$

**Remark 4.11.** We do not know if there are convergent $G \in \mathcal{C}(\hat{S})$ and $H \in \mathcal{C}^{\mathsf{c}}(\hat{S})$ such that $F = GH$ when $F$ is convergent. Note that the formal decomposition exists.

Recall that for $j = 1, \ldots, p$, we define

$$Z_j \colon \xi' = \xi, \quad \eta'_k = \eta_k,\ k \neq j, \quad \eta'_j = -\eta_j.$$

We have seen in section 3 how invariant functions of $Z_j$ play a role in constructing normal form of quadrics. In section 7, we will also need a centralizer for non linear maps (see Lemma 7.2) to obtain normal forms for two families of involutions. Therefore, let us first record here the following description of centralizer of $Z_1, \ldots, Z_p$.

**Lemma 4.12.** *The centralizer,* $\mathcal{C}(Z_1, \ldots, Z_p)$*, consists of formal mappings*

$$(\xi, \eta) \to (U(\xi,\eta), \ldots, \eta_1 V_1(\xi,\eta), \ldots, \eta_p V_p(\xi,\eta))$$

*such that* $U(\xi,\eta), V(\xi,\eta)$ *are even in each* $\eta_j$*. Let* $\mathcal{C}^{\mathsf{c}}(Z_1, \ldots, Z_p)$ *denote the set of mappings* $I + (U, V)$ *which are tangent to the identity such that*

$$U_{j,PQ} = V_{j,P(e_j+Q')} = 0, \quad Q, Q' \in 2\mathbf{N}^p,\ |P| + |Q| > 1,\ |P| + |Q'| > 1. \tag{4.15}$$

*Let* $\psi \in \mathcal{C}(Z)$ *be tangent to the identity. There exist unique* $\psi_0 \in \mathcal{C}(Z_1, \ldots, Z_p)$ *and* $\psi_1 \in \mathcal{C}^{\mathsf{c}}(Z_1, \ldots, Z_p)$ *such that* $\psi = \psi_1 \psi_0^{-1}$*. Moreover, if* $\psi$ *is convergent, then* $\psi_0$ *and* $\psi_1$ *are convergent.*

*Proof.* The lemma follows immediately from Lemma 4.8 in which $\hat{H}$ is the $\mathbf{R}$ linear space of mappings $(U, V)$ without constant or linear terms, which satisfy (4.15). The projection $\pi$ is the unique projection onto $\hat{H}$ (i.e. $\pi^2 = \pi$, and $\pi$ is the identity on $\hat{H}$) such that $\pi$ is linear and preserves degrees, and $\pi(E) = 0$ if $E(\xi, \eta) = O(|(\xi, \eta)|^2)$ and $E \in \mathcal{C}_2(Z_1, \ldots, Z_p)$. □

## 5. Formal normal forms of the reversible map $\sigma$

Let us first describe our plans to derive the normal forms of $M$. We would like to show that two families of involutions $\{\tau_{1j}, \tau_{2j}, \rho\}$ and $\{\tilde{\tau}_{1j}, \tilde{\tau}_{2j}, \tilde{\rho}\}$ are holomorphically equivalent, if their corresponding normal forms are equivalent under a much smaller set of changes of coordinates. Ideally, we would like to conclude that $\{\tilde{\tau}_{1j}, \tilde{\tau}_{2j}, \tilde{\rho}\}$ are holomorphically equivalent if and only if their corresponding normal forms are the same, or if they are the same under a change of coordinates with finitely many parameters. For instance the Moser-Webster normal form for real analytic surfaces ($p = 1$) with non-vanishing elliptic Bishop invariant falls into the former situation, while the Chern-Moser theory [CM74] for real analytic hypersurfaces with non-degenerate Levi-form is an example for the latter. Such a normal form will tell us if the real manifolds have infinitely many invariants or not. One of our goals is to understand if the normal form so achieved can be realized by a convergent normalizing transformation. We will see soon that we can achieve our last goal under some assumptions on the family of involutions. Alternatively and perhaps for simplicity of the normal form theory, we would like to seek normal forms which are dynamically or geometrically significant.

Recall that for each real analytic manifold that has $2^p$, the maximum number of, commuting deck transformations $\{\tau_{1j}\}$, we have found a unique set of generators $\tau_{11}, \ldots, \tau_{1p}$ so that each $\mathrm{Fix}(\tau_{1j})$ has codimension 1. More importantly $\tau_1 = \tau_{11} \cdots \tau_{1p}$ is the unique deck transformation of which the set of fixed points has dimension $p$. Let $\tau_2 = \rho\tau_1\rho$ and $\sigma = \tau_1\tau_2$. To normalize $\{\tau_{1j}, \tau_{2j}, \rho\}$, we will choose $\rho$ to be the standard anti-holomorphic involution determined by the linear parts of $\sigma$. Then we normalize $\sigma = \tau_1\tau_2$ under formal mapping commuting with $\rho$. This will determine a normal form for $\{\tau_1^*, \tau_2^*, \rho\}$. This part of normalization is analogous to the Moser-Webster normalization. When $p = 1$, Moser and Webster obtained a unique normal form by a simple argument. However, this last step of simple normalization is not available when $p > 1$. By assuming $\log \hat{M}$ associated to $\hat{\sigma}$ is tangent to the identity, we will obtain a unique formal normal form $\hat{\sigma}, \hat{\tau}_1, \hat{\tau}_2$ for $\sigma, \tau_1, \tau_2$. Next, we need to construct the normal form for the families of involutions. We first ignore the reality condition, by finding $\Phi$ which transforms $\{\tau_{1j}\}$ into a set of involutions $\{\hat{\tau}_{1j}\}$ which is decomposed canonically according to $\hat{\tau}_1$. This allows us to express $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$ via $\{\hat{\tau}_1, \hat{\tau}_2, \Phi, \rho\}$, as in the classification of the families of linear involutions. Finally, we further normalize $\{\hat{\tau}_1, \hat{\tau}_2, \Phi, \rho\}$ to get our normal form.

**Definition 5.1.** Throughout this section and next, we denote $\{h\}_d$ the set of coefficients of $h_P$ with $|P| \leq d$ if $h(x)$ is a map or function in $x$ as power series. We denote by $\mathcal{A}_P(t), \mathcal{A}(y; t)$, etc., a universal *polynomial* whose coefficients and degree depend on a multiindex. The variables in these polynomials will involve a collection of Taylor coefficients

of various mappings. The collection will also depend on $|P|$. As such dependency (or independency to coefficients of higher degrees) is crucial to our computation, we will remind the reader the dependency when emphasis is necessary.

For instance, let us take two formal mappings $F, G$ from $\mathbf{C}^n$ into itself. Suppose that $F = \mathrm{I} + f$ with $f(x) = O(|x|^2)$ and $G = LG + g$ with $g(x) = O(|x|^2)$ and $LG$ being linear. For $P \in \mathbf{N}^n$ with $|P| > 1$, we can express

$$(F^{-1})_P = -f_P + \mathcal{F}_P(\{f\}_{|P|-1}), \tag{5.1}$$

$$(G \circ F)_P = g_P + ((LG) \circ f)_P + \mathcal{G}_P(LG; \{f, g\}_{|P|-1}), \tag{5.2}$$

$$(F^{-1} \circ G \circ F)_P = g_P - (f \circ (LG))_P + ((LG) \circ f)_P + \mathcal{H}_P(LG; \{f, g\}_{|P|-1}). \tag{5.3}$$

5.1. **Formal normal forms of pair of involutions $\{\tau_1, \tau_2\}$.** We first find a normal form for $\sigma$ in $\mathcal{C}(S)$.

**Proposition 5.2.** *Let $\sigma$ be a holomorphic map. Suppose that $\sigma$ has the linear part*

$$\hat{S}\colon \xi_j' = \mu_j \xi_j, \quad \eta_j = \mu_j^{-1}\eta_j, \quad 1 \le j \le p$$

*and $\mu_1, \dots, \mu_p$ are non-resonant. Then there exists a unique normalized formal map $\Psi \in \mathcal{C}^{\mathsf{c}}(\hat{S})$ such that $\sigma^* = \Psi^{-1}\sigma\Psi \in \mathcal{C}(\hat{S})$. Moreover, $\tilde{\sigma} = \psi_0^{-1}\sigma^*\psi_0 \in \mathcal{C}(\hat{S})$, if and only if $\psi_0 \in \mathcal{C}(\hat{S})$ and it is invertible. Let*

$$\begin{aligned} \sigma^*\colon \xi_j' &= M_j(\xi\eta)\xi_j, \quad \eta_j' = N_j(\xi\eta)\eta_j, \\ \tilde{\sigma}\colon \xi_j' &= \tilde{M}_j(\xi\eta)\xi_j, \quad \eta_j' = \tilde{N}_j(\xi\eta)\eta_j, \\ \psi_0\colon \xi_j' &= a_j(\xi\eta)\xi_j, \quad \eta_j' = b_j(\xi\eta)\eta_j. \end{aligned}$$

(*i*) *Assume that $\tau_1, \tau_2$ are holomorphic involutions and $\sigma = \tau_1\tau_2$. Then $\sigma^* = \tau_1^*\tau_2^*$ with*

$$\begin{gathered} \tau_i^* = \Psi^{-1}\tau_i\Psi\colon \xi_j' = \Lambda_{ij}(\xi\eta)\eta_j, \quad \eta_j' = \Lambda_{ij}^{-1}(\xi\eta)\xi_j; \\ N_j = M_j^{-1}, \quad M_j = \Lambda_{1j}\Lambda_{2j}^{-1}. \end{gathered} \tag{5.4}$$

*Let the linear part of $\tau_i$ be given by*

$$\hat{T}_i\colon \xi_j' = \lambda_{ij}\eta_j, \quad \eta_j' = \lambda_{ij}^{-1}\xi_j.$$

*Suppose that $\lambda_{2j}^{-1} = \lambda_{1j}$. There exists a unique $\psi_0 \in \mathcal{C}^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$ such that*

$$\begin{gathered} \tilde{\tau}_i = \psi_0^{-1}\tau_i^*\psi_0\colon \xi_j' = \tilde{\Lambda}_{ij}(\xi\eta)\eta_j, \quad \eta_j' = \tilde{\Lambda}_{ij}^{-1}(\xi\eta)\xi_j; \\ \tilde{M}_j = \tilde{\Lambda}_{1j}^2 = \tilde{N}_j^{-1}, \quad \tilde{\Lambda}_{2j} = \tilde{\Lambda}_{1j}^{-1}. \end{gathered} \tag{5.5}$$

*Let $\psi_1$ be a formal biholomorphic map. Then $\{\psi_1^{-1}\tilde{\tau}_1\psi_1, \psi_1^{-1}\tilde{\tau}_2\psi\}$ has the same form as of $\{\tilde{\tau}_1, \tilde{\tau}_2\}$ if and only if $\psi_1 \in \mathcal{C}(\hat{T}_1, \hat{T}_2)$; moreover, $\tilde{\Lambda}_{ij}(\xi\eta)$, $\tilde{M}_j(\xi\eta)$ are transformed into*

$$\tilde{\Lambda}_{ij} \circ \tilde{\psi}_1, \quad \tilde{M}_j \circ \tilde{\psi}_1. \tag{5.6}$$

*Here $\tilde{\psi}_1(\zeta) = (\operatorname{diag} c(\zeta))^2\zeta$ and $\psi_1(\xi, \eta) = ((\operatorname{diag} c(\xi\eta))\xi, (\operatorname{diag} c(\xi\eta))\eta)$.*

(*ii*) *Assume further that* $\tau_2 = \rho\tau_1\rho$, *where* $\rho$ *is defined by* (3.7). *Let*

$$\rho_z\colon \zeta_j \to \overline{\zeta}_j, \quad 1 \le j \le e_* + h_*; \quad \zeta_s \to \overline{\zeta}_{s+s_*}, \quad e_* + h_* < s \le p - s_*.$$

*Then* $\rho\Psi = \Psi\rho$, $\tau_2^* = \rho\tau_1^*\rho$, *and* $(\sigma^*)^{-1} = \rho\sigma^*\rho$. *The last two identities are equivalent to*

$$\Lambda_{2e}^{-1} = \overline{\Lambda_{1e} \circ \rho_z}, \qquad \overline{M_e \circ \rho_z} = M_e, \qquad 1 \le e \le e_*; \tag{5.7}$$

$$\Lambda_{2h} = \overline{\Lambda_{1h} \circ \rho_z}, \qquad \overline{M_h \circ \rho_z} = M_h^{-1}, \qquad e_* < h \le h_* + e_*; \tag{5.8}$$

$$\Lambda_{2(s)} = \overline{\Lambda_{1(s_*+s)} \circ \rho_z}, \tag{5.9}$$

$$\Lambda_{2(s_*+s)} = \overline{\Lambda_{1s} \circ \rho_z}, \qquad \overline{M_s^{-1} \circ \rho_z} = M_{s_*+s}, \quad h_* + e_* < s \le p - s_*. \tag{5.10}$$

*Let* $\psi_0$ *and* $\tilde{\tau}_i = \psi_0^{-1}\tau_i^*\psi_0$ *be as in (i). Then* $\rho\psi_0 = \psi_0\rho$, *and* $\hat{\tau}_1, \hat{\tau}_2$ *satisfy*

$$\tilde{\Lambda}_{ie} = \overline{\tilde{\Lambda}_{ie} \circ \rho_z}, \quad \tilde{\Lambda}_{ih}^{-1} = \overline{\tilde{\Lambda}_{ih} \circ \rho_z}, \quad \tilde{\Lambda}_{is+s_*} = \overline{\tilde{\Lambda}_{is}^{-1} \circ \rho_z}. \tag{5.11}$$

*Proof.* We will use the Taylor formula

$$f(x+y) = f(x) + \sum_{k=1}^{m} \frac{1}{k!} D_k f(x;y) + R_{m+1} f(x;y)$$

with $D_k f(x;y) = \{\partial_t^k f(x+ty)\}|_{t=0}$ and

$$R_{m+1} f(x;y) = (m+1) \int_0^1 (1-t)^m \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \partial^\alpha f(x+ty) y^\alpha \, dt. \tag{5.12}$$

Set $D = D_1$. Let $\sigma$ be given by

$$\xi_j' = M_j^0(\xi\eta)\xi_j + f_j(\xi,\eta), \quad \eta_j' = N_j^0(\xi\eta)\eta_j + g_j(\xi,\eta)$$

with

$$(f,g) \in \mathcal{C}_2^{\mathsf{c}}(\hat{S}). \tag{5.13}$$

We need to find $\Phi \in \mathcal{C}^{\mathsf{c}}(S)$ such that $\Psi^{-1}\sigma\Psi = \sigma^*$ is given by

$$\xi_j' = M_j(\xi\eta)\xi_j, \quad \eta_j' = N_j(\xi\eta)\eta_j.$$

By definition, $\Psi$ has the form

$$\xi_j' = \xi_j + U_j(\xi,\eta), \quad \eta_j' = \eta_j + V_j(\xi,\eta), \quad U_{j,(P+e_j)P} = V_{j,P(P+e_j)} = 0.$$

The components of $\Psi\sigma^*$ are

$$\xi_j' = M_j(\xi\eta)\xi_j + U_j(M(\xi\eta)\xi, N(\xi\eta)\eta), \tag{5.14}$$

$$\eta_j' = N_j(\xi\eta)\eta_j + V_j(M(\xi\eta)\xi, N(\xi\eta)\eta). \tag{5.15}$$

To derive the normal form, we only need Taylor theorem in order one. This can also demonstrate small divisors in the normalizing transformation; however, one cannot see the small divisors in the normal forms. Later we will show the existence of divergent normal

forms. This requires us to use Taylor formula whose remainder has order two. By the Taylor theorem, we write the components of $\sigma\Psi$ as

$$\begin{aligned}\xi_j' &= (M_j^0(\xi\eta) + DM_j^0(\xi\eta)(\eta U + \xi V + UV))(\xi_j + U_j) \\ &\quad + f_j(\xi,\eta) + Df_j(\xi,\eta)(U,V) + A_j(\xi,\eta),\end{aligned} \tag{5.16}$$

$$\begin{aligned}\eta_j' &= (N_j^0(\xi\eta) + DN_j^0(\xi\eta)(\eta U + \xi V + UV))(\eta_j + V_j) \\ &\quad + g_j(\xi,\eta) + Dg_j(\xi,\eta)(U,V) + B_j(\xi,\eta).\end{aligned} \tag{5.17}$$

Recall our notation that $UV = (U_1(\xi,\eta)V_1(\xi,\eta), \dots, U_p(\xi,\eta)V_p(\xi,\eta))$. The second order remainders are

$$A_j(\xi,\eta) = R_2M_j^0(\xi\eta; \xi U + \eta V + UV)(\xi_j + U_j) + R_2f_j(\xi,\eta;U,V), \tag{5.18}$$

$$B_j(\xi,\eta) = R_2N_j^0(\xi\eta; \xi U + \eta V + UV)(\eta_j + V_j) + R_2g_j(\xi,\eta;U,V). \tag{5.19}$$

Note that the remainder $R_2M^0$ is independent of the linear part of $M^0$. Thus

$$R_2M_j^0 = R_2(M_j^0 - LM_j^0), \quad R_2N_j^0 = R_2(N_j^0 - LN_j^0).$$

Let us calculate the largest degrees of coefficients of $M^0 - LM^0, (U,V,f,g)$ on which $A_{j,PQ}$ depend. We denote the two degrees by $w, d$, respectively. Since $\mathrm{ord}(f,g,U,V) \geq 2$, we have

$$2(w-2) + (d+4) + 1 \leq |P| + |Q|, \quad \text{or} \quad (d-2) + 2d \leq |P| + |Q|,$$

where the first inequality is obtain from the first term on the right-hand side of (5.18) and the second term yields the second inequality. Since $M^0 - LM^0, (U,V)$, and $(f,g)$ do not have linear terms, we have $w \geq 2$ and $d \geq 2$. Thus, we have crude bounds

$$d \leq |P| + |Q| - 1, \quad w \leq \frac{|P| + |Q| - 1}{2}.$$

Analogously, we can estimate the degrees of coefficients of $N^0$. We obtain

$$\begin{aligned}A_{j,PQ} &= \mathcal{A}_{j,PQ}(\{M^0 - LM^0\}_{\frac{|P|+|Q|-1}{2}}; \{f,U,V\}_{|P|+|Q|-1}), \\ B_{j,QP} &= \mathcal{B}_{j,QP}(\{N^0 - LN^0\}_{\frac{|P|+|Q|-1}{2}}; \{g,U,V\}_{|P|+|Q|-1}).\end{aligned}$$

Recall our notation that $\{f,U,V\}_d$ is the set of coefficients of $f_{PQ}, U_{PQ}, V_{PQ}$ with $|P|+|Q| \leq d$. Here $\mathcal{A}_{j,PQ}(t';t''), \mathcal{B}_{j,QP}(t';t'')$ are polynomials of which each has coefficients that depend only on $j, P, Q$ and they vanish at $t'' = 0$.

To finish the proof of the proposition, we will not need the explicit expressions involving $DM_j^0, DN_j^0, Df_j, Dg_j$. We will use these derivatives in the proof of Lemma 6.1. So we derive derive these expression in this proof too.

We apply the projection (5.14)-(5.15) and (5.16)-(5.17) onto $\mathcal{C}_2^\varsigma(S)$, via monomials in each component of both sides of the identities. The images of the mappings

$$\begin{aligned}&(\xi,\eta) \mapsto (U(M(\xi\eta)\xi, N(\xi\eta)\eta), V(M(\xi\eta)\xi, N(\xi\eta)\eta)), \\ &(\xi,\eta) \mapsto (M^0(\xi\eta)U(\xi,\eta), N^0(\xi,\eta)V(\xi,\eta))\end{aligned}$$

under the projection are 0. We obtain from (5.14)-(5.17) and (5.18)-(5.19)

$$(\mu^{P-Q}-\mu_j)U_{j,PQ}=f_{j,PQ}+\mathcal{U}_{j,PQ}(\{M^0\}_{\frac{|P|+|Q|-1}{2}};\{f,U,V\}_{|P|+|Q|-1}), \tag{5.20}$$

$$(\mu^{Q-P}-\mu_j^{-1})V_{j,QP}=g_{j,QP}+\mathcal{V}_{j,QP}(\{N^0\}_{\frac{|P|+|Q|-1}{2}};\{g,U,V\}_{|P|+|Q|-1}) \tag{5.21}$$

for $\mu^{P-Q}\neq\mu_j$, which is always solvable. Next, we project (5.14)-(5.15) and (5.16)-(5.17) onto $\mathcal{C}_2(\hat{S})$, via monomials in each component of both sides of the identities. Using (5.13) we obtain

$$M_P=M_P^0+\mathcal{M}_P(\{M^0\}_{|P|-1};\{f,U,V\}_{2|P|-1}), \tag{5.22}$$

$$N_P=N_P^0+\mathcal{N}_P(\{N^0\}_{|P|-1};\{f,U,V\}_{2|P|-1}). \tag{5.23}$$

Here $\mathcal{M}_P,\mathcal{N}_P$ are polynomials of which each has coefficients that depend only on $P$, and $\{M^0\}_d$ stands for the set of coefficients $M_P^0$ with $|P|\leq d$. Note that $\mathcal{U}_{j,PQ}=\mathcal{V}_{j,QP}=0$ when $|P|+|Q|=2$, or $\mathrm{ord}(f,g)>|P|+|Q|$. And $\mathcal{M}_P=\mathcal{N}_P=0$ when $\mathrm{ord}(f,g)>2|P|$, by (5.13). Inductively, by using (5.20)-(5.21) and (5.22)-(5.23), we obtain unique solutions $U,V,M,N$. Moreover, the solutions and their dependence on the coefficients of $f,g$ and small divisors have the form

$$U_{j,PQ}=(\mu^{P-Q}-\mu_j)^{-1}\left\{f_{j,PQ}+\mathcal{U}^*_{j,PQ}(\delta_{d-1},\{M^0,N^0\}_{[\frac{d-1}{2}]};\{f,g\}_{d-1})\right\}, \tag{5.24}$$

$$V_{j,QP}=(\mu^{Q-P}-\mu_j^{-1})^{-1}\left\{g_{j,QP}+\mathcal{V}^*_{j,QP}(\delta_{d-1},\{M^0,N^0\}_{[\frac{d-1}{2}]};\{f,g\}_{d-1})\right\}, \tag{5.25}$$

where $d=|P|+|Q|$ and $\mu^{P-Q}\neq\mu_j$, and $\delta_{d-1}$ is the union of $\{\mu_1,\mu_1^{-1},\dots,\mu_p,\mu_p^{-1}\}$ and

$$\left\{\frac{1}{\mu^{A-B}-\mu_j}:\,|A|+|B|\leq d-1,A,B\in\mathbf{N}^p\right\}.$$

This shows that for any $M^0,N^0$ there exists a unique mapping $\Psi$ transforms $\sigma$ into $\sigma^*$. Furthermore, $\mathcal{U}^*_{j,PQ}(t';t''),\mathcal{V}^*_{j,QP}(t';t'')$ are polynomials of which each has coefficients that depend only on $j,P,Q$, and they vanish at $t''=0$.

For later purpose, let us express $M,N$ in terms of $f,g$. We substitute expressions (5.24)-(5.25) for $U,V$ in (5.22)-(5.23) to obtain

$$M_P=M_P^0+\mathcal{M}^*_P(\delta_{2|P|-1},\{M^0,N^0\}_{|P|-1};\{f,g\}_{2|P|-1}), \tag{5.26}$$

$$N_P=N_P^0+\mathcal{N}^*_P(\delta_{2|P|-1},\{M^0,N^0\}_{|P|-1};\{f,g\}_{2|P|-1}) \tag{5.27}$$

with $f,g$ satisfying (5.13).

Assume that $\tilde{\sigma}=\psi_0^{-1}\sigma^*\psi_0$ commutes with $\hat{S}$. By Corollary 4.9, we can decompose $\psi_0=HG^{-1}$ with $G\in\mathcal{C}(\hat{S})$ and $H\in\mathcal{C}^{\mathsf{c}}(\hat{S})$. Furthermore, $G^{-1}\tilde{\sigma}G$ commutes with $\hat{S}$ and $H^{-1}\sigma^*H$. By the uniqueness conclusion for the above $\psi_0$, $H$ must be the identity. This shows that $\psi_0\in\mathcal{C}(\hat{S})$.

(i). Assume that we have normalized $\sigma$. We now use it to normalize the pair of involutions. Assume that $\sigma=\tau_1\tau_2$ and $\tau_j^2=I$. Then $\sigma^*=\tau_1^*\tau_2^*$. Let $T_0(\xi,\eta):=(\eta,\xi)$. We have $T_0(\sigma^*)^{-1}T_0=T_0\tau_1^*\sigma^*\tau_1^*T_0$. By the above normalization, $T_0(\sigma^*)^{-1}T_0$ commutes with $\hat{S}$. Therefore, $\tau_1^*T_0$ belongs to the centralizer of $\hat{S}$ and it must be of the form $(\xi,\eta)\to(\xi\Lambda_1(\xi\eta),\eta\Lambda_1^*(\xi\eta))$. Then $(\tau_1^*)^2=I$ implies that

$$\Lambda_1(\xi\eta(\Lambda_1\Lambda_1^*)(\xi\eta))\Lambda_1^*(\xi\eta)=1.$$

The latter implies, by induction on $d>1$, that $\Lambda_1\Lambda_1^*=1+O(d)$ for all $d>1$, i.e. $\Lambda_1\Lambda_1^*=1$.

Let $\tau_i^*$ be given by (5.4). We want to achieve $\tilde\Lambda_{1j}\tilde\Lambda_{2j}=1$ for $\tilde\tau_i=\psi_0^{-1}\tau_i^*\psi_0$ by applying a transformation $\psi_0$ in $\mathcal{C}^{\mathsf{c}}(\hat T_1,\hat T_2)$ that commutes with $\hat S$. According to Definition 4.5, it has the form

$$\psi_0\colon \xi_j=\tilde\xi_j(1+a_j(\tilde\zeta)),\quad \eta_j=\tilde\eta_j(1-a_j(\tilde\zeta))$$

with $a_j(0)=0$. Here $\tilde\zeta_j:=\tilde\xi_j\tilde\eta_j$ and $\tilde\zeta:=(\tilde\zeta_1,\dots,\tilde\zeta_p)$. Computing the products $\zeta$ in $\tilde\zeta$ and solving $\tilde\zeta$ in $\zeta$, we obtain

$$\psi_0^{-1}\colon \tilde\xi_j=\xi_j(1+b_j(\zeta))^{-1},\quad \tilde\eta_j=\eta_j(1-b_j(\zeta))^{-1}.$$

Note that $(a_j^2)_P=\mathcal{A}_{j,P}(\{a\}_{|P|-1})$, and

$$\xi_j\eta_j=\tilde\xi_j\tilde\eta_j(1-a_j^2(\tilde\zeta)),\quad \tilde\xi_j\tilde\eta_j=\xi_j\eta_j(1-b_j^2(\zeta))^{-1}.$$

¿From $\psi_0^{-1}\psi_0=I$, we get

$$b_j(\zeta)=a_j(\tilde\zeta),\quad b_{j,P}=a_{j,P}+\mathcal{B}_{j,P}(\{a\}_{|P|-1}). \tag{5.28}$$

By a simple computation we see that $\tilde\tau_i=\psi_0^{-1}\tau_i^*\psi_0$ is given by

$$\tilde\xi_j'=\tilde\eta_j\tilde\Lambda_{ij}(\tilde\zeta),\quad \tilde\eta_j'=\tilde\xi_j\tilde\Lambda_{ij}^{-1}(\tilde\zeta)$$

with

$$\tilde\Lambda_{1j}\tilde\Lambda_{2j}(\tilde\zeta)=(\Lambda_{1j}\Lambda_{2j})(\zeta)(1+b_j(\zeta'))^{-2}(1-a_j(\tilde\zeta))^2.$$

Here $\zeta_j'=\zeta_j(1-a_j^2(\tilde\zeta))$. Using (5.28) and the implicit function theorem, we determine $a_j$ uniquely to achieve $\tilde\Lambda_{1j}\tilde\Lambda_{2j}=1$.

To identify the transformations that preserve the form of $\tilde\tau_1,\tilde\tau_2$, we first verify that each element $\psi_1\in\mathcal{C}(\hat T_1,\hat T_2)$ preserves that form. According to (4.11), we have

$$\begin{gathered}\psi_1\colon \xi_j=\tilde\xi_j\tilde a_j(\tilde\zeta),\quad \eta_j=\tilde\eta_j\tilde a_j(\tilde\zeta),\\ \psi_1^{-1}\colon \tilde\xi_j=\xi_j\tilde b_j(\zeta),\quad \tilde\eta_j=\eta_j\tilde b_j(\zeta),\\ \tilde b_j(\zeta)\tilde a_j(\tilde\zeta)=1.\end{gathered}$$

This shows that $\psi_1^{-1}\tilde\tau_i$ is given by

$$\tilde\xi_j'=\tilde\Lambda_{ij}(\zeta)\tilde b_j(\zeta)\eta_j,\quad \tilde\eta_j'=\tilde\Lambda_{ij}^{-1}(\zeta)\tilde b_j(\zeta)\xi_j.$$

Then $\psi_1^{-1}\tilde\tau_i\psi_1$ is given by

$$\tilde\xi_j'=\tilde\Lambda_{ij}(\zeta)\tilde\eta_j,\quad \tilde\eta_j'=\tilde\Lambda_{ij}^{-1}(\zeta)\tilde\xi_j.$$

Since $\zeta_j=\tilde\zeta_j\tilde a_j^2(\tilde\zeta)$, then $\psi_1^{-1}\tilde\tau_i\psi_1$ still satisfy (5.5). Conversely, suppose that $\psi_1$ preserves the forms of $\tilde\tau_1,\tilde\tau_2$. We apply Corollary 4.9 to decompose $\psi_1=\phi_1\phi_0^{-1}$ with $\phi_0\in\mathcal{C}(\hat T_1,\hat T_2)$ and $\phi_1\in\mathcal{C}^{\mathsf{c}}(\hat T_1,\hat T_2)$. Since we just proved that each element in $\mathcal{C}(\hat T_1,\hat T_2)$ preserves the form of $\tilde\tau_i$, then $\phi_1=\psi_1\phi_0$ also preserves the forms of $\tilde\tau_1,\tilde\tau_2$. On the other hand, we have shown that there exists a unique mapping in $\mathcal{C}^{\mathsf{c}}(\hat T_1,\hat T_2)$ which transforms $\{\tau_1^*,\tau_2^*\}$ into $\{\tilde\tau_1,\tilde\tau_2\}$. This shows that $\phi_0=I$. We have verified all assertions in (i).

(ii). According to Remark 4.6, $\mathcal{C}^{\mathsf{c}}(\hat S)$ and $\mathcal{C}^{\mathsf{c}}(\hat T_1,\hat T_2)$ are invariant under conjugacy by $\rho$. We have $\Psi^{-1}\sigma\Psi=\sigma^*$ and $\Psi\in\mathcal{C}^{\mathsf{c}}(\hat S)$. Note that $\rho\sigma\rho=\sigma^{-1}$ and $\rho\sigma^*\rho$ have the same form as of $(\sigma^*)^{-1}$, i.e. they are in $\mathcal{C}(\hat S)$ and have the same linear part. We have

$\rho\Psi\rho\sigma\rho\Psi^{-1}\rho = \rho(\sigma^*)^{-1}\rho$. The uniqueness of $\Psi$ implies that $\rho\Psi\rho = \Psi$ and $\tau_2^* = \rho\tau_1^*\rho$. Thus, we obtain relations (5.7)-(5.10). Analogously, $\rho\psi_0\rho$ is still in $\mathcal{C}^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$, and $\rho\phi_0\rho$ preserves the form of $\tilde{\tau}_1, \tilde{\tau}_2$. Thus $\rho\psi_0\rho = \psi_0$ and $\tilde{\tau}_2 = \rho\tilde{\tau}_1\rho$, which gives us (5.11). □

We will also need the following uniqueness result.

**Corollary 5.3.** *Suppose that $\sigma$ has linear part $\hat{S}$ with nonresonant $\mu_1, \dots, \mu_p$. Let $\Psi$ be the unique formal mapping in $\mathcal{C}^{\mathsf{c}}(\hat{S})$ such that $\Psi^{-1}\sigma\Psi \in \mathcal{C}(\hat{S})$. If $\tilde{\Psi} \in \mathcal{C}^{\mathsf{c}}(\hat{S})$ is a polynomial map of degree at most $d$ such that $\tilde{\Psi}^{-1}\sigma\tilde{\Psi}(\xi,\eta) = \tilde{\sigma}(\xi,\eta) + O(|(\xi,\eta)|^{d+1})$ and $\tilde{\sigma} \in \mathcal{C}(\hat{S})$, then $\tilde{\Psi}$ is unique. In fact, $\Psi - \tilde{\Psi} = O(d+1)$.*

*Proof.* The proof is contained in the proof of Proposition 5.2. Let us recap it by using (5.24)-(5.25) and the proposition. We take a unique normalized mapping $\Phi$ such that $\Phi^{-1}\tilde{\Psi}^{-1}\sigma\tilde{\Psi}\Phi \in \mathcal{C}(\hat{S})$. By (5.24)-(5.25), $\Phi = I + O(d+1)$. From Proposition 5.2 it follows that $\psi_0 := \tilde{\Psi}\Phi\Psi^{-1} \in \mathcal{C}(\hat{S})$. We obtain $\tilde{\Psi}\Phi = \psi_0\Psi$. Thus $\psi_0\Psi = \tilde{\Psi} + O(d+1)$. Since $\psi_0 \in \mathcal{C}(\hat{S})$, and $\Psi$, $\tilde{\Psi}$ are in $\mathcal{C}^{\mathsf{c}}(\hat{S})$, we conclude that $\Psi = \tilde{\Psi} + O(d+1)$. □

When $p = 1$, Proposition 5.2 is due to Moser and Webster. In fact, they achieved

$$\tilde{M}_1(\zeta_1) = e^{\delta(\xi_1\eta_1)^s}.$$

Here $\delta = 0, \pm 1$ for the elliptic case and $\delta = 0, \pm i$ for the hyperbolic case when $\mu_1$ is not a root of unity, i.e. $\gamma$ is *non-exceptional*. In particular the normal form is always convergent, although the normalizing transformations are generally divergent for the hyperbolic case.

Let us find out further normalization that can be performed to preserve the form of $\sigma^*$. In Proposition 5.2, we have proved that if $\sigma$ is tangent to $\hat{S}$, there exists a unique $\Psi \in \mathcal{C}^{\mathsf{c}}(\hat{S})$ such that $\Psi^{-1}\sigma\Psi$ is an element $\sigma^*$ in the centralizer of $\hat{S}$. Suppose now that $\sigma = \tau_1\tau_2$ while $\tau_i$ is tangent to $\hat{T}_i$. Let $\tau_i^* = \Psi^{-1}\tau_i\Psi$. We have also proved that there is a unique $\psi_0 \in \mathcal{C}^{\mathsf{c}}(\hat{T}_1, \hat{T}_2)$ such that $\tilde{\tau}_i = \psi_0^{-1}\tau_i^*\psi_0$, $i = 1, 2$, are of the form (5.5), i.e.

$$\tilde{\tau}_i \colon \xi_j' = \tilde{\Lambda}_{ij}(\zeta)\eta_j, \quad \eta_j' = \tilde{\Lambda}_{ij}^{-1}(\zeta)\xi_j;$$
$$\tilde{\sigma} \colon \xi_j' = \tilde{M}_j(\zeta)\xi_j, \quad \eta_j' = \tilde{M}_j^{-1}(\zeta)\eta_j.$$

Here $\zeta = (\xi_1\eta_1, \dots, \xi_p\eta_p)$, $\tilde{\Lambda}_{2j} = \tilde{\Lambda}_{1j}^{-1}$ and $\tilde{M}_j = \tilde{\Lambda}_{1j}^2$. We still have freedom to further normalize $\tilde{\tau}_1, \tilde{\tau}_2$ and to preserve their forms. However, any new coordinate transformation must be in $\mathcal{C}(\hat{T}_1, \hat{T}_2)$, i.e. it must have the form

$$\psi_1 \colon \xi_j \to a_j(\xi\eta)\xi_j, \quad \eta_j \to a_j(\xi\eta)\eta_j.$$

When $\tau_{2j} = \rho\tau_{1j}\rho$, we require that $\psi_1$ commutes with $\rho$, i.e.

$$a_e = \overline{a}_e, \quad a_h = \overline{a}_h, \quad a_s = \overline{a}_{s+s_*}.$$

In $\zeta$ coordinates, the transformation $\psi_1$ has the form

$$\varphi \colon \zeta_j \to b_j(\zeta)\zeta_j, \quad 1 \le j \le p \tag{5.29}$$

with $b_j = a_j^2$. Therefore, the mapping $\varphi$ needs to satisfy

$$b_e > 0, \quad b_h > 0, \quad b_s = \overline{b}_{s+s_*}.$$

Recall from (5.7)-(5.10) the reality conditions on $\tilde{M}_j$

$$\begin{aligned}
\overline{\tilde{M}_e \circ \rho_z} &= \tilde{M}_e, && 1 \le e \le e_*;\\
\overline{\tilde{M}_h \circ \rho_z} &= \tilde{M}_h^{-1}, && e_* < h \le h_* + e_*;\\
\tilde{M}_{s_*+s} &= \overline{\tilde{M}_s^{-1} \circ \rho_z}, && h_* + e_* < s \le p - s_*.
\end{aligned}$$

Here

$$\rho_z \colon \zeta_j \to \overline{\zeta}_j, \quad \zeta_s \to \overline{\zeta}_{s+s_*}, \quad \zeta_{s+s_*} \to \overline{\zeta}_s \tag{5.30}$$

for $1 \le j \le e_* + h_*$ and $e_* + h_* < s \le p - s_*$.

Therefore, our normal form problem leads to another normal form problem which is interesting in its own right. To formulate a new normalization problem, let us define

$$(\log \tilde{M})_j(\zeta) := \begin{cases} \log(\tilde{M}_j(\zeta)/\tilde{M}_j(0)), & 1 \le j \le e_*,\\ -i \log(\tilde{M}_j(\zeta)/\tilde{M}_j(0)), & e_* < j \le p. \end{cases} \tag{5.31}$$

Let $F = \log \tilde{M} := ((\log \tilde{M})_1, \dots, (\log \tilde{M})_p)$. Then the reality conditions on $\tilde{M}$ become

$$F = \rho_z F \rho_z. \tag{5.32}$$

The transformations (5.29) will then satisfy

$$\rho_z \varphi \rho_z = \varphi, \quad b_j(0) > 0, \quad 1 \le j \le e_* + h_*. \tag{5.33}$$

Therefore, when $F'(0)$ is furthermore diagonal and invertible and its $j$th diagonal entry is positive for $j = e, h$, we apply a dilation $\varphi$ satisfying the above condition so that $F$ is tangent to the identity. Then any further change of coordinates must be tangent to the identity too. Thus, we need to normalize the formal holomorphic mapping $F$ by composition $F \circ \varphi$, for which we study in next subsection.

5.2. **A normal form for maps tangent to the identity.** Let us consider a germ of holomorphic mapping $F(\zeta)$ in $\mathbf{C}^p$ with an invertible linear part $\mathbf{A}\zeta$ at the origin. According to the inverse function theorem, there exists a holomorphic mapping $\Psi$ with $\Psi(0) = 0$, $\Psi'(0) = I$ such that $F \circ \Psi(\zeta) = \mathbf{A}\zeta$. On the other hand, if we impose some restrictions on $\Psi$, we can no longer linearize $F$ in general.

To focus on applications to CR singularity and to limit the scope of our investigation, we now deliberately restrict our analysis to the simplest case : $F$ is tangent to the identity. We shall apply our result to $F = \log \tilde{M}$ as defined in the previous subsection. In what follows, we shall devise a normal form of such an $F$ under right composition by $\Psi$ that *preserve all coordinate hyperplanes*, i.e. $\Psi_j(\zeta) = \zeta_j \psi_j(\zeta)$, $j = 1, \dots, p$.

**Lemma 5.4.** *Let $F$ be a formal holomorphic map of $\mathbf{C}^p$ that is tangent to the identity at the origin.*

(*i*) *There exists a unique formal biholomorphic map $\psi$ which preserves all $\zeta_j = 0$ such that $\hat{F} := F \circ \psi$ has the form*

$$\hat{F} = I + \hat{f}, \quad \hat{f}(\zeta) = O(|\zeta|^2); \quad \partial_{\zeta_j} \hat{f}_j = 0, \quad 1 \le j \le p. \tag{5.34}$$

(*ii*) *If $F$ is convergent, the $\psi$ in (i) is convergent. If $F$ commutes with $\rho_z$, so does the $\psi$.*

(*iii*) *The formal normal form in (i) has the form*

$$\hat{f}_{j,Q} = f_{j,Q} + \mathcal{F}_{j,Q}(\{f\}_{|Q|-1}), \quad q_j = 0, \quad |Q| > 1. \tag{5.35}$$

*Here $\mathcal{F}_{j,Q}$ are universal polynomials depending only on $F'(0)$ and they vanish at 0.*

*Proof.* (i) Write $F = I + f$ and

$$\psi\colon \zeta_j' = \zeta_j + \zeta_j g_j(\zeta), \quad g_j(0) = 0.$$

For $\hat{F} = F \circ \psi$, we need to solve for $\hat{f}, g$ from

$$\hat{f}_j(\zeta) = \zeta_j g_j(\zeta) + f_j \circ \psi(\zeta).$$

Fix $Q = (q_1, \dots, q_p) \in \mathbf{N}^p$ with $|Q| > 1$. We obtain unique solutions

$$g_{j,Q-e_j} = -\{f_j(\psi(\zeta))\}_Q, \quad q_j > 0, \tag{5.36}$$

$$\hat{f}_{j,Q} := \{f_j(\psi(\zeta))\}_Q, \quad q_j = 0. \tag{5.37}$$

(ii) Assume that $F$ is convergent. Define $\overline{h}(\zeta) = \sum |h_Q| \zeta^Q$. We obtain for every multi-index $Q = (q_1, \dots, q_p)$ and for every $j$ satisfying $q_j \geq 1$

$$\overline{g}_{j,Q-e_j} \leq \left\{\overline{f_j}(\zeta_1 + \zeta_1\overline{g}_1(\zeta), \dots, \zeta_p + \zeta_p\overline{g}_p(\zeta))\right\}_Q.$$

Set $w(\zeta) = \sum \zeta_k \overline{g}_k(\zeta)$. We obtain

$$w(\zeta) \prec \sum \overline{f_j}(\zeta_1 + w(\zeta), \dots, \zeta_p + w(\zeta)).$$

Note that $f_j(\zeta) = O(|\zeta|^2)$ and $w(0) = 0$. By the Cauchy majorization and the implicit function theorem, $w$ and hence $g, \psi, \hat{f}$ are convergent.

Assume that $\rho_z F \rho_z = F$. Then $\rho_z L F \rho_z$ is normalized, $\rho_z \psi \rho_z$ is tangent to the identity, and the $j$th component of $\rho_z \hat{F} \rho_z(\zeta) - LF(\zeta)$ is independent of $\zeta_j$. Thus $\rho_z \psi \rho_z$ normalizes $F$ too. By the uniqueness of $\psi$, we obtain $\rho_z \psi \rho_z = \psi$.

(iii) By rewriting (5.37), we obtain

$$\hat{f}_{j,Q} = f_{j,Q} + \{f_j(\psi) - f_j\}_Q = f_{j,Q} + \mathcal{F}'_{j,Q}(\{f\}_{|Q|-1}, \{g\}_{|Q|-2}). \tag{5.38}$$

From (5.36), it follows that

$$g_{k,Q-e_k} = -f_{k,Q} + \mathcal{G}_{k,Q-e_k}(\{f\}_{|Q|-1}, \{g\}_{|Q|-2}), \quad |Q| > 1.$$

Note that $\{g\}_0 = 0$ and $\{f\}_1 = 0$. Using the identity repeatedly, we obtain $g_{k,Q-e_k} = -f_{k,Q} + \mathcal{G}^*_{k,Q-e_k}(\{f\}_{|Q|-1})$. Therefore, we can rewrite (5.38) as (5.35). □

5.3. **A unique formal normal form of a reversible map $\sigma$.** We now state a normal form for $\{\tau_1, \tau_2, \rho\}$ under a condition on the third-order invariants of $\sigma$.

**Theorem 5.5.** *Let $\tau_1$, $\tau_2$ be a pair of holomorphic involutions with linear parts $\hat{T}_i$. Let $\sigma = \tau_1\tau_2$. Assume that the linear part of $\sigma$ is*

$$\hat{S}\colon \xi_j' = \mu_j \xi_j, \quad \eta_j = \mu_j^{-1}\eta_j, \quad 1 \leq j \leq p$$

*and $\mu_1, \dots, \mu_p$ are non-resonant. Let $\Psi \in \mathcal{C}^{\mathsf{c}}(\hat{S})$ be the unique formal mapping such that*

$$\tau_i^* = \Psi^{-1}\tau_i\Psi\colon \xi_j' = \Lambda_{ij}(\xi\eta)\eta_j, \quad \eta_j' = \Lambda_{ij}(\xi\eta)^{-1}\xi_j;$$
$$\sigma^* = \Psi^{-1}\sigma\Psi\colon \xi_j' = M_j(\xi\eta)\xi_j, \quad \eta_j' = M_j(\xi\eta)^{-1}\eta_j$$

*with $M_j = \Lambda_{1j}\Lambda_{2j}^{-1}$. Suppose that $\sigma$ satisfies the condition that $\log M$ is tangent to the identity.*

*(i) Then there exists an invertible formal map $\psi_1 \in \mathcal{C}(\hat{S})$ such that*

$$\hat{\tau}_i = \psi_1^{-1}\tau_i^*\psi_1\colon \xi_j' = \hat{\Lambda}_{ij}(\xi\eta)\eta_j, \quad \eta_j' = \hat{\Lambda}_{ij}(\xi\eta)^{-1}\xi_j; \tag{5.39}$$

$$\hat{\sigma} = \psi_1^{-1}\sigma^*\psi_1\colon \xi_j' = \hat{M}_j(\xi\eta)\xi_j, \quad \eta_j' = \hat{M}_j(\xi\eta)^{-1}\eta_j. \tag{5.40}$$

*Here $\hat{\Lambda}_{2j} = \hat{\Lambda}_{1j}^{-1}$, and $\hat{T}_i$ is the linear part of $\hat{\tau}_i$. Moreover, $\log \hat{M}$ is tangent to the identity at the origin.*

*(ii) The centralizer of $\{\hat{\tau}_1, \hat{\tau}_2\}$ consists of $2^p$ dilations $(\xi, \eta) \to (a\xi, a\eta)$ with $a_j = \pm 1$. And $\hat{\Lambda}_{ij}$ are unique. If $\Lambda_{ij}$ are convergent, then $\psi_1$ is convergent too.*

*(iii) Suppose that $\hat{\sigma}$ is divergent. If $\sigma$ is formally equivalent to a mapping $\tilde{\sigma} \in \mathcal{C}(\hat{S})$ then $\tilde{\sigma}$ must be divergent too.*

*(iv) Let $\rho$ be given by (3.7) and let $\tau_2 = \rho\tau_1\rho$. Then the above $\Psi$ and $\psi_1$ commute with $\rho$. Moreover, $\hat{\tau}_i$, $\hat{\sigma}$ are unique.*

*Proof.* Assertions in (i) and (ii) are direct consequences of Proposition 5.2 and Lemma 5.4 in which $F$ is the $\tilde{M}$ in Proposition 5.2. The assertion on the centralizer of $\{\hat{\tau}_1, \hat{\tau}_2\}$ is obtained from (5.6) of Proposition 5.2 in which $\tilde{\Lambda}_{ij} = \hat{\Lambda}_{ij}$. Now (iii) follows from (ii) too. Indeed, suppose $\sigma$ is formally equivalent to some convergent

$$\tilde{\sigma}\colon \xi_j = \tilde{M}_j(\xi\eta)\xi_j, \quad \eta_j' = \tilde{M}_j(\xi\eta)^{-1}\eta_j.$$

Then by the assumption on the linear part of $\log M$, we can apply a dilation to achieve that $(\log \tilde{M})'(0)$ is tangent to the identity. By Lemma 5.4, there exists a unique convergent mapping $\varphi\colon \zeta_j' = b_j(\zeta)\zeta_j$ $(1 \le j \le p)$ with $b_j(0) = 1$ such that $\log \tilde{M} \circ \varphi$ is in the normal form $\log M_*$. Then

$$(\xi_j', \eta_j') = (b_j^{1/2}(\xi\eta)\xi_j, b_j^{1/2}(\xi\eta)\eta_j), \quad 1 \le j \le p$$

transforms $\tilde{\sigma}$ into a convergent mapping $\sigma_*$. Since the normal form for $\log M$ is unique, then $\hat{\sigma} = \sigma_*$. In particular, $\hat{\sigma}$ is convergent.

(iv). Note that $\rho\sigma\rho = \sigma^{-1}$. Also $\rho(\sigma^*)^{-1}\rho$ has the same form as $\sigma^*$. By $(\rho\Psi^{-1}\rho)\sigma(\rho\Psi\rho) = (\rho\sigma^*\rho)^{-1}$, we conclude that $\rho\Psi\rho = \Psi$. The rest of assertions can be verified easily. □

Under the condition that $\log M$ is tangent to the identity, the above theorem completely settles the formal classification of $\{\tau_1, \tau_2, \rho\}$. It also says that **the normal form $\hat{\tau}_1, \hat{\tau}_2$ can be achieved by a convergent transformation, if and only if $\sigma^*$ can be achieved by some convergent transformation**, i.e. the $\Psi$ in the theorem is convergent.

However, we would like state clear that our results do not rule out the case where a refined normal form for $\{\tau_1^*, \tau_2^*, \rho\}$ is achieved by convergent transformation, while $\Psi$ is divergent, when $\log M$ is tangent to the identity.

5.4. **An algebraic manifold with linear $\sigma$.** We conclude the section showing that when $\tau_1, \tau_2$ are normalized as in this section, $\{\tau_{ij}\}$ might still be very general; in particular $\{\tau_{1j}, \rho\}$ cannot always be simultaneously linearized even at the formal level. This is one of main differences between $p = 1$ and $p > 1$.

**Example 5.6.** Let $p = 2$. Let $\phi$ be a holomorphic mapping of the form

$$\phi\colon \xi_i' = \xi_i + q_i(\xi, \eta), \quad \eta_i' = \eta_i + \lambda_i^{-1} q_i(T_1(\xi, \eta)),\ i = 1, 2.$$

Here $q_i$ is a homogeneous quadratic polynomial map and

$$T_1(\xi, \eta) = (\lambda_1\eta_1, \lambda_2\eta_2, \lambda_1^{-1}\xi_1, \lambda_2^{-1}\xi_2).$$

Let $\tau_{1j} = \phi T_{1j}\phi^{-1}$ and $\tau_{2j} = \rho\tau_{1j}\rho$. Then $\phi$ commutes with $T_1$ and $\tau_1 = T_1$. In particular $\tau_2 = \rho T_1 \rho$ and $\sigma = \tau_1\tau_2$ are in linear normal forms. However, $\tau_{11}$ is given by

$$\begin{aligned}
\xi_1' &= \lambda_1\eta_1 - q_1(\lambda\eta, \lambda^{-1}\xi) + q_1(\lambda_1\eta_1, \xi_2, \lambda_1^{-1}\xi_1, \eta_2) + O(3),\\
\xi_2' &= \xi_2 - q_2(\xi, \eta) + q_2(\lambda_1\eta_1, \xi_2, \lambda_1^{-1}\xi_1, \eta_2) + O(3),\\
\eta_1' &= \lambda_1^{-1}\xi_1 - \lambda_1^{-1} q_1(\xi, \eta) + \lambda_1^{-1} q_1(\xi_1, \lambda_2\eta_2, \eta_1, \lambda_2^{-1}\xi_2) + O(3),\\
\eta_2' &= \eta_2 - \lambda_2^{-1} q_2(\lambda\eta, \lambda^{-1}\xi) + \lambda_2^{-1} q_2(\xi_1, \lambda_2\eta_2, \eta_1, \lambda_2^{-1}\xi_2) + O(3).
\end{aligned}$$

Notice that the common zero set $V$ of $\xi_1\eta_1$ and $\xi_2\eta_2$ is invariant under $\tau_1, \tau_2, \sigma$ and $\rho$. In fact, they are linear on $V$. However, for $(\xi', \eta') = \tau_{11}(\xi, \eta)$, we have

$$\begin{aligned}
\xi_1'\eta_1' = &-\eta_1 q_1(0, \xi_2, \eta) + \eta_1 q_1(0, \lambda_2\eta_2, \eta_1, \lambda_2^{-1}\xi_2) - \lambda_1^{-1}\xi_1 q_1(0, \lambda_2\eta_2, \lambda^{-1}\xi)\\
&+ \lambda_1^{-1}\xi_1 q_1(0, \xi_2, \lambda_1^{-1}\xi_1, \eta_2) \mod (\xi_1\eta_1, \xi_2\eta_2, O(4)).
\end{aligned}$$

For a generic $q$, $\tau_{11}$ does not preserve $V$.

By a simple computation, we can verify that $\sigma_j = \tau_{1j}\tau_{2j}$ for $j = 1, 2$ do not commute with each other. In fact, we will prove in section 9 that if the $\mu_1, \ldots, \mu_p$ are nonresonant, $\sigma_j$ commute pairwise, and $\sigma$ is linear as above, then $\tau_{1j}$ must be linear.

## 6. Divergence of all normal forms of a reversible map $\sigma$

Unlike the Birkhoff normal form for a Hamiltonian system, the Poincaré-Dulac normal form is not unique for a general $\sigma$; it just belongs to the centralizer of the linear part $S$ of $\sigma$. One can obtain a divergent normal form easily from any non-linear Poincaré-Dulac normal form of $\sigma = \tau_1\tau_2$ by conjugating with a divergent transformation in the centralizer of $S$; see (5.6). We have seen how the small divisors enter in the computation of the normalizing transformations via (5.24)-(5.25), but they have not yet appeared in (5.22)-(5.23) in the computation of the normal forms. To see the effect of small divisors on normal forms, we first assume a condition, to be achieved later, on the third order invariants of $\sigma$ and then we shall need to modify the normalization procedure. We will use two sequences of normalizing mappings to normalize $\sigma$. The composition of normalized mappings might not be normalized. Therefore, the new normal form $\tilde\sigma$ might not be the $\sigma^*$ in Proposition 5.2. We will show that this $\tilde\sigma$, after it is transformed into the normal form $\hat\sigma$ in Theorem 5.5 (i), is divergent. Using the divergence of $\hat\sigma$, we will then show that any other normal forms of $\sigma$ that are in the centralizer of $S$ must be divergent too. This last step requires a convergent solution given by Lemma 5.4.

Our goal is to see a small divisor in a normal form $\tilde{\sigma}$; however they appear as a product. This is more complicated than the situation for the normalizing transformations, where a small divisor appears in a much simple way. In essence, a small divisor problem occurs naturally when one applies a Newton iteration scheme for a convergence proof. For a small divisor to show up in the normal form, we have to go beyond the Newton iteration scheme, measured in the degree or order of approximation in power series. Therefore, we first refine the formulae (5.22).

**Lemma 6.1.** *Let $\sigma$ be a holomorphic mapping, given by*

$$\xi_j' = M_j^0(\xi\eta)\xi_j + f_j(\xi,\eta), \quad \eta_j' = N_j^0(\xi\eta)\eta_j + g_j(\xi,\eta), \quad 1 \le j \le p.$$

*Here $M_j^0(0) = \mu_j = N_j^0(0)^{-1}$. Suppose that $\operatorname{ord}(f,g) \ge d$, $d \ge 4$, and $I + (f,g) \in \mathcal{C}^{\mathsf{c}}(S)$. Assume that $\mu_1, \dots, \mu_p$ are non-resonant. Assume that*

$$(M^0)'(0) = \operatorname{diag}(\mu_1, \dots, \mu_p).$$

*There exist unique polynomials $U, V$ of degree at most $2d-1$ such that $\Psi = I + (U,V) \in \mathcal{C}^{\mathsf{c}}(S)$ transforms $\sigma$ into*

$$\sigma^*\colon \xi' = M(\xi\eta)\xi + \tilde{f}(\xi,\eta), \quad \eta' = N(\xi\eta)\eta + \tilde{g}(\xi,\eta)$$

*with $I + (\tilde{f},\tilde{g}) \in \mathcal{C}^{\mathsf{c}}(S)$ and $\operatorname{ord}(\tilde{f},\tilde{g}) \ge 2d$. Moreover,*

$$U_{j,PQ} = (\mu^{P-Q} - \mu_j)^{-1}\left\{ f_{j,PQ} + \mathcal{U}^*_{j,PQ}(\delta_{\ell-1}, \{M^0,N^0\}_{[\frac{\ell-1}{2}]}; \{f,g\}_{\ell-1})\right\}, \tag{6.1}$$

$$V_{j,QP} = (\mu^{Q-P} - \mu_j^{-1})^{-1}\left\{ g_{j,QP} + \mathcal{V}^*_{j,QP}(\delta_{\ell-1}, \{M^0,N^0\}_{[\frac{\ell-1}{2}]}; \{f,g\}_{\ell-1})\right\}, \tag{6.2}$$

*for $2 \le |P| + |Q| = \ell \le 2d-1$ and $\mu^{P-Q} \ne \mu_j$. In particular, $\operatorname{ord}(U,V) \ge d$. For $|P| = d$ and $|P'| < d$,*

$$M_{j,P'} = M^0_{j,P'}, \tag{6.3}$$

$$M_{j,P} = M^0_{j,P} + \mu_j\left\{2(U_jV_j)_{PP} + (U_j^2)_{(P+e_j)(P-e_j)}\right\} + \{Df_j(\xi,\eta)(U,V)\}_{(P+e_j)P}. \tag{6.4}$$

**Remark 6.2.** Formula (6.4) gives us an effective way to compute the Poincaré-Dulac normal form. It tells us that under the above conditions, the coefficients of $M_{jP}(\xi\eta)\xi_j$ of degree $2|P|+1$ do not depend on coefficients of $f(\xi,\eta), g(\xi,\eta)$ of degree $\ge 2|P|$, if $2|P| > 3$.

*Proof.* Identities (6.1)-(6.3) follow directly from (5.24)-(5.26), where by notation in Definition 5.1

$$\mathcal{U}^*_{j,PQ}(\cdot;0) = \mathcal{V}^*_{j,QP}(\cdot;0) = \mathcal{M}^*_P(\cdot;0) = 0.$$

Let $D_i$ denote $\partial_{\zeta_i}$. Let $Du(\xi,\eta)$ and $Dv(\zeta)$ denote the gradients of two functions. The right-hand sides of (5.14) and (5.16) give us

$$\begin{aligned} M_j(\xi\eta)\xi_j + U_j(M(\xi\eta)\xi, N(\xi\eta)\eta) &= f_j(\xi,\eta) + Df_j(\xi,\eta)(U,V) + A_j(\xi,\eta) \\ &\quad + (M_j^0(\xi\eta) + DM_j^0(\xi\eta)(\eta U + \xi V + UV))(\xi_j + U_j). \end{aligned} \tag{6.5}$$

We recall from (5.18) the remainders

$$A_j(\xi,\eta) = R_2M_j^0(\xi\eta; \xi U + \eta V + UV)(\xi_j + U_j) + R_2f_j(\xi,\eta;U,V).$$

Here by (5.12), we have the Taylor remainder formula

$$R_2 f(x;y) = 2\int_0^1 (1-t)\sum_{|\alpha|=2}\frac{1}{\alpha!}\partial^\alpha f(x+ty)y^\alpha\,dt.$$

Since $\operatorname{ord}(U,V)\geq d$, $\operatorname{ord}(f,g)\geq d$, and $d\geq 4$, then $A_j$, defined by (5.18), satisfies

$$A_j(\xi,\eta) = O(|(\xi,\eta)|^{2d+2}).$$

Recall that $f_j(\xi,\eta)$ and $U_j(\xi,\eta)$ do not contain terms of the form $\xi_j\xi^P\eta^P$, while $g_j(\xi,\eta)$ and $V_j(\xi,\eta)$ do not contain terms of the form $\eta_j\xi^P\eta^P$. Assume that $i\neq j$. Then $D_iM_j^0(\xi\eta) = O(|\xi\eta|)$. We see that $D_iM_j^0(\xi\eta)\eta_iU_i(\xi,\eta)$ and $D_iM_j^0(\xi\eta)\xi_iV_i(\xi,\eta)$ do not contain terms of $\xi^P\eta^P$, and

$$D_iM_j^0(\xi\eta)\xi_iU_i(\xi,\eta)V_i(\xi,\eta) = O(2d+3).$$

Comparing both sides of (6.5) for coefficients of $\xi_j\xi^P\eta^P$, we get (6.4). □

Set $|\delta_N(\mu)| := \max\{|\nu|\colon \nu\in\delta_N(\mu)\}$ for

$$\delta_N(\mu) = \bigcup_{j=1}^{p}\left\{\mu_i,\mu_i^{-1},\frac{1}{\mu^P-\mu_j}\colon P\in\mathbf{Z}^p, P\neq e_j, |P|\leq N\right\}. \tag{6.6}$$

**Definition 6.3.** We say that $\mu^{P-Q}-\mu_j$ and $\mu^{Q-P}-\mu_j^{-1}$ are small divisors of *height* $N$, if there exists a partition

$$\bigcup_j\left\{|\mu^{P-Q}-\mu_j|\colon P,Q\in\mathbf{N}^p, |P|+|Q|\leq N, \mu^{P-Q}\neq\mu_j\right\} = S_N^0\cup S_N^1$$

with $|\mu^{P-Q}-\mu_j|\in S_N^0$ and $S_N^1\neq\emptyset$ such that

$$\max S_N^0 < C\min S_N^0,$$
$$\max S_N^0 < (\min S_N^1)^{L_N} < 1.$$

Here $C$ depends only on an upper bound of $|\mu|$ and $|\mu|^{-1}$ and

$$L_N\geq N.$$

If $|\mu^{P-Q}-\mu_j|$ is in $S_N^0$ and if $P,Q\in\mathbf{N}^p$, we call $|P-Q|$ the *degree* of the small divisors $\mu^{P-Q}-\mu_j$ and $\mu^{Q-P}-\mu_j^{-1}$.

To avoid confusion, let us call $\mu^{P-Q}-\mu_j$ that appear in $S_N^0$ the *exceptional* small divisors. These small divisors have played important roles in Siegel's works [Sie54, Sie41]. Siegel's small divisors technic was extended to a construction of divergent Birkhoff normal form in [Gon12] (see also [PM03] for related problems). The degree and height play different roles in computation. The height serves as the maximum degree of all small divisors that need to be considered in computation.

Roughly speaking, the quantities in $S_N^0$ are comparable but they are much smaller than the ones in $S_N^1$. We will construct $\mu$ for any prescribed sequence of positive integers $L_N$ so that

$$\max S_N^0 < (\min S_N^1)^{L_N} < 1$$

for a subsequence $N = N_k$ tending to $\infty$. Furthermore, to use the small divisors we will identify all exceptional small divisors of height $2N_k + 1$ and all degrees of the exceptional small divisors with $N_k$ being the smallest.

We start with the following lemma which gives us small divisors that decay as rapidly as we wish.

**Lemma 6.4.** *Let $L_k$ be an increasing sequence of positive integers such that $L_k$ tends to $\infty$ as $k \to \infty$. There exist a real number $\nu \in (0, 1/2)$ and a sequence $(p_k, q_k) \in \mathbf{N}^2$ such that $e, 1, \nu$ are linearly independent over $\mathbf{Q}$, and*

$$|q_k\nu - p_k - e| \le \Delta(p_k, q_k)^{(p_k+q_k)L_k}, \tag{6.7}$$

$$\Delta(p_k, q_k) = \min\Big\{\frac{1}{2}, |q\nu - p - re| \colon 0 < |r| + |q| < 3(q_k + 1), \\ (p, q, r) \ne 0, \pm(p_k, q_k, 1), \pm 2(p_k, q_k, 1)\Big\}. \tag{6.8}$$

*Proof.* We consider two increasing sequences $\{m_k\}_{k=1}^\infty, \{n_k\}_{k=1}^\infty$ of positive integers, which are to be chosen. For $k = 1, 2, \ldots$, we set

$$\nu = \nu_k + \nu_k', \quad \nu_k = \sum_{\ell=1}^{k} \frac{1}{m_\ell!} \sum_{j=0}^{n_\ell} \frac{1}{j!}, \quad \nu_k' = \sum_{\ell>k} \frac{1}{m_\ell!} \sum_{j=0}^{n_\ell} \frac{1}{j!},$$
$$q_k = m_k!.$$

We choose $m_k > (m_\ell)!(n_\ell!)$ for $\ell < k$ and decompose

$$q_k\nu = p_k + e_k + e_k',$$
$$p_k = m_k!\nu_{k-1} \in \mathbf{N}, \quad e_k = \sum_{\ell=0}^{n_k} \frac{1}{k!}, \quad e_k' = m_k!\nu_k'.$$

We have $e_k' < m_k! \sum_{\ell>k} \frac{e}{m_\ell!}$ and

$$q_k\nu = p_k + e + e_k' - \sum_{\ell=n_k+1}^{\infty} \frac{1}{\ell!},$$

$$|q_k\nu - p_k - e| \le m_k!\nu_k' + \sum_{\ell=n_k+1}^{\infty} \frac{1}{\ell!} < \big\{12(3(q_k+1)^3)!\big\}^{-(p_k+q_k)L_k}. \tag{6.9}$$

Here $(6.9)_k$ is achieved by choosing $(m_2, n_1)$, $\ldots$, $(m_{k+1}, n_k)$ successively. Clearly we can get $0 < \nu < 1/2$ if $m_1$ is sufficiently large.

Next, we want to show that $re + p + q\nu \ne 0$ for all integers $p, q, r$ with $(p, q, r) \ne (0, 0, 0)$. Otherwise, we rewrite $-m_k!p = m_k!(q\nu + re)$ as

$$-m_k!p = qp_k + r\sum_{j=0}^{m_k} \frac{m_k!}{j!} + qe + q\left(e_k' - \sum_{\ell=n_k+1}^{\infty} \frac{1}{\ell!}\right) + r\sum_{j>m_k} \frac{m_k!}{j!}.$$

The left-hand side is an integer. On the right-hand side, the first two terms are integers, $qe$ is a fixed irrational number, and the rest terms tend to 0 as $k \to \infty$. We get a contradiction.

To verify (6.7), we need to show that for each tuple $(p, q, r)$ satisfying (6.8),

$$|q\nu - p - re| \geq |q_k\nu - p_k - e|^{\frac{1}{(p_k+q_k)L_k}}. \tag{6.10}$$

We first note the following elementary inequality

$$|p + qe| \geq \frac{1}{(q-1)!} \min\left\{3 - e, \frac{1}{q+1}\right\}, \quad p, q \in \mathbf{Z}, \quad q \geq 1. \tag{6.11}$$

Indeed, the inequality holds for $q = 1$. For $q \geq 2$ we have $q!e = m + \epsilon$ with $m \in \mathbf{N}$ and

$$\epsilon := \sum_{k=q+1}^{\infty} \frac{q!}{k!} > \frac{1}{q+1}.$$

Furthermore, $1 - \epsilon > 1 - \frac{2}{q+1} = \frac{q-1}{q+1}$ as

$$\epsilon < \frac{1}{q+1} + \sum_{k \geq q+2} \frac{1}{k(k+1)} = \frac{2}{q+1}.$$

We may assume that $q \geq 0$. If $q = 0$, then $|r| < 3q_k + 3$ and hence $|p + re| \geq \frac{1}{(3q_k+4)!}$. Now (6.10) follows from (6.9). Assume that $q > 0$. We have

$$\begin{aligned} | - q\nu + p + re| &\geq | - q\frac{p_k + e}{q_k} + p + re| - q\frac{|e + p_k - q_k\nu|}{q_k} \\ &= \left|\frac{q_kp - qp_k}{q_k} + \frac{rq_k - q}{q_k}e\right| - q\frac{|e + p_k - q_k\nu|}{q_k}. \end{aligned} \tag{6.12}$$

We first verify that $q_kp - qp_k$ and $q - rq_k$ do not vanish simultaneously. Assume that both are zero. Then $(p, q, r) = r(p_k, q_k, 1)$. Thus $|r| \neq 1, 2$, and $|r| \geq 3$ by conditions in (6.8); we obtain $|r| + |q| \geq 3(|q_k| + 1)$, a contradiction. Therefore, either $q_kp - qp_k$ or $rq_k - q$ is not zero. By (6.11) and (6.12),

$$\begin{aligned} | - q\nu + p + re| &\geq \frac{1}{q_k} \cdot \frac{1}{3} \cdot \frac{1}{(|rq_k - q| + 1)!} - q\frac{|e + p_k - q_k\nu|}{q_k} \\ &\geq \frac{1}{(3q_k + 4)^2!} - 4|e + p_k - q_k\nu|. \end{aligned}$$

Using (6.9) twice, we obtain the next two inequalities:

$$| - q\nu + p + re| \geq \frac{1}{2}\left\{(3q_k + 4)^2!\right\}^{-1} \geq |p_k + e - q_k\nu|^{\frac{1}{(p_k+q_k)L_k}}.$$

The two ends give us (6.10). □

We now reformulate the above lemma as follows.

**Lemma 6.5.** *Let $L_k$ be an increasing sequence of positive integers such that $L_k$ tends to $+\infty$ as $k \to \infty$. Let $\nu \in (0, 1/2)$, and let $p_k$ and $q_k$ be positive integers as in Lemma* 6.4.

*Set* $(\mu_1, \mu_2, \mu_3) := (e^{-1}, e^{\nu}, e^{e})$. *Then*

$$|\mu^{P_k} - \mu_3| \le (C\Delta^*(P_k))^{|P_k|L_k}, \quad P_k = (p_k, q_k, 0), \tag{6.13}$$

$$\Delta^*(P_k) = \min_j \Big\{ |\mu^R - \mu_j| \colon R \in \mathbf{Z}^3, |R| \le 2(q_k + p_k) + 1, \\ R - e_j \ne 0, \pm(p_k, q_k, -1), \pm 2(p_k, q_k, -1) \Big\}. \tag{6.14}$$

*Here* $C$ *does not depend on* $k$. *Moreover, all exceptional small divisors of height* $2|P_k| + 1$ *have degree at least* $|P_k|$. *Moreover,* $\mu^{P_k} - \mu_3$ *is the only exceptional small divisor of degree* $|P_k|$ *and height* $2|P_k| + 1$.

In the definition of $\Delta^*(P_k)$, equivalently we require that

$$R \ne P_k,\ R_k^1,\ R_k^2,\ R_k^3$$

with $R_k^1 := -P_k + 2e_3, R_k^2 := 2P_k - e_3$, and $R_k^3 := -2P_k + 3e_3$. Note that $|R_k^1| = |P_k| + 2, |R_k^2| = 2|P_k| + 1$, and $|R_k^3| = 2|P_k| + 3$ are bigger than $|P_k|$, i.e. the degree of the exceptional small divisor $\mu^{P_k} - \mu_3$. Each $\mu^{R_k^i} - \mu_3$ is a small divisor comparable with $\mu^{P_k} - \mu_3$. Finally, $\Delta^*(P_k)$ tends to zero as $|P_k| \to \infty$. Let us set $N := 2|P_k| + 1$, and

$$S_N^0 := \left\{ |\mu^{P_k} - \mu_3|, |\mu^{R_k^1} - \mu_3|, |\mu^{R_k^2} - \mu_3|, |\mu^{R_k^3} - \mu_3| \right\},$$

$$S_N^1 := \bigcup_j \Big\{ |\mu^R - \mu_j| \colon R \in \mathbf{Z}^3, |R| \le 2(q_k + p_k) + 1, \\ R - e_j \ne 0,\ \pm(p_k, q_k, -1),\ \pm 2(p_k, q_k, -1) \Big\}.$$

This implies that the last paragraph of Lemma 6.5 holds when the $L_N$ in Definition 6.3, denoted it by $L'_N$, takes the value $L'_N = \frac{1}{2}|P_k|L_k$ and $k$ is sufficiently large, while $L_k$ is given in Lemma 6.5.

*Proof.* By Lemma 6.4, we find a real number $\nu \in (0, 1/2)$ and positive integers $p_k, q_k$ such that $e, 1, \nu$ are linearly independent over $\mathbf{Q}$ and

$$|q_k\nu - e - p_k| \le \Delta(p_k, q_k)^{|P_k|L_k}, \tag{6.15}$$

$$\Delta(p_k, q_k) = \min \{ |q\nu - re - p| \colon 0 < |r| + |q| < 3(q_k + 1), \\ (p, q, r) \ne 0, \pm(p_k, q_k, 1), \pm 2(p_k, q_k, 1) \}.$$

Note that $\mu_1, \mu_2, \mu_3$ are positive and non-resonant. We have

$$|\mu^{P_k} - \mu_3| = |\mu_3| \cdot |e^{q_k\nu - p_k - e} - 1|.$$

Let $\nu^* := (-1, \nu, e)$. If $|R \cdot \nu^* - \nu_j^*| < 2$, then by the intermediate value theorem

$$e^{-2}|\mu_j||R \cdot \nu^* - \nu_j^*| \le |\mu^R - \mu_j| \le e^2|\mu_j||R \cdot \nu^* - \nu_j^*|.$$

If $R \cdot \nu^* - \nu_j^* > 2$ or $R \cdot \nu^* - \nu_j^* < -2$, we have

$$|\mu^R - \mu_j| \ge e^{-2}|\mu_j|.$$

Thus, we can restate the properties of $\nu^*$ as follows:

$$|\mu^{-(p_k,q_k,0)} - \mu_3| \le C'(C'\tilde{\Delta}(p_k,q_k))^{|P_k|L_k},$$
$$\tilde{\Delta}(p_k,q_k) = \min\left\{|\mu^{(p,q,r)} - 1| \colon 0 < |r| + |q| < 3(q_k+1), \right.$$
$$\left. (p,q,r) \neq 0, \pm(p_k,q_k,-1), \pm 2(p_k,q_k,-1)\right\}.$$

Recall that $0 < \nu < 1/2$. By (6.15), we have $|q_k\nu - e - p_k| < 1$. Since $p_k, q_k$ are positive, then $p_k < \nu q_k < q_k/2$. Assume that $|\mu^R - \mu_j| = \Delta^*(P_k)$, $|R| \le 2(p_k+q_k)+1$, and

$$R - e_j \neq 0, \pm(p_k,q_k,-1), \pm 2(p_k,q_k,-1).$$

Set $R' := R - e_j$ and $(p,q,r) := R'$. Then $\Delta^*(P_k) = |\mu_j||\mu^{R'} - 1|$. Also, $|r| + |q| \le |R'| \le |R| + 1 \le 2(p_k+q_k)+2 \le q_k + 2q_k + 2 < 3(q_k+1)$. This shows that $|\mu^{Q'} - 1| \ge \tilde{\Delta}(p_k,q_k)$. We obtain $\Delta^*(P_k) \ge \mu_j\Delta(p,q_k)$. We have verified (6.13). For the remaining assertions, see the remark following the lemma. □

In the above we have retained $\mu_j > 0$ which are sufficient to realize $\mu_1, \mu_2, \mu_3, \mu_1^{-1}, \mu_2^{-1}, \mu_3^{-1}$ as eigenvalues of $\sigma$ for an elliptic complex tangent. Indeed, with $0 < \mu_1 < 1$, interchanging $\xi_1$ and $\eta_1$ preserves $\rho$ and changes the $(\xi_1, \eta_1)$ components of $\sigma$ into $(\mu_1^{-1}\xi_1, \mu_1\eta_1)$.

We are ready to prove Theorem 1.4, which is restated here:

**Theorem 6.6.** *There exists a non-resonant elliptic real analytic 3-submanifold $M$ in $\mathbf{C}^6$ such that $M$ admits the maximum number of deck transformations and all Poincaré-Dulac normal forms of the $\sigma$ associated to $M$ are divergent.*

*Proof.* We will not construct the real analytic submanifold $M$ directly. Instead, we will construct a family of involutions $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$ so that all Poincaré-Dulac normal forms of $\sigma$ are divergent. By the realization in Proposition 2.10, we get the desired submanifold.

We first give an outline of the proof. To prove the theorem, we first deal with the associated $\sigma$ and its normal form $\tilde{\sigma}$, which belongs to the centralizer of $S$, the linear part of $\sigma$ at the origin. Thus $\sigma^*$ has the form

$$\sigma^* \colon \xi' = M(\xi\eta)\xi, \quad \eta' = N(\xi\eta)\eta.$$

We assume that $\log M$ is tangent to identity at the origin. We then normalize $\sigma^*$ into the normal form $\hat{\sigma}$ stated in Theorem 5.5 (i). (In Lemma 6.1 we take $F = \log M$ and $\hat{F} = \log \hat{M}$.) We will show that $\hat{\sigma}$ is divergent if $\sigma$ is well chosen. By Theorem 5.5 (iii), all normal forms of $\sigma$ in the centralizer of $S$ are divergent. To get $\sigma^*$, we use the normalization of Proposition 5.2 (i). To get $\hat{\sigma}$, we normalize further using Lemma 5.4. To find a divergent $\hat{\sigma}$, we need to tie the normalizations of two formal normal forms together, by keeping track of the small divisors in the two normalizations.

We will start with our initial pair of involutions $\{\tau_1^0, \tau_2^0\}$ satisfying $\tau_2^0 = \rho\tau_1^0\rho$ such that $\sigma^0$ is a third order perturbation of $S$. We require that $\tau_1^0$ be the composition of $\tau_{11}^0, \ldots, \tau_{1p}^0$. The latter can be realized by a real analytic submanifold by using Proposition 2.10. We will then perform a sequence of holomorphic changes of coordinates $\varphi_k$ such that $\tau_1^k = \varphi_k^{-1}\tau_1^{k-1}\varphi_k$, $\tau_2^k = \rho\tau_1^k\rho$, and $\sigma^k = \tau_1^k\tau_2^k$. Each $\varphi_k$ is tangent to the identity to order $d_k$. For a suitable choice of $\varphi_k$, we want to show that the coefficients of order $d_k$ of the normal form of $\sigma^k$ increase rapidly to the effect that the coefficients of the normal form of the limit mapping

$\sigma^\infty$ increase rapidly too. Here we will use the exceptional small divisors to achieve the rapid growth of the coefficients of the normal forms.

We now present the proof. Let $\sigma^0 = \tau_1^0\tau_2^0$, $\tau_2^0 = \rho\tau_1^0\rho$, and

$$\begin{aligned} \tau_1^0\colon\ & \xi_j' = \Lambda_{1j}^0(\xi\eta)\eta_j, \qquad \eta_j' = (\Lambda_{1j}^0(\xi\eta))^{-1}\xi_j, \\ \sigma^0\colon\ & \xi_j' = (\Lambda_{1j}^0(\xi\eta))^2\xi_j, \quad \eta_j' = (\Lambda_{1j}^0(\xi\eta))^{-2}(\xi\eta)\eta_j. \end{aligned}$$

Since we consider the elliptic case, we require that $(\Lambda_{1j}^0(\xi\eta))^2 = \mu_j e^{\xi_j\eta_j}$. So $\zeta \to (\Lambda_1^0)^2(\zeta)$ is biholomorphic. Recall that $\sigma^0$ can be realized by $\{\tau_{11}^0, \dots, \tau_{1p}^0, \rho\}$. We will take

$$\varphi_k\colon \xi_j' = (\xi + h^{(k)}(\xi), \eta), \quad \operatorname{ord} h^{(k)} = d_k > 3, \tag{6.16}$$

$$d_k \geq 2d_{k-1}, \quad |h_P^{(k)}| \leq 1. \tag{6.17}$$

We will also choose each $h_j^{(k)}(\xi)$ to have one monomial only. Let $\Delta_r := \Delta_r^3$ denote the polydisc of radius $r$. Let $\|\cdot\|$ be the sup norm on $\mathbf{C}^3$. Let $H^{(k)}(\xi) = \xi + h^{(k)}(\xi)$ and we first verify that $H_k = H^{(k)} \circ \cdots \circ H^{(1)}$ converges to a holomorphic function on the polydisc $\Delta_{r_1}$ for $r_1 > 0$ sufficiently small; consequently, $\varphi_k \circ \cdots \circ \varphi_1$ converges to a germ of holomorphic map at the origin. Note that $H^{(k)}$ sends $\Delta_{r_k}$ into $\Delta_{r_{k+1}}$ for $r_{k+1} = r_k + r_k^{d_k}$. We want to show that when $r_1$ is sufficiently small,

$$r_k \leq s_k := (2 - \frac{1}{k})r_1. \tag{6.18}$$

It holds for $k = 1$. Let us show that $r_{k+1}/r_k - 1 \leq \theta_k := s_{k+1}/s_k - 1$, i.e.

$$r_k^{d_k - 1} \leq \theta_k = \frac{1}{(k+1)(2k-1)}.$$

We have $(2r_1)^{d_k-1} \leq (2r_1)^k$ when $0 < r_1 < 1/2$. Fix $r_1$ sufficiently small such that $(2r_1)^k < \frac{1}{(k+1)(2k-1)}$ for all $k$. By induction, we obtain (6.18) for all $k$. In particular, we have $\|h^{(k)}(\xi)\| \leq \|\xi\| + \|H^{(k)}(\xi)\| \leq 2r_{k+1}$ for $\|\xi\| < r_k$. To show the convergence of $H_k$, we write $H_k - H_{k-1}(\xi) = h^{(k)} \circ H_{k-1}$. By the Schwarz lemma, we obtain

$$\|h^{(k)} \circ H_{k-1}(\xi)\| \leq \frac{2r_{k+1}}{r_1^{d_k}}\|\xi\|^{d_k}, \quad \|\xi\| < r_1.$$

Therefore, $H_k$ converges to a holomorphic function on $\|\xi\| < r_1$.

Throughout the proof, we make initial assumptions that $d_k$ and $h^{(k)}$ satisfy (6.16)-(6.17), $e^{-1} \leq \mu_j \leq e^e$, and $\mu^Q \neq 1$ for $Q \in \mathbf{Z}^3$ with $Q \neq 0$. Set $\sigma^k = \tau_1^k\tau_2^k$, $\tau_2^k = \rho\tau_1^k\rho$, and

$$\tau_1^k = \varphi_k^{-1}\tau_1^{k-1}\varphi_k.$$

We want $\sigma^k$ not to be holomorphically equivalent to $\sigma^{(k-1)}$. Thus we have chosen a $\varphi_k$ that does not commute with $\rho$ in general. Note that $\sigma^k$ is still generated by a real analytic submanifold; indeed, when $\tau_i^{k-1} = \tau_{i1}^{(k-1)} \cdots \tau_{ip}^{(k-1)}$ and $\tau_{2j}^{k-1} = \rho\tau_{1j}^{k-1}\rho$, we still have the same identities if the superscript $(k-1)$ is replaced by $k$ and $\tau_{1j}^{(k)}$ equals $\varphi_k^{-1}\tau_{1j}^{(k-1)}\varphi_k$. It is clear that $\sigma^k = \sigma^{k-1} + O(d_k)$. As power series, we have

$$\sigma^\ell = \sigma^{k-1} + O(d_k), \quad k \leq \ell \leq \infty.$$

We know that $\sigma^\infty$ does not have a unique normal form in the centralizer $S$. Therefore, we will choose a procedure that arrives at a unique formal normal form in $S$. We show that this unique normal form is divergent; and hence by Theorem 5.5 (iii) any normal form of $\sigma$ that is in the centralizer of $S$ must diverge.

We now describe the procedure. For a formal mapping $F$, we have a unique decomposition

$$F = NF + N^{\mathsf{c}}F, \quad NF \in \mathcal{C}(S), \quad N^{\mathsf{c}}F \in \mathcal{C}^{\mathsf{c}}(S).$$

Set $\hat\sigma_0^\infty = \sigma^\infty$. For $k = 0, 1, \ldots$, we take a normalized polynomial map $\Phi_k \in \mathcal{C}_2^{\mathsf{c}}(S)$ of degree less than $d_k$ such that $\sigma_k^\infty := \Phi_k^{-1}\hat\sigma_k^\infty\Phi_k$ is normalized up to degree $d_k - 1$. Specifically, we require that

$$\deg \Phi_k \le d_k - 1, \quad \Phi_k \in \mathcal{C}^{\mathsf{c}}(S); \qquad N^c\sigma_k^\infty(\xi,\eta) = O(d_k).$$

Take a normalized polynomial map $\Psi_{k+1}$ such that $\Psi_{k+1}$ and $\hat\sigma_{k+1}^\infty := \Psi_{k+1}^{-1}\sigma_k^\infty\Psi_{k+1}$ satisfy

$$\deg \Psi_{k+1} \le 2d_k - 1; \quad \Psi_{k+1} \in \mathcal{C}_2^{\mathsf{c}}(S), \quad N^c\hat\sigma_{k+1}^\infty = O(2d_k).$$

We can repeat this for $k = 0, 1, \ldots$. Thus we apply two sequences of normalization as follows

$$\hat\sigma_{k+1}^\infty = \Psi_{k+1}^{-1} \circ \Phi_k^{-1} \cdots \Psi_1^{-1} \circ \Phi_0^{-1} \circ \sigma^\infty \circ \Phi_0 \circ \Psi_1 \cdots \Phi_k \circ \Psi_{k+1}.$$

We will show that $\Psi_{k+1} = I + O(d_k)$ and $\Phi_k = I + O(2d_{k-1})$. This shows that the sequence $\Phi_0\Psi_1\cdots\Phi_k\Psi_{k+1}$ defines a formal biholomorphic mapping $\Phi$ so that

$$\hat\sigma^\infty := \Phi^{-1}\sigma^\infty\Phi \tag{6.19}$$

is in a normal form. Finally, we need to combine the above normalization with the normalization for the unique normal form in Lemma 5.4. We will show that the unique normal form diverges.

Let us recall previous results to show that $\Phi_k, \Psi_{k+1}$ are uniquely determined. Set

$$\hat\sigma_k^\infty\colon \begin{cases} \xi' = \hat M^{(k)}(\xi\eta)\xi + \hat f^{(k)}(\xi,\eta), \\ \eta' = \hat N^{(k)}(\xi\eta)\eta + \hat g^{(k)}(\xi,\eta), \end{cases} \tag{6.20}$$

$$(\hat f^{(k)}, \hat g^{(k)}) \in \mathcal{C}_2^{\mathsf{c}}(S). \tag{6.21}$$

Recall that $\hat\sigma_0 = \sigma^\infty$. Assume that we have achieved

$$(\hat f^{(k)}, \hat g^{(k)}) = O(2d_{k-1}). \tag{6.22}$$

Here we take $d_{-1} = 2$ so that (6.20)-(6.22) hold for $k = 0$. By Proposition 5.2, there is a unique normalized polynomial mapping $\tilde\Phi_k$ that transforms $\hat\sigma_k^\infty$ into a normal form. We denote by $\Phi_k$ the truncated polynomial mapping of $\tilde\Phi_k$ of degree $d_k - 1$. We write

$$\Phi_k\colon \xi' = \xi + U^{(k)}(\xi,\eta), \quad \eta' = \eta + V^{(k)}(\xi,\eta),$$
$$(U^{(k)}, V^{(k)}) = O(2), \quad \deg(U^{(k)}, V^{(k)}) \le d_k - 1.$$

By Corollary 5.3, $\Phi_k$ satisfies

$$\sigma_k^\infty = \Phi_k^{-1}\hat\sigma_k^\infty\Phi_k\colon \begin{cases} \xi' = M^{(k)}(\xi\eta)\xi + f^{(k)}(\xi,\eta), \\ \eta' = N^{(k)}(\xi\eta)\eta + g^{(k)}(\xi,\eta), \end{cases}$$

$$(f^{(k)}, g^{(k)}) \in \mathcal{C}_2^{\mathsf{c}}(S), \quad \operatorname{ord}(f^{(k)}, g^{(k)}) \ge d_k. \tag{6.23}$$

In fact, by (5.24)-(5.25) (or (5.20)-(5.21)), we have

$$U^{(k)}_{j,PQ} = (\mu^{P-Q} - \mu_j)^{-1}\left\{\hat f^{(k)}_{j,PQ} + \mathcal{U}_{j,PQ}(\delta_{d-1}, \{\hat M^{(k)}, \hat N^{(k)}\}_{[\frac{d-1}{2}]}; \{\hat f^{(k)}, \hat g^{(k)}\}_{d-1})\right\}, \tag{6.24}$$

$$V^{(k)}_{j,QP} = (\mu^{Q-P} - \mu_j^{-1})^{-1}\left\{\hat g^{(k)}_{j,QP} + \mathcal{V}_{j,QP}(\delta_{d-1}, \{\hat M^{(k)}, \hat N^{(k)}\}_{[\frac{d-1}{2}]}; \{\hat f^{(k)}, \hat g^{(k)}\}_{d-1})\right\}, \tag{6.25}$$

for $|P|+|Q| = d < d_k$ and $\mu^{P-Q} \neq \mu_j$. By (5.26)-(5.27) (or (5.22)-(5.23)), we have

$$M^{(k)}_P = \hat M^{(k)}_P + \mathcal{M}_P(\delta_{2|P|-1}, \{\hat M^{(k)}, \hat N^{(k)}\}_{|P|-1}; \{\hat f^{(k)}, \hat g^{(k)}\}_{2|P|-1}), \tag{6.26}$$

$$N^{(k)}_P = \hat N^{(k)}_P + \mathcal{N}_P(\delta_{2|P|-1}, \{\hat M^{(k)}, \hat N^{(k)}\}_{|P|-1}; \{\hat f^{(k)}, \hat g^{(k)}\}_{2|P|-1}) \tag{6.27}$$

for $2|P|-1 < d_k$. Recall that $\mathcal{U}_{j,PQ}, \mathcal{V}_{j,QP}, \mathcal{M}_{j,P}$, and $\mathcal{N}_{j,P}$ are universal polynomials in their variables. In notation defined by Definition 5.1,

$$\mathcal{U}_{j,PQ}(\cdot;0) = \mathcal{V}_{j,QP}(\cdot;0) = 0, \quad \mathcal{M}_P(\cdot;0) = \mathcal{N}_P(\cdot;0) = 0.$$

Since $d_k \geq 2d_{k-1}$, we apply (6.24)-(6.25) for $d < 2d_{k-1} \leq d_k$ and (6.26)-(6.27) for $2|P|-1 < 2d_{k-1} \leq d_k$ to obtain

$$\Phi_k - I = (U^{(k)}, V^{(k)}) = O(2d_{k-1}), \tag{6.28}$$

$$M^{(k)}_P = \hat M^{(k)}_P, \quad N^{(k)}_P = \hat N^{(k)}_P, \quad |P| \leq d_{k-1}. \tag{6.29}$$

By Lemma 6.1, there is a unique normalized polynomial mapping

$$\Psi_{k+1}(\xi,\eta) = (\xi + \hat U^{(k+1)}(\xi,\eta), \eta + \hat V^{(k+1)}(\xi,\eta)),$$
$$(\hat U^{(k+1)}, \hat V^{(k+1)}) \in \mathcal{C}^{\mathfrak{c}}_2(S),$$
$$(\hat U^{(k+1)}, \hat V^{(k+1)}) = O(2), \quad \deg(\hat U^{(k+1)}, \hat V^{(k+1)}) \leq 2d_k - 1$$

such that $\hat\sigma^\infty_{k+1} = \Psi^{-1}_{k+1}\Phi^{-1}_k\sigma^\infty_k\Phi_k\Psi_{k+1}$ satisfies the following:

$$\hat\sigma^\infty_{k+1}\colon \xi' = \hat M^{(k+1)}(\xi\eta)\xi + \hat f^{(k+1)}, \quad \eta' = \hat N^{(k+1)}(\xi\eta)\eta + \hat g^{(k+1)},$$
$$(\hat f^{(k+1)}, \hat g^{(k+1)}) \in \mathcal{C}^{\mathfrak{c}}_2(S), \quad \mathrm{ord}(\hat f^{(k+1)}, \hat g^{(k+1)}) \geq 2d_k.$$

By (6.1)-(6.2), we know that

$$\hat U^{(k+1)}_{j,PQ} = (\mu^{P-Q} - \mu_j)^{-1}\left\{f^{(k)}_{j,PQ} + \mathcal{U}^*_{j,PQ}(\delta_{\ell-1}, \{M^{(k)}, N^{(k)}\}_{[\frac{\ell-1}{2}]}; \{f^{(k)}, g^{(k)}\}_{\ell-1})\right\}, \tag{6.30}$$

$$\hat V^{(k+1)}_{j,QP} = (\mu^{Q-P} - \mu_j^{-1})^{-1}\left\{g^{(k)}_{j,QP} + \mathcal{V}^*_{j,QP}(\delta_{\ell-1}, \{M^{(k)}, N^{(k)}\}_{[\frac{\ell-1}{2}]}; \{f^{(k)}, g^{(k)}\}_{\ell-1})\right\}, \tag{6.31}$$

for $d_k \leq |P|+|Q| = \ell \leq 2d_k - 1$ and $\mu^{P-Q} \neq \mu_j$. Recall that $\mathcal{U}^*_{j,PQ}$ and $\mathcal{V}^*_{j,QP}$ are universal polynomials in their variables. In notation defined by Definition 5.1, $\mathcal{U}^*_{j,PQ}(\cdot;0) = \mathcal{V}^*_{j,QP}(\cdot;0) = 0$. Thus

$$\Psi_{k+1} - I = (\hat U^{(k+1)}, \hat V^{(k+1)}) = O(d_k), \tag{6.32}$$

$$\hat U^{(k+1)}_{j,PQ} = \frac{f^{(k)}_{j,PQ}}{\mu^{P-Q} - \mu_j}, \quad \hat V^{(k+1)}_{j,QP} = \frac{g^{(k)}_{j,QP}}{\mu^{Q-P} - \mu_j^{-1}}, \quad |P|+|Q| = d_k. \tag{6.33}$$

Here $\mu^{P-Q} \neq \mu_j$. By (6.3)-(6.4), we have

$$\hat{M}_{j,P'}^{(k+1)} = M_{j,P'}^{(k)}, \quad |P'| < d_k, \tag{6.34}$$

$$\hat{M}_{j,P}^{(k+1)} = M_{j,P}^{(k)} + \mu_j \left\{ 2(\hat{U}_j^{(k+1)} \hat{V}_j^{(k+1)})_{PP} + ((\hat{U}_j^{(k+1)})^2)_{(P+e_j)(P-e_j)} \right\} + \left\{ Df_j^{(k)}(\xi,\eta)(\hat{U}^{(k+1)}, \hat{V}^{(k+1)}) \right\}_{(P+e_j)P}, \quad |P| = d_k. \tag{6.35}$$

As mentioned in Remark 6.2, one of consequence of the above formula is that the coefficients of $\hat{M}_j^{(k+1)}(\xi\eta)\xi_j$ of degree $2d_k + 1$ do not depend on the coefficients of $f^{(k)}, g^{(k)}$ of degree $\geq 2d_k$, provided $(f^{(k)}, g^{(k)}) = O(d_k)$ is in $\mathcal{C}_2^{\mathsf{c}}(S)$ as we assume.

Next, we need to estimate the size of coefficients of $M^{(k)}$ that appear in (6.34)-(6.35). Recall that we apply two sequences of normalization. We have

$$\hat{\sigma}_{k+1}^{\infty} = \Psi_{k+1}^{-1} \circ \Phi_k^{-1} \cdots \Psi_1^{-1} \circ \Phi_0^{-1} \circ \sigma^{\infty} \circ \Phi_0 \circ \Psi_1 \cdots \Phi_k \circ \Psi_{k+1}.$$

Thus, $M^{(k)}, N^{(k)}$ depend only on $\sigma^{\infty}$, $\Phi_0, \Psi_1, \Phi_1, \ldots, \Psi_{k-1}, \Phi_k$.

Recall that if $u_1, \ldots, u_m$ are power seres, then $\{u_1, \ldots, u_m\}_d$ denotes the set of their coefficients of degree at most $m$, and $|\{u_1, \ldots, u_m\}_d|$ denotes the sup norm. We choose $\sigma^{\infty}$ in such a way that its coefficients of degree $m$ satisfy

$$|\{\sigma^k\}_m| + |\{\sigma^{\infty}\}_m| \leq C^m.$$

Here $C$ does not dependent on $k$, $\mu_1, \mu_2, \mu_3$, $d_k$, and $h^{(k)}$. We also need some crude estimates on the growth of Taylor coefficients. If $F = I + f$ and $f = O(2)$ is a map in formal power series, then (5.1)-(5.3) imply

$$|(F^{-1})_P| \leq |F_P| + (2 + |\{f\}_{m-1}|)^{\ell_m}, \tag{6.36}$$

$$|(G \circ F)_P| \leq |((LG) \circ F, G)_P| + (2 + |\{f, G\}_{m-1}|)^{\ell_m}, \tag{6.37}$$

$$|(F^{-1} \circ G \circ F)_P| \leq |(G, (LG) \circ F, F \circ LG)_P| + (2 + |\{f, G\}_{m-1}|)^{\ell_m}, \tag{6.38}$$

for $m = |P| > 1$ and some positive integer $\ell_m$. Inductively, let us show that for $k = 0, 1, 2, \ldots$,

$$|\{\hat{M}^{(k)}, \hat{N}^{(k)}\}_P| \leq \delta_{d_{k-1}-1}^{L_m}, \quad m = 2|P| + 1 < 2d_{k-1}, \tag{6.39}$$

$$|\{\hat{\sigma}_k^{\infty}\}_{PQ}| \leq \delta_{2d_{k-1}-1}^{L_m}, \quad m = |P| + |Q| \geq 2d_{k-1}. \tag{6.40}$$

We emphasize that here and in what follows $L_m$ does not depend on the choices of $\mu_j, d_k, h^{(k)}$ which satisfy the initial conditions but are arbitrary otherwise. The above estimates hold trivially for $k = 0$ and $d_{-1} = 2$, since $\hat{\sigma}_0^{\infty} = \sigma^{\infty}$ is convergent. For induction, we assume that (6.39)-(6.40) hold. We need to find possibly larger $L_m$ for $m \geq 2d_{k-1}$ in order to verify

$$|\{\hat{M}^{(k+1)}, \hat{N}^{(k+1)}\}_P| \leq \delta_{d_k-1}^{L_m}, \quad m = 2|P| + 1 < 2d_k, \tag{6.41}$$

$$|\{\hat{\sigma}_{k+1}^{\infty}\}_{PQ}| \leq \delta_{2d_k-1}^{L_m}, \quad m = |P| + |Q| \geq 2d_k. \tag{6.42}$$

The $\Phi_k = I + (U^{(k)}, V^{(k)})$ is a polynomial mapping. Its degree is at most $d_k - 1$ and its coefficients are polynomials in $\{\hat{\sigma}_k\}_{d_k-1}$ and $\delta_{d_k-1}$; see (6.24)-(6.25). Hence

$$|U_{j,PQ}^{(k)}| + |V_{j,QP}^{(k)}| \leq \delta_{d_k-1}^{L_m}, \quad m = |P| + |Q|. \tag{6.43}$$

Note that the inequality holds trivially for $m < 2d_{k-1}$, in which case the left-hand side is zero and we do not change the value of $L_m$. For $m \geq 2d_{k-1}$, we might have to increase $L_m$ if necessary to obtain (6.43). Applying (6.38) to $\sigma_k^\infty = \Phi_k^{-1}\hat{\sigma}_k^\infty\Phi_k$, we obtain from (6.40) and (6.43) that

$$|M_{j,P}^{(k)}| + |N_{j,P}^{(k)}| \leq \delta_{d_k-1}^{L_m}, \quad m = 2|P| + 1, \tag{6.44}$$

$$|f_{j,PQ}^{(k)}| + |g_{j,QP}^{(k)}| \leq \delta_{d_k-1}^{L_m}, \quad m = |P| + |Q|. \tag{6.45}$$

Here we use that fact that since $d_k \geq 2d_{k-1}$, the small divisors in $\delta_{2d_{k-1}-1}$ appear in $\delta_{d_k-1}$ too. Furthermore, (6.45) holds trivially when $m < 2d_{k-1}$, as in this case its left-hand side is 0. If $|P| < d_{k-1}$ (i.e. $m = 2|P| + 1 < 2d_{k-1} + 1$), from (6.29) and (6.39) it follows (6.44) for the same $L_m$ in (6.39). For (6.45) with $m \geq 2d_{k-1}$ and for (6.44) with $m \geq 2d_{k-1} + 1$, we might have to increase the value of $L_m$ so that they are valid. We further remark that for possibly increased $L_m$, (6.39)-(6.40) remain valid. To obtain (6.41)-(6.42), we note that $\Psi_{k+1}$ is a polynomial map that has degree at most $2d_k - 1$ and the coefficients of degree $m$ bounded by $\delta_{2d_k-1}^{L_m}$; see (6.30)-(6.31). This shows that

$$|\hat{U}_{j,PQ}^{(k+1)}| + |\hat{V}_{j,QP}^{(k+1)}| \leq \delta_{2d_k-1}^{L_m}, \quad |P| + |Q| = m. \tag{6.46}$$

The argument to obtain (6.41)-(6.42) for $\hat{\sigma}_{k+1}^\infty = \Psi_{k+1}^{-1}\sigma_k^\infty\Psi_{k+1}$ is similar to the one to obtain (6.44)-(6.45) for $\sigma_k^\infty = \Phi_k^{-1}\hat{\sigma}_k^\infty\Phi_k$. Of course, we still use (6.38), while replacing (6.28), (6.29), (6.39), and (6.40) by (6.32), (6.34), (6.44), and (6.45), respectively. We emphasize that the sequence $L_m$ can be chosen consistently, as for $d_k \to \infty$, we only increase each $L_m$ for finitely many times.

Let us summarize the above computation for $\hat{\sigma}^\infty$ defined by (6.19). We know that $\hat{\sigma}^\infty$ is the unique power series such that $\hat{\sigma}^\infty - \hat{\sigma}_k^\infty = O(d_k)$ for all $k$, and $\hat{\sigma}^\infty$ is a formal formal form of $\sigma^\infty$. Let us write

$$\hat{\sigma}^\infty\colon \begin{cases} \xi' = \hat{M}^\infty(\xi\eta)\xi, \\ \eta' = \hat{N}^\infty(\xi\eta)\eta. \end{cases}$$

Let $|P| \leq d_k$. By (6.29), we get $\hat{M}_P^{(k+1)} = M_P^{(k+1)}$; by (6.34) in which $k$ is replaced by $k+1$, we get $\hat{M}_P^{(k+2)} = M_P^{(k+1)}$ as $|P| \leq d_k < d_{k+1}$. Therefore,

$$\hat{M}_P^\infty = \hat{M}_P^{(k+1)}, \quad |P| \leq d_k. \tag{6.47}$$

For $|P| < d_k$, (6.34) says that $\hat{M}_{j,P}^{(k+1)} = M_P^{(k)}$; by (6.44) that holds for any $P$, we obtain

$$|\hat{M}_P^\infty| = |\hat{M}_{j,P}^{(k+1)}| \leq \delta_{d_k-1}^{L_m}, \quad |P| < d_k, \quad m = 2|P| + 1. \tag{6.48}$$

To obtain rapid increase of coefficients of $\hat{M}_{j,P}^{(k+1)}$, we want to use both small divisors hidden in $\hat{U}_{j,PQ}^{(k)}$ and $\hat{V}_{j,QP}^{(k)}$ in (6.35). Therefore, if $M_{j,P}^{(k)}$ is already sufficiently large for $|P| = d_k$ that will be specified later, we take $\varphi_k$ to be the identity, i.e. $\tau_1^k = \tau_1^{k-1}$. Otherwise, we need to achieve it by choosing

$$\tau_1^k = \varphi_k^{-1}\tau_1^{k-1}\varphi_k.$$

Therefore, we examine the effect of a coordinate change by $\varphi_k$ on these coefficients.

Recall that we are in the elliptic case. We have $\rho(\xi,\eta)=(\overline{\eta},\overline{\xi})$ and $\tau_2^k=\rho\tau_1^k\rho$. Recall that

$$\varphi_k\colon \xi_j'=(\xi+h^{(k)}(\xi),\eta),\quad \operatorname{ord}h^{(k)}=d_k>3.$$

By a simple computation, we obtain

$$\begin{aligned}\tau_1^k(\xi,\eta)&=\tau_1^{k-1}(\xi,\eta)+(-h^{(k)}(\lambda\eta),\lambda^{-1}h^{(k)}(\xi))+O(|(\xi,\eta)|^{d_k+1}),\\ \tau_2^k(\xi,\eta)&=\tau_2^{k-1}(\xi,\eta)+(\lambda^{-1}\overline{h^{(k)}}(\eta),-\overline{h^{(k)}}(\lambda\xi))+O(|(\xi,\eta)|^{d_k+1}).\end{aligned}$$

Then we have

$$\begin{aligned}\sigma^k&=\sigma^{k-1}+(r^{(k)},s^{(k)})+O(d_k+1);\\ r^{(k)}(\xi,\eta)&=-\lambda\overline{h^{(k)}}(\lambda\xi)-h^{(k)}(\lambda^2\xi),\\ s^{(k)}(\xi,\eta)&=\lambda^{-2}\overline{h^{(k)}}(\eta)+\lambda^{-1}h^{(k)}(\lambda^{-1}\eta).\end{aligned}\tag{6.49}$$

Since $\sigma^k$ converges to $\sigma^\infty$, from (6.49) it follows that

$$\sigma^\infty=\sigma^{k-1}+(r^{(k)},s^{(k)})+O(d_k+1).\tag{6.50}$$

For $|P|+|Q|=d_k$, we have

$$\begin{aligned}r_{j,PQ}^{(k)}&=\left\{-\lambda_j\overline{h_j^{(k)}}(\lambda\xi)-h_j^{(k)}(\lambda^2\xi)\right\}_{PQ},\\ s_{j,QP}^{(k)}&=\left\{\lambda_j^{-2}\overline{h_j^{(k)}}(\eta)+\lambda_j^{-1}h_j^{(k)}(\lambda^{-1}\eta)\right\}_{QP}.\end{aligned}$$

We obtain

$$r_{j,P0}^{(k)}=-\lambda^{P+e_j}\overline{h_{j,P}^{(k)}}-\lambda^{2P}h_{j,P}^{(k)},\tag{6.51}$$

$$s_{j,0P}^{(k)}=\lambda_j^{-2}\overline{h_{j,P}^{(k)}}+\lambda^{-P-e_j}h_{j,P}^{(k)},\quad |P|=d_k,\tag{6.52}$$

$$r_{j,PQ}^{(k)}=s_{j,QP}^{(k)}=0,\qquad |P|+|Q|=d_k,\ Q\neq 0.\tag{6.53}$$

The above computation is actually sufficient to construct a divergent normal form $\tilde\sigma\in\mathcal{C}(S)$. To show that all normal forms of $\sigma$ in $\mathcal{C}(S)$ are divergent, We need to related it to the normal form $\hat\sigma$ in Theorem 5.5, which is unique. This requires us to keep track of the small divisors in the normalization procedure in the proof of Lemma 5.4.

Recall that $F^{(k+1)}=\log\hat M^{(k+1)}$ is defined by

$$F_j^{(k+1)}(\zeta)=\log(\mu_j^{-1}\hat M_j^{(k+1)}(\zeta))=\zeta_j+a_j^{(k+1)}(\zeta),\quad 1\le j\le 3.\tag{6.54}$$

We also have $F^\infty=\log\hat M^\infty$ with $F_j^\infty(\zeta)=\zeta_j+a_j^\infty(\zeta)$. Then by (6.47),

$$a_{j,P}^\infty=a_{j,P}^{(k+1)},\quad |P|\le d_k.\tag{6.55}$$

By (5.2) and (6.54), we have

$$a_{j,P}^{(k+1)}(\zeta)=\mu_j^{-1}\hat M_{j,P}^{(k+1)}+\mathcal{A}_{j,P}(\{\hat M_j^{(k+1)}\}_{|P|-1}),\quad |P|>1.$$

By (6.48), we have

$$|\mathcal{A}_{j,P}(\{\hat M_j^{(k+1)}\}_{|P|-1})|\le\delta_{d_k-1}^{L_m},\quad |P|=d_k,\quad m=2|P|+1.\tag{6.56}$$

Recall from the formula (5.35) that $F^{(k+1)}$, $F^\infty$ have the normal forms $\hat{F}^{(k+1)} = I + \hat{a}^{(k+1)}$ and $\hat{F}^\infty = I + \hat{a}^\infty$, respectively. The coefficients of $\hat{a}^{(k+1)}_{j,Q}$ and $\hat{a}^\infty_{j,Q}$ are zero, except the ones given by

$$\begin{aligned}\hat{a}^{(k+1)}_{j,Q} &= a^{(k+1)}_{j,Q} + \mathcal{B}_{j,Q}(\{a^{(k+1)}\}_{|Q|-1}),\\ \hat{a}^{(\infty)}_{j,Q} &= a^{(\infty)}_{j,Q} + \mathcal{B}_{j,Q}(\{a^{(\infty)}\}_{|Q|-1}),\end{aligned}$$

for $Q = (q_1, \dots, q_p), q_j = 0$, and $|Q| > 1$. Derived from the same normalization, the $\mathcal{B}_{j,Q}$ in both formulae stands for the same polynomial. Hence $\hat{a}^{(\infty)}_{j,P} = \hat{a}^{(k+1)}_P$ for $|P| \le d_k$, by (6.55). Combining (6.35) and (6.47) yields

$$\begin{aligned}\hat{a}^\infty_{3,P_k} = \hat{a}^{(k+1)}_{3,P_k} &= 2(\hat{U}^{(k+1)}_3 \hat{V}^{(k+1)}_3)_{P_k P_k} + ((\hat{U}^{(k+1)}_3)^2)_{(P_k+e_3)(P_k-e_3)} + \mu_3^{-1} M^{(k)}_{3,P_k}\\ &\quad + \{Df^{(k)}_3(\xi,\eta)(\hat{U}^{(k+1)}, \hat{V}^{(k+1)})\}_{(P_k+e_3)P_k} + \mathcal{A}_{k,P_k}(\{\hat{M}^{(k+1)}\}_{|P_k|-1}).\end{aligned} \tag{6.57}$$

We regard $\hat{a}^\infty_{3,P_k}$ as polynomials in $(\mu^{P-Q} - \mu_j)^{-1}$. The above formula holds for any $P_k$ with $|P_k| = d_k$.

To examine the effect of small divisors, we assume that $P_k = (p_k, q_k, 0)$ are given by Lemma 6.5, so are $\mu_1, \mu_2$, and $\mu_3$. Then the second term in (6.57) is 0 as the third component of $P_k - e_3$ is negative. We apply the above computation to

$$|P_k| = d_k.$$

Taking a subsequence of $P_k$ if necessary, we may assume that $d_k \ge 2d_{k-1}$ and $d_{k-1} > 3$ for all $k \ge 1$. The 4 exceptional small divisors of height $2|P_k| + 1$ in (6.14) are

$$\mu^{P_k} - \mu_3, \quad \mu^{-P_k} - \mu_3^{-1}, \quad \mu^{2P_k - e_3} - \mu_3, \quad \mu^{-2P_k + e_3} - \mu_3^{-1}.$$

The last two cannot show up in $\hat{a}^\infty_{3,P_k}$, since their degree, $2d_k + 1$, is larger than the degrees of Taylor coefficients in $\hat{a}_{3,P_k}$ . We have 3 products of the two exceptional small divisors of height $2|P_k| + 1$ and degree $|P_k|$, which are

$$(\mu^{P_k} - \mu_3)(\mu^{-P_k} - \mu_3^{-1}), \quad (\mu^{P_k} - \mu_3)(\mu^{P_k} - \mu_3), \quad (\mu^{-P_k} - \mu_3^{-1})(\mu^{-P_k} - \mu_3^{-1}).$$

The first product, but none of the other two, appears in $(\hat{U}^{(k+1)}_3 \hat{V}^{(k+1)}_3)_{P_k P_k}$. The third term and $f^{(k)}_3$ in $\hat{a}^\infty_{3,P_k}$ do not contain exceptional small divisors of degree $|P_k| = d_k > 2d_{k-1} - 1$. Since $f^{(k)}_3 = O(d_k)$ by (6.23), the exceptional small divisors of height $2|P_k| + 1$ can show up at most once in the fourth term of $\hat{a}^\infty_{3,P_k}$. Therefore, we arrive at

$$\begin{aligned}\hat{a}^\infty_{3,P_k} &= 2\hat{U}^{(k+1)}_{3,P_k 0} \hat{V}^{(k+1)}_{3,0P_k} + \hat{\mathcal{A}}^1_k(\delta_{d_k-1}, \frac{1}{\mu^{P_k} - \mu_3}; \{f^{(k)}, g^{(k)}\}_{d_k})\\ &\quad + \hat{\mathcal{A}}^2_k(\delta_{d_k-1}; \{f^{(k)}, g^{(k)}\}_{d_k}) + \mu_3^{-1} M^{(k)}_{3,P_k} + \mathcal{A}_{k,P_k}(\{\hat{M}^{(k+1)}\}_{|P_k|-1}),\\ \hat{\mathcal{A}}^1_k(\delta_{d_k-1}, \frac{1}{\mu^{P_k} - \mu_3}; \{f^{(k)}, g^{(k)}\}_{d_k}) &= (\hat{U}^{(k+1)}_{3,P_k 0}, \hat{V}^{(k+1)}_{3,0P_k}) \cdot \hat{\mathcal{A}}^3_k(\delta_{d_k-1}; \{f^{(k)}, g^{(k)}\}_{d_k}).\end{aligned}$$

By (6.44) and (6.56), we obtain $|M^{(k)}_{3,P_k}| + |\mathcal{A}_{k,P_k}(\{\hat{M}^{(k+1)}\}_{|P_k|-1})| \le \delta^{L_m}_{d_k-1}$ for $m = 2d_k + 1$. Omitting the arguments in the polynomial functions, we obtain from (6.43)-(6.46), and

(6.47) that

$$|\hat{\mathcal{A}}_k^1| + |\hat{\mathcal{A}}_k^2| + |M_{3,P_k}^{(k)}| + |\mathcal{A}_{k,P_k}| \le \frac{|\delta_{d_k-1}(\mu)|^{L_m}}{|\mu^{P_k} - \mu_3|}, \quad m = 2|P_k| + 1,$$

for a possibly larger $L_m$. We remark that although each term in the inequality depends on the choices of the sequences $\mu_i, d_j, h^{(\ell)}$, the $L_m$ does not depend on the choices, provided that $\mu_j, d_k, h^{(i)}$ satisfy our initial conditions. Therefore, we have

$$|\hat{a}_{3,P_k}^{\infty}| \ge 2|\hat{U}_{3,P_k0}^{(k+1)}\hat{V}_{3,0P_k}^{(k+1)}| - |\delta_{d_k-1}(\mu)|^{L_{2|P_k|+1}}|\mu^{P_k} - \mu_3|^{-1}.$$

Recall that $\sigma_k^{\infty} = \Phi_k^{-1}\Psi_{k-1}^{-1}\cdots\Phi_0^{-1}\sigma^{\infty}\Phi_0\Psi_1\cdots\Phi_k$. Set

$$\tilde{\sigma}_k^{\infty} := \Phi_k^{-1}\Psi_{k-1}^{-1}\cdots\Phi_0^{-1}\sigma^{k-1}\Phi_0\Psi_1\cdots\Phi_k.$$

By (6.50), we get

$$\sigma_k^{\infty} = \tilde{\sigma}_k^{\infty} + (r^{(k)}, s^{(k)}) + O(d_k + 1). \tag{6.58}$$

Recall that $\Phi_k$ depends only on coefficients of $\hat{\sigma}_{k-1}^{\infty} = \Psi_{k-1}^{-1}\sigma_{k-2}^{\infty}\Psi_{k-1}$ of degree less than $d_k$, while $\Psi_{k-1}$ depends only on coefficients of $\sigma_{k-1}^{\infty} = \Phi_{k-1}^{-1}\hat{\sigma}_{k-1}\Phi_{k-1}$ of degree at most $2d_{k-1}-1$ which is less than $d_k$ too. Therefore, $\Phi_k, \Psi_{k-1}, \ldots, \Phi_0$ depend only on coefficients of $\sigma^{\infty}$ of degree less than $d_k$. On the other hand, $\sigma^{\infty} = \sigma^{k-1} + O(d_k)$. Therefore, $\tilde{\sigma}_k^{\infty}$ depends only on $\sigma^{k-1}$, and hence it depends only on $h^{(\ell)}$ for $\ell < k$. By (6.58), we can express

$$f_{j,PQ}^{(k)} = \tilde{f}_{j,PQ}^{(k)} + r_{j,PQ}^{(k)}, \quad g_{j,QP}^{(k)} = \tilde{g}_{j,QP}^{(k)} + s_{j,QP}^{(k)}, \tag{6.59}$$

where $|P| + |Q| = d_k$ and $\tilde{f}_{j,PQ}^{(k)}, \tilde{g}_{j,QP}^{(k)}$ depend only on $h^{(\ell)}$ for $\ell < k$. Collecting (6.33), (6.59), and (6.51)-(6.53), we obtain

$$|\hat{a}_{3,P_k}^{\infty}| \ge 2\frac{|T_k|}{|\mu^{P_k} - \mu_3||\mu^{-P_k} - \mu_3^{-1}|} - \frac{|\delta_{d_k-1}(\mu)|^{L_{2d_k+1}}}{|\mu^{P_k} - \mu_3|}$$

with

$$\begin{aligned} T_k &= (-\lambda^{P_k+e_3}\overline{h_{3,P_k}^{(k)}} - \lambda^{2P_k}h_{3,P_k}^{(k)} + \tilde{f}_{3,P_k0}^{(k-1)})(\lambda_3^{-2}\overline{h_{3,P_k}^{(k)}} + \lambda^{-P_k-e_3}h_{3,P_k}^{(k)} + \tilde{g}_{3,0P_k}^{(k-1)}) \\ &= -\lambda^{2P_k-2e_3}(\lambda^{e_3-P_k}\overline{h_{3,P_k}^{(k)}} + h_{3,P_k}^{(k)} - \lambda^{-2P_k}\tilde{f}_{3,P_k0}^{(k-1)})(\lambda^{e_3-P_k}h_{3,P_k}^{(k)} + \overline{h_{3,P_k}^{(k)}} + \lambda_3^2\tilde{g}_{3,0P_k}^{(k-1)}). \end{aligned}$$

Set $\tilde{T}_k(h_{3,P_k}^{(k)}) := -\lambda^{2e_3-2P_k}T_k$. We are ready to choose $h_{3,P_k}^{(k)}$ to get a divergent normal form. We have either $|\lambda^{P_k-e_3} + 1| \ge 1$ or $|\lambda^{P_k-e_3} - 1| \ge 1$. When the first case occurs, one of $|\tilde{T}_k(0)|, |\tilde{T}_k(1)|, |\tilde{T}_k(-1)|$ is at least $1/4$; otherwise, we would have

$$2|\lambda^{P_k-e_3} + 1|^2 = |\tilde{T}_k(1) + \tilde{T}_k(-1) - 2\tilde{T}_k(0)| < 1,$$

which is a contradiction. Here the first identity follows from the fact that $\tilde{f}_{j,PQ}^{(k)}, \tilde{g}_{j,QP}^{(k)}$ depend only on $h^{(\ell)}$ for $\ell < k$. When the second case occurs, we conclude that one of $|\tilde{T}_k(0)|$, $|\tilde{T}_k(i)|$, $|\tilde{T}_k(-i)|$ is at least $1/4$. This shows that by taking $h_{3,P_k}^{(k)}$ to be one of $0, 1, -1, i, -i$, we can achieve

$$|T_k| \ge \frac{1}{4}\mu^{2P_k-2e_3}.$$

Therefore,

$$|\hat{a}^{\infty}_{3,P_k}| \geq \frac{\mu^{2P_k-2e_3}}{2|\mu^{P_k}-\mu_3||\mu^{-P_k}-\mu_3^{-1}|} - \frac{|\delta_{d_k-1}(\mu)|^{L_{2d_k+1}}}{|\mu^{P_k}-\mu_3|}.$$

Recall that $\mu_3 = e^e$. If $|\mu^{P_k}-\mu_3| < 1$ then $1/2 < \mu^{P_k-e_3} < 2$. The above inequality implies

$$|\hat{a}^{\infty}_{3,P_k}| \geq \frac{\mu^{3P_k-3e_3}}{4|\mu^{P_k}-\mu_3|^2}, \tag{6.60}$$

provided

$$|\mu^{P_k}-\mu_3| \leq \frac{1}{32}|\delta_{d_k-1}(\mu)|^{-L_{2d_k+1}}, \quad |P_k| = d_k.$$

For the last inequality to hold, it suffices have

$$|\mu^{P_k}-\mu_3| \leq |\delta_{d_k-1}(\mu)|^{-L_{2d_k+1}-1}, \quad |\delta_{d_k-1}(\mu)|^{-1} < 1/4. \tag{6.61}$$

Note that the sequence $L_m$ does not depend on the choice of $\lambda$. The existence of $\mu_1, \mu_2, \mu_3$ is ensured by Lemma 6.5 as follows: We choose the sequence $L_m$ in Lemma 6.5, denoted by $L'_m$ now, so that $|P_k|L'_k > 2L_{2d_k+1} + 2$. Then (6.61) follows from (6.13), the definitions of $\delta_{d_k-1}(\mu)$ by (6.6) and of $\Delta^*(P_k)$ by (6.14); indeed

$$\begin{aligned}|\mu^{P_k}-\mu_3| &\leq (C\Delta^*(P_k))^{|P_k|L'_k} \leq (\Delta^*(P_k)^{1/2})^{|P_k|L'_k} \\ &\leq (\delta_{d_k-1}(\mu))^{-|P_k|L'_k/2} \leq |\delta_{d_k-1}(\mu)|^{-L_{2d_k+1}-1}.\end{aligned}$$

Here the second inequality follows from $C(\Delta^*(P_k))^{1/2} < 1$ when $k$ is sufficiently large. The third inequality is obtained as follows. The definition of $\Delta^*(P_k)$ and $|P_k| = d_k$ imply that any small divisor in $\delta_{d_k-1}(\mu)$ is contained in $\Delta^*(P_k)$. Also, $\Delta^*(P_k) < \mu_i^{-1}$ for $i = 1, 2, 3$ and $k$ sufficiently large. Hence, $\Delta^*(P_k) \leq \delta^{-1}_{d_k-1}(\mu)$, which gives us the third inequality. Without loss of generality, we may assume that $L_k > k$. From (6.60) and (6.61) it follows that

$$|\hat{a}^{\infty}_{3,P_k}| > \delta^{d_k+1}_{d_{k-1}}(\mu) = \delta^{|P_k|+1}_{d_{k-1}}(\mu),$$

for $k$ sufficiently large. As $\delta_{d_k}(\mu) \to +\infty$, this shows that the divergence of $\hat{F}_3$ and the divergence of the normal form $\hat{\sigma}$.

As mentioned earlier, Theorem 5.5 (iii) implies that any normal form of $\sigma$ that is in the centralizer of $\hat{S}$ must diverge. □

## 7. A unique formal normal form of a real submanifold

Recall that we consider submanifolds of which the complexifications admit the maximum number of deck transformations. The deck transformations of $\pi_1$ are generated by $\{\tau_{i1}, \ldots, \tau_{1p}\}$. We also set $\tau_{2j} = \rho\tau_{1j}\rho$. Each of $\tau_{i1}, \ldots, \tau_{ip}$ fixes a hypersurface and $\tau_i = \tau_{11}\cdots\tau_{1p}$ is the unique deck transformation whose set of fixed points has the smallest dimension. We first normalize the composition $\sigma = \tau_1\tau_2$. This normalization is reduced to two normal form problems. In Proposition 5.2 we obtain a transformation $\Psi$ to transform $\tau_1, \tau_2$, and $\sigma$ into

$$\begin{aligned}\tau_i^*&: \xi'_j = \Lambda_{ij}(\xi\eta)\eta_j, \quad \eta'_j = \Lambda^{-1}_{ij}(\xi\eta)\xi_j, \\ \sigma^*&: \xi'_j = M_j(\xi\eta)\xi_j, \quad \eta'_j = M_j^{-1}(\xi\eta)\eta_j, \quad 1 \leq j \leq p.\end{aligned}$$

Here $\Lambda_{2j} = \Lambda_{1j}^{-1}$ and $M_j = \Lambda_{1j}^2$ are power series in the product $\zeta = (\xi_1\eta_1, \dots, \xi_p\eta_p)$. We also normalize the map $M\colon \zeta \to M(\zeta)$ by a transformation $\varphi$ which preserves all coordinate hyperplanes. This is the second normal form problem, which is solved formally in Theorem 5.5 under the condition on the normal form of $\sigma$, namely, that $\log \hat{M}$ is tangent to the identity. This gives us a map $\Psi_1$ which transforms $\tau_1, \tau_2$, and $\sigma$ into $\hat{\tau}_1, \hat{\tau}_2, \hat{\sigma}$ of the above form where $\Lambda_{ij}$ and $M_j$ become $\hat{\Lambda}_{ij}, \hat{M}_j$.

In this section, we derive a **unique formal normal form for $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ under the above condition on** $\log \hat{M}$. In this case, we know from Theorem 5.5 that $\mathcal{C}(\hat{\sigma})$ consists of only $2^p$ dilatations

$$R_\epsilon\colon (\xi_j, \eta_j) \to (\epsilon_j\xi_j, \epsilon_j\eta_j), \quad \epsilon_j = \pm 1, \quad 1 \le j \le p. \tag{7.1}$$

We will consider two cases. In the first case, we impose no restriction on the linear parts of $\{\tau_{ij}\}$ but the coordinate changes are restricted to mappings that are tangent to the identity. The second is for the family $\{\tau_{ij}\}$ that arises from a higher order perturbation of a product quadric, while no restriction is imposed on the changes of coordinates. We will show that in both cases, if the normal form of $\sigma$ can be achieved by a convergent transformation, the normal form of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ can be achieved by a convergent transformation too.

We now restrict our real submanifolds to some classes. First, we assume that $\sigma$ and $\tau_1, \tau_2$ are already in the normal form $\hat{\sigma}$ and $\hat{\tau}_1, \hat{\tau}_2$ such that

$$\hat{\tau}_i\colon \xi' = \hat{\Lambda}_i(\xi\eta)\eta, \quad \eta' = \hat{\Lambda}_i(\xi\eta)^{-1}\xi, \quad \hat{\Lambda}_2 = \hat{\Lambda}_1^{-1}, \tag{7.2}$$

$$\hat{\sigma}\colon \xi' = \hat{M}(\xi\eta)\xi, \quad \eta' = \hat{M}(\xi\eta)^{-1}\eta, \quad \hat{M} = \hat{\Lambda}_1^2. \tag{7.3}$$

Let us start with the general situation without imposing the restriction on the linear part of $\log M$. Assume that $\hat{\sigma}$ and $\hat{\tau}_i$ are in the above forms. Recall that $\mathbf{Z}_j = \operatorname{diag}(1, \dots, -1, \dots, 1)$ with $-1$ at the $(p+j)$-th place, and $\mathbf{Z} := \mathbf{Z}_1 \cdots \mathbf{Z}_p$. Let $Z_j$ (resp. $Z$) be the linear transformation with the matrix $\mathbf{Z}_j$ (resp. $\mathbf{Z}$). We also use notation

$$\mathbf{B}_* = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \quad \mathbf{E}_{\hat{\mathbf{\Lambda}}_i} = \begin{pmatrix} \mathbf{I} & \hat{\mathbf{\Lambda}}_i \\ -\hat{\mathbf{\Lambda}}_i^{-1} & \mathbf{I} \end{pmatrix}. \tag{7.4}$$

Here $\mathbf{B}$, as well as $\hat{\mathbf{\Lambda}}_i$ given by (7.2), is a non-singular complex $(p \times p)$ matrix. Define two transformations

$$(B_i)_*\colon \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to (\mathbf{B}_i)_* \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad E_{\hat{\mathbf{\Lambda}}_i}\colon \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \begin{pmatrix} \mathbf{I} & \hat{\mathbf{\Lambda}}_i(\xi\eta) \\ -\hat{\mathbf{\Lambda}}_i^{-1}(\xi\eta) & \mathbf{I} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{7.5}$$

Let us assume that in suitable linear coordinates, the linear parts of two families of involutions $\{\tau_{i1}, \dots, \tau_{ip}\}$ for $i = 1, 2$ are given by

$$L\tau_{ij} = T_{ij}, \quad T_{ij} = E_{\mathbf{\Lambda}_i}(B_i)_* Z_j (B_i)_*^{-1} E_{\mathbf{\Lambda}_i}^{-1}, \quad \mathbf{\Lambda_i} = \hat{\mathbf{\Lambda}}_i(0). \tag{7.6}$$

Here $T_{ij}$ are in the normal forms described in Lemma 3.5 or in Proposition 3.9.

Note that $(B_i)_*$ commutes with $Z$. Also, $E_{\hat{\mathbf{\Lambda}}_i} \circ \hat{\tau}_i = Z \circ E_{\hat{\mathbf{\Lambda}}_i}$. Let us set

$$\hat{\tau}_{ij} := E_{\hat{\mathbf{\Lambda}}_i} \circ (B_i)_* \circ Z_j \circ (B_i)_*^{-1} \circ E_{\hat{\mathbf{\Lambda}}_i}^{-1} \tag{7.7}$$

and we have $\hat{\tau}_1 = \hat{\tau}_{11} \cdots \hat{\tau}_{1p}$. The following lemma is analogous to the scheme used to classify the quadrics with the maximum number of deck transformations. The lemma

provides a way to represent all involutions $\{\tau_{11},\dots,\tau_{12^p},\rho\}$ provided that we already have a normal form for $\sigma$.

**Lemma 7.1.** *Let $\{\tau_{1j}\}$ and $\{\tau_{2j}\}$ be two families of formal holomorphic commuting involutions. Let $\tau_i=\tau_{i1}\cdots\tau_{ip}$ and $\sigma=\tau_1\tau_2$. Suppose that*

$$\tau_i=\hat\tau_i\colon\ \xi_j'=\hat\Lambda_{ij}(\xi\eta)\eta_j,\quad \eta_j'=\hat\Lambda_{ij}(\xi\eta)^{-1}\xi_j;$$
$$\sigma=\hat\sigma\colon\ \xi_j'=\hat M_j(\xi\eta)\xi_j,\quad \eta_j'=\hat M_j(\xi\eta)^{-1}\eta_j$$

*with $\hat M_j=\hat\Lambda_{1j}^2$. Assume further that the linear parts $T_{ij}$ of $\tau_{ij}$, given by* (7.6) *are in normal forms in Lemma* 3.5 *or Proposition* 3.9. *Then we have the following* :

(*i*) *For $i=1,2$ there exists $\Phi_i\in\mathcal{C}(\hat\tau_i)$, which is tangent to the identity, such that $\Phi_i^{-1}\tau_{ij}\Phi_i=\hat\tau_{ij}$ for $1\le j\le p$.*

(*ii*) *Let $\{\tilde\tau_{1j}\}$ and $\{\tilde\tau_{2j}\}$ be two families of formal holomorphic commuting involutions. Suppose that $\tilde\tau_i=\hat\tau_i$ and $\tilde\sigma=\hat\sigma$ and $\tilde\Phi_i^{-1}\tilde\tau_{ij}\tilde\Phi_i=\widehat{\tilde\tau}_{ij}$ with $\tilde\Phi_i\in\mathcal{C}(\hat\tau_i)$ being tangent to the identity and*

$$\widehat{\tilde\tau}_{ij}=E_{\hat{\mathbf{\Lambda}}_i}\circ(\tilde B_i)_*\circ Z_j\circ(\tilde B_i)_*^{-1}\circ E_{\hat{\mathbf{\Lambda}}_i}^{-1}.$$

*Here for $i=1,2$, the matrix $\tilde{\mathbf{B}}_i$ of $\tilde B_i$ is non-singular. Then*

$$\Upsilon^{-1}\tau_{ij}\Upsilon=\tilde\tau_{i\nu_i(j)},$$

*if and only if there exist $\Upsilon\in\mathcal{C}(\hat\tau_1,\hat\tau_2)$ and $\Upsilon_i\in\mathcal{C}(\hat\tau_i)$ such that*

$$\tilde\Phi_i=\Upsilon^{-1}\circ\Phi_i\circ\Upsilon_i,\quad i=1,2,\tag{7.8}$$
$$\Upsilon_i^{-1}\hat\tau_{ij}\Upsilon_i=\widehat{\tilde\tau}_{i\nu_i(j)},\quad 1\le j\le p.$$

*Here each $\nu_i$ is a permutation of $\{1,\dots,p\}$.*

(*iii*) *Assume further that $\tau_{2j}=\rho\tau_{1j}\rho$ with $\rho$ being defined by* (3.7). *Define $\hat\tau_{1j}$ by* (7.7) *and let*

$$\hat\tau_{2j}:=\rho\hat\tau_{1j}\rho.$$

*Then we can choose $\Phi_2=\rho\Phi_1\rho$ for* (*i*). *Suppose that $\tilde\Phi_2=\rho\tilde\Phi_1\rho$ where $\tilde\Phi_1$ is as in* (*ii*). *Then $\{\tilde\tau_{1j},\rho\}$ is equivalent to $\{\tau_{1j},\rho\}$ if and only if there exist $\Upsilon_i$, $\nu_i$ with $\nu_2=\nu_1$, and $\Upsilon$ satisfying the conditions in* (*ii*) *and $\Upsilon_2=\rho\Upsilon_1\rho$. The latter implies that $\Upsilon\rho=\rho\Upsilon$.*

*Proof.* (i) Note that $\hat\tau_{ij}$ is conjugate to $Z_j$ via the map $E_{\hat{\mathbf{\Lambda}}_i}\circ(B_i)_*$. Fix $i$. Each $\hat\tau_{ij}$ is an involution and its set of fixed-point is a hypersurface. Furthermore, $\mathrm{Fix}(\tau_{11}),\dots,\mathrm{Fix}(\tau_{1p})$ intersect transversally at the origin. By Lemma 2.6 there exists a formal mapping $\psi_i$ such that $\psi_i^{-1}\tau_{ij}\psi_i=L\tau_{ij}$. Now $L\psi_i$ commutes with $L\tau_{ij}$, Replacing $\psi_i$ by $\psi_i(L\psi_i)^{-1}$, we may assume that $\psi_i$ is tangent to the identity. We also find a formal mapping $\hat\psi_i$, which is tangent to the identity, such that $\hat\psi_i^{-1}\hat\tau_{ij}\hat\psi_i=L\hat\tau_{ij}=L\tau_{ij}$. Then $\Phi_i=\psi_i\hat\psi_i^{-1}$ fulfills the requirements.

(ii) Suppose that

$$\tau_{ij}=\Phi_i\hat\tau_{ij}\Phi_i^{-1},\quad \tilde\tau_{ij}=\tilde\Phi_i\widehat{\tilde\tau}_{ij}\tilde\Phi_i^{-1}.$$

Assume that there is a formal biholomorphic mapping $\Upsilon$ that transforms $\{\tau_{ij}\}$ into $\{\tau_{ij}\}$ for $i = 1, 2$. Then

$$\Upsilon^{-1}\tau_{ij}\Upsilon = \tilde{\tau}_{i\nu_i(j)}, \quad j = 1, \ldots, p, \ i = 1, 2. \tag{7.9}$$

Here $\nu_i$ is a permutation of $\{1, \ldots, p\}$. Then

$$\Upsilon\hat{\tau}_i = \hat{\tau}_i\Upsilon, \quad \Upsilon\hat{\sigma} = \hat{\sigma}\Upsilon. \tag{7.10}$$

Set $\Upsilon_i := \Phi_i^{-1}\Upsilon\tilde{\Phi}_i$. We obtain

$$\Upsilon_i^{-1}\hat{\tau}_{ij}\Upsilon_i = \widehat{\tilde{\tau}}_{i\nu_i(j)}, \quad 1 \le j \le p, \tag{7.11}$$

$$\tilde{\Phi}_i = \Upsilon^{-1}\Phi_i\Upsilon_i, \quad i = 1, 2. \tag{7.12}$$

Conversely, assume that (7.10)-(7.12) are valid. Then (7.9) holds as

$$\Upsilon^{-1}\tau_{ij}\Upsilon = \Upsilon^{-1}\Phi_i\hat{\tau}_{ij}\Phi_i^{-1}\Upsilon = \tilde{\Phi}_i\Upsilon_i^{-1}\hat{\tau}_{ij}\Upsilon_i\tilde{\Phi}_i^{-1} = \tilde{\tau}_{\nu_i(j)}.$$

(iii) Assume that we have the reality assumption $\tau_{2j} = \rho\tau_{1j}\rho$ and $\tilde{\tau}_{2j} = \rho\tilde{\tau}_{1j}\rho$. As before, we take $\Phi_1$, tangent to the identity, such that $\tau_{1j} = \Phi_1\hat{\tau}_{1j}\Phi_1^{-1}$. Let $\Phi_2 = \rho\Phi_1\rho$. By $\hat{\tau}_{2j} = \rho\hat{\tau}_{1j}\rho$, we get $\tau_{2j} = \rho\tau_{1j}\rho = \Phi_2\hat{\tau}_{2j}\Phi_2^{-1}$ for $\nu_2 = \nu_1$. Suppose that $\tilde{\Phi}_i$ satisfy the analogous properties for $\tilde{\tau}_{1j}$ and $\rho$. Suppose that $\Upsilon^{-1}\tau_{ij}\Upsilon = \tilde{\tau}_{i\nu_i(j)}$, $\nu_2 = \nu_1$, and $\Upsilon\rho = \rho\Upsilon$. Letting $\Upsilon_i = \Phi_i^{-1}\Upsilon\tilde{\Phi}_i$ we get $\Upsilon_2 = \rho\Upsilon_1\rho$. Conversely, if $\Upsilon_1$ and $\Upsilon_2$ satisfy $\Upsilon_2 = \rho\Upsilon_1\rho$, then

$$\rho\Upsilon\rho = \rho\Phi_1\Upsilon_1\tilde{\Phi}_1^{-1}\rho = \Phi_2\Upsilon_2\tilde{\Phi}_2^{-1} = \Upsilon.$$

This shows that $\Upsilon$ satisfies the reality condition. □

Now we assume that $\hat{F} = \log\hat{M}$ is tangent to the identity and is in the normal form (5.34). Recall the latter means that the $j$th component of $\hat{F} - I$ is independent of the $j$ variable. We assume that the linear part $T_{ij}$ of $\tau_{ij}$ are given by (7.6), where the non-singular matrix $\mathbf{B}$ is arbitrary. As mentioned earlier in this section, the group of formal biholomorphisms that preserve the form of $\hat{\sigma}$ consists of only linear involutions $R_\epsilon$ defined by (7.1). This restricts the holomorphic equivalence classes of the quadratic parts of $M$. By Proposition 3.9, such quadrics are classified by a more restricted equivalence relation, namely, $(\tilde{\mathbf{B}}_\mathbf{1}, \tilde{\mathbf{B}}_2) \sim (\mathbf{B}_1, \mathbf{B}_2)$, if and only if

$$\tilde{\mathbf{B}}_i = \mathbf{R}_\epsilon^{-1}\mathbf{B}_i \operatorname{diag}_{\nu_i}\mathbf{d}, \quad i = 1, 2.$$

For simplicity, we will now fix a representative $\mathbf{B}_1, \mathbf{B}_2$ for its equivalence class.

Using the normal form $\{\hat{\tau}_1, \hat{\tau}_2\}$ and the matrices $\mathbf{B}_1, \mathbf{B}_2$, we first decompose $\hat{\tau}_i = \hat{\tau}_{11} \cdots \hat{\tau}_{1p}$. By Lemma 7.1 (i), we then find $\Phi_i$ such that

$$\tau_{ij} = \Phi_i\hat{\tau}_{ij}\Phi_i^{-1}, \quad 1 \le j \le p.$$

For each $i$, $\Phi_i$ commutes with $\hat{\tau}_i$. It is within this family of $\Phi_i \in \mathcal{C}(\hat{\tau}_i)$ for $i = 1, 2$ that we will find a normal form for $\{\tau_{ij}\}$. When restricted to $\tau_{2j} = \rho\tau_{1j}\rho$, the classification of the real submanifolds is within the family of $\{\{\tau_{1j}\}, \{\tau_{2j}\}\}$ as described above and such that

$$\Phi_2 = \rho\Phi_1\rho.$$

From Lemma 7.1 (ii), the equivalence relation on $\mathcal{C}(\hat{\tau}_i)$ is given by

$$\tilde{\Phi}_i = \Upsilon^{-1}\Phi_i\Upsilon_i, \quad i = 1, 2.$$

Here $\Upsilon_i$ and $\Upsilon$ satisfy

$$\Upsilon_i^{-1}\hat\tau_{ij}\Upsilon_i=\hat\tau_{i\nu_i(j)},\quad 1\le j\le p;\quad \Upsilon\hat\tau_i\Upsilon^{-1}=\hat\tau_i,\quad i=1,2.$$

We now construct a normal form for $\{\tau_{ij}\}$ within the above family. Let us first use the centralizer of $\mathcal{C}^{\mathsf{c}}(Z_1,\dots,Z_p)$, described in Lemma 4.12, to define the complement of the centralizer of the family of non-linear commuting involutions $\{\hat\tau_{11},\dots,\hat\tau_{1p}\}$. Recall that the mappings $E_{\hat{\mathbf{\Lambda}}_i}$ and $(B_i)_*$ are defined by (7.5). According to Lemma 4.12, we have the following.

**Lemma 7.2.** *Let $i=1,2$. Let $\{\hat\tau_{i1},\dots,\hat\tau_{ip}\}$ be given by (7.7). Set*

$$\mathcal{E}_i:=E_{\hat{\mathbf{\Lambda}}_i}\circ(B_i)_*.$$

*Then $\mathcal{C}(\hat\tau_{i1},\dots,\hat\tau_{ip})=\left\{\mathcal{E}_i\phi_0\mathcal{E}_i^{-1}\colon \phi_0\in\mathcal{C}(Z_1,\dots,Z_p)\right\}$. Set*

$$\mathcal{C}^{\mathsf{c}}(\hat\tau_{i1},\dots,\hat\tau_{ip}):=\left\{\mathcal{E}_i\psi_1\mathcal{E}_i^{-1}\colon \psi_1\in\mathcal{C}^{\mathsf{c}}(Z_1,\dots,Z_p)\right\}.$$

*Each formal biholomorphic mapping $\psi$ admits a unique decomposition $\psi_1\psi_0^{-1}$ with*

$$\psi_1\in\mathcal{C}^{\mathsf{c}}(\hat\tau_{i1},\dots,\hat\tau_{ip}),\quad \psi_0\in\mathcal{C}(\hat\tau_{i1},\dots,\hat\tau_{ip}).$$

*If $\hat\tau_{ij}$ and $\psi$ are convergent, then $\psi_0,\psi_1$ are convergent. Assume further that $\tau_{2j}=\rho\tau_{1j}\rho$ with $\rho$ being given by (3.7). Then $\rho\phi_1\rho\in\mathcal{C}^{\mathsf{c}}(\hat\tau_{11},\dots,\hat\tau_{1p})$ for $\phi_1\in\mathcal{C}^{\mathsf{c}}(\hat\tau_{21},\dots,\hat\tau_{2p})$.*

**Proposition 7.3.** *Let $\hat\tau_i,\hat\sigma$ be given by (7.2)-(7.3) in which $\log\hat M$ is in the formal normal form (5.34). Let $\{\hat\tau_{ij}\}$ be given by (7.7). Assume further that the linear parts $T_{ij}$ of $\hat\tau_{ij}$ are in normal forms in Lemma 3.5 or Proposition 3.9. Suppose that*

$$\tau_{ij}=\Phi_i\hat\tau_{ij}\Phi_i^{-1},\quad \tilde\tau_{ij}=\tilde\Phi_i\hat\tau_{ij}\tilde\Phi_i^{-1}\quad 1\le j\le p, \tag{7.13}$$

$$\Phi_i\in\mathcal{C}(\hat\tau_i),\quad \tilde\Phi\in\mathcal{C}(\hat\tau_i),\quad \tilde\Phi_i'(0)=\Phi_i'(0)=\mathbf{I},\quad i=1,2. \tag{7.14}$$

*Then $\{\Upsilon^{-1}\tau_{ij}\Upsilon\}=\{\tilde\tau_{ij}\}$ for $i=1,2$ and for some $\Upsilon\in\mathcal{C}(\hat\tau_1,\hat\tau_2)$, if and only if there exist formal biholomorphisms $\Upsilon,\Upsilon_1^*,\Upsilon_2^*$ such that*

$$\Upsilon^{-1}\circ((B_i)_*\circ Z_j\circ(B_i)_*^{-1})\circ\Upsilon^{-1}=(B_i)_*\circ Z_{\nu_i(j)}\circ(B_i)_*^{-1}, \tag{7.15}$$

$$\tilde\Phi_i=\Upsilon^{-1}\Phi_i\Upsilon_i^*\Upsilon,\quad \Upsilon_i^*\in\mathcal{C}(\hat\tau_{i1},\dots,\hat\tau_{ip}),\quad i=1,2, \tag{7.16}$$

$$\Upsilon\hat\sigma\Upsilon^{-1}=\hat\sigma, \tag{7.17}$$

*where each $\nu_i$ is a permutation of $\{1,\dots,p\}$. Assume further that $\hat\tau_{2j}=\rho\hat\tau_{1j}\rho$ and $\Phi_2=\rho\Phi_1\rho$ and $\tilde\Phi_2=\rho\tilde\Phi_1\rho$. If $\Upsilon$ commutes with $\rho$, one can take $\Upsilon_2^*=\rho\Upsilon_1^*\rho$ and $\nu_2=\nu_1$.*

*Proof.* Recall that

$$\tau_{ij}=\Phi_i\hat\tau_{ij}\Phi_i^{-1},\quad \Phi_i\in\mathcal{C}(\hat\tau_i);\qquad \tilde\tau_{ij}=\tilde\Phi_i\widehat\tau_{ij}\tilde\Phi_i^{-1},\quad \tilde\Phi_i\in\mathcal{C}(\hat\tau_i).$$

Suppose that

$$\Upsilon^{-1}\tau_{ij}\Upsilon=\tilde\tau_{i\nu_i(j)},\quad j=1,\dots,p,\ i=1,2. \tag{7.18}$$

By Lemma 7.1, there are invertible $\Upsilon_i$ such that

$$\Upsilon_i^{-1}\hat\tau_{ij}\Upsilon_i=\widehat\tau_{i\nu_i(j)},\quad 1\le j\le p, \tag{7.19}$$

$$\tilde\Phi_i=\Upsilon^{-1}\circ\Phi_i\circ\Upsilon_i,\quad i=1,2.$$

Let us simplify the equivalence relation. By Theorem 5.5, $\mathcal{C}(\hat{\tau}_1, \hat{\tau}_2)$ consists of $2^p$ dilations $\Upsilon$ of the form $(\xi, \eta) \to (a\xi, a\eta)$ with $a_j = \pm 1$. Since $\Phi_i, \tilde{\Phi}_i$ are tangent to the identity, then $D\Upsilon_i(0)$ is diagonal too. In fact the linear part of $\Upsilon$ at the origin is

$$L\Upsilon_i = \Upsilon.$$

Clearly, $\Upsilon$ commutes with each non-linear transformation $E_{\hat{\mathbf{\Lambda}}_i}$. Simplifying the linear parts of both sides of (7.19), we get

$$\Upsilon^{-1} \circ ((B_i)_* \circ Z_j \circ (B_i)_*^{-1}) \circ \Upsilon^{-1} = (\tilde{B}_i)_* \circ Z_{\nu_i(j)} \circ (\tilde{B}_i)_*^{-1}. \tag{7.20}$$

From the commutativity of $\Upsilon$ and $E_{\hat{\Lambda}_i}$ again and the above identity, it follows that

$$\Upsilon^{-1} \circ \hat{\tau}_{ij} \circ \Upsilon = \widehat{\tau}_{\nu_i(j)}, \quad j = 1, \ldots, p, \ i = 1, 2. \tag{7.21}$$

Using (7.13) and (7.21), we can rewrite (7.18) as

$$\Upsilon^{-1}\Phi_i\hat{\tau}_{ij}\Phi_i^{-1}\Upsilon = \tilde{\Phi}_i\Upsilon^{-1}\hat{\tau}_{ij}\Upsilon\tilde{\Phi}_i^{-1}.$$

It is equivalent to $\Upsilon_i^*\hat{\tau}_{ij} = \hat{\tau}_{ij}\Upsilon_i^*$, where

$$\Upsilon_i^* := \Phi_i^{-1}\Upsilon\tilde{\Phi}_i\Upsilon^{-1}.$$

Therefore, by (7.8), in $\mathcal{C}(\hat{\tau}_i)$, $\tilde{\Phi}_i$ and $\Phi_i$ are equivalent, if and only if

$$\tilde{\Phi}_i = \Upsilon^{-1}\Phi_i\Upsilon_i^*\Upsilon, \quad \Upsilon_i^* \in \mathcal{C}(\hat{\tau}_{i1}, \ldots, \hat{\tau}_{ip}), \quad i = 1, 2.$$

Conversely, if $\Upsilon_i^*$ satisfy the above identities, we take $\Upsilon_i = \Upsilon_i^*\Upsilon$. Note that (7.17) ensures that $\Upsilon$ commutes with $\hat{\tau}_i$ and $E_{\hat{\mathbf{\Lambda}}_\mathbf{i}}$. Then (7.21), or equivalently (7.20) as $\Upsilon$ commutes with $E_{\hat{\mathbf{\Lambda}}_i}$, gives us (7.19). □

**Proposition 7.4.** *Let $\{\tau_{ij}\}$, $\{\tilde{\tau}_{ij}\}$, $\Phi_i$, and $\tilde{\Phi}_i$ be as in Proposition 7.3. Decompose $\Phi_i = \Phi_{i1} \circ \Phi_{i0}^{-1}$ with $\Phi_{i1} \in \mathcal{C}^{\mathsf{c}}(\hat{\tau}_{i1}, \ldots, \hat{\tau}_{1p})$ and $\Phi_{i0} \in \mathcal{C}(\hat{\tau}_{i1}, \ldots, \hat{\tau}_{1p})$, and decompose $\tilde{\Phi}_i$ analogously. Then $\{\{\tau_{1j}\}, \{\tau_{2j}\}\}$ and $\{\{\tilde{\tau}_{1j}\}, \{\tilde{\tau}_{2j}\}\}$ are equivalent under a mapping that is tangent to the identity if and only if $\Phi_{i1} = \tilde{\Phi}_{i1}$ for $i = 1, 2$. Assume further that $\tau_{2j} = \rho\tau_{1j}\rho$ and $\tilde{\tau}_{2j} = \rho\tilde{\tau}_{1j}\rho$. Then two families are equivalent under a mapping that is tangent to the identity and commutes with $\rho$ if and only if $\Phi_{i1} = \tilde{\Phi}_{i1}$.*

*Proof.* When restricting to changes of coordinates that are tangent to the identity, we have $\Upsilon = I$ in (7.18). Also (7.15) is the same as $\nu_i$ being the identity. By the uniqueness of the decomposition $\Phi_i = \Phi_{i1}\Phi_{i0}^{-1}$, (7.16) becomes $\Phi_{i1} = \tilde{\Phi}_{i1}$. □

We consider the following special case without restriction on coordinate changes. We will assume that $M$ is a higher order perturbation of non-resonant product quadric. Let us recall $\hat{\sigma}$ be given by (7.3) and define $\hat{\tau}_{ij}$ as follows:

$$\hat{\sigma} : \begin{cases} \xi_j' = \hat{M}_j(\xi\eta)\xi_i \\ \eta_j' = \hat{M}_j^{-1}(\xi\eta)\eta_j, \end{cases} \qquad \hat{\tau}_{ij} : \begin{cases} \xi_i' = \hat{\Lambda}_{ij}(\xi\eta)\eta_j \\ \eta_j' = \hat{\Lambda}_{ij}^{-1}(\xi\eta)\xi_j \\ \xi_k' = \xi_k \\ \eta_k' = \eta_k, \quad k \neq j \end{cases} \tag{7.22}$$

with $\hat{\Lambda}_{2j} = \hat{\Lambda}_{1j}^{-1}$ and $\hat{M}_j = \hat{\Lambda}_{1j}^2$. Let $\hat{\tau}_i = \hat{\tau}_{i1} \cdots \hat{\tau}_{1p}$. Recall that $E_{\hat{\mathbf{\Lambda}}_i}$ in (7.5). Set

$$\mathcal{C}^{\mathsf{c}}(\hat{\tau}_{11}, \ldots, \hat{\tau}_{1p}) := \left\{ E_{\hat{\mathbf{\Lambda}}_1} \psi E_{\hat{\mathbf{\Lambda}}_1}^{-1} \colon \psi \in \mathcal{C}^{\mathsf{c}}(Z_1, \ldots, Z_p) \right\}. \tag{7.23}$$

**Proposition 7.5.** *Let $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$ be the family of involutions associated with a real analytic submanifold $M$ that is a higher order perturbation of a non-resonant product quadric. Assume further that the linear parts $T_{ij}$ of $\tau_{ij}$ are in normal forms in Lemma 3.5. Let $\hat{\sigma}$ be the formal normal form $\hat{\sigma}$ of the $\sigma$ associated to $M$ that is given by (7.3) in which $\log \hat{M}$ is in the formal normal form (5.34). In suitable formal coordinates the involutions $\tau_{ij}$ of $M$ have the form*

$$\tau_{1j} = \Psi \hat{\tau}_{ij} \Psi^{-1}, \quad \tau_{2j} = \rho \tau_{1j} \rho, \qquad \Psi \in \mathcal{C}(\hat{\tau}_1) \cap C^c(\hat{\tau}_{11}, \ldots, \hat{\tau}_{1p}). \tag{7.24}$$

*Moreover, if $\tilde{\tau}_{11}, \ldots, \tilde{\tau}_{1p}$ have the form (7.24) in which $\Phi$ is replaced by $\tilde{\Phi}$. Then there exists a formal mapping $R$ commuting with $\rho$ and transforms the family $\{\tilde{\tau}_{11}, \ldots, \tilde{\tau}_{1p}\}$ into $\{\tau_{11}, \ldots, \tau_{1p}\}$ if and only if $R$ is an $R_\epsilon$ defined by (7.1) and*

$$\tilde{\Psi} = R_\epsilon^{-1} \Psi R_\epsilon. \tag{7.25}$$

*In particular, $\{\tau_{11}, \ldots, \tau_{1p}, \rho\}$ is formally equivalent to $\{\hat{\tau}_{11}, \ldots, \hat{\tau}_{1p}, \rho\}$ if and only if $\Psi$ in (7.24) is the identity map.*

*Proof.* We apply Proposition 7.3 with $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{I}$. We need to refine the equivalence relation (7.15)-(7.17). First we know that (7.17) means that $\Upsilon$ is some $R_\epsilon$. Since $R_\epsilon$ is diagonal, then (7.15) is always true for $\nu_1 = \nu_2 = I$. It remains to refine (7.16). We have $\Phi_2 = \rho \Phi_1 \rho$. We know that $\Upsilon$ is a dilation of the form

$$\xi_j \to \epsilon_j \xi_j, \quad \eta_j \to \epsilon_j \eta_j, \quad 1 \le j \le p, \quad \epsilon_j = \pm 1.$$

Since $\mathbf{B}_1 = \mathbf{I}$, then $\Phi_1 \in \mathcal{C}^{\mathsf{c}}(\hat{\tau}_{11}, \ldots, \hat{\tau}_{1p})$ implies that $\Upsilon^{-1} \Phi_1 \Upsilon \in \mathcal{C}^{\mathsf{c}}(\hat{\tau}_{11}, \ldots, \hat{\tau}_{1p})$; and $\Upsilon$ commutes with each $\hat{\tau}_{1j}$. By the uniqueness of decomposition, (7.16) becomes

$$\tilde{\Phi}_{11} = \Upsilon^{-1} \Phi_{11} \Upsilon, \quad \tilde{\Phi}_{10}^{-1} = \Upsilon^{-1} \Phi_{10}^{-1} \Upsilon^* \Upsilon.$$

The second equation defines $\Upsilon_1^*$ that is in $\mathcal{C}(\hat{\tau}_{11}, \ldots, \hat{\tau}_{1p})$ as $\Upsilon, \Phi_{i0}, \tilde{\Phi}_{i0}$ are in the centralizer. Rename $\Phi_{11}, \tilde{\Phi}_{11}$ by $\Psi, \tilde{\Psi}$. This shows that the equivalence relation is reduced to (7.25). $\square$

We now derive the following formal normal form.

**Theorem 7.6.** *Let $M$ be a real analytic submanifold that is a higher order perturbation of a non-resonant product quadric. Assume that the formal normal form $\hat{\sigma}$ of the $\sigma$ associated to $M$ is given by (7.3) in which $\log \hat{M}$ is tangent to the identity and in the formal normal form (5.34). Let $E_{\hat{\mathbf{\Lambda}}_1}$ be defined by (7.4). Then $M$ is formally equivalent to a formal submanifold in the $(z_1, \ldots, z_{2p})$-space defined by*

$$\tilde{M} \colon z_{p+j} = (\lambda_j^{-1} U_j(\xi, \eta) - V_j(\xi, \eta))^2, \quad 1 \le j \le p,$$

*where $(U, V) = E_{\hat{\mathbf{\Lambda}}_1(0)} E_{\hat{\mathbf{\Lambda}}_1}^{-1} \Psi^{-1}$, $\Psi$ is in $\mathcal{C}(\hat{\tau}_1)$ and $\mathcal{C}^{\mathsf{c}}(\hat{\tau}_{11}, \ldots, \hat{\tau}_{1p})$, defined by (7.23), and $\xi, \eta$ are solutions to*

$$z_j = U_j(\xi, \eta) + \lambda_j V_j(\xi, \eta), \quad \overline{z}_j = \overline{U_j \circ \rho(\xi, \eta)} + \overline{\lambda}_j \overline{V_j \circ \rho(\xi, \eta)}, \quad 1 \le j \le p.$$

*Furthermore, the $\Psi$ is uniquely determined up to conjugacy $R_\epsilon \Psi R_\epsilon^{-1}$ by an involution $R_\epsilon\colon \xi_j \to \epsilon_j \xi_j, \eta_j \to \epsilon_j \eta_j$ for $1 \le j \le p$. The formal holomorphic automorphism group of $\hat{M}$ consists of involutions of the form*

$$L_\epsilon\colon z_j \to \epsilon_j z_j, \quad z_{p+j} \to z_{p+j}, \quad 1 \le j \le p$$

*with $\epsilon$ satisfying $R_\epsilon \Psi = \Psi R_\epsilon$. If the $\sigma$ associated to $M$ is holomorphically equivalent to a Poincaré-Dulac normal form, then $\tilde{M}$ can be achieved by a holomorphic transformation too.*

*Proof.* We fist choose linear coordinates so that the linear parts of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ are in the normal form in Lemma 3.2. We apply Proposition 7.5 and assume that $\tau_{ij}$ are already in the normal form. The rest of proof is essentially in Proposition 4.2 and we will be brief. Write $T_{1j} = E_{\hat{\mathbf{\Lambda}}_1(0)} \circ Z_j \circ E^{-1}_{\hat{\mathbf{\Lambda}}_1(0)}$. Let $\psi = (U, V)$ with $U, V$ being given in the theorem. We obtain

$$\tau_{1j} = \psi T_{1j} \psi^{-1}, \quad 1 \le j \le p.$$

Let $f_j = \xi_j + \lambda_j \eta_j$ and $h_j = (\lambda_j \xi_j - \eta_j)^2$. The invariant functions of $\{T_{11}, \dots, T_{1p}\}$ are generated by $f_1, \dots, f_p, h_1, \dots, h_p$. This shows that the invariant functions of $\{\tau_{11}, \dots, \tau_{1p}\}$ are generated by $f_1 \circ \psi, \dots, f_p \circ \psi, h_1 \circ \psi, \dots, h_p \circ \psi$. Set $g := \overline{f \circ \psi \circ \rho}$. We can verify that $\phi = (f, g)$ is biholomorphic. Now $\phi \rho \phi^{-1} = \rho_0$. Then $\tilde{M}$ is defined by

$$z_{p+j} = E_j(z', \overline{z}'), \quad 1 \le j \le p,$$

where $E_j = h_j \circ \phi^{-1}$. Then $E_j \circ \phi$ and $z_j \circ \phi = f_j$ are invariant by $\{\tau_{1k}\}$. This shows that $\{\phi \tau_{ij} \phi^{-1}\}$ has the same invariant functions as deck transformations of $\pi_1$ of the complexification of $\tilde{M}$. By Lemma 2.7, $\phi \tau_{1j} \phi^{-1}$ agrees with the unique set of generators for the deck transformations of $\pi_1$. Then $\hat{M}$ is a realization of $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$. □

## 8. Normal forms of completely integrable commuting biholomorphisms

In this section, we shall consider a family of commuting germs of holomorphic diffeomorphisms at a common fixed point, say $0 \in \mathbf{C}^n$. We shall give conditions that ensure that the family can be transformed simultaneously and holomorphically to a normal form. This means that there exists a germ of biholomorphism at the origin which conjugates each germ of biholomorphism in the family to a mapping that commutes with the linear part of every mapping of the family. We can achieve this under two conditions:

*a)* *The family is "formally completely integrable".* This means that the the normal form of the family has the "same resonances" as the normal form of the family of the linear parts.

*b)* *The family of linear parts is of "Poincaré type".* In general, individually, each linear part might not satisfy these conditions. They are satisfied, collectively, by the family.

For our convergence proof, both conditions will be essential. To be more specific, let $\mathbf{D}_1 := \operatorname{diag}(\mu_{11}, \dots, \mu_{1n}), \dots, \mathbf{D}_\ell := \operatorname{diag}(\mu_{\ell 1}, \dots, \mu_{\ell n})$ be diagonal invertible matrices of $\mathbf{C}^n$. Let us consider a family $F := \{F_i\}_{i=1}^\ell$ of germs of holomorphic diffeomorphisms of $(\mathbf{C}^n, 0)$ of which the linear of $F_i(x)$ at the origin is

$$D_i\colon x \to \mathbf{D}_i x.$$

Let us set $D := \{D_i\}_{i=1,\dots\ell}$. Thus

$$F_i(x) = \mathbf{D}_i x + f_i(x), \quad f_i(0) = 0, \quad Df_i(0) = 0.$$

The group of germs of (resp. formal) biholomorphisms tangent to identity acts on the family $F$ by $\Phi_* F := \{\Phi^{-1} \circ F_i \circ \Phi \colon 1 \le i \le \ell\}$.

Let us denote $\mathcal{O}_n$ (resp. $\widehat{\mathcal{O}}_n$) the ring of germs of holomorphic functions at the origin (resp. ring of formal power series) of $\mathbf{C}^n$. Let $Q = (q_1, \dots, q_n) \in \mathbf{N}^n$ and $x = (x_1, \dots, x_n) \in \mathbf{C}^n$, we shall write

$$|Q| := q_1 + \cdots + q_n, \quad x^Q := x_1^{q_1} \cdots x_n^{q_n}.$$

Let us specialize to a family $\{F_i\}_{i=1,\dots\ell}$ of **commuting germs of holomorphic diffeomorphisms**, that is that $F_i \circ F_j = F_j \circ F_i$ for all $1 \le i, j \le \ell$. Since it generates an abelian group, such a family is said to be **abelian**. We emphasize that the family does not necessarily form a group and

$$\ell < \infty.$$

Let us recall a result by M. Chaperon (see theorem 4 in [Cha86], page 132):

**Proposition 8.1.** *If the family of diffeomorphisms is abelian then there exists a formal diffeomorphism $\Phi$, which is tangent to the identity, such that*

$$\widehat{F}_i(\mathbf{D}_j x) = \mathbf{D}_j \widehat{F}_i(x), \quad 1 \le i, j \le \ell$$

*where $\widehat{F}_i := \Phi_* F_i$, for $1 \le i \le \ell$. We call the family $\{\widehat{F}_i\}$ a formal normal form of the family $F$ with respect to the family $D$ of linear maps.*

As mentioned above, for convenience, we have restricted ourselves to changes of holomorphic coordinates that are tangent to the identity. Also $\Phi_*\{F_i\}_{i=1}^{\ell} = \{\tilde{F}_i\}_{i=1}^{\ell}$ means that

$$\Phi_* F_i = \tilde{F}_i, \quad 1 \le i \le \ell.$$

These restrictions will be removed by mild changes. For instance, if $\Phi$ transforms a family $F$ into a family $\hat{F}$ that commutes with $LF$, the family of the linear part of the $F$, then $(L\Phi)^{-1}(LF_i)L\Phi = L\hat{F}_i$. Therefore, $\Phi(L\Phi)^{-1}$ is tangent to the identity and transforms $F$ into $(L\Phi)\hat{F}(L\Phi)^{-1}$ which commutes with $LF$.

Let $\widehat{\mathcal{O}}_n^D$ be the ring of formal invariants of the family $D$, that is

$$\widehat{\mathcal{O}}_n^D := \{f \in \widehat{\mathcal{O}}_n \mid f(\mathbf{D}_i x) = f(x),\ i = 1, \dots, \ell\}.$$

As defined in Definition 4.5, $\mathcal{C}_2(D)$ is the "non-linear formal centralizer" of $D$, that is

$$\mathcal{C}_2(D) = \{H \in (\widehat{\mathfrak{M}}_n^2)^n \mid H(\mathbf{D}_i x) = \mathbf{D}_i H(x),\ i = 1, \dots, \ell\}.$$

Here $\widehat{\mathfrak{M}}_n$ denotes the maximal ideal of the ring $\widehat{\mathcal{O}}_n$ of formal power series, that is the set of formal power series vanishing at the origin of $\mathbf{C}^n$. Let $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ denote the $j$th unit vector of $\mathbf{C}^n$. If $Q \in \mathbf{N}^n$ with $|Q| > 0$, then $x^Q \in \widehat{\mathcal{O}}_n^D$ if and only if

$$\mu_i^Q = 1, \quad \forall\, 1 \le i \le \ell.$$

Here $\mu_i^Q := \mu_{i1}^{q_1} \cdots \mu_{in}^{q_n}$. If $|Q| > 1$, then $x^Q e_j \in \mathcal{C}_2(D)$ if and only if

$$\mu_i^Q = \mu_{ij}, \quad \forall\, 1 \le i \le \ell.$$

It can be shown (as in proposition 5.3.2 of [Sto00, GW05]) that $\widehat{\mathfrak{M}}_n^D$ is a ring generated by a finite number of monomials $x^{R_1}, \ldots, x^{R_p}$ ($R_i \in \mathbf{N}^n$) and that the non-linear centralizer $\mathcal{C}_2(D)$ of $D$ is a module over $\widehat{\mathfrak{M}}_n^D$ of finite type.

**Definition 8.2.** A formal normal form $\{\hat{F}_i\}_{i=1,\ldots,\ell}$ is said to be **completely integrable** if

(1) each $\hat{F}_i$ has the form

$$x_j' = \hat{\mu}_{ij}(x)x_j, \quad j = 1, \ldots, n$$

where $\hat{\mu}_{ij}$ are invariant by $D$ (i.e. $\hat{\mu}_{ij}(x) \in \widehat{\mathcal{O}}_n^D$) and satisfy $\hat{\mu}_{ij}(0) = \mu_{ij}$;

(2) for each $(j, Q) \in \{1, \ldots, n\} \times \mathbf{N}^n$ with $|Q| \geq 2$,

$$\hat{\mu}_i(x)^Q \equiv \hat{\mu}_{ij}(x) \text{ for all } i = 1, \ldots, \ell, \quad \text{if and only if} \quad \mu_i^Q = \mu_{ij} \text{ for all } i = 1, \ldots, \ell.$$

**Definition 8.3.** A commutative family of germs of diffeomorphisms $F$ is said to be formally (resp. holomorphically) **completely integrable** if it is formally (resp. holomorphically) conjugated to a completely integrable normal form.

**Remark 8.4.** For a completely integrable normal form, we have that for each $Q \in \mathbf{N}^n$, $\mu_i^Q = 1$ for all $1 \leq i \leq \ell$, if and only if $\hat{\mu}_i(x)^Q = 1$ for all $i = 1, \ldots, \ell$. Indeed, if $\hat{\mu}_i(x)^Q = 1$ for all $i = 1, \ldots, \ell$, then evaluation at zero give the result. On the other hand, if $\mu_i^Q = 1$ for all $1 \leq i \leq \ell$, then $\mu_{ij}\mu_i^Q = \mu_{ij}$ for all $1 \leq i \leq \ell$. Hence, according to the definition, $\hat{\mu}_{ij}(x)\hat{\mu}_i^Q(x) = \hat{\mu}_{ij}(x)$ for all $1 \leq i \leq \ell$, which gives $\hat{\mu}_i^Q(x) = 1$ for all $1 \leq i \leq \ell$.

We recall from Definition 4.5 (iii) that a formal diffeomorphism $\Phi$, tangent to identity, is *normalized* (w.r.t. $D$) if it does not have components along the centralizer of $D$, i.e. $\Phi_{j,Q} = 0$ if $\mu_i^Q = \mu_{ij}$ for all $i$, $Q \in \mathbf{N}^n$ with $|Q| \geq 2$. Let $\mathcal{C}^{\mathsf{c}}(D)$ denote the set of the normalized mappings, and let $\mathcal{C}_2^{\mathsf{c}}(D)$ denote the set of mappings $\Phi - I$ with $\Phi \in \mathcal{C}^{\mathsf{c}}(D)$.

**Lemma 8.5.** *Any formal diffeomorphism $\Phi$ of $(\mathbf{C}^n, 0)$, tangent to identity, can be written uniquely as $\Phi = \Phi_1 \circ \Phi_0^{-1}$ with $\Phi_1 \in \mathcal{C}^{\mathsf{c}}(D)$ and $\Phi_0 \in \mathcal{C}(D)$. Furthermore, $\Phi_0, \Phi_1$ are convergent when $\Phi$ is convergent.*

*Proof.* This follows from Lemma 4.8, where $\hat{\mathcal{H}}$ is replaced by $\mathcal{C}_2(D)$ and $\pi$ is defined by

$$\pi\left(\sum f_{j,Q}x^Q e_j\right) = \sum_j \sum_{x^Q e_j \in \mathcal{C}_2(D)} f_{j,Q}x^Q e_j. \qquad \square$$

**Lemma 8.6.** *Let $\hat{F} := \{\hat{F}_i\}$ be a formal normal form of the abelian family of diffeomorphisms $F := \{F_i\}$. Let $\tilde{F} := \{\tilde{F}_i\}$ be another formal normal form of $F$. Then, there exists a formal diffeomorphism $\Phi$, tangent to identity at the origin, such that $\Phi \in \mathcal{C}(D)$ and $\Phi \circ \tilde{F}_i = \hat{F}_i \circ \Phi$. Furthermore, there is a unique $\Phi \in \mathcal{C}^{\mathsf{c}}(D)$ that transforms the family $F$ into a normal form.*

*Proof.* Since both $\hat{F}$ and $\tilde{F}$ are normal forms of $F$, there exists a formal diffeomorphism $\Phi$, tangent to identity at the origin, such that $\tilde{F}_i \circ \Phi = \Phi \circ \hat{F}_i$. According to Lemma 8.5, we can decompose $\Phi = \Phi_1 \circ \Phi_0^{-1}$ where $\Phi_0 \in \mathcal{C}(D)$ and $\Phi_1 \in \mathcal{C}^{\mathsf{c}}(D)$. Hence, we have $\Phi_1^{-1} \circ \tilde{F}_i \circ \Phi_1 = \Phi_0^{-1} \circ \hat{F}_i \circ \Phi_0$. Let us set $G_i := \Phi_0^{-1} \circ \hat{F}_i \circ \Phi_0$. Then $G_i$ is a formal diffeomorphism satisfying $G_i(x) - \mathbf{D}_i x \in \mathcal{C}(D)$. Let us show by induction on $N \geq 2$ that if

$\Phi_1 = I + \Phi_1^N + O(N+1)$ with $\Phi_1^N$ being homogeneous of degree $N$, then $\Phi_1^N = 0$. Indeed, a computation shows that

$$\{G_i\}_N = \{\hat{F}_i\}_N + D_i \circ \Phi_1^N - \Phi_1^N \circ D_i.$$

Express $\Phi_1^N$ as sum of monomial mappings. The monomial mappings are not in $\mathcal{C}(D)$, while those of $F_i$ and $G_i$ are. We obtain $\Phi_1^N = 0$.

To verify the last assertion, assume that $\Psi_* F = \hat{F}$ and $\tilde{\Psi}_* F = \tilde{F}$ are in the normal form. Suppose that $\Psi, \tilde{\Psi}$ are normalized. Then $(\Psi^{-1}\tilde{\Psi})_* \hat{F} = \tilde{\Psi}_*(\Psi^{-1})_* \hat{F}$ is in the normal form. Write $\Psi^{-1}\tilde{\Psi} = \psi_1 \psi_0^{-1}$ with $\psi_1 \in \mathcal{C}^{\mathsf{c}}(D)$ and $\psi_0 \in \mathcal{C}(D)$. Then $(\psi_1)_* \hat{F}$ is in a normal form. From the above proof, we know that $\psi_1 = I$. Now $\Psi = \tilde{\Psi}\psi_0$, which implies that $\Psi = \tilde{\Psi}$. □

**Lemma 8.7.** *If a formal normal form of $F$ is completely integrable so are all other normal forms of $F$; in particular, the unique $\Phi$ in Lemma 8.6 transforms $F$ into a completely integrable normal form.*

*Proof.* By Lemma 8.6, we transform a normal form $\{\hat{F}_i\}$ into another one $\{\tilde{F}_i\}$ by applying a transformation $\Phi$ that commutes with each $D_j$. Hence, we have $\tilde{F}_i := \Phi^{-1} \circ \hat{F}_i \circ \Phi$, for all $i = 1, \ldots, \ell$. Let us write $\Phi(x) = \sum_{Q \in \mathbf{N}^n,\, 1 \le j \le n} \phi_{j,Q} x^Q e_j$. Then

$$\Phi \circ \hat{F}_i(x) = \sum_{Q \in \mathbf{N}^n} \phi_{j,Q} \mu_i(x)^Q x^Q e_j.$$

Suppose that $\{\hat{F}_i\}$ is completely integrable and $\Phi$ commutes with each $D_j$. Then

$$\Phi \circ \hat{F}_i(x) = \operatorname{diag}(\mu_{i1}(x), \ldots, \mu_{in}(x)) \cdot \Phi(x).$$

The conjugacy equation leads to

$$\operatorname{diag}(\mu_{i1}(x), \ldots, \mu_{in}(x)) \cdot \Phi(x) = \begin{pmatrix} \tilde{F}_{i1}(\Phi(x)) \\ \vdots \\ \tilde{F}_{in}(\Phi(x)) \end{pmatrix}.$$

As a consequence, we have

$$\tilde{F}_i(x) = \operatorname{diag}((\tilde{\mu}_{i1}(x), \ldots, \tilde{\mu}_{in}(x)) \cdot x$$

with

$$(\tilde{\mu}_{ij} \circ \Phi(x)) \cdot \Phi_j(x) = \mu_{ij}(x) \cdot \Phi_j(x), \qquad \text{i.e. } \tilde{\mu}_{ij} = \mu_{ij} \circ \Phi^{-1}.$$

Each function $\tilde{\mu}_{ij}$ is an invariant function of $D$ since

$$\tilde{\mu}_{ij}(\mathbf{D}_k x) = \mu_{ij} \circ \Phi^{-1}(\mathbf{D}_k x) = \mu_{ij} \circ D_k(\Phi^{-1}(x)) = \mu_{ij} \circ \Phi^{-1}(x).$$

The second and third conditions of the definition of the complete integrability is obviously satisfied by $\{\tilde{F}_i\}$ since $\tilde{\mu}_{ij} = \mu_{ij} \circ \Phi^{-1}$. □

**Lemma 8.8.** *If a formal normal form of $F$ is linear so are all other normal forms of $F$.*

*Proof.* According to Lemma 8.6, we transform a linear normal form $\{\hat{F}_i\}$ into another one $\{\tilde{F}_i\}$ by applying a transformation $\Phi$ that commutes with each $D_j$. Since $\hat{F}_i(x) = \mathbf{D}_i x$, we have $\tilde{F}_i = \Phi^{-1}(D_i\Phi(x))$, for all $i = 1, \ldots, \ell$. Since $\Phi$ commutes with each map $x \mapsto \mathbf{D}_i x$, then

$$\tilde{F}_i = \Phi^{-1}(D_i\Phi(x)) = \Phi^{-1}(\Phi(\mathbf{D}_i x)) = \mathbf{D}_i x. \qquad \square$$

**Definition 8.9.** We say that *the family $D$ is of Poincaré type* if there exist constants $d > 1$ and $c > 0$ such that, for each $(j, Q) \in \{1, \ldots, n\} \times \mathbf{N}^n$ that satisfies $\mu_m^Q - \mu_{mj} \neq 0$ for some $m$, there exists $(i, Q') \in \{1, \ldots, n\} \times \mathbf{N}^n$ such that $\mu_k^{Q'} = \mu_k^Q$ for all $1 \leq k \leq \ell$, $\mu_i^{Q'} - \mu_{ij} \neq 0$, and

$$\max\left(|\mu_i^{Q'}|, |\mu_i^{-Q'}|\right) > c^{-1}d^{|Q'|}, \qquad Q' - Q \in \mathbf{N}^n \cup (-\mathbf{N}^n).$$

Such a condition has appeared in the definition of the good set in [BHV10].

**Definition 8.10.** Let $f = \sum_{Q\in\mathbf{N}^n} f_Q x^Q$ and $g = \sum_{Q\in\mathbf{N}^n} g_Q x^Q$ be two formal power series. We say that $g$ *majorizes* $f$, written as $f \prec g$, if $g_Q \geq 0$ and $|f_Q| \leq g_Q$ for all $Q \in \mathbf{N}^n$. Set

$$\bar{f} := \sum_{Q\in\mathbf{N}^n} |f_Q| x^Q.$$

**Theorem 8.11.** *Let $F$ be an abelian family of germs of holomorphic diffeomorphisms at the origin of $\mathbf{C}^n$. Assume that it is formally completely integrable and that its linear part at the origin is of Poincaré type. Then $F$ is holomorphically conjugated to a normal form $\hat{F} = \{\hat{F}_1, \ldots, \hat{F}_\ell\}$ so that each $\hat{F}_i$ is defined by*

$$x_j' = \mu_{ij}(x)x_j, \quad j = 1, \ldots, n$$

*where $\mu_{ij}(x)$ are germs of holomorphic functions invariant under $D$ and $\mu_{ij}(0) = \mu_{ij}$. In fact, the unique normalized mapping $\Phi$ in Lemma 8.6 is convergent.*

The last assertion follows from Lemma 8.5 and Lemma 8.7. Such a result for commuting germs of vector fields is known [Sto00] under a Brjuno-type of small divisor conditions. Such an integrability result for a single germ of two-dimensional hyperbolic real analytic area-preserving mapping was proved by Moser [Mos56]. For a single germ of reversible biholomorphism of very special type, this result was due to Moser-Webster [MW83]; indeed, as shown by Moser-Webster [MW83][lemma 3.2], a germ of (hyperbolic reversible) mapping of the form $\phi = \tau_1\tau_2$ where the $\tau_1, \tau_2$ are germs holomorphic involutions, is formally completely integrable under some condition on the linear parts at the origin of $\tau_1, \tau_2$. Our proof is inspired from these proofs. However, in Moser-Webster situation, there is only two eigenvalues $\mu$ and $\mu^{-1}$ and the remaining eigenvalues are 1 with multiplicity. The Poincaré type condition in the above theorem, that is $|\mu| \neq 1$, is necessary to obtain the convergence as demonstrated by Moser-Webster. We shall use our result in the next section in order to normalize a special kind of CR-singularities.

*Proof.* Let us conjugate, simultaneously, each $F_i = \mathbf{D}_i x + f_i$ to $\hat{F}_i := \hat{\mathbf{D}}_i(x)x$ by the action of $\Phi(x) = x + \phi(x)$ where $\phi(0) = 0$ and $\phi'(0) = 0$. Here, $\hat{\mathbf{D}}_i(x)$ denotes the matrix $\mathrm{diag}(\hat{\mu}_{i1}(x), \ldots, \hat{\mu}_{in}(x))$ and each $\hat{\mu}_{ij}(x)$ is a formal power series invariant under $D$, i.e. $\hat{\mu}_{ij}(x) \in \widehat{\mathcal{O}}_n^D$. We can assume that $\Phi$ does not have a non-zero component along the

centralizer of $D$; indeed, by Lemma 8.7, we can assume that $\Phi$ is normalized w.r.t $D$. Then, for each $i = 1, \ldots, \ell$, we have

$$F_i \circ \Phi(x) = \mathbf{D}_i x + f_i(\Phi)(x) + \mathbf{D}_i \phi(x), \quad \Phi \circ \hat{F}_i(x) = \hat{\mathbf{D}}_i(x) x + \phi(\hat{F}_i)(x).$$

Equation $F_i \circ \Phi = \Phi \circ \hat{F}_i$ reads

$$\left(\phi(\hat{\mathbf{D}}_i(x)x) - \mathbf{D}_i\phi(x)\right) + \left(\hat{\mathbf{D}}_i(x) - \mathbf{D}_i\right) x = f_i(\Phi)(x) \quad i = 1, \ldots, \ell. \tag{8.1}$$

Our convergence proof is based on two conditions: the existence of a formal $\phi \in \mathcal{C}^{\mathsf{c}}(D)$ that satisfies the above equation, and the Poincaré type condition on the linear part $D$. We already know that $\phi$ is unique. We shall project equation (8.1) along the "non-resonant" space (i.e. the space $\mathcal{C}^{\mathsf{c}}(D)$ of normalized mappings w.r.t. $\mathbf{D}$). The mapping $\phi$ also solves this last equation and we shall majorize it using that projected equation.

Let us first decompose these equations along the "resonant" and "non-resonant" parts, i.e. $\mathcal{C}_2(D)$ and $\mathcal{C}_2^{\mathsf{c}}(D)$. Since $\phi = \sum_{Q \in \mathbf{N}^n, |Q| \geq 2} \phi_{j,Q} x^Q e_j$ is normalized then $\phi_{j,Q} = 0$ for some $Q \in \mathbf{N}^n$, $|Q| \geq 2$ and $1 \leq j \leq n$, if we have $\mu_m^Q = \mu_{mj}$ for all $m$. We recall that, since each $\mathbf{D}_i$ is a diagonal matrix, then a map belongs to the centralizer of $D$ if and only if each monomial map of its Taylor expansion at the origin belongs to this centralizer as well. Since the $\hat{\mu}_{ij}$ is a formal invariant function then

$$\phi(\hat{\mathbf{D}}_i(x)x) = \sum_{Q \in \mathbf{N}^n, |Q| \geq 2} \phi_{j,Q} \hat{\mu}_i^Q(x) x^Q e_j =: \sum_{Q' \in \mathbf{N}^n, |Q'| \geq 2} \psi_{j,Q'} x^Q e_j.$$

The latter contains only non-resonant terms, that is that if $\mu_i^{Q'} = \mu_{ij}$ for all $i$, then $\psi_{j,Q'} = 0$. Indeed, $\hat{\mu}_i^Q(x)$ contains monomials of the form $x^P$ with $\mu_i^P = 1$ for all $1 \leq i \leq \ell$. Hence, $\psi_{j,Q'}$ is a linear combination of $\phi_{j,Q}$ such that $Q' = Q + P$ with $\mu_i^P = 1$ for all $i$. Therefore, if $\mu_i^{Q'} = \mu_{ij}$ for all $i$, then for all these $Q$'s, we have $\mu_i^Q = \mu_i^{Q'} = \mu_{ij}$ for all $i$ so that $\phi_{j,Q} = 0$; that is $\psi_{j,Q'} = 0$.

Hence, the projection on the resonant mappings in $\mathcal{C}_2(D)$ leads to

$$\left(\hat{\mathbf{D}}_i(x) - \mathbf{D}_i\right) x = \{f_i(\Phi)(x)\}_{res}, \quad i = 1, \ldots, \ell. \tag{8.2}$$

Here for any formal mapping $g(x) = O(|x|^2)$ on $\mathbf{C}^n$, we define the projection on $\mathcal{C}_2(D)$ by

$$(g(x))_{res} = \sum_j \sum_{\forall i, \mu_i^Q = \mu_{ij}} g_{j,Q} x^Q e_j. \tag{8.3}$$

The projection $g$ onto $\mathcal{C}_2^{\mathsf{c}}(D)$ is defined as $g(x) - (g(x))_{res}$, i.e. it is the projection of $g$ on the non-resonant mappings.

Let us consider the projection on the non-resonant mappings. We first need to decompose power series according to a non-homogeneous equivalence relation on their coefficients. Let us define the equivalence relation on $\{1, \ldots, n\} \times \mathbf{N}^n$ by

$$(j, Q) \sim (\tilde{j}, \tilde{Q}), \text{ if } \mu_{ij} - \hat{\mu}_i^Q(x) = \mu_{i\tilde{j}} - \hat{\mu}_i^{\tilde{Q}}(x) \text{ for all } 1 \leq i \leq \ell.$$

Here the identities hold as formal power series. Let $\Delta$ be the set of the equivalent classes on the non-resonant multiindex set

$$\left\{(j, Q) \in \{1, \ldots, n\} \times \mathbf{N}^n \colon (\mu_1^Q - \mu_{1j}, \ldots, \mu_\ell^Q - \mu_{\ell j}) \neq 0, |Q| > 1\right\}.$$

If $\mu_k^Q - \mu_{kj} \neq 0$ for some $k$, clearly $\hat{\mu}_k^Q - \mu_{kj}$ is not identically zero. We can decompose any formal power series map $f$ along these equivalent classes and the resonant part of the mapping. Let $\delta \in \Delta$ and $f = \sum_{Q=\in \mathbf{N}^n, 1\leq j\leq n} f_{j,Q} x^Q e_j$ with $f = O(2)$. We can write

$$f_\delta(x) := \sum_{(j,Q)\in\delta} f_{j,Q} x^Q e_j, \quad \sum_{\delta\in\Delta} f_\delta(x) \prec \overline{f}(x). \tag{8.4}$$

We denote by $\widehat{\mathfrak{M}}_{n,\delta}^n$ the vector space of such maps. To a given equivalent class $\delta$, we now associate a representative $(j_\delta, Q_\delta)$, and later we shall identify an equation among $n$ equations in (8.1) for estimation.

Since $\phi$ contains no resonant mappings, then

$$\phi = \sum_{\delta\in\Delta} \phi_\delta. \tag{8.5}$$

Let us decompose the projection onto non-resonant mappings in $\mathcal{C}_2^{\mathsf{c}}(D)$ of equation (8.1) along each equivalent class $\delta$ as follows. Using the definition of the equivalence class $\Delta$, we obtain

$$\left[\hat{\mu}_i^{Q_\delta}(x) - \mu_{ij_\delta}\right] \phi_\delta(x) = \left\{f_i(\Phi)\right\}_\delta(x), \quad \forall i = 1, \ldots, \ell \tag{8.6}$$

where $\{f\}_\delta$ denotes the projection of $f$ on $\widehat{\mathfrak{M}}_{n,\delta}^n$, defined by (8.4).

For each $(j_\delta, Q_\delta) \in \Delta$, we know that $\mu_k^{Q_\delta} - \mu_{kj_\delta} \neq 0$ for some $k$. By the Poincaré type condition, there exist $i$ and $Q'_\delta \in \mathbf{N}^n$ such that

$$\mu_i^{Q'_\delta} - \mu_{ij_\delta} \neq 0; \quad \mu_m^{Q'_\delta} = \mu_m^{Q_\delta}, \quad \forall 1 \leq m \leq \ell; \quad Q'_\delta - Q_\delta \in \mathbf{N}^n \cup (-\mathbf{N}^n) \tag{8.7}$$

and, furthermore, one of the following holds:

$$|\mu_i^{Q'_\delta}| \leq c d^{-|Q'_\delta|}, \tag{8.8}$$

$$|\mu_i^{-Q'_\delta}| \leq c d^{-|Q'_\delta|}. \tag{8.9}$$

Here, $d > 1$ does not depend on $Q_\delta$. So, let us use the $i$th equation of (8.6) to estimate $\phi_\delta$. We have, for that $i$,

$$\phi_\delta = \left[\hat{\mu}_i^{Q_\delta} - \mu_{ij_\delta}\right]^{-1} \left\{f_i(\Phi)\right\}_\delta. \tag{8.10}$$

Therefore, we have established the uniqueness of $\phi$ under (8.5) and (8.10), and under the condition that $\phi$ satisfies the equation when (8.1) is projected onto $\mathcal{C}^{\mathsf{c}}(D)$. The existence of $\phi$ is ensured by assumption. We now consider the convergence of $\phi$. By (8.7) and Remark 8.4, we obtain $\hat{\mu}_i^{Q'_\delta - Q_\delta} = 1$. This allows us to rewrite (8.10) as

$$\phi_\delta = \left[\hat{\mu}_i^{Q'_\delta} - \mu_{ij_\delta}\right]^{-1} \left\{f_i(\Phi)\right\}_\delta. \tag{8.11}$$

We majorize this power series.

Recall that $\hat{\mu}_{ij}(0) = \mu_{ij}$. Let us set

$$M_{ij}(x) := \mu_{ij}^{-1} \hat{\mu}_{ij}(x).$$

We have $M_{ij}(0) = 1$ and we decompose

$$M_{ij}(x) = \sum_{Q \in \mathbf{N}^n} M_{ij,Q} x^Q.$$

Let us set $\mu^* := \max_{ij}\{|\mu_{ij}|, |\mu_{ij}^{-1}|\}$, and

$$m_i = \sum_{Q \in \mathbf{N}^n} \max_{1 \le j \le n} |M_{ij,Q}| x^Q, \quad m = \sum_{Q \in \mathbf{N}^n} \max_{1 \le i \le \ell,\, 1 \le j \le n} |M_{ij,Q}| x^Q.$$

Note that $m(0) = 1$. Then $M_{ij} \prec m$ and

$$M_{ij}^{-1} = \frac{1}{1 + (M_i - 1)} \prec \frac{1}{1 - (m-1)} = \frac{1}{2 - m}.$$

Here and in what follows, if $f(x)$ is a formal power series with $f(0) = 0$, then for any number $a \neq 0$, $\frac{1}{a - f(x)}$ stands for the formal power series in $x$ for

$$\frac{1}{a}\left\{1 + \sum_{n=1}^{\infty} (a^{-1} f(x))^n\right\}.$$

To simplify notation in (8.11), let us write $Q$ for $Q'_\delta$ and $j$ for $j_\delta$. Fix $d_1$ with $1 < d_1 < d$. We consider the first case that $\mu^* c d^{-|Q|} > d_1^{-|Q|}$. Since $d > d_1$, we have only finitely many such $Q's$ (recall that each $Q$ has the form $Q'_\delta$). The function $M_i \mapsto \mu_{ij} - \mu_i^Q M_i^Q$ is holomorphic in $M_i \in \mathbf{C}^p$ at $M_i = (1, \ldots, 1)$ and does not vanish at this point. Hence, the function

$$(\mu_{ij} - \hat{\mu}_i^Q)^{-1} = (\mu_{ij} - \mu_i^Q M_i^Q)^{-1}$$

is also holomorphic at $M_i = (1, \ldots, 1)$. For all $Q's$ in the first case, we have

$$(\mu_{ij} - \hat{\mu}_i^Q)^{-1} \prec \frac{C}{1 - C(\overline{M}_{i1} - 1 + \cdots + \overline{M}_{in} - 1)} \prec \frac{C}{1 - nC(m-1)}.$$

We now consider the second case that $\mu^* c d^{-|Q|} \le d_1^{-|Q|}$. In case (8.8), we obtain

$$\begin{aligned}
(\hat{\mu}_i^Q - \mu_{ij})^{-1} &= -\mu_{ij}^{-1}(1 - \mu_{ij}^{-1}\mu_i^Q M_i^Q)^{-1} \\
&\prec \mu^* \left[1 - \mu^* c d^{-|Q|} m^{|Q|}\right]^{-1} \\
&\prec \mu^* \left[1 - d_1^{-|Q|} m^{|Q|}\right]^{-1} \\
&\prec \mu^* \left[1 - d_1^{-1} m\right]^{-1}.
\end{aligned}$$

In case (8.9), we have

$$\begin{aligned}
(\hat{\mu}_i^Q - \mu_{ij})^{-1} &= -\mu_i^{-Q} M_i^{-Q} \left[1 - \mu_{ij} \hat{\mu}_i^{-Q} M_i^{-Q}\right]^{-1} \\
&\prec c d^{-|Q|} (2 - m)^{-|Q|} \left[1 - \mu^* c d^{-|Q|} (2 - m)^{-|Q|}\right]^{-1} \\
&\prec (\mu^*)^{-1} d_1^{-|Q|} (2 - m)^{-|Q|} \left[1 - d_1^{-|Q|} (2 - m)^{-|Q|}\right]^{-1} \\
&\prec (\mu^*)^{-1} \left[1 - d_1^{-1} (2 - m)^{-1}\right]^{-1}.
\end{aligned}$$

We have obtained the estimates for the second case. Therefore, we have shown that for any $Q = Q'_\delta$ and $1 \leq j \leq \ell$,

$$(\hat{\mu}_i^Q - \mu_{ij})^{-1} \prec S(m-1). \tag{8.12}$$

Here $S(t)$ is a convergent power series in $t$ that is independent of all $Q's$ of the form $Q'_\delta$.

Let us set

$$f^* := \sum_{Q \in \mathbf{N}^n} \max_{1 \leq i \leq \ell,\ 1 \leq j \leq n} |f_{ij,Q}| x^Q e_j.$$

By the definition of the equivalence relation on multiindices, we have

$$\sum_{\delta \in \Delta} f^*_\delta \prec f^*. \tag{8.13}$$

According to (8.11) and (8.12), we have

$$\phi_\delta \prec S(m-1) \left\{ f^*(\bar{\Phi}) \right\}_\delta .$$

Now (8.4) and (8.13) imply

$$\phi \prec S(m-1) f^*(\bar{\Phi}). \tag{8.14}$$

Let us project (8.2) onto the $k$th components of $\mathcal{C}_2(D)$ as follows. For a power series map $g$, we define

$$g_{res,k}(x) = \sum_{\mu^Q = \mu_k} g_{k,Q} x^Q.$$

By the definition of $g_{res}$ in (8.3), $g_{res} = (g_{res,1}, \ldots, g_{res,n})$. We have

$$\mu_{ik} \left( M_{i,k}(x) - 1 \right) x_k = \left( \hat{\mu}_{ik}(x) - \mu_{ik} \right) x_k = \{ f_{ik}(\Phi) \}_{res,k}(x).$$

Therefore, for all $1 \leq k \leq n$,

$$(m-1) x_k \prec \frac{1}{\min_{i,j} |\mu_{i,j}|} f^*(\bar{\Phi}). \tag{8.15}$$

Let us set $\mu_* := \frac{1}{\min_{i,j} |\mu_{ij}|}$. We set $x_1 = t, \ldots, x_n = t$ in $\overline{\Phi}(x)$ and $m(x)$. Let $\phi(t)$, $\overline{\Phi}(t)$, and $m(t)$ still denote $\phi(t, \ldots, t)$, $\overline{\Phi}(t, \ldots, t)$, and $m(t, \ldots, t)$, respectively. Let

$$tW(t) := \phi(t) + (m(t) - 1)t.$$

We have $W(0) = 0$, $\phi(t) \prec tW(t)$, and $(m(t) - 1) \prec W(t)$. From estimates (8.14) and (8.15), we obtain

$$tW(t) \prec \mu_* f^*(\bar{\Phi}(t)) + S(m(t) - 1) f^*(\bar{\Phi}(t)). \tag{8.16}$$

Since $f_{ij}(x) = O(|x|^2)$, there exists a constant $c_1$ such that

$$f^*(x) \prec \frac{c_1 (\sum_j x_j)^2}{1 - c_1 (\sum_j x_j)}.$$

Hence, estimate (8.16) reads

$$tW(t) \prec (\mu_* + S(m(t)-1))\,\frac{c_1(n(t+\phi))^2}{1-c_1n(t+\phi)} \prec (\mu_* + S(W(t)))\,\frac{c_1t^2(n(1+W(t)))^2}{1-c_1nt(1+W(t))}. \tag{8.17}$$

Let us consider the equation in the unknown $U$ with $U(0)=0$ :

$$U(t)(1-c_1nt(1+U(t))) = (\mu_* + S(W(t)))\,c_1t(n(1+U(t)))^2. \tag{8.18}$$

According to the implicit function theorem, there exists a unique germ of holomorphic function $U(t)$, solution to (8.18) with $U(0)=0$. According to inequality (8.17), the function $W$ is dominated by $U$ : $W(t) \prec U(t)$. This can be seen by induction on the degree of the Taylor polynomials at the origin. Therefore, $W$ converges at the the origin. The theorem is proved. □

## 9. Real manifolds with an abelian CR-singularity

Let us consider a real analytic manifold $M$ with a CR-singularity at the origin, as in section 2. We assume that its complexification $\mathcal{M}$ has the maximum number of deck transformations with respect to each projection $\pi_1$ and $\pi_2$. The deck transformations are then generated by germs of holomorphic involutions of $(\mathbf{C}^{2p},0)$, which are denoted by

$$\{\tau_{11},\dots,\tau_{1p}\},\quad \{\tau_{21},\dots,\tau_{2p}\}.$$

Recall that both families are abelian, that is that

$$\tau_{ij}\circ\tau_{ik} = \tau_{ik}\circ\tau_{ij}.$$

They are intertwined by the antiholomorphic involution $\rho$:

$$\tau_{2j} = \rho\circ\tau_{1j}\circ\rho.$$

Let us consider the following germs of holomorphic diffeomorphisms :

$$\sigma_i := \tau_{1i}\circ\tau_{2i},\quad 1\le i\le e_*+h_*, \tag{9.1}$$

$$\sigma_s := \tau_{1s}\circ\tau_{2(s_*+s)},\quad \sigma_{s+s_*} = \tau_{1(s+s_*)}\circ\tau_{2s},\quad e_*+h_* < s \le p-s_*. \tag{9.2}$$

Notice that the above property holds for quadrics of the complex case by Proposition 2.13. The family $\{\sigma_i\}$ is reversible with respect to $\rho$. More precisely, we have the following relations

$$\sigma_i^{-1} = \rho\sigma_i\rho,\quad 1\le i\le e_*+h_*;\quad \sigma_{s+s_*}^{-1} = \rho\sigma_s\rho,\quad e_*+h_* < s \le p-s_*.$$

**Definition 9.1.** We say that the manifold $M$ has an **abelian CR-singularity** at the origin if its complexification $\mathcal{M}$ has the maximum number of deck transformation and if the family $\{\sigma_1,\dots,\sigma_p\}$ of germs of biholomorphisms at the origin of $\mathbf{C}^{2p}$ is abelian, i.e.

$$\sigma_i\sigma_j = \sigma_j\sigma_i.$$

**Definition 9.2.** A *product quadric* is a submanifold in $\mathbf{C}^{2p}$ defined by

$$\begin{aligned} z_{p+e} &= (z_e + 2\gamma_e \overline{z}_e)^2, && 1 \le e \le e_* \\ z_{p+h} &= (z_h + 2\gamma_h \overline{z}_h)^2, && e_*+1 \le h \le e_*+h_* \\ z_{p+s} &= (z_s + 2\gamma_s \overline{z}_{s+s_*})^2, \\ z_{p+s+s_*} &= (z_{s+s_*} + 2(1-\overline{\gamma}_s)\overline{z}_s)^2, && e_*+h_* < s \le p-s_* \end{aligned}$$

with $0 < \gamma_e < 1/2, \gamma_h > 1/2$, and $\gamma_s \in (1/2,\infty) \times i(0,\infty)$.

In what follows, we assume that $M$ has an *abelian CR-singularity* at the origin and that $M$ is a *higher order perturbation of a product quadric.*

The aim of this section is to show that such an analytic perturbation with an abelian CR-singularity and no hyperbolic component is holomorphically conjugate to a normal form. We shall give two proofs of this result. The first one rests on Moser-Webster result [MW83][theorem 4.1] applied successively to each $\sigma_i$. The other one is based on the fact that the family $\{\sigma_i\}$ is formally completely integrable and their linear part is of Poincaré type. We then apply Theorem 8.11.

## 9.1. **Normal forms for real submanifolds with an abelian CR singularity.**

**Theorem 9.3.** *Let $M$ be a germ of real analytic submanifold in $\mathbf{C}^n$ at an abelian CR-singularity at the origin. Suppose that $M$ is a higher order perturbation of a product quadric of which $\gamma_1, \dots, \gamma_p$ satisfy* (1.2)*. Assume that the associated $\sigma$ of $M$ has distinct eigenvalues. Suppose that $M$ does not have a hyperbolic component (i.e. $e_* \ge 0, s_* \ge 0, h_* = 0$). Then there exists a germ of biholomorphism $\psi$ that commutes with $\rho$ and such that, for $1 \le i \le p$ and $k = 1, 2$*

$$\psi^{-1} \circ \sigma_i \circ \psi : \begin{cases} \xi_i' = M_i(\xi\eta)\xi_i \\ \eta_i' = M_i^{-1}(\xi\eta)\eta_i \\ \xi_j' = \xi_j \\ \eta_j' = \eta_j, \quad j \ne i, \end{cases} \qquad \psi^{-1} \circ \tau_{ki} \circ \psi : \begin{cases} \xi_i' = \Lambda_{ki}(\xi\eta)\eta_i \\ \eta_i' = \Lambda_{ki}^{-1}(\xi\eta)\xi_i \\ \xi_j' = \xi_j \\ \eta_j' = \eta_j, \quad j \ne i. \end{cases} \tag{9.3}$$

*Moreover, we have*

$$\Lambda_{1e} = \overline{\Lambda_{1e} \circ \rho_z}, \qquad 1 \le e \le e_* \tag{9.4}$$

$$\Lambda_{1s} = \overline{\Lambda_{1(s+s_*)}^{-1} \circ \rho_z}, \quad e_* < s \le p - s_* \tag{9.5}$$

$$\Lambda_{2j} = \Lambda_{1j}^{-1}, \qquad 1 \le j \le p. \tag{9.6}$$

*Proof.* We will present two convergence proofs: one is based on a convergent theorem of Moser and Webster and another is based on Theorem 8.11. We first use some formal results obtained by Moser and Webster [MW83] and some results in section 8.

Since $M$ is a higher order perturbation of a product quadric, there are linear coordinates such that, for $1 \le i \le p$ and $k = 1, 2$, $\tau_{k,i}$ and $\sigma_i$ are higher order perturbations of

$$S_i : \begin{cases} \xi_i' = \mu_i \xi_i \\ \eta_i' = \mu_i^{-1} \eta_i \\ \xi_j' = \xi_j \\ \eta_j' = \eta_j, \quad j \neq i, \end{cases} \qquad T_{ki} : \begin{cases} \xi_i' = \lambda_{ki} \eta_i \\ \eta_i' = \lambda_{ki}^{-1} \xi_i \\ \xi_j' = \xi_j \\ \eta_j' = \eta_j, \quad j \neq i. \end{cases}$$

For elliptic coordinates, this was computed in [MW83] and recalled in (2.24). For complex coordinates, this is computed in (2.27),(2.29). Recall that

$$\sigma_m = \tau_{1m} \circ \tau_{2m}, \quad 1 \le m \le p.$$

Since $|\mu_1| \neq 1$, then by theorem 4.1 of Moser-Webster ([MW83]), there is a unique convergent transformation $\psi_1$ normalized w.r.t. $S_1$ such that $\sigma_1^* := \psi_1^{-1} \circ \sigma_1 \circ \psi_1$ and $\tau_{i1}^* := \psi_1^{-1} \circ \tau_{i1} \circ \psi_1$ are given by

$$\sigma_1^* : \begin{cases} x_1' = M_1(\xi, \eta) \xi_1 \\ \eta_1' = M_1^{-1}(\xi, \eta) \eta_1 \\ \xi_j' = \xi_j \\ \eta_j' = \eta_j, \quad j \neq 1, \end{cases} \qquad \tau_{k1}^* : \begin{cases} \xi_1' = \Lambda_{k1}(\xi, \eta) \eta_1 \\ \eta_1' = \Lambda_{k1}^{-1}(\xi, \eta) \xi_1 \\ \xi_j' = \xi_j \\ \eta_j' = \eta_j, \quad j \neq 1. \end{cases} \tag{9.7}$$

Here $k = 1, 2$. It is a simple fact (e.g. see Lemma 8.7, $D = \{S_1\}$) that there is a unique $\phi_1 \in \mathcal{C}^c(S_1)$ such that $\phi^{-1} \sigma_1 \phi$ is in the centralizer of $S_1$. Therefore, $\phi_1 = \psi_1$ is also convergent.

Furthermore, we have $M_1(\xi, \eta) = \Lambda_{11}(\xi, \eta) \Lambda_{21}^{-1}(\xi, \eta)$; and $\Lambda_{11}, \Lambda_{21}, M_1$ are invariant by $S_1$. In the new coordinates, let us denote $\tau_{im}, \sigma_m$ by the same symbols for $m > 1$. However, $\sigma_1 = \sigma_1^*$ and $\tau_{k1} = \tau_{k1}^*$. Since each $\sigma_m$ commutes with $\sigma_1$, then $\sigma_m$ is in the centralizer of $S_1$. Indeed, according to [MW83][Lemma 3.1](or Lemma 8.5 with $D = \{S_1\}$), we can decompose $\sigma_m = \sigma_m^1 \sigma_m^0$ where $\sigma_m^1$ is normalized w.r.t $S_1$ and $\sigma_m^0$ is in the centralizer of $S_1$. Write $\sigma_1 \sigma_m = \sigma_m \sigma_1$ as

$$(\sigma_m^1)^{-1} \sigma_1 \sigma_m^1 = \sigma_m^0 \sigma_1 (\sigma_m^0)^{-1}.$$

Since $\sigma_m^0 \sigma_1 (\sigma_m^0)^{-1}$ belongs to $\mathcal{C}(S_1)$, so does $(\sigma_m^1)^{-1} \sigma_1 \sigma_m^1$. Then applying the uniqueness of $\psi_1$ stated earlier to $\sigma_m^1$, we conclude that $\sigma_m^1 = I$ and $\sigma_m = \sigma_m^0$ is in the centralizer of $S_1$.

Let us verify that $\sigma_m^0$ or in general each (formal) transformation $\varphi$ in $\mathcal{C}(S_1)$ preserves the form of $\sigma_1^*$ and $\tau_{i1}^*$. Indeed, $\varphi^{-1}$ commutes with $S_1$ too. Thus $\varphi^{-1} \sigma_1^* \varphi$ commutes with $S_1$ and its linear part is $S_1$. The linear part of $\varphi_1$ must preserve the eigenspaces of $S_1$ and hence it is given by

$$\xi_1 \to a \xi_1, \quad \eta_1 \to b \eta_1, \quad (\xi_*, \eta_*) \to \phi(\xi_*, \eta_*)$$

for $\xi_* = (\xi_2, \ldots, \xi_n)$ and $\eta_* = (\eta_2, \ldots, \eta_n)$. By a simple computation, the linear part of $\varphi^{-1} \tau_{k1}^* \varphi$ still has the form (9.7). According to [MW83][lemma 3.2], there a unique normalized mapping $\Psi$ that normalizes $\varphi^{-1} \sigma_1^* \varphi$ and the $\varphi^{-1} \tau_{k1}^* \varphi$'s. According to the uniqueness property of Lemma 8.6, $\Psi = Id$. Therefore, $\varphi$ preserves the forms of $\tau_{i1}^*$ and $\sigma_1^*$.

Let $\psi_2$ be the unique biholomorphic map normalized w.r.t. $S_2$ such that $\psi_2^{-1}\sigma_2\psi_2 = \sigma_2^*$ and $\psi_2^{-1}\tau_{k2}\psi_2 = \tau_{k2}^*$ are in the normal form :

$$\sigma_2^*: \begin{cases} \xi_2' = M_2(\xi,\eta)\xi_2 \\ \eta_2' = M_2^{-1}(\xi,\eta)\eta_2 \\ \xi_j' = \xi_j \\ \eta_j' = \eta_j, \quad j \neq 2, \end{cases} \qquad \tau_{k2}^*: \begin{cases} \xi_2' = \Lambda_{k2}(\xi,\eta)\eta_2 \\ \eta_2' = \Lambda_{k2}^{-1}(\xi,\eta)\xi_2 \\ \xi_j' = \xi_j \\ \eta_j' = \eta_j, \quad j \neq 2. \end{cases} \tag{9.8}$$

Here $k = 1, 2$, and $M_2$ and $\Lambda_{k2}$ are invariant by $S_2$. Since $\sigma_2$ commutes with $S_1$, we have

$$(S_1^{-1}\psi_2 S_1)^{-1} \circ \sigma_2 \circ (S_1^{-1}\psi_2 S_1) = S_1^{-1}\sigma_2^* S_1.$$

Note that $S_1^{-1}\sigma_2^* S_1$ (resp. $S_1^{-1}\tau_{k2}^* S_1$) has the form (9.8) in which $M_2$ (resp. $\Lambda_{k2}$) is replaced by $M_2 \circ S_1$ (resp. $\Lambda_{k2} \circ S_1$). In other words $S_1^{-1}\sigma_2^* S_1$ and $S_1^{-1}\tau_{k2}^* S_1$ are still of the form (9.8). Since $S_1$ is diagonal, then $S_1^{-1}\psi_2 S_1$ remains normalized w.r.t. $S_2$. Applying the above uniqueness on $\psi_2$ for $\sigma_2$, we conclude that $\psi_2 = S_1^{-1}\psi_2 S_1$. This shows that $\psi_2$ preserves the forms of $\tau_{k1}^*$ and $\sigma_1^*$. By the same argument as above, we have $\sigma_m^* \in \mathcal{C}(S_1, S_2)$.

In summary, we have found holomorphic coordinates so that $\tau_{ij} = \tau_{im}^*$ and $\sigma_m = \sigma_m^*$ for $m = 1, 2$. As mentioned previously, we know that $\sigma_1^*, \sigma_2^*, \sigma_3, \ldots, \sigma_m$ commute with $S_1$ and $S_2$. In particular, $M_1, M_2$ are invariant by $S_1, S_2$. Repeating this procedure, we find a holomorphic map $\phi$ so that all $\phi^{-1}\sigma_j\phi = \sigma_j^*$ and $\phi^{-1}\tau_{kj}\phi = \tau_{kj}^*$ are in the normal forms. Furthermore, $M_i$ and $\Lambda_{k,i}$ are invariant by $\{S_1, \ldots, S_p\}$.

By Lemma 8.5, we decompose $\phi = \phi_1\phi_0^{-1}$ where $\phi_1$ is normalized w.r.t. $\{S_1, \ldots, S_p\}$ and $\phi_0$ is in the centralizer of $\{S_1, \ldots, S_p\}$. Then $\phi_1^{-1}\sigma_j\phi_1 = \sigma_j^*$ and $\phi_1^{-1}\tau_{ij}\phi_1 = \tau_{ij}^*$ are in the normal forms, since $\phi_0$ commutes with $S_j$. We want to show that $\phi_1$ commutes with $\rho$.

Note that $\sigma_e^{-1} = \rho\sigma_e\rho$ and $\sigma_{s+s_*}^{-1} = \rho\sigma_s\rho$. Thus $(\rho\phi_1\rho)^{-1}\sigma_j(\rho\phi_1\rho) = \tilde{\sigma}_j^*$ where $\tilde{\sigma}_e^* := \rho(\sigma_e^*)^{-1}\rho$ and $\tilde{\sigma}_s^* := \rho(\sigma_{s+s*}^*)^{-1}\rho$. According to (3.7), we see that $\rho\phi_1\rho$ is still normalized w.r.t. $\{S_1, \ldots, S_p\}$. By Lemma 8.6, we know that there is a unique normalized formal mapping $\phi_1$ such that $\phi_1^{-1}\sigma_j\phi_1$ are in the centralizer of $\{S_1, \ldots, S_p\}$. Since $\tilde{\sigma}_j^*$ belongs to the centralizer of $\{S_1, \ldots, S_p\}$, then we have $\rho\phi_1\rho = \phi_1$.

Now, $\tau_{2j}^* = \rho\tau_{1j}^*\rho$ follows from $\tau_{2j} = \rho\tau_{1j}\rho$. This shows that

$$\begin{aligned} \Lambda_{2e} &= \overline{\Lambda_{1e}^{-1} \circ \rho}, \quad 1 \le e \le e_*, \\ \Lambda_{2s} &= \overline{\Lambda_{1(s+s_*)} \circ \rho}, \\ \Lambda_{2(s+s_*)} &= \overline{\Lambda_{1s} \circ \rho}, \quad e_* + h_* < s \le p - s_*. \end{aligned}$$

Let $\phi_2$ be defined by

$$\xi_j' = (\Lambda_{1j}^{1/2} M_j^{1/4})(\xi\eta)\xi_j, \quad \eta_j' = (\Lambda_{1j}^{-1/2} M_j^{-1/4})(\xi\eta)\eta_j, \quad 1 \le j \le p.$$

For a suitable choice of the roots, we have $\phi_2\rho = \rho\phi_2$. Furthermore, $\phi_2$ preserves all invariant functions of $\{S_1, \ldots, S_p\}$. Hence, each $\phi_2^{-1} \circ \phi_1^{-1} \circ \tau_{ki} \circ \phi_1 \circ \phi_2$ has the form $\tau_{kj}^*$ stated in Theorem 9.3.

We now present another proof by using the more general Theorem 8.11.

Note that the above proof is valid at the formal level without using the convergence result of Moser and Webster. More specifically, if $\tau_{ij}$ are given by formal power series with $\sigma_1, \ldots, \sigma_p$ commuting pairwise, there exists a formal map $\psi$ that is tangent to the identity

and commutes with $\rho$ such that (9.3) holds. Since each $\mu_j$ is not a root of unity, then (9.3) implies that the conjugate family $\{\sigma_m^*\}$ is a completely integrable normal form.

Let $\sigma_i$ be defined as above. Let $S_i$ be its linear part at the origin of $\mathbf{C}^n$. The eigenvalues $\{\mu_{ij}\}_{1\le j\le n}$ of $S_i$ are either $\mu_i$, $\mu_i^{-1}$ or 1. More precisely, if $Q\in\mathbf{N}^n$, $|Q|\ge 2$ then

$$\mu_m^Q-\mu_{mj}=\mu_m^{q_m-q_{m+p}}-\begin{cases}\mu_m & \text{if } j=m\\ \mu_m^{-1} & \text{if } j=m+p\\ 1 & \text{otherwise.}\end{cases} \tag{9.9}$$

We need to verify the condition that the family of linear part $\{S_1,\dots,S_p\}$ is of the Poincaré type. So we can apply Theorem 8.11.

Suppose that $(j,Q)\in\{1,\dots,2p\}\times\mathbf{N}^{2p}$ satisfies

$$\mu_l^Q-\mu_{lj}\neq 0$$

for some $1\le l\le 2p$. Set $d=\{\min_i\max(|\mu_i|,|\mu_i^{-1}|)\}^{1/(2p)}$. We define

$$Q'=Q-\sum_{i=1}^p\min(q_i,q_{i+p})(e_i+e_{i+p}):=(q_1',\dots,q_{2p}').$$

Then $\mu_i^Q=\mu_i^{Q'}$ for all $i$. Take $i=l$ if $|Q'|\le 2p$. In this case, we easily get

$$\mu_i^{Q'}-\mu_{ij}\neq 0,\quad |\mu_i^{Q'}|>c^{-1}d^{|Q'|} \tag{9.10}$$

by choosing a sufficiently large $c$. Assume that $|Q'|>2p$. Take $i$ such that

$$q_i+q_{i+p}=\max_k(q_k+q_{k+p}).$$

Then $q_i+q_{i+p}\ge|Q'|/p>1$. By (9.9), we get the first inequality in (9.10). We note that $(q_i',q_{i+p}')=(q_i,0)$ or $(0,q_{i+p})$. Thus

$$\max(|\mu_i^{Q'}|,|\mu_i^{-Q'}|)=(\max(|\mu_i|,|\mu_i|^{-1}))^{q_i+q_{i+p}}\ge d^{|Q'|}.$$

This shows that $\{D\sigma_1(0),\dots,D\sigma_p(0)\}$ is of the Poincaré type.

We now apply Theorem 8.11 as follows. We decompose $\psi=\psi_1\psi_0^{-1}$ such that $\psi_1\in\mathcal{C}^{\mathsf{c}}(S_1,\dots,S_p)$ and $\psi_0\in\mathcal{C}(S_1,\dots,S_p)$. Then each $\sigma_i^*=\psi_1^{-1}\sigma_i\psi_1$ still has the form in (9.3); in particular, $\{\sigma_1^*,\dots,\sigma_p^*\}$ is a completely integrable formal normal form. By Theorem 8.11, $\psi_1$ is convergent. Now, $\psi_1^{-1}\tau_{kj}\psi_1=\psi_0^{-1}(\psi^{-1}\tau_{kj}\psi)\psi_0$ are still of the form (9.3); however (9.4)-(9.6) might not hold. As in the first proof, we can verify that $\psi_1\rho=\rho\psi_1$. Applying another change of coordinates that commutes with $\rho$ and each $S_j$ as before, we achieve (9.3)-(9.6). The proof of the theorem is complete. □

**Remark 9.4.** When $M$ is non-resonant and $\log\hat{M}$ is tangent to the identity, we apply Theorem 5.5 to obtain a further holomorphic change of coordinates so that $(M_1,\dots,M_p)$ are uniquely determined by the real analytic submanifold. Then by Proposition 7.5, $\{\hat{\tau}_{i1},\dots,\tau_{ip}\}$, $i=1,2$, are formally equivalent to $\{\hat{\tau}_{i1},\dots,\hat{\tau}_{ip}\}$, $i=1,2$, defined by (7.22), if and only if the formal map $\Psi$ in Proposition 7.5 is the identity. In other words, $M$ has an abelian CR singularity at the origin if and only if $\Phi-I$ vanishes.

As a corollary of Theorem 9.3, we have the following normal form for real submanifolds. In order to study the holomorphic flatness and hull of holomorphy, we choose a realization similar to the case of Moser-Webster for $p = 1$.

**Theorem 9.5.** *Let $M$ be a germ of real analytic submanifold at an abelian CR singularity. Assume that $M$ is a higher order perturbation of a product quadric of which $\gamma_1, \dots, \gamma_p$ satisfy* (1.2)*. Suppose that $M$ has no hyperbolic component of complex tangent at the origin. Suppose that the associated $\sigma$ of $M$ has distinct eigenvalues $\mu_1, \dots, \mu_p$, $\mu_1^{-1}, \dots, \mu_p^{-1}$. Then $M$ is holomorphically equivalent to*

$$\widehat{M}\colon z_{p+j} = \Lambda_{1j}(\zeta)\zeta_j, \quad 1 \le j \le p, \tag{9.11}$$

*where $\zeta = (\zeta_1, \dots, \zeta_p)$ are the convergent solutions to*

$$\zeta_e = \frac{1 + \Lambda_{1e}^2(\zeta)}{(1 - \Lambda_{1e}^2(\zeta))^2}|z_e|^2 - \frac{\Lambda_{1e}(\zeta)}{(1 - \Lambda_{1e}^2(\zeta))^2}(z_e^2 + \overline{z}_e^2), \tag{9.12}$$

$$\zeta_s = \frac{\Lambda_{1s}(\zeta) + \Lambda_{1s}^3(\zeta)}{(1 - \Lambda_{1,s}^2(\zeta))^2} z_s \overline{z}_{s+s_*} - \frac{\Lambda_{1s}(\zeta)}{(1 - \Lambda_{1s}^2(\zeta))^2}(z_s^2 + \Lambda_{1s}^2(\zeta)\overline{z}_{s+s_*}^2), \tag{9.13}$$

$$\begin{aligned} \zeta_{s+s_*} = {} & \frac{\Lambda_{1(s+s_*)}(\zeta) + \Lambda_{1(s+s_*)}^3(\zeta)}{(1 - \Lambda_{1(s+s_*)}^2(\zeta))^2} \overline{z}_s z_{s+s_*} \\ & - \frac{\Lambda_{1(s+s_*)}(\zeta)}{(1 - \Lambda_{1(s+s_*)}^2(\zeta))^2}(z_{s+s_*}^2 + \Lambda_{1(s+s_*)}^2(\zeta)\overline{z}_s^2). \end{aligned} \tag{9.14}$$

*Here $\Lambda_{1j}(\zeta) = \lambda_j + O(\zeta)$ $(1 \le j \le p)$ satisfy* (9.4)-(9.5)*. In particular, $\widehat{M}$ is contained in $z_{p+e} = \overline{z}_{p+e}$ and $z_{p+s} = \overline{z}_{p+s+s_*}$.*

By Lemma 11.2, that $\sigma$ has distinct eigenvalues is equivalent to $\gamma_1, \dots, \gamma_p$ being distinct.

*Proof.* We use a realization which is different from (2.23). We assume that $M$ already has the normal form as in Theorem 9.3. Thus for $j = 1, \dots, p$, we have

$$\tau_{1j}\colon \xi_j' = \Lambda_{1j}(\xi\eta)\eta_j, \quad \eta_j' = \Lambda_{1j}^{-1}(\xi\eta)\xi_j, \quad (\xi_k', \eta_k') = (\xi_k, \eta_k), \quad k \ne j. \tag{9.15}$$

Let us define

$$f_j(\xi, \eta) = \xi_j + \xi_j \circ \tau_{1j}, \quad g_j = \overline{f_j \circ \rho}, \quad 1 \le j \le p.$$

The latter implies that the biholomorphic mapping $\varphi(\xi, \eta) = (f(\xi, \eta), g(\xi, \eta))$ transforms $\rho$ into the standard complex conjugation $(z', w') \to (\overline{w}', \overline{z}')$. Define

$$F_j(\xi, \eta) = \xi_j \circ \tau_{1j}(\xi, \eta)\xi_j, \quad 1 \le j \le p.$$

Using the expressions of $\tau_{1j}$ given by (9.15), we verify that $f_j$ and $F_j$ are invariant by $\tau_{1k}$. Note that the linear part of $f_j(\xi, \eta)$ is $\xi_j + \lambda_j\eta_j$ for $1 \le j \le p$, and the quadratic part of $F_j(\xi, \eta)$ is $\lambda_j\xi_j^2$. By Lemma 2.7, $f_1, \dots, f_p$ and $F_1, \dots, F_p$ generate all invariant functions of $\{\tau_{11}, \dots, \tau_{1p}\}$.

Next using $\overline{\Lambda_{1e}\circ\rho_z}=\Lambda_{1e}$ and $\overline{\Lambda_{1s}\circ\rho_z}=\Lambda^{-1}_{1(s+s_*)}$, we rewrite $z_j=f_j(\xi,\eta), w_j=g_j(\xi,\eta)$ as

$$\begin{aligned}
\xi_e &= \frac{z_e-\Lambda_{1e}(\xi\eta)w_e}{1-\Lambda^2_{1e}}, & \eta_e &= \frac{w_e-\Lambda_{1e}(\xi\eta)z_e}{1-\Lambda^2_{1e}},\\
\xi_s &= \frac{z_s-\Lambda^2_{1s}(\xi\eta)w_{s+s_*}}{1-\Lambda^2_{1s}(\xi\eta)}, & \eta_s &= \frac{\Lambda_{1s}(\xi\eta)(w_{s+s_*}-z_s)}{1-\Lambda^2_{1s}(\xi\eta)},\\
\xi_{s+s_*} &= \frac{z_{s+s_*}-\Lambda^2_{1(s+s_*)}(\xi\eta)w_s}{1-\Lambda^2_{1(s+s_*)}(\xi\eta)}, & \eta_{s+s_*} &= \frac{\Lambda_{1(s+s_*)}(\xi\eta)(w_s-z_{s+s_*})}{1-\Lambda^2_{1(s+s_*)}(\xi\eta)}.
\end{aligned}$$

Using the above formulae and $w_j=\overline{z}_j$, we compute $\zeta_j=\xi_j\eta_j$ to obtain (9.12)-(9.14).

Note that $F_j(\xi,\eta)=\zeta_j\Lambda_{1j}(\zeta)$. This shows that $z_{p+j}=F_j\circ\varphi^{-1}(z',\overline{z}')$ have the form (9.11). Again, we use the formula of $\tau_{1,k}$ to verify that $z=(z',z'')$ are invariant by all $\varphi\tau_{1k}\varphi^{-1}$. On the other hand, $z=(z',z'')$ generate invariant functions of the deck transformations of $\pi_1$ for the complexification of $\hat{M}$ given by (9.11). This shows that $\{\varphi\tau_{11}\varphi^{-1},\dots,\varphi\tau_{1p}\varphi^{-1}\}$ and the deck transformations of $\pi_1$, of which each family consists of commuting involutions, have the same invariant functions. By Lemma 2.7, we know that the two families must be identical. This shows that (9.11) is a realization for $\{\tau_{11},\dots,\tau_{1p},\rho\}$.

To verify the last assertion of the theorem, we set $z^*_{p+e}=\overline{z}_{p+e}$, $z^*_{p+s}=\overline{z}_{p+s+s_*}$, and $z^*_{p+s+s_*}=\overline{z}_{p+s}$. Set $\zeta^*_e=\overline{\zeta}_e$, $\zeta^*_s=\overline{\zeta}_{s+s_*}$, and $\zeta^*_{s+s_*}=\overline{\zeta}_s$. We take complex conjugate on identities (9.11)-(9.14). By (9.4)-(9.5), we have

$$\Lambda_{1e}=\overline{\Lambda_{1e}\circ\rho_z},\quad \Lambda_{1s}=\overline{\Lambda^{-1}_{1(s+s_*)}\circ\rho_z}.$$

We verify that $z^*,\zeta^*$ still satisfy (9.11)-(9.14), if $z_{j+p},\zeta_j$ are replaced by $z^*_{p+j},\zeta^*_j$, respectively, and $z_j$ are unchanged for $1\le j\le p$. By the uniqueness of solutions $\zeta$ to (9.12)-(9.14), we conclude that $\zeta^*_j=\zeta_j$. Therefore, $z_{p+j}=z^*_{p+j}$. The proof is complete. □

9.2. **Hull of holomorphy of real submanifolds with an abelian CR singularity.** Let $X$ be a subset of $\mathbf{C}^n$. We define the hull of holomorphy of $X$, denoted by $\mathcal{H}(X)$, to be the intersection of domains of holomorphy in $\mathbf{C}^n$ that contain $X$.

We assume that $M$ is real analytic and has a non-resonant complex tangent at the origin of elliptic type only. By Theorem 9.5, we may assume that $M$ is given by

$$M\colon z_{p+j}=\Lambda_{1j}(\zeta)\zeta_j,\quad 1\le j\le p,$$

where $\zeta_j=\zeta_j(z')$ $(j=1,\dots,p)$ are the convergent real-valued solutions to

$$\zeta_j=\frac{1+\Lambda^2_{1j}(\zeta)}{(1-\Lambda^2_{1j})^2(\zeta)}|z_j|^2-\frac{\Lambda_{1j}(\zeta)}{(1-\Lambda^2_{1j}(\zeta))^2}(z_j^2+\overline{z}_j^2),\quad 1\le j\le p. \tag{9.16}$$

For $\zeta\in\mathbf{R}^p$ with small $|\zeta|$, we know that $\Lambda_{1j}(\zeta)>1$.

In a neighborhood of the origin in $\mathbf{R}^p$, let us define the following germ of real analytic diffeomorphism:

$$R\colon \zeta\to(\Lambda_{11}(\zeta)\zeta_1,\dots,\Lambda_{1p}(\zeta)\zeta_p)\,.$$

If $\epsilon$ is small enough, for each $x''\in[0,\epsilon]^p$, we can define $\zeta=R^{-1}(x'')$. Note that $R$ sends $\zeta_j=0$ into $x_{p+j}=0$ for each $j$. We can write

$$R^{-1}(x'')=(x_{p+1}S_1(x''),\dots,x_{2p}S_p(x''))$$

with $S_j(0) > 0$. Then $M \cap \{z'' = x''\}$ is given by (9.16). For $x'' \in [0,\epsilon]^p$ let $D_j(x'')$ be the compact set in the $z_j$ plane whose boundary is defined by the $j$th equation in (9.16) where $\zeta = R^{-1}(x'')$. When $x_{p+j} > 0$, the boundary of $D_j(x'')$ is an ellipse with

$$D_j(x'') \subset \Delta_{C_1\sqrt{x_{p+j}}}. \tag{9.17}$$

Here and in what follows constants will depend only on $\lambda_1, \ldots, \lambda_p$. Thus

$$D(x'') := D_1(x'') \times \cdots \times D_p(x'') \times \{x''\} \subset \mathbf{C}^p \times \mathbf{R}^p$$

is a product of ellipses and its dimension equals the number of positive numbers among $x_{p+1}, \ldots, x_{2p}$. We will call $D(x'')$ an *analytic polydisc* and $\partial^* D(x'') := \partial D_1(x'') \times \cdots \times \partial D_p(x'') \times \{x''\}$ its distinguished boundary. Note that $\partial^* D(x'')$ is contained in $M$. In fact, $M$ is foliated by $\partial^* D(x'')$ as $x''$ vary in $[0,\epsilon]^p$ and $\epsilon$ is sufficiently small. We will specify the value of $\epsilon$ later. We will use this foliation and Hartogs' figures in analytic polydiscs to find the local hull of holomorphy of $M$ at the origin.

As $x''$ vary in $[0,\epsilon]^p$, let $M^\epsilon$ be the union of $\partial^* D(x'')$, and $\mathcal{H}^\epsilon$ the union of $D(x'')$. Both $\mathcal{H}^\epsilon$ and $M^\epsilon$ are compact subsets in $\mathbf{C}^{2p}$. For any open ball $B^{\epsilon_*}$ in $\mathbf{C}^{2p}$ centered at the origin with radius $\epsilon_*$,

$$B^{\epsilon_*} + M^\epsilon := \{a + b\colon a \in B^{\epsilon_*}, b \in M^\epsilon\}$$

is contained in a given neighborhood of $M^\epsilon$, if $\epsilon_*$ is sufficiently small. Analogously, $B^{\epsilon_*} + \mathcal{H}^\epsilon$ is a connected open neighborhood of $\mathcal{H}^\epsilon$. Let us first verify that a function that is holomorphic in a connected neighborhood of $M^\epsilon$ in $\mathbf{C}^n$ extends holomorphically to a neighborhood of $\mathcal{H}^\epsilon$. Assume that $f$ is holomorphic in a neighborhood $\mathcal{U}$ of $\partial^*_\epsilon D := \cup_{x'' \in [0,\epsilon]^p} \partial^* D(x'')$. We first note that $\mathcal{H}^\epsilon$ is defined by

$$A_j(x'')|z_j|^2 - B_j(x'')(z_j^2 + \overline{z}_j^2) \le x_{p+j}, \quad 1 \le j \le p; \tag{9.18}$$

$$y'' = 0, \quad x'' \in [0,\epsilon]^p \tag{9.19}$$

with

$$A_j(x'') = \frac{1 + \Lambda_{1j}^2(R^{-1}(x''))}{S_j(x'')(1 - \Lambda_{1j}^2(R^{-1}(x'')))}, \quad B_j(x'') = \frac{\Lambda_{1j}(R^{-1}(x''))}{S_j(x'')(1 - \Lambda_{1j}^2(R^{-1}(x'')))}.$$

Let $\delta$ be a small positive number. For $x'' \in [-\delta,\epsilon]^p$, let $D_j^\delta(x'') \subset \mathbf{C}$ be defined by

$$A_j(x'')|z_j|^2 - B_j(x'')(z_j^2 + \overline{z}_j^2) \le x_{p+j} + \delta.$$

Fix $\delta > 0$ sufficiently small. Let $P_{\epsilon,\delta}$ (resp. $\partial^* P_{\epsilon,\delta}$) be the set of $z = (z', z'')$ satisfying the following: $y'' \in [-\delta,\delta]^p$, $x'' \in [-\delta,\epsilon]^p$, and $z_j \in D_j^\delta(x'')$ (resp. $z_j \in \partial D_j^\delta(x'')$) for $1 \le j \le p$. Let $\mathcal{U}_{\epsilon,\delta}$ (resp. $\mathcal{U}_{\epsilon,\delta_1}$) be a small neighborhood of $P_{\epsilon,\delta}$ (resp. $P_{\epsilon,\delta_1}$). Assume that $0 < \delta_1 < \delta$ and $\delta_1$ is sufficiently small. We may also assume that $\mathcal{U}_{\epsilon,\delta_1}$ is contained in $\mathcal{U}_{\epsilon,\delta}$ and $\partial^* P_{\epsilon,\delta} \subset \mathcal{U}$. Thus, for $(z', z'') \in \mathcal{U}_{\epsilon,\delta_1}$, we can define

$$F(z', z'') = \int_{\zeta_1 \in \partial D_1^\delta(x'')} \cdots \int_{\zeta_p \in \partial D_p^\delta(x'')} \frac{f(\zeta', z'')\, d\zeta_1 \cdots d\zeta_p}{(\zeta_1 - z_1) \cdots (\zeta_p - z_p)}.$$

When $z$ is sufficiently small, $F(z) = f(z)$ as $f$ is holomorphic near the origin. Fix $z_0 \in \mathcal{U}_{\epsilon,\delta_1}$. We want to show that $F$ is holomorphic at $z_0$. So $F$ is a desired extension of $f$. By continuity, when $z = (z_1, \ldots, z_{2p})$ tends to $z_0$, $x''$ tends to $x_0''$ and $\partial D_j^\delta(x'')$ tends to $\partial D_j^\delta(x_0'')$,

while $z_j \in D_j^\delta(x_0'')$ when $z$ is sufficiently close to $z_0$. By Cauchy theorem, for $z$ sufficiently close to $z_0$ we change contour integrals successively to get

$$F(z',z'') = \int_{\zeta_1\in\partial D_1^\delta(x_0'')}\int_{\zeta_2\in\partial D_2^\delta(x_0'')}\cdots\int_{\zeta_p\in\partial D_p^\delta(x_0'')}\frac{f(\zeta',z'')\,d\zeta_1\cdots d\zeta_p}{(\zeta_1-z_1)\cdots(\zeta_p-z_p)}.$$

The set of integration is fixed. The integrand is holomorphic in $z$. Hence $F$ is holomorphic at $z=z_0$.

Next we want to show that $\mathcal{H}^\epsilon$ is the hull of holomorphy of $M^\epsilon$ in $B^{2p}_{\epsilon_0}$ for suitable $\epsilon,\epsilon_0$ that can be arbitrarily small.

Let us first show that $\mathcal{H}^\epsilon$ is the intersection of domains of holomorphy in $\mathbf{C}^n$. Recall that $\mathcal{H}^\epsilon$ is defined by (9.18)-(9.19). Next, we define for $\delta' := (\delta_1,\dots,\delta_p)$ with $\delta_j>0$

$$\begin{aligned}\rho_j^{\delta'} &= A_j(x'')|z_j|^2 - B_j(x'')(z_j^2+\overline{z}_j^2) - x_{p+j} + (\delta_1^{-1}+\cdots+\delta_p^{-1})\sum_{i=1}^p y_{p+i}^2\\ &\quad+\sum_{i\neq j}\delta_i^{-1}\left\{A_i(x'')|z_i|^2 - B_i(x'')(z_i^2+\overline{z}_i^2) - x_{p+i}\right\}.\end{aligned}$$

When $p=1$, the last summation is 0. The complex Hessian of $\rho_j^{\delta'}$ is

$$\begin{aligned}\sum_{\alpha,\beta=1}^{2p}\frac{\partial^2\rho_j^{\delta'}}{\partial z_\alpha\overline{z}_\beta}t_\alpha\overline{t}_\beta &= A_j(x'')|t_j|^2 + \frac{\delta_1^{-1}+\cdots+\delta_p^{-1}}{2}\sum_i|t_{p+i}|^2 + \sum_{i\neq j}\frac{1}{\delta_i}A_i(x'')|t_i|^2\\ &\quad+\mathrm{Re}\sum_k a_{jk}(x'';z_j)t_j\overline{t}_{p+k} + \mathrm{Re}\sum_{k,\ell}b_{j,k\ell}(x'';z_j)t_{p+k}\overline{t}_{p+\ell}\\ &\quad+\mathrm{Re}\sum_{i\neq j}\sum_k\frac{1}{\delta_i}c_{j,ik}(x'';z_i)t_i\overline{t}_{p+k} + \mathrm{Re}\sum_{i\neq j}\sum_{k,\ell}\frac{1}{\delta_i}d_{j,k\ell}(x'';z_i)t_{p+k}\overline{t}_{p+\ell}.\end{aligned}$$

Here $a_{jk}(x'';0)=b_{j,kl}(x'';0)=c_{j,ik}(x'';0)=d_{j,kl}(x'';0)=0$, and $i,j,k,\ell$ are in $\{1,\dots,p\}$. From the Cauchy-Schwarz inequality, it follows that for $|z|<\epsilon_0$ with $\epsilon_0>0$ sufficiently small and $0<\delta_j<1$,

$$\sum_{\alpha,\beta=1}^{2p}\frac{\partial^2\rho_j^{\delta'}}{\partial z_\alpha\overline{z}_\beta}t_\alpha\overline{t}_\beta \geq \frac{1}{2}A_j(x'')|t_j|^2 + \frac{\delta_1^{-1}+\cdots+\delta_p^{-1}}{4}\sum_j|t_{p+j}|^2 + \frac{1}{2}\sum_{i\neq j}\delta_i^{-1}A_i(x'')|t_i|^2.$$

Therefore, each $\rho_j^{\delta'}$ is strictly plurisubharmonic on $|z|<\epsilon_0$ for all $0<\delta_i<1$. Hence for $\delta=(\delta_0,\dots,\delta_p)=(\delta_0,\delta')\in(0,1)^{p+1}$,

$$\rho^\delta(z) = \max_j\{\rho_j^{\delta'}, |y''|^2-\delta_0^2, x_{p+j}^2-\epsilon^2\}$$

is plurisubharmonic on the ball $B^{2p}_{\epsilon_0}$. By (9.17), $D(x'')$ is contained in $B^{2p}_{C_2\epsilon^{1/2}}$ for $x''\in[0,\epsilon]^p$. We now fix $\epsilon<(\epsilon_0/C_2)^2$ to ensure

$$D(x'')\subset B^{2p}_{\epsilon_0},\quad \forall x''\in[0,\epsilon]^p. \tag{9.20}$$

This shows that

$$\mathcal{H}^\epsilon_\delta := \{z\in B^{2p}_{\epsilon_0}\mid : \rho^\epsilon_\delta(z)<0\}$$

is a domain of holomorphy.

Let us verify that

$$\mathcal{H}^\epsilon = \bigcap_{\delta_0>0,\dots,\delta_p>0} \mathcal{H}^\epsilon_\delta.$$

Fix $z \in \mathcal{H}^\epsilon$. From (9.20) we get $z \in B^{2p}_{\epsilon_0}$. We have $y'' = 0$. Hence (9.18) hold and $x^2_{p+j} \le \epsilon^2$. Clearly, $\rho^\delta_j(z) < 0$ for each $j$ and $\delta \in (0,1)^p$. This shows that $z \in \mathcal{H}^\epsilon$ is in the intersection. For the other inclusion, let us assume that $z$ is in the intersection. Then $y'' = 0$ and $x'' \in [0,\epsilon]^p$. So (9.19) holds. With $\rho^\delta_j(z) < 0$, we let $\delta_i$ tend to 0 for $i \neq j$. We conclude

$$A_i(x'')|z_i|^2 - B_i(x'')(z_i^2 + \overline{z}_i^2) \le x_{p+i}$$

for all $i \neq j$, and hence for all $i$ as $p > 1$. When $p = 1$ the above inequality can be obtained directly from $\rho^\delta_1$. We have verified (9.18). This shows that $z \in \mathcal{H}^\epsilon$.

In view of (9.18)-(9.19), the boundary of $\mathcal{H}^\epsilon$ is the union $\cup_{j=1}^p \mathcal{H}^\epsilon_j$ with $\mathcal{H}^\epsilon_j$ being defined by

$$\begin{aligned} A_j(x'')|z_j|^2 - B_j(x'')(z_j^2 + \overline{z}_j^2) &= x_{p+j}, \\ A_i(x'')|z_i|^2 - B_i(x'')(z_i^2 + \overline{z}_i^2) &\le x_{p+i}, \quad 1 \le i \le p,\ i \neq j; \\ y'' = 0, \qquad x_{p+j} &\le \epsilon \quad 1 \le j \le p. \end{aligned}$$

Therefore, we have proved the following theorem.

**Theorem 9.6.** *Let $M$ be a germ of real analytic submanifold at an abelian CR singularity. Assume that the complex tangent of $M$ is purely elliptic and has distinct eigenvalues at the origin. There is a base of neighborhoods $\{U_j\}$ of the origin in $\mathbf{C}^n$ which satisfies the following: For each $U_j$, the local hull of holomorphy $H(M \cap U_j)$ of $M \cap U_j$ is foliated by embedding complex submanifolds with boundaries. Furthermore, near the origin $H(M \cap U_j)$ is the transversal intersection of $p$ real analytic submanifolds of dimension $3p$ with boundary. The boundary of $H(M \cap U_j)$ contains $M \cap U_j$; and two sets are the same if and only if $p = 1$.*

**Remark 9.7.** The proof shows that the hull of $H(M \cap U_j)$ is foliated by analytic polydiscs, where an analytic polydisc is a biholomorphic embedding of closed unit polydisc in some $\mathbf{C}^k$ with $1 \le k \le p$.

## 10. Rigidity of product quadrics

The aim of this section is to prove the following rigidity theorem: Let us consider a higher order analytic perturbation of a product quadric. If this manifold is formally equivalent to the product quadric, then under a small divisors condition, it is also holomorphically equivalent to it.

The proof goes as follows : Since the manifold is formally equivalent to the quadric, the associated sets of involutions $\{\tau_{1i}\}$ and $\{\tau_{2i}\}$ are simultaneously linearizable by a formal biholomorphism that commutes with $\rho$. In particular, $\sigma_1, \dots, \sigma_p$, as defined by (9.1) and (9.2), are formally linearizable and they commute pairwise. These are germs of biholomorphisms with a diagonal linear part. According to [Sto13][theorem 2.1], this abelian family can be holomorphically linearized under a collective Brjuno type condition (11.32). Furthermore, the transformation commutes with $\rho$. Then, we linearize simultaneously and

holomorphically both $\tau_1 := \tau_{11}\circ\cdots\circ\tau_{1p}$ and $\tau_2 := \tau_{21}\circ\cdots\circ\tau_{2p}$ by a transformation that commutes with both $\rho$ and $\mathcal{S}$, the family of linear parts of the $\sigma_1, \ldots, \sigma_p$. Finally, we linearize simultaneously and holomorphically both families $\{\tau_{1i}\}$ and $\{\tau_{2i}\}$ by a transformation that commutes with $\rho$, $\mathcal{S}$, $T_1$ and $T_2$.

These last two steps will be obtained through a majorant method and the application of a holomorphic implicit function theorem. This is done in Proposition 10.6. They first require a complete description of the various centralizers and their associated normalized mappings, i.e. suitable complements. This is a goal of Proposition 10.3.

Throughout this section, we do not assume that $\mu_1, \ldots, \mu_p$ are non resonant in the sense that $\mu^Q \neq 1$ if $Q \in \mathbf{Z}^p$ and $Q \neq 0$. In fact, we will apply our results to $M$ which might be resonant. However, we will retain the assumption that $\sigma$ has distinct eigenvalues when we apply the results to the manifolds.

We recall from (9.1) and (9.2), the definition of germs of holomorphic diffeomorphisms :

$$\sigma_i := \tau_{1i}\circ\tau_{2i}, \quad 1 \leq i \leq e_* + h_*; \tag{10.1}$$

$$\sigma_s := \tau_{1s}\circ\tau_{2(s_*+s)}, \tag{10.2}$$

$$\sigma_{s+s_*} := \tau_{1(s+s_*)}\circ\tau_{2s}, \quad e_* + h_* < s \leq p - s_*. \tag{10.3}$$

They satisfy

$$\sigma_i^{-1} = \rho\sigma_i\rho, \quad 1 \leq i \leq e_* + h_*; \quad \sigma_{s+s_*}^{-1} = \rho\sigma_s\rho, \quad e_* + h_* < s \leq p - s_*.$$

Recall the linear maps

$$S\colon \xi_j' = \mu_j\xi_j, \quad \eta_j' = \mu_j^{-1}\eta_j;$$

$$S_j\colon \xi_j' = \mu_j\xi_j, \quad \eta_j' = \mu_j^{-1}\eta_j, \quad \xi_k' = \xi_k, \quad \eta_k' = \eta_k, \quad k \neq j; \tag{10.4}$$

$$T_{ij}\colon \xi_j' = \lambda_{ij}\eta_j, \quad \eta_j' = \lambda_{ij}^{-1}\xi_j, \quad \xi_k' = \xi_k, \quad \eta_k' = \eta_k, \quad k \neq j; \tag{10.5}$$

$$\rho\colon \begin{cases} (\xi_e', \eta_e', \xi_h', \eta_h') = (\overline{\eta}_e, \overline{\xi}_e, \overline{\xi}_h, \overline{\eta}_h), \\ (\xi_s', \xi_{s+s_*}', \eta_s', \eta_{s+s_*}') = (\overline{\xi}_{s+s_*}, \overline{\xi}_s, \overline{\eta}_{s+s_*}, \overline{\eta}_s). \end{cases} \tag{10.6}$$

We need to introduce notation for the indices to describe various centralizers regarding $T_{1j}, S_j$ and $\rho$. We first introduce index sets for the centralizer of $\mathcal{S}, T_1, \rho$. We recall that $\mathcal{S}$ and $\mathcal{T}_i$ denote the families $\{S_1, \ldots, S_p\}$ and $\{T_{i1}, \ldots, T_{ip}\}$, respectively. Also, $T_i = T_{i1}\circ\cdots\circ T_{ip}$ .

Let $(P, Q) \in \mathbf{N}^p \times \mathbf{N}^p$ and $1 \leq j \leq p$. By definition, $\xi^P\eta^Q e_j$ belongs to the centralizer of $\mathcal{S}$ if and only if it commutes with each $S_i$. In other words, $\xi^P\eta^Q e_j \in \mathcal{C}(\mathcal{S})$ if and only if

$$\mu_k^{p_k - q_k} = 1, \quad \forall k \neq j; \quad \mu_j^{p_j - q_j} = \mu_j. \tag{10.7}$$

Note that the same condition holds for $\xi^Q\eta^P e_{p+j}$ to belong to $\mathcal{C}(\mathcal{S})$. This leads us to define the set of multiindices

$$\mathcal{R}_j := \{(P, Q) \in \mathbf{N}^{2p}\colon \mu_j^{p_j - q_j} = \mu_j,\ \mu_i^{p_i - q_i} = 1, \forall i \neq j\}, \quad 1 \leq j \leq p.$$

We observe that if $(P,Q)\in\mathcal{R}_j$, then

$$p_j=q_j+1,\quad j\neq h;\quad p_i=q_i,\quad \forall i\neq j,h;$$
$$\lambda_h^{p_h-q_h}=\pm 1,\quad h\neq j.$$

For convenience, we define for $P=(p_e,p_h,p_s,p_{s+s_*})$ and $Q=(q_e,q_h,q_s,q_{s+s_*})$

$$\rho(PQ):=(q_e,p_h,p_{s+s_*},p_s,p_e,q_h,q_{s+s_*},q_s),$$

$$\rho_a(PQ):=(q_e,p_h,p_{s+s_*},p_s), \tag{10.8}$$

$$\rho_b(PQ):=(p_e,q_h,q_{s+s_*},q_s), \tag{10.9}$$

$$\overline{f}_{\rho(PQ)}:=(\overline{f\circ\rho})_{PQ}=:\overline{f_{\rho(PQ)}}.$$

Hence, we have $\xi^P\eta^Q\circ\rho=\overline{\xi}^{\rho_a(PQ)}\overline{\eta}^{\rho_b(PQ)}$ as well as $\rho(PQ)=(\rho_a(PQ),\rho_b(PQ))$.

According to (10.7) and equation (10.6) of $\rho$, the restriction of $\rho$ to $\mathcal{R}_h$ is an involution, which will be denoted by $\rho_h$. Moreover, $\rho$ is a bijection $\rho_s$ from $\mathcal{R}_s$ onto $\mathcal{R}_{s+s_*}$. We define an involution on $\mathcal{R}_e$ by

$$\rho_e(PQ):=(\rho_b(PQ),\rho_a(PQ)).$$

Note that $\rho_e$ is not a restriction of $\rho$, and $\rho_s$ is not an involution either.

Next, we introduce sets of indices to be used to compute the centralizers on $\mathcal{T}_1,\mathcal{T}_2,\rho$. Set

$$\mathcal{N}_j:=\mathcal{R}_j\cap\{(P,Q)\colon p_i\geq q_i,\quad \forall i\neq j\},\quad 1\leq j\leq p.$$

Note that when $p=1$, $\mathcal{N}_j=\mathcal{R}_j$ for $j=e$ or $h$. Let us set

$$A_{jk}(P,Q):=\max\{p_k,q_k\},\quad k\neq j,\quad A_{jj}(P,Q)=p_j;$$
$$B_{jk}(P,Q):=\min\{p_k,q_k\},\quad k\neq j,\quad B_{jj}(P,Q)=q_j.$$

We define a mapping

$$(A_j,B_j)\colon\mathcal{R}_j\to\mathcal{N}_j$$

with

$$A_j:=(A_{j1},\dots,A_{jp}),\quad B_j:=(B_{j1},\dots,B_{jp}).$$

We notice that, for $(P,Q)\in\mathcal{N}_j$, $A_j\circ\rho_j(P,Q)=(p_e,p_h,p_{s+s^*},p_s)$ and $B_j\circ\rho_j(P,Q)=(q_e,q_h,q_{s+s^*},q_s)$. In other words, on $\mathcal{N}_j$ for $j=e$ or $h$, $A_j\circ\rho_j$ just interchanges the $s$th and the $(s+s_*)$th coordinates for each $s$, so does $B_j\circ\rho_j$.

Finally, for $(P,Q)\in\mathcal{R}_j$ we define

$$\nu_{PQ}:=\begin{cases}\prod_{h'}\lambda_{h'}^{q_{h'}-p_{h'}}, & j\neq h,\\ \lambda_h^{p_h-q_h-1}\prod_{h'\neq h}\lambda_{h'}^{q_{h'}-p_{h'}}, & j=h;\end{cases} \tag{10.10}$$

$$\nu_{PQ}^+:=\begin{cases}\prod_{h'|q_{h'}>p_{h'}}\lambda_{h'}^{q_{h'}-p_{h'}}, & j\neq h;\\ \prod_{h'\neq h,q_{h'}>p_{h'}}\lambda_{h'}^{q_{h'}-p_{h'}}, & j=h.\end{cases} \tag{10.11}$$

Here $e_*<h',h\leq e_*+h_*$. Note that $\nu_{PQ}^+$ is only defined for $(P,Q)\in\mathcal{R}_j$. For convenience, we however define

$$\nu_{QP}:=\nu_{PQ},\quad (P,Q)\in\mathcal{R}_j.$$

If $p=1$ we set $\nu_{PQ}^+=1$.

**Lemma 10.1.** *Let $(P,Q)\in\mathcal{R}_j$. Then*

$$\nu_{PQ}=\pm1,\quad (P,Q)\in\mathcal{R}_j,\ j\neq h;\qquad \nu^+_{PQ}=\pm1; \tag{10.12}$$

$$\nu_{\rho_e(PQ)}=\nu_{PQ},\quad (P,Q)\in\mathcal{R}_e;\qquad \nu_{\rho(PQ)}=\nu_{PQ},\quad \forall(P,Q); \tag{10.13}$$

$$\nu^+_{\rho_e(PQ)}=\nu_{PQ}\nu^+_{PQ},\quad (P,Q)\in\mathcal{R}_e;\qquad \nu^+_{\rho(PQ)}=\nu^+_{PQ},\quad (P,Q)\in\mathcal{R}_h\cup\mathcal{R}_{s+s_*}. \tag{10.14}$$

*Proof.* From the definition of $\mathcal{R}_j$, we have $(\lambda_i^{p_i-q_i})^2=\mu_i^{p_i-q_i}=1$ for $i=h'$ in (10.10)-(10.11). We also have $\mu_h^{p_h-q_h-1}=1$ for terms in (10.10)-(10.11). Thus

$$\lambda_{h'}^{p_{h'}-q_{h'}}=\pm1,\quad \lambda_h^{p_h-q_h-1}=\pm1.$$

Thus we obtain (10.12); the rest identities follow from the definition of $\rho_e$, $\rho$, and the above identities. □

**Lemma 10.2.** *For all multiindices $(P,Q)\in\mathcal{R}_e\cup\mathcal{R}_h$, we have*

$$\overline{\lambda^{\rho_a(P,Q)-\rho_b(P,Q)}}=\lambda^{Q-P},\quad \overline{\mu^{\rho_b-\rho_a}}=\mu^{P-Q}, \tag{10.15}$$

$$\xi^P\eta^Q\circ\rho\circ T_1=\lambda^{Q-P}\overline{\xi}^{\rho_b(PQ)}\overline{\eta}^{\rho_a(PQ)}, \tag{10.16}$$

$$\xi^P\eta^Q\circ\rho\circ S^{-1}=\mu^{P-Q}\overline{\xi}^{\rho_a(PQ)}\overline{\eta}^{\rho_b(PQ)}. \tag{10.17}$$

*Proof.* The first identity in (10.15) follows from (10.8)-(10.9) and the fact that $\lambda_e$ and $\mu_e$ are reals, $\lambda_h^{-1}=\overline{\lambda}_h$, $p_s=q_s$, and $p_{s+s_*}=q_{s+s_*}$. This gives us the first identity in (10.15), and the second identity follows from the first. A direct computation shows that

$$\xi^P\eta^Q\circ\rho\circ T_1=\overline{\lambda}^{\rho_a-\rho_b}\overline{\xi}^{\rho_b(PQ)}\overline{\eta}^{\rho_a(PQ)},\quad \xi^P\eta^Q\circ\rho\circ S^{-1}=\overline{\mu}^{\rho_b-\rho_a}\overline{\xi}^{\rho_a(PQ)}\overline{\eta}^{\rho_b(PQ)}.$$

The result follows from (10.15). □

It is tedious to find necessary and sufficient conditions to describe the centralizer of $\mathcal{T}_1,\mathcal{T}_2,\rho$, as the mappings in the families are non diagonal. There are different ways to described these conditions too. To keep computation relatively simple, we do not aim a minimum set of conditions. Of course, when we use the centralizers we will verify all the sufficient conditions.

**Proposition 10.3.** *Let $\mathcal{S}=\{S_1,\dots,S_p\}$, $\mathcal{T}_i=\{T_{i1},\dots,T_{ip}\}$ and $\rho$ be given by* (10.4)-(10.6). *Let $\varphi=I+(U,V)$ be a formal biholomorphic map that is tangent to the identity.*

*(i) $\varphi\in\mathcal{C}(\mathcal{S})$ if and only if*

$$U_{j,PQ}=0=V_{j,QP},\quad \forall(P,Q)\notin\mathcal{R}_j. \tag{10.18}$$

*Also, $\varphi\in\mathcal{C}(\mathcal{S},\rho)$ if and only if additionally*

$$U_{h,PQ}=\overline{U}_{h,\rho(PQ)},\ (P,Q)\in\mathcal{R}_h;\quad U_{s+s_*,PQ}=\overline{U}_{s,\rho(PQ)},\ (P,Q)\in\mathcal{R}_{s+s_*};$$

$$V_{e,QP}=\overline{U}_{e,\rho(PQ)},\quad (P,Q)\in\mathcal{R}_e;$$

$$V_{h,QP}=\overline{V}_{h,\rho(QP)},\ (P,Q)\in\mathcal{R}_h;\quad V_{s+s_*,QP}=\overline{V}_{s,\rho(QP)},\ (P,Q)\in\mathcal{R}_{s+s_*}.$$

*(ii) $\varphi\in\mathcal{C}(\mathcal{S},T_1)$ if and only if* (10.18) *holds and*

$$V_{j,QP}=\lambda_j^{-1}\lambda^{P-Q}U_{j,PQ},\quad \forall(P,Q)\in\mathcal{R}_j. \tag{10.19}$$

*Also, $\varphi \in \mathcal{C}(\mathcal{S}, T_1, \rho)$ if and only if in addition to* (10.18) *and* (10.19)

$$U_{e,PQ} = \nu_{PQ}\overline{U}_{e,\rho_e(PQ)}, \quad (P,Q) \in \mathcal{R}_e; \tag{10.20}$$

$$U_{h,PQ} = \overline{U}_{h,\rho(PQ)}, \qquad (P,Q) \in \mathcal{R}_h; \tag{10.21}$$

$$U_{s,PQ} = \overline{U}_{s+s^*,\rho(PQ)}, \qquad (P,Q) \in \mathcal{R}_s. \tag{10.22}$$

*(iii) $\varphi \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2)$ if and only if in addition to* (10.18) *and* (10.19)

$$U_{j,PQ} \;=\; \nu^+_{PQ} U_{j,(A_j,B_j)(P,Q)}, \quad (P,Q) \in \mathcal{R}_j \setminus \mathcal{N}_j. \tag{10.23}$$

*Also, $\varphi \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2, \rho)$ if and only if additionally*

$$U_{j,PQ} = \nu^+_{PQ}\overline{U}_{j,(A_j,B_j)\circ\rho_j(PQ)}, \quad (P,Q) \in \mathcal{N}_j, \quad j = e, h; \tag{10.24}$$

$$U_{s+s_*,PQ} = \nu^+_{PQ}\overline{U}_{s,(A_s,B_s)\circ\rho(PQ)}, \quad (P,Q) \in \mathbf{N}_{s+s_*}. \tag{10.25}$$

We remark that condition (10.23) holds trivially when $(P,Q) \in \mathcal{N}_j$, in which case it becomes $U_{j,PQ} = U_{j,PQ}$.

*Proof.* To simplify notation, we abbreviate

$$\rho_a = \rho_a(PQ), \quad \rho_b = \rho_b(PQ), \quad A_j = A_j(P,Q), \quad B_j = B_j(P,Q).$$

Recall that $\lambda_e = \overline{\lambda}_e$, $\lambda_h = \overline{\lambda}_h^{-1}$ and $\lambda_{s+s_*} = \overline{\lambda}_s^{-1}$. By definition,

$$S_e = T_{1e}T_{2e}, \quad S_h = T_{1h}T_{2h}, \quad S_s = T_{1s}T_{2(s+s_*)}, \quad S_{s+s_*} = T_{1(s+s_*)}T_{2s}.$$

In the proof, we will use the fact that $S_j$ is reversible by both involutions in the composition for $S_j$. In particular,

$$T_{1j}S_jT_{1j} = S_j^{-1}, \quad \forall j. \tag{10.26}$$

However, we have $T_{2(s+s_*)}S_sT_{2(s+s_*)} = S_s^{-1}$ and $T_{2s}S_{s+s_*}T_{2s} = S_{s+s_*}^{-1}$. For simplicity, we will derive identities by using (10.26) and

$$S_e^{-1} = \rho S_e \rho, \quad S_h^{-1} = \rho S_h \rho, \quad S_{s+s_*}^{-1} = \rho S_s \rho. \tag{10.27}$$

Finally, we need one more identity. Recall that

$$\begin{aligned} &T_{1e}T_{2j} = T_{2j}T_{1e}, \quad j \neq e; && T_{1h}T_{2j} = T_{2j}T_{1h}, \quad j \neq h;\\ &T_{1s}T_{2j} = T_{2j}T_{1s}, \quad j \neq s+s_*; && T_{1(s+s_*)}T_{2j} = T_{2j}T_{1(s+s_*)}, \quad j \neq s. \end{aligned}$$

Therefore, for any $j$ we have the identity

$$T_1S_jT_1 = S_j^{-1}. \tag{10.28}$$

In what follows, we will derive all identities by using (10.26), (10.27) and (10.28), as well as $S_iS_j = S_jS_i$, $T_{1i}T_{1j} = T_{1j}T_{1i}$ and $T_2 = \rho T_1 \rho$.

(i) The centralizer of $\mathcal{S}$ is easy to describe. Namely, $\varphi \in \mathcal{C}(\mathcal{S})$ if and only if

$$\begin{aligned} &U_j \circ S_j = \mu_j U_j, \quad U_j \circ S_k = U_j, \quad k \neq j,\\ &V_j \circ S_j = \mu_j^{-1} V_j, \quad V_j \circ S_k = V_j, \quad k \neq j. \end{aligned}$$

For $\varphi\rho = \rho\varphi$, we need

$$U_h = \overline{U_h \circ \rho}, \quad U_{s+s_*} = \overline{U_s \circ \rho}, \tag{10.29}$$

$$V_e = \overline{U_e \circ \rho}, \quad V_h = \overline{V_h \circ \rho}, \quad V_{s+s_*} = \overline{V_s \circ \rho}. \tag{10.30}$$

(ii) Suppose that $\varphi \in \mathcal{C}(\mathcal{S}, T_1)$. Then, it also belongs to $\mathcal{C}(S, T_1)$. Hence, it satisfies

$$V_j = \lambda_j^{-1} U_j \circ T_1. \tag{10.31}$$

This implies (10.19).

Assume furthermore that $\varphi \in \mathcal{C}(\mathcal{S}, T_1, \rho)$. Eliminating $V_e$ in (10.31) with (10.30), we obtain

$$U_e = \lambda_e \overline{U_e \circ \rho \circ T_1}.$$

According to (10.16), we obtain

$$U_{e,\rho_b\rho_a} = \lambda_e \overline{\lambda}^{Q-P} \overline{U}_{e,PQ}.$$

If $(P,Q) \in \mathcal{R}_e$ and since $\mu_e, \mu_s, \mu_{s+s^*}$ are of norm greater than 1, then we have $p_{s+s_*} = q_{s+s^*}$, $p_s = q_s$ and $p_e = q_e + 1$. By $\lambda_e \overline{\lambda}^{Q-P} = \overline{\nu}_{PQ} = \nu_{PQ}^{-1}$ we get (10.20).

Using (10.31), we eliminate $V_j$ from (10.30) and (10.29) to obtain

$$\lambda_h^{-1} U_h \circ T_1 = \overline{\lambda_h^{-1} U_h \circ T_1 \circ \rho}, \quad \lambda_{s+s_*}^{-1} U_{s+s_*} \circ T_1 = \overline{\lambda_s^{-1} U_s \circ T_1 \circ \rho}.$$

Since $T_1 \rho T_1 = \rho T_2 T_1 = \rho S^{-1}$, the previous equalities read

$$U_h = \lambda_h^2 \overline{U_h \circ \rho \circ S^{-1}}, \quad U_{s+s_*} = \lambda_{s+s_*}^2 \overline{U_s \circ \rho \circ S^{-1}}.$$

We recall that $\lambda_{s+s_*} = \overline{\lambda}_s^{-1}$. According to (10.17), we obtain

$$U_{h,\rho(PQ)} = \lambda_h^2 \overline{\mu}^{P-Q} \overline{U}_{h,PQ}, \quad U_{s+s_*,\rho(PQ)} = \lambda_{s+s_*}^2 \overline{\mu}^{P-Q} \overline{U}_{s,PQ}.$$

If $(P,Q) \in \mathcal{R}_h$, then $\overline{\mu}^{P-Q} = \overline{\mu}_h = \lambda_h^{-2}$. If $(P,Q) \in \mathcal{R}_s$, then $\overline{\mu}^{P-Q} = \overline{\mu}_s = \lambda_{s+s_*}^{-2}$. The result then follows.

(iii) Let $\varphi \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2)$. Then, in particular, we have

$$U_j = U_j(T_{1k}), \quad k \neq j; \quad V_j = \lambda_j^{-1} U_j \circ T_1.$$

Let $(P,Q) \in \mathcal{R}_j \setminus \mathcal{N}_j$. For each $k$ such that $q_k > p_k$, we compose $U_j$ by $T_{1k}$. We emphasize that when $(P,Q) \in \mathcal{R}_j$, such a $k$ is a hyperbolic index. Using the previous identity, we obtain

$$U_{j,PQ} = L_{j,PQ} U_{j,A_jB_j} \tag{10.32}$$

with

$$L_{j,PQ} := \prod_{k \neq j, p_k < q_k} \lambda_k^{q_k - p_k}.$$

By the definition of $\nu_{PQ}^+$, we conclude

$$L_{j,PQ} = \nu_{PQ}^+, \quad (P,Q) \in \mathcal{R}_j. \tag{10.33}$$

If $(P,Q) \in \mathcal{N}_j$, then $(A_j, B_j) = (P,Q)$ and we have $L_{j,PQ} = \nu_{PQ}^+ = 1$, so that the relation (10.32) just becomes the identity $U_{j,PQ} = U_{j,PQ}$.

Assume now that $\varphi \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2, \rho)$. In addition to the previous conditions, we have (10.29) and (10.30). Hence, (10.20)-(10.22) and (10.32) lead to:

$$\begin{aligned}
\nu_{PQ}\overline{U}_{e,\rho_e(PQ)} = U_{e,PQ} = L_{e,PQ}U_{e,A_eB_e}, &\quad (P,Q) \in \mathcal{R}_e;\\
\overline{U}_{h,\rho_h(PQ)} = U_{h,PQ} = L_{h,PQ}U_{h,A_hB_h}, &\quad (P,Q) \in \mathcal{R}_h;\\
\overline{U}_{s+s_*,\rho(PQ)} = U_{s,PQ} = L_{s,PQ}U_{s,A_sB_s}, &\quad (P,Q) \in \mathcal{R}_s.
\end{aligned}$$

Since $\rho_e, \rho_h$ are involutions on $\mathcal{R}_e$ and $\mathcal{R}_h$, respectively, and since $\rho$ is a bijection from $\mathcal{R}_s$ onto $\mathcal{R}_{s+s_*}$, we obtain

$$\begin{aligned}
\nu_{\rho_e(PQ)}\overline{U}_{e,PQ} = L_{e,\rho_e(PQ)}U_{e,(A_e,B_e)\circ\rho_e(PQ)}, &\quad (P,Q) \in \mathcal{R}_e;\\
\overline{U}_{h,PQ} = L_{h,\rho_h(AB)}U_{h,(A_h,B_h)\circ\rho_h(PQ)}, &\quad (P,Q) \in \mathcal{R}_h(PQ);\\
\overline{U}_{s+s_*,PQ} = L_{s,\rho(AB)}U_{s,(A_s,B_s)\circ\rho(PQ)}, &\quad (P,Q) \in \mathcal{R}_{s+s_*}.
\end{aligned}$$

By (10.33), we copy the values $L_{j,\rho(PQ)} = \nu^+_{\rho(PQ)}$ from (10.14). We have

$$\begin{aligned}
\nu^+_{\rho_j(PQ)} &= \nu^+_{PQ}, && \text{if } j \neq e, \text{ and } (P,Q) \in \mathcal{R}_j;\\
\nu^+_{\rho_e(PQ)} &= \nu_{PQ}\nu^+_{PQ}, && \text{if } (P,Q) \in \mathcal{R}_e;\\
\nu_{\rho_e(PQ)} &= \nu_{PQ}, && \text{if } (P,Q) \in \mathcal{R}_e.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
U_{j,PQ} &= \nu^+_{PQ}\overline{U}_{j,(A_j,B_j)\circ\rho_j(PQ)}, \quad (P,Q) \in \mathcal{R}_j, \quad j = e,h;\\
U_{s+s_*,PQ} &= \nu^+_{PQ}\overline{U}_{s,(A_s,B_s)\circ\rho(PQ)}, \quad (P,Q) \in \mathcal{R}_{s+s_*}.
\end{aligned}$$

Therefore, we have derived necessary conditions for the centralizers. Let us verify that the conditions are also sufficient. Of course, the verification for (i) is straightforward. Furthermore, that $\varphi = I + (U,V)$ commutes with $S_1, \ldots, S_p$ is equivalent to $U_{j,PQ} = V_{j,QP} = 0$ for all $(P,Q) \in \mathcal{R}_j$, which is also trivial in cases (ii) and (iii).

For (ii), (10.18) and (10.19) imply that $\varphi$ commutes with $T_1$. We verify that $\varphi$ commutes with $\rho$. Write $\rho\varphi\rho = (\tilde{U}, \tilde{V})$. Applying (10.19) and (10.20) each twice, we get for $(P,Q) \in \mathcal{R}_e$

$$\tilde{U}_{e,PQ} = \overline{V}_{e,\rho(PQ)} = \lambda_e^{-1}\lambda^{\rho_b-\rho_a}\overline{U}_{e,\rho_e(PQ)} = \lambda_e^{-1}\lambda^{\rho_b-\rho_a}\nu_{PQ}U_{e,PQ}.$$

We get $\tilde{U}_{e,PQ} = U_{e,PQ}$. The identities for hyperbolic and complex components of $\rho\varphi\rho = \varphi$ are easy to verify.

For (iii), let us verify that (10.23), (10.18), and (10.19) are sufficient conditions for $\varphi \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2)$. By (10.19), we get $\varphi T_1 = T_1\varphi$. Also, for $\varphi \in \mathcal{C}(\mathcal{T}_1)$ it remains to show that for $(P,Q) \in \mathcal{R}_j$

$$(U_j \circ T_{1k})_{PQ} = U_{j,PQ}, \quad k \neq j; \quad (U_j \circ T_{1j})_{QP} = \lambda_j V_{j,QP}. \tag{10.34}$$

We introduce $(P_j, Q_j)$ via $\xi^P\eta^Q \circ T_{1j} = \lambda_j^{p_j-q_j}\xi^{P_j}\eta^{Q_j}$ and also denote $(P_j, Q_j)$ by $(P,Q)_j$.

We first remark that (10.23) also holds for $(P,Q) \in \mathcal{N}_j$. Therefore, we will use (10.23) for all $(P,Q) \in \mathcal{R}_j$.

For $k \neq j, h$, we have $(P_k, Q_k) = (P, Q)$. Thus in this case we immediately get the first identity in (10.34). Using (10.23) twice, we obtain for $j \neq h$

$$\begin{aligned}(U_j \circ T_{1h})_{PQ} &= \lambda_h^{p_h - q_h} U_{j,(PQ)_h} = \lambda_h^{p_h - q_h} \nu^+_{(PQ)_h} U_{j,(A_j,B_j)(P,Q)} \\ &= \lambda^{p_h - q_h} \nu^+_{(PQ)_h} \overline{\nu}^+_{PQ} U_{j,PQ} = U_{j,PQ}.\end{aligned}$$

Combining with the identities which we have proved, we get $(U_j \circ T_{1j})_{QP} = (U_j \circ T_1)_{QP} = (\lambda_j V_j)_{QP}$ for $j \neq h$. This gives us all the identities in (10.34) for $(P, Q) \in \mathcal{R}_j$. These identities are trivial when $(P, Q)$ is not in $\mathcal{R}_j$. Therefore, we have shown that these conditions are sufficient for $\varphi \in \mathcal{C}(\mathcal{T}_1, \mathcal{T}_2)$.

Finally, we need to verify that (10.18), (10.19), and (10.23)-(10.25) imply that $\varphi$ and $\rho$ commute.

To shorten operations applied to multiindices, let us introduce the follow notation. For $(P, Q) \in N_j$, define

$$\iota_j \colon (P, Q) \mapsto (A_j, B_j) \circ \rho_j(P, Q), \quad j = e, h; \quad \iota_s \colon (P, Q) \mapsto (A_s, B_s) \circ \rho(P, Q).$$

Then $\iota_j$ is an involution on $\mathcal{N}_j$ when $j = e, h$, and it is a bijection from $\mathcal{N}_{s+s_*}$ onto $\mathcal{N}_s$ when $j = s$. Furthermore, the inverse of $\iota_s$ is given by

$$\iota_{s+s_*} \colon (P, Q) \mapsto (A_{s+s_*}, B_{s+s_*}) \circ \rho(P, Q).$$

Fix $(P, Q) \in \mathcal{R}_e$. By (10.23) and (10.24), we have

$$U_{e,PQ} = \nu^+_{PQ} U_{e,(A_e,B_e)(P,Q)} = \nu^+_{PQ} \nu^+_{(A_e,B_e)(P,Q)} \overline{U}_{e,\iota_e \circ (A_e,B_e)(P,Q)}. \tag{10.35}$$

We know that $p_j = q_j$ when $j \neq h$ or $j$ does not equal the $e$ ($j$ can represent other elliptic components). We know that $p_e = q_e + 1$ for the $e$. By treating case by case for $p_h \geq q_h$ or $p_h < q_h$, i.e. $2^{h_*}$ cases in total, we verify that

$$\iota_e \circ (A_e, B_e)(P, Q) = \iota_e(P, Q), \quad \nu^+_{PQ} \nu^+_{(A_e,B_e)(P,Q)} = \lambda_e^{-1} \lambda^{\rho_b - \rho_a} \nu^+_{\rho_b \rho_a}.$$

This allows us to apply (10.19) and (10.23) to rewrite the right-hand side of (10.35) as $\overline{V}_{e,\rho(PQ)}$. We repeat a simpler procedure for $U_{h,PQ}$ with $(P, Q) \in \mathcal{R}_h$: We apply $(A_h, B_h)$ to the multi-index $(P, Q)$ and use (10.24) once. We then check the multiindex and the coefficient to conclude that the result is $\overline{U}_{h,\rho(PQ)}$. (Here we do not need apply (10.19).) For $U_{s+s_*,PQ}$ with $(P, Q) \in \mathcal{R}_{s+s_*}$, we apply $(A_{s+s_*}, B_{s+s_*})$ to $(P, Q)$ and use (10.25) once. The result is $\overline{U}_{s,\rho(PQ)}$. With $U_{h,PQ} = \overline{U}_{h,\rho(PQ)}$ and $U_{s+s_*,PQ} = \overline{U}_{s,\rho(PQ)}$, we apply (10.19) to obtain $V_{h,PQ} = \overline{V}_{h,\rho(PQ)}$ and $V_{s+s_*,PQ} = \overline{V}_{s,\rho(PQ)}$. This shows that $\varphi$ commutes with $\rho$. The proof is complete. □

We have described the conditions on centralizers. We now determine complements of these conditions to define normalized mappings.

**Definition 10.4.** Let $\varphi = I + (U, V)$ be a formal mapping tangent to the identity.

($i$) We say that $\varphi$ is *normalized* with respect to $S_1, \ldots, S_p$ if

$$U_{j,PQ} = 0 = V_{j,QP}, \quad \text{if } (P, Q) \in \mathcal{R}_j, \quad \forall j.$$

Furthermore, $\rho\varphi\rho$ is normalized w.r.t. $S_1, \ldots, S_p$ if and only if $\varphi$ is.

(*ii*) We say that $\varphi$ is *normalized* with respect to $\{\mathcal{S}, T_1, \rho\}$ if

$$U_{h,PQ} = -\overline{U}_{h,\rho(PQ)}, \qquad \forall (P,Q) \in \mathcal{R}_h; \tag{10.36}$$

$$U_{s+s_*,PQ} = -\overline{U}_{s,\rho(PQ)}, \qquad \forall (P,Q) \in \mathcal{R}_{s+s_*}; \tag{10.37}$$

$$U_{e,PQ} = -\nu_{PQ}\overline{U}_{e,\rho_e(PQ)}, \qquad \forall (P,Q) \in \mathcal{R}_e; \tag{10.38}$$

$$V_{j,QP} = -\lambda_j^{-1}\lambda^{P-Q}U_{j,PQ}, \qquad \forall (P,Q) \in \mathcal{R}_j. \tag{10.39}$$

(*iii*) We say that $\varphi$ is *normalized* w.r.t. $\{\mathcal{T}_1, \mathcal{T}_2, \rho\}$ if

$$U_{j,PQ} = -\nu^+_{PQ}U_{j,(A_j,B_j)(P,Q)}, \qquad \forall (P,Q) \in \mathcal{R}_j \setminus \mathcal{N}_j, \tag{10.40}$$

$$U_{j,PQ} = -\nu^+_{PQ}\overline{U}_{j,(A_j,B_j)\circ\rho_j(P,Q)}, \quad \forall (P,Q) \in \mathcal{N}_j,\ j = e,h; \tag{10.41}$$

$$U_{s+s_*,PQ} = -\nu^+_{PQ}\overline{U}_{s,(A_s,B_s)\circ\rho(P,Q)}, \quad \forall (P,Q) \in \mathcal{N}_{s+s_*}. \tag{10.42}$$

**Lemma 10.5.** *Let $F$ be a formal map which is tangent to the identity. There exists a unique formal decomposition $F = HG^{-1}$ with $G \in \mathcal{C}(\mathcal{S}, T_1, \rho)$ (resp. $\mathcal{C}(\mathcal{T}_1, \mathcal{T}_2, \rho)$) and $H \in \mathcal{C}^{\mathsf{c}}(\mathcal{S}, T_1, \rho)$ (resp. $\mathcal{C}^{\mathsf{c}}(\mathcal{T}_1, \mathcal{T}_2, \rho)$)). If $F$ is convergent, then $G$ and $H$ are also convergent.*

*Proof.* We will apply Lemma 4.8 as follows. Let $\hat{H}$ be the set of mappings in $\mathcal{C}^{\mathsf{c}}_2(\mathcal{S}, \mathcal{T}_1, \rho)$. Note that $\hat{H}$ is a $\mathbf{R}$-linear subspace of $(\widehat{\mathfrak{M}}^2_n)^n$. We will define a $\mathbf{R}$-linear projection $\pi$ from $(\widehat{\mathfrak{M}}^2_n)^n$ onto $\hat{H}$ such that $\pi$ preserves the degree of $F$ if $F$ is homogeneous. We will show that $\hat{G} = (\mathrm{I} - \pi)\hat{H}$ agrees with $\mathcal{C}^{\mathsf{c}}_2(\mathcal{S}, \mathcal{T}_1, \rho)$. We will derive estimates on $\pi$ stated in Lemma 4.8, from which we conclude the convergence of $H, G$.

The same argument will be applied to the second case of $\mathcal{C}(\mathcal{T}_1, \rho)$ and $\mathcal{C}^{\mathsf{c}}(\mathcal{T}_1, \rho)$.

For the first case, let us define a projection $\pi\colon (\widehat{\mathfrak{M}}^2_n)^n \to \hat{H}$. We decompose

$$(U, V) = (U' + U'', V' + V''), \quad \pi(U, V) = (U', V').$$

We first define

$$U'_{j,PQ} = U_{j,PQ}, \quad V'_{j,PQ} = V_{j,PQ}, \quad U''_{j,PQ} = 0, \quad V''_{j,PQ} = 0,$$

for $(P,Q) \notin \mathcal{R}_j$. Suppose that $(P,Q) \in \mathcal{R}_e$. We have

$$U_{e,PQ} = U'_{e,PQ} + U''_{e,PQ},$$
$$U_{e,\rho_e(PQ)} = U'_{e,\rho_e(PQ)} + U''_{e,\rho_e(PQ)}.$$

According to (10.38) and (10.20), we need to seek solutions that satisfy

$$U'_{e,PQ} + \nu_{PQ}\overline{U}'_{e,\rho_e(PQ)} = 0, \quad U''_{e,PQ} - \nu_{PQ}\overline{U}''_{e,\rho_e(PQ)} = 0. \tag{10.43}$$

Hence, for $(P,Q) \in \mathcal{R}_e$ we choose

$$U'_{e,PQ} = \frac{1}{2}(U_{e,PQ} - \nu_{PQ}\overline{U}_{e,\rho_e(PQ)}),\ U''_{e,PQ} = \frac{1}{2}(U_{e,PQ} + \nu_{PQ}\overline{U}_{e,\rho_e(PQ)}).$$

We verify directly that the solutions satisfy (10.43) as follows:

$$\begin{aligned} U'_{e,PQ} + \nu_{PQ}\overline{U}'_{e,\rho_e(PQ)} &= \frac{1}{2}(U_{e,PQ} - \nu_{PQ}\overline{U}_{e,\rho_e(PQ)}) \\ &\quad + \frac{1}{2}(\nu_{PQ}\overline{U}_{e,\rho_e(PQ)} - \nu_{PQ}\nu_{\rho_e(PQ)}\overline{U}_{e,PQ}) = 0. \end{aligned}$$

Here we have used the fact that $\rho_e$ is an involution and $\nu_{\rho_e(PQ)}\nu_{PQ}=1$ from (10.13).

Analogously, for $(P,Q)\in\mathcal{R}_h$, we achieve (10.36) and (10.21) by taking

$$U'_{h,PQ}=\frac{1}{2}(U_{h,PQ}-\overline{U}_{h,\rho_h(PQ)}),\quad U''_{h,PQ}=\frac{1}{2}(U_{h,PQ}+\overline{U}_{h,\rho_h(PQ)}).$$

For $(P,Q)\in\mathcal{R}_{s+s_*}$, we achieve (10.22) and (10.37) by taking

$$U'_{s+s_*,PQ}=\frac{1}{2}(U_{s+s_*,PQ}-\overline{U}_{s,\rho(PQ)}),\quad U''_{s+s_*,PQ}=\frac{1}{2}(U_{s+s_*,PQ}+\overline{U}_{s,\rho(PQ)}).$$

We have determined coefficients for $U'_{j,PQ},U''_{j,PQ}$ with $(P,Q)\in\mathcal{R}_j$. Let us set for $(P,Q)\in\mathcal{R}_j$,

$$V'_{j,QP} = -\lambda_j^{-1}\lambda^{P-Q}U'_{j,PQ}, \tag{10.44}$$

$$V''_{j,QP} = \lambda_j^{-1}\lambda^{P-Q}U''_{j,PQ}. \tag{10.45}$$

This fulfills the conditions on $V'_j$ and $V''_j$ easily. Note that the last identity means that $(U'',V'')$ commutes with $T_1$. We have obtained the required formal decomposition.

To prove the convergence, we start with

$$\lambda_j^{-1}\lambda^{P-Q}=\nu_{PQ}=\pm 1 \tag{10.46}$$

for $(P,Q)\in\mathcal{R}_j$. So $\pi$ is indeed an $\mathbf{R}$-linear projection which preserves degrees. Since $|\nu_{PQ}|=1$, we have that

$$|U'_{PQ}|\le\max_{(P',Q')}|U_{P'Q'}|.$$

Here $(P',Q')$ runs over all permutations of $(P,Q)$ in $2p$ coordinates. The same holds for $V'$. Hence, with the notation of Lemma 4.8, we have

$$\{\pi(U,V)\}_{sym}\prec(U,V)_{sym}.$$

The existence and uniqueness as well as the convergence also follow from Lemma 4.8.

We now consider the second case of $\mathcal{C}(\mathcal{T}_1,\mathcal{T}_2,\rho)$ by minor changes. Let us define a projection $\pi\colon((\widehat{\mathfrak{M}}_n^2)^n\to\hat{H}$. Here $\hat{H}$ is the space associated with the mappings satisfying the normalized conditions (10.40)-(10.42). Let $\hat{G}=(\mathrm{I}-\pi)\hat{H}$. We decompose as above

$$(U,V)=(U'+U'',V'+V''),\quad \pi(U,V)=(U',V').$$

We choose :

$$U''_{j,PQ}=\frac{1}{2}(U'_{j,PQ}+\nu^+_{PQ}U_{j,(A_j,B_j)(P,Q)}),\qquad (P,Q)\in\mathcal{R}_j\setminus\mathcal{N}_j, \tag{10.47}$$

$$U'_{j,PQ}=\frac{1}{2}(U'_{j,PQ}-\nu^+_{PQ}U_{j,(A_j,B_j)(P,Q)}),\qquad (P,Q)\in\mathcal{R}_j\setminus\mathcal{N}_j, \tag{10.48}$$

$$U''_{j,PQ}=\frac{1}{2}(U_{j,PQ}+\nu^+_{PQ}\overline{U}_{j,(A_j,B_j)\circ\rho_j(PQ)}),\qquad (P,Q)\in\mathcal{N}_j,\ j=e,h, \tag{10.49}$$

$$U'_{j,PQ}=\frac{1}{2}(U_{j,PQ}-\nu^+_{PQ}\overline{U}_{j,(A_j,B_j)\circ\rho_j(PQ)}),\qquad (P,Q)\in\mathcal{N}_j,\ j=e,h, \tag{10.50}$$

$$U''_{s+s_*,PQ}=\frac{1}{2}(U_{s+s_*,PQ}+\nu^+_{PQ}\overline{U}_{s,(A_s,B_s)\circ\rho(PQ)}),\quad (P,Q)\in\mathcal{N}_{s+s_*}, \tag{10.51}$$

$$U'_{s+s_*,PQ}=\frac{1}{2}(U_{s+s_*,PQ}-\nu^+_{PQ}\overline{U}_{s,(A_s,B_s)\circ\rho(PQ)}),\quad (P,Q)\in\mathcal{N}_{s+s_*}. \tag{10.52}$$

We set $U''_{j,PQ} = 0 = V''_{j,QP}$ for $(P,Q) \notin \mathcal{R}_j$. Let us verify that $\pi(U,V) = (U',V')$ is in $\hat{H}$. Recall that

$$\iota_e \colon (P,Q) \to (A_e,B_e) \circ \rho_e(PQ) = (A_e,B_e)(\rho_b(P,Q),\rho_a(P,Q)), \quad (P,Q) \in \mathcal{N}_e.$$

To verify (10.41) for $j = e$, via (10.50) we compute

$$\begin{aligned} U'_{e,PQ} + \nu^+_{PQ}\overline{U}'_{e,(A_e,B_e)\circ\rho_e(PQ)} &= \frac{1}{2}(U_{e,PQ} - \nu^+_{PQ}\overline{U}_{e,\iota_e(PQ)}) \\ &\quad + \frac{\nu^+_{PQ}}{2}(U_{e,\iota_e(PQ)} - \nu^+_{\iota_e(PQ)}\overline{U}_{e,PQ}) = 0. \end{aligned}$$

Here we have used the fact that $\iota_e \colon \mathcal{N}_e \to \mathcal{N}_e$ is an involution and

$$\nu_{PQ}\nu_{\iota_e(PQ)} = 1, \quad \nu^+_{PQ}\nu^+_{\iota_e(PQ)} = 1, \quad (P,Q) \in N_e.$$

Recall that

$$\iota_j(P,Q) = (A_j,B_j)(\rho_a(PQ),\rho_b(PQ)), \quad j = h,s.$$

We also know that $\iota_h$ is an involution on $\mathcal{N}_h$ and $\iota_s$ is a bijection from $\mathcal{N}_{s+s_*}$ onto $\mathcal{N}_s$. Analogously, we verify (10.41) for $U'_h$ and (10.42) via (10.50) and (10.52). Note that $(P,Q) \to (A_j,B_j)(P,Q)$ is a projection on $\mathcal{R}_j$. Analogously, we verify (10.40) via (10.48). This shows that $\pi(U,V)$ is in $\hat{H}$. We can also verify that $(U'',V'') = (\mathrm{I} - \pi)(U,V)$ satisfies the conditions on the centralizer, i.e. it is in $\hat{G}$.

As before, we have

$$|U'_{j,PQ}|, |U''_{j,PQ}| \le \max_i \max_{(P',Q')\text{permutation of } (P,Q)} |U_{i,P'Q'}|.$$

Equations (10.44), (10.45) lead to the same inequality for $V', V''$. Hence, again the result follows from Lemma 4.8. □

**Proposition 10.6.** *Assume that the family of involutions $\{\mathcal{T}_1, \mathcal{T}_2, \rho\}$ is formally linearizable. Assume further that $\sigma_1, \dots, \sigma_p$ defined by* (10.1)-(10.3)*, are linear.*

(*i*) *There is a biholomorphic mapping in the centralizer of $\{\mathcal{S}, \rho\}$ which linearizes $\tau_1$ and $\tau_2$.*

(*ii*) *Assume further that $\tau_1 = T_1$ and $\tau_2 = T_2$. Then $\{\tau_{11}, \dots, \tau_{1p}, \rho\}$ is holomorphically linearizable.*

*Proof.* (i) Suppose that $\Psi$ is a formal mapping satisfying

$$\Psi^{-1}\tau_{1j}\Psi = T_{1i_j}, \quad \Psi\rho = \rho\Psi.$$

Then $T_{1j} = (L\Psi) \circ T_{1i_j} \circ (L\Psi)^{-1}$, and $L\Psi$ commutes with $\rho$. Replacing $\Psi$ by $\Psi \circ L\Psi^{-1}$, we may assume that $\Psi$ is tangent to the identity and $i_j = j$. We decompose $\Psi = \Psi_1\Psi_0^{-1}$, where $\Psi_1$ is normalized w.r.t. $\mathcal{S}, T_1, \rho$ and $\Psi_0$ is in the centralizer of $\mathcal{S}, T_1, \rho$. Since $\Psi, \Psi_0$ commute with $S_j$ and $\rho$, then $\Psi_1$ commutes with $S_j, \rho$ too. We now let $\Psi$ denote $\Psi_1$.

To be more specific, let us write

$$\tau_1 \colon \begin{cases} \xi'_i = \lambda_i\eta_i + f_i(\xi,\eta) & i = 1,\dots,p, \\ \eta'_i = \lambda_i^{-1}\xi_i + g_i(\xi,\eta) & i = 1,\dots,p, \end{cases}$$

and

$$\Psi\colon \begin{cases} \xi_i' = \xi_i + U_i(\xi,\eta) & i=1,\dots,p,\\ \eta_i' = \eta_i + V_i(\xi,\eta) & i=1,\dots,p. \end{cases}$$

Let us write that $\Psi$ conjugates $\tau_1$ to

$$T_1\colon \xi_i' = \lambda_i\eta_i, \quad \eta_i' = \lambda_i^{-1}\xi_i, \quad i=1,\dots,p.$$

We have $\Psi\circ T_1 = \tau_1\circ\Psi$; that is

$$\lambda_i V_i - U_i\circ T_1 = -f_i\circ\Psi(\xi,\eta) \quad i=1,\dots,p, \tag{10.53}$$

$$\lambda_i^{-1}U_i - V_i\circ T_1 = -g_i\circ\Psi(\xi,\eta) \quad i=1,\dots,p. \tag{10.54}$$

Since $\Psi$ commutes with each $S_j$, then $U_{j,PQ} = V_{j,QP} = 0$ for $(P,Q)\notin\mathcal{R}_j$. Let us find an equation involving only the unknown $U_e, U_h, V_h, U_s, V_s$. By the reality conditions, they determine $U, V$ completely.

Since the normalized mapping $\Psi$ commutes with $\rho$, we have

$$\begin{gathered} U_{h,PQ} = \overline{U}_{h,\rho(PQ)},\ (P,Q)\in\mathcal{R}_h, \quad U_{(s+s_*),PQ} = \overline{U}_{s,\rho(PQ)},\ (P,Q)\in\mathcal{R}_{s+s_*},\\ V_{e,QP} = \overline{U}_{e,\rho_e(QP)}, \quad (P,Q)\in\mathcal{R}_e, \\ V_{h,QP} = \overline{V}_{h,\rho(PQ)},\ (P,Q)\in\mathcal{R}_h, \quad V_{s+s_*,QP} = \overline{V}_{s,\rho(QP)},\ (P,Q)\in\mathcal{R}_{s+s_*}. \end{gathered} \tag{10.55}$$

Let us combine the above identities with the (first two) normalizing conditions

$$\begin{aligned} U_{h,PQ} &= -\overline{U}_{h,\rho(PQ)}, && (P,Q)\in\mathcal{R}_h,\\ U_{s+s_*,PQ} &= -\overline{U}_{s,\rho(PQ)}, && (P,Q)\in\mathcal{R}_{s+s_*},\\ U_{e,PQ} &= -\nu_{PQ}\overline{U}_{e,\rho_e(PQ)}, && (P,Q)\in\mathcal{R}_e,\\ V_{j,QP} &= -\lambda_j^{-1}\lambda^{P-Q}U_{j,PQ}, && (P,Q)\in\mathcal{R}_j. \end{aligned}$$

Recall that $\Psi$ belongs to the centralizer of $\mathcal{S}$ so that $U_{j,PQ} = V_{j,QP} = 0$ for $(PQ)\notin\mathcal{R}_j$ and all $j$. We then immediately see that $U_h, U_s, U_{s+s_*}, V_h, V_s, V_{s+s_*}$ are 0.

We now use the two last conditions to determine $U_e, V_e$ and majorize them. By (10.54), (10.55) and (10.46), we obtain

$$U_{e,PQ} - \nu_{PQ}\overline{U}_{e,\rho_e(QP)} = -\lambda_e\{g_e\circ\Psi\}_{PQ}.$$

Using (10.38), we obtain that, for $(P,Q)\in\mathcal{R}_e$,

$$U_{e,PQ} = -\frac{1}{2}\lambda_e\{g_e\circ\Psi\}_{PQ},$$

as well as

$$V_{e,QP} = \frac{1}{2}\nu_{PQ}\lambda_e\{g_e\circ\Psi\}_{PQ}.$$

Therefore, we have

$$|V_{e,QP}|, |U_{e,PQ}| \le C\left|\{g_e\circ\Psi\}_{PQ}\right|.$$

In view of (4.12), we then have

$$\psi_{sym} \prec Cg_{sym}\circ\Psi_{sym} = g_{sym}\circ(I_{sym}+\psi_{sym}).$$

Therefore, $\psi_{sym}$ is convergent at the origin and so is $\Psi$.

(ii) Assume now that $\sigma = S, \tau_1 = T_1, \tau_2 = T_2$ are linear. Suppose that $\Psi$ linearizes the $\{\tau_{ij}\}$ and commutes with $\rho$. We decompose $\Psi = \Psi_1\Psi_0^{-1}$ with $\Psi_1$ being normalized w.r.t. $\mathcal{S}, T_1, T_2, \rho$ and with $\Psi_0$ being in the centralizer of $\mathcal{S}, T_1, T_2, \rho$. From (i), we know that $\Psi$ is diagonal and $\Psi^{-1}\tau_{ij}\Psi = T_{ij}$. We have

$$\Psi_1^{-1}\tau_{ij}\Psi_1 = \Psi_0^{-1}T_{ij}\Psi_0 = T_{ij}.$$

Hence, $\Psi_1$ linearizes the $\tau_{ij}$ and is normalized w.r.t $\mathcal{S}, T_1, T_2, \rho$. Since $\Psi, \Psi_1$ commute with $\mathcal{S}$ and $\rho$, so does $\Psi_1$. Let us denote $\Psi = \Psi_1$ and let us write $\Phi = I + (U, V)$.

We recall

$$T_{1j}\colon \begin{cases} \xi_j' = \lambda_j\eta_j \\ \eta_j' = \lambda_j^{-1}\xi_i \\ \xi_k' = \xi_k, \quad k \neq j \\ \eta_k' = \eta_k, \quad k \neq j, \end{cases} \qquad \tau_{1j}\colon \begin{cases} \xi_j' = \lambda_j\eta_j + f_{jj}(\xi,\eta) \\ \eta_j' = \lambda_j^{-1}\xi_i + g_{jj}(\xi,\eta) \\ \xi_k' = \xi_k + f_{jk}(\xi,\eta), \quad k \neq j \\ \eta_k' = \eta_k + g_{jk}(\xi,\eta), \quad k \neq j. \end{cases}$$

Since we have $\Psi \circ T_{1j} = \tau_{1j} \circ \Psi$, we obtain the following relations

$$\begin{cases} \lambda_j V_j - U_j \circ T_{1j} = -f_{jj} \circ \Psi \\ \lambda_j^{-1}U_j - V_j \circ T_{1j} = -g_{jj} \circ \Psi \\ U_k - U_k \circ T_{1j} = -f_{jk} \circ \Psi, \quad k \neq j \\ V_k - V_k \circ T_{1j} = -g_{jk} \circ \Psi, \quad k \neq j. \end{cases} \tag{10.56}$$

According to (10.19), the left-hand side of the two first equations are zero. We shall use the two last ones to obtain estimates. According to the normalizing conditions, we find as above, that $U_h, U_s, U_{s+s_*}, V_h, V_s, V_{s+s_*}$ are 0. Thus we only have to show that $U_e, V_e$ are convergent.

In the second last identity in (10.56) with $k = e$, let us compose on the right by all $T_{1j}$ with $j \neq e$. We have for $j' \neq e$ and $j \neq e$,

$$U_e \circ T_{1j} - U_e \circ T_{1j} \circ T_{1j'} = -f_{j,e} \circ \Psi \circ T_{1j'} = f_{j,e} \circ \tau_{1j'} \circ \Psi.$$

Repeating this for all $T_{1j}$ except for $j = e$ and taking summation, we get

$$U_e - U_e \circ T_{1e}^{-1} \circ T_1 = -\left\{\sum_{i=1}^{p} f_{j,e} \circ \tau_{11} \cdots \circ \widehat{\tau_{1e}} \circ \cdots \circ \tau_{1i}\right\} \circ \Psi.$$

Here $\widehat{\tau_{1e}}$ means that $\tau_{1e}$ is not included in composition if $i \geq e$. Thus

$$U_e \circ T_{1e} - U_e \circ T_1 = -\left\{\sum_{i=1}^{p} f_{j,e} \circ \tau_{11} \cdots \circ \widehat{\tau_{1e}} \circ \cdots \circ \tau_{1i}\right\} \circ \tau_{1e} \circ \Psi.$$

Combining with the first identity in (10.56) and eliminating $U_e \circ T_{1e}$, we obtain

$$\lambda_e V_e - U_e \circ T_1 = \tilde{f}_e \circ \Psi$$

for a convergent power series $\tilde{f}_e$. The normalizing condition (10.39) says that $\lambda_1^{Q-P}U_{e,PQ} = -\lambda_e V_{e,QP}$ for $(P,Q) \in R_e$. We obtain

$$V_{e,QP} = \frac{1}{2\lambda_e}\{\tilde{f}_e \circ \Psi\}_{QP} \prec \frac{1}{2\lambda_e}\{\overline{\tilde{f}_e} \circ \overline{\Psi}\}_{QP}, \quad (P,Q) \in \mathcal{R}_e.$$

If $(P,Q)$ is not in $\mathcal{R}_e$, the above still holds as $V_{e,QP}=0$.

Indeed this is the key point, if $U_{j,PQ}\neq 0$ then $(P,Q)\in\mathcal{R}_j$ so that $\mu_j^{p_j-q_j}=\mu_j$ and $\mu_\ell^{p_\ell-q_\ell}=1$, $\ell\neq j$. As we have observed and since $j\neq h$ ($j$ is actually $e$), this implies that $p_j=q_j+1$ and $p_\ell=q_\ell$, $\ell\neq j,h$. Since the hyperbolic $\lambda_h$ are of modulus one, we have either $|\lambda^{P-Q}|=\lambda_e$. Thus

$$|U_{e,PQ}|=|V_{e,QP}|\leq\frac{1}{2\lambda_e}\{\overline{\tilde{f}_e}\circ\overline{\Psi}\}_{QP}\leq\frac{1}{2\lambda_e}\{\overline{\tilde{f}_e}\circ\overline{\Psi}_{sym}\}_{PQ}.$$

We obtain

$$\psi_{sym}\prec C\left(\overline{\tilde{f}}_{sym}\circ(I_{sym}+\psi_{sym}\right).$$

Therefore, $U_e,V_e$ are convergent at the origin since they are majorized by a solution of an analytic implicit function theorem. □

**Remark 10.7.** The results obtained so far in this section does not require that $\sigma$ has distinct eigenvalues. To apply the results to the real manifolds, we impose it again as in previous sections.

**Theorem 10.8.** *Let $M$ be a germ of analytic submanifold that is a third order perturbation of a product quadric $Q$ in $\mathbf{C}^{2p}$. Suppose that $M$, i.e. its $\sigma$, has $n$ distinct eigenvalues. Suppose that $M$ is formally equivalent to the product quadric $Q$. Suppose that each hyperbolic component has an eigenvalue $\mu_h$ which is either a root of unity or satisfies the Brjuno condition (11.32). Then $M$ is holomorphically equivalent to the product quadric.*

*Proof.* We first apply Theorem 11.8 with $\mathcal{I}=0$ ([Sto13]) that linearize simultaneously and holomorphically the $\sigma_1,\ldots,\sigma_p$. Note that the small divisor condition in this special case is equivalent that each $\mu_h$ is either a root of unity or a Brjuno number. Then, we apply successively the two assertions of Proposition 10.6. Hence, in good holomorphic coordinates, $\{\tau_{11},\ldots,\tau_{1p},\rho\}$ are linear. Then, by Proposition 2.10, the manifold is holomorphically equivalent to the quadric. □

We present two convergence proofs for Theorem 9.3: one is based on normalization for each member of the family $\{\sigma_1,\ldots,\sigma_p\}$, and another is based on simultaneous normalization for the whole family. Besides the simultaneous linearization in a more general frame work [Sto13] used above, the first approach by linearizing the family $\{\sigma_1,\ldots,\sigma_p\}$ one by one is still valid. Here it is crucial that the linear maps of $\{\sigma_1,\ldots,\sigma_p\}$ have a very simple structure. Indeed, let $\phi_1$ be a holomorphic mapping that linearizes $\sigma_1$; the existence of such a convergent $\phi_1$ is ensured [Rüs02]. With the transformation by $\phi_1$, we may assume that $\sigma_1$ is the linear $\hat{S}_1$. Let $\phi_2$ be the unique holomorphic mapping that is normalized w.r.t. $\hat{S}_2$ and linearizes $\sigma_2$. Since $\hat{S}_1$ and $\sigma_2$ commute, we verify that $\hat{S}_1\phi_2\hat{S}_1^{-1}$ is normalized w.r.t. $\hat{S}_2$ and linearizes $\sigma_2$. Then $\phi_2$ commutes with $\hat{S}_1$ and linearizes $\sigma_2$. Inductively, we find a biholomorphic mapping that linearizes all $\sigma_1,\ldots,\sigma_p$. The remaining argument is as in the proof of the theorem.

## 11. Existence of attached complex manifolds

We are interested in complex submanifolds $K$ in $\mathbf{C}^{2p}$ that intersect the real submanifold $M$ at the origin. Recall that $M$ has real dimension $2p$. Generically, the origin is an isolated

intersection point if $\dim K = p$. Let us consider the situation when the intersection has dimension $p$. Without further restrictions, there are many such complex submanifolds; for instance, we can take a $p$-dimensional totally real and real analytic submanifold $K_1$ of $M$. We then let $K$ be the complexification of $K_1$. To ensure the uniqueness or finiteness of the complex submanifolds $K$, we therefore introduce the following.

**Definition 11.1.** Let $M$ be a formal real submanifold of dimension $2p$ in $\mathbf{C}^n$. We say that a formal complex submanifold $K$ is *attached* to $M$ if $K \cap M$ contains at least two germs of totally real and formal submanifolds $K_1, K_2$ of dimension $p$ that intersect transversally at the origin. Such a pair $\{K_1, K_2\}$ are called a pair of *asymptotic* formal submanifolds of $M$.

Before we present the details, let us describe the main steps to derive the results. We first derive the results at the formal level. We then apply the results of [Pös86] and [Sto13]. The proof of the co-existence of convergent and divergent attached submanifolds will rely on a theorem of Pöschel on stable invariant submanifolds and Siegel's small divisor technique used in the proof of the divergent normal form in section 6. However, the argument for the divergent part will be simpler.

We now describe the formal results. When $p = 1$ and $M$ has a non-resonant hyperbolic complex tangent, it admits a unique attached formal holomorphic curve [Kli85]. When $p > 1$, new situations arise. First, we show that there are obstructions to attach formal submanifolds. However, the formal obstructions disappear when $M$ admits the maximum number of deck transformations and $M$ is non-resonant. These two conditions allow us to express $M$ in an equivalent form (4.4). This equivalent form for $M$, which has not been used so far, will play an essential role in our proof for $p > 1$.

We will consider a real submanifold $M$ which is a higher order perturbation of a non-resonant product quadrics. By adapting the proof of Klingenberg [Kli85] to the manifold $M$ (4.4), we will show the existence of a unique attached formal submanifold for a prescribed non-resonance condition. As in [Kli85], we also show that the complexification of $K$ in $\mathcal{M}$ is a pair of invariant formal submanifolds $\mathcal{K}_1, \mathcal{K}_2$ of $\sigma$. Furthermore, $K$ is convergent if and only if $\mathcal{K}_1$ is convergent.

Let us first recall the values of the Bishop invariants. The types of the invariants play an important role for the existence and the convergence of attached formal complex submanifolds. From (3.34), and (3.36), we recall that

$$\gamma_e = \frac{1}{\lambda_e + \lambda_e^{-1}}, \quad \gamma_h = \frac{1}{\lambda_h + \overline{\lambda}_h}, \quad \gamma_s = \frac{1}{1 + \lambda_s^{-2}}, \tag{11.1}$$

$$0 < \gamma_e < 1/2, \quad \gamma_h > 1/2, \quad \gamma_s \in (1/2, \infty) + i(0, \infty), \quad \gamma_{s+s_*} = 1 - \overline{\gamma}_s. \tag{11.2}$$

As in Lemma 3.2, we normalize

$$\lambda_e > 1, \quad |\lambda_h| = 1, \quad |\lambda_s| > 1, \quad \lambda_{s+s_*} = \overline{\lambda}_s^{-1}; \tag{11.3}$$

$$\arg \lambda_h \in (0, \pi/2), \quad \arg \lambda_s \in (0, \pi/2). \tag{11.4}$$

Recall that $\mu_j = \lambda_j^2$. By (11.1), we have

$$\gamma_j^2 = \frac{\mu_j}{(1+\mu_j)^2}, \quad j = e, h; \qquad \gamma_s \overline{\gamma}_{s+s_*} = \frac{\mu_s}{(1+\mu_s)^2}. \tag{11.5}$$

We first verify the following.

**Lemma 11.2.** *Let $\gamma_j, \lambda_j$ be given by* (11.1)-(11.4). *Let $\mu_j = \lambda_j^2$. Assume that $\mu_1$, $\mu_1^{-1}, \ldots,$ $\mu_p^{-1}$ are distinct. Then*

$$\gamma_e^2, \quad \gamma_h^2, \quad \overline{\gamma}_s \gamma_{s+s_*}, \quad \gamma_s \overline{\gamma}_{s+s_*}$$

*are distinct p numbers. The latter is equivalent to $\gamma_1, \ldots, \gamma_p$ being distinct.*

*Proof.* Note that $x^{-1} + x$ and $x^{-1}$ decrease strictly on $(0, 1)$. So $\gamma_e^2, \gamma_h^2$ are distinct. We also have

$$\gamma_s \overline{\gamma}_{s+s_*} = \gamma_s - \gamma_s^2.$$

If $a, b$ are complex numbers, then $a - a^2 = b - b^2$ if and only if $a = b$ or $a + b = 1$. Since $\gamma_s$ is not real, then $\gamma_s \overline{\gamma}_{s+s_*}$ are different from $\gamma_e^2$ and $\gamma_h^2$. For any distinct complex numbers $a_1, a_2$ in $(0, \infty) + i(1/2, \infty)$. We have $1 - a_2 \neq 1 - a_1, a_1, a_2$. The lemma is proved. □

Let us first investigate the numbers of pairs of formal asymptotic submanifolds and attached formal submanifolds.

**Lemma 11.3.** *Let $M$ be a formal submanifold that is a third order perturbation of a product quadric $Q$ in $\mathbf{C}^{2p}$. Assume that the associated $S$ of $Q$ has distinct eigenvalues*

$$\mu_1, \ldots, \mu_p, \quad \mu_1^{-1}, \ldots, \mu_p^{-1}.$$

(*i*) *If $M$ admits an attached formal submanifold, its CR singularity has no elliptic component.*

(*ii*) *If $Q$ has no elliptic components, then $Q$ has at least $2^{h_*+s_*-1}$ pairs of asymptotic totally real and real analytic submanifolds and all of them are contained in a single attached complex submanifold.*

(*iii*) *There is no formal submanifold attached to*

$$M\colon z_3 = (z_1 + 2\gamma_1 \overline{z}_1)^2 + (z_2 + 2\gamma_2 \overline{z}_2)^3, \quad z_4 = (z_2 + 2\gamma_2 \overline{z}_2)^2.$$

*Here $M$ has a hyperbolic complex tangent at the origin.*

(*iv*) *Assume that $M$ has no elliptic component and it admits the maximum number of formal deck transformations. Let*

$$\nu = \mu_\epsilon = (\mu_1^{\epsilon_1}, \ldots, \mu_p^{\epsilon_p}), \quad \epsilon_j = \pm 1, \quad \epsilon_{s+s_*} = \epsilon_s. \tag{11.6}$$

*Suppose that*

$$\nu^Q \neq \nu_j^{-1}, \quad \forall Q \in \mathbf{N}^p, \quad |Q| > 0, \quad 1 \leq j \leq p. \tag{11.7}$$

*Then $M$ admits a unique pair of asymptotic formal submanifolds $K_1, K_2$ such that each $K_i$ is defined by $z' = \rho_i(z')$ for a formal anti-holomorphic involution $\rho_i$ and the linear part of $\overline{\rho}_2^{-1} \overline{\rho}_1$ has eigenvalues $\nu_1, \ldots, \nu_p$. In particular, if* (11.7) *holds for each $\nu$ of the form* (11.6) *then $M$ admits exactly $2^{h_*+s_*-1}$ pairs of asymptotic formal submanifolds.*

*Proof.* (i) Let $M$ be defined by

$$z_{p+j} = Q_j(z', \overline{z}') + H_j(z', \overline{z}'), \quad 1 \leq j \leq p$$

where $H_j(z', \overline{z'}) = O(|z'|^3)$ and each $Q_j$ is quadratic. Let $\{K_1, K_2\}$ be a pair of asymptotic formal submanifolds of $M$. We know that $K_1, K_2$ are tangent to $M$ at the origin. Let

$K_i'$ be the projection of $K_i$ onto the $z'$-subspace. Since $T_0M$ is a $p$-dimensional complex subspace, then $K_1', K_2'$ are still totally real. Let $K_1'$ be defined by

$$K_1' \colon \overline{z}' = \mathbf{A}z' + R(z'), \quad \overline{\mathbf{A}}\mathbf{A} = \mathbf{I}, \quad R(z') = O(2)$$

such that $\rho_1(z') := \overline{\mathbf{A}}\overline{z}' + \overline{R}(\overline{z}')$ defines anti-holomorphic formal involutions. Let $K_2$ be the (formal) fixed-point set of the anti-holomorphic involution $\rho_2(z') = \overline{\tilde{\mathbf{A}}}\overline{z}' + \overline{\tilde{R}(z')}$ with $\tilde{R}(z') = O(2)$. Then $K_1, K_2$ intersect transversally at the origin if and only if

$$\det(\tilde{\mathbf{A}} - \mathbf{A}) \neq 0.$$

Let us define holomorphic mappings

$$\overline{\rho}_i(z') := \overline{\rho_i(z')}, \quad i = 1, 2.$$

Then $K$ is given by

$$z''_{p+j} = Q_j(z', \overline{\rho}_i(z')) + H_j(z', \overline{\rho}_i(z')), \quad i = 1, 2, \quad j = 1, \ldots, p.$$

The two equations agree, if and only if

$$Q_j(z', \overline{\rho}_1(z')) + H_j(z', \overline{\rho}_1(z')) = Q_j(z', \overline{\rho}_2(z')) + H_j(z', \overline{\rho}_2(z')), \quad 1 \leq j \leq p. \tag{11.8}$$

Recall that

$$\begin{aligned} Q_j(z', \overline{z}) &= (z_j + 2\gamma_j \overline{z}_j)^2, \quad j = e, h; \\ Q_s(z', \overline{z}') &= (z_{s+s_*} + 2\gamma_{s+s_*} \overline{z}_s)^2, \\ Q_{s+s_*}(z', \overline{z}') &= (z_s + 2\gamma_s \overline{z}_{s+s_*})^2. \end{aligned}$$

Let us first find necessary conditions on the linear parts of $\rho_i$ for (11.8) to be solvable. Let $w' = \mathbf{A}z'$ and $\tilde{w}' = \tilde{\mathbf{A}}z'$. Comparing the quadratic terms in (11.8) for $i = 1, 2$, we see that

$$\begin{aligned} (z_j + 2\gamma_j w_j)^2 &= (z_j + 2\gamma_j \tilde{w}_j)^2, \\ (z_{s+s_*} + 2\gamma_{s+s_*} w_s)^2 &= (z_{s+s_*} + 2\gamma_{s+s_*} \tilde{w}_s)^2, \\ (z_s + 2\gamma_s w_{s+s_*})^2 &= (z_s + 2\gamma_s \tilde{w}_{s+s_*})^2. \end{aligned}$$

Here $\gamma_{s+s_*} = 1 - \overline{\gamma}_s$, by (11.2). For each $j$, $w_j \neq \tilde{w}_j$. Otherwise, the fixed points of $\rho_1$ and $\rho_2$ do not intersect transversally. Therefore, the above 3 identities can be written as

$$\begin{aligned} z_j + 2\gamma_j w_j &= -(z_j + 2\gamma_j \tilde{w}_j), \\ z_{s+s_*} + 2\gamma_{s+s_*} w_s &= -(z_{s+s_*} + 2\gamma_{s+s_*} \tilde{w}_s), \\ z_s + 2\gamma_s w_{s+s_*} &= -(z_s + 2\gamma_s \tilde{w}_{s+s_*}). \end{aligned}$$

In the matrix form, we get $\tilde{\mathbf{A}} = -\boldsymbol{\gamma}^{-1} - \mathbf{A}$ with

$$\boldsymbol{\gamma} := \begin{pmatrix} \boldsymbol{\gamma}_{e_*} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\gamma}_{h_*} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\gamma}_{s_*} \\ \mathbf{0} & \mathbf{0} & \tilde{\boldsymbol{\gamma}}_{s_*} & \mathbf{0} \end{pmatrix}.$$

Here $\tilde{\boldsymbol{\gamma}}_{s_*} = \mathbf{I}_{s_*} - \overline{\boldsymbol{\gamma}}_{s_*}$. Let us express in block matrices

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{e_*e_*} & \mathbf{A}_{e_*h_*} & \mathbf{A}_{e_*s_*} & \mathbf{A}_{e_*(2s_*)} \\ \mathbf{A}_{h_*e_*} & \mathbf{A}_{h_*h_*} & \mathbf{A}_{h_*s_*} & \mathbf{A}_{h_*(2s_*)} \\ \mathbf{A}_{s_*e_*} & \mathbf{A}_{s_*h_*} & \mathbf{A}_{s_*s_*} & \mathbf{A}_{s_*(2s_*)} \\ \mathbf{A}_{(2s_*)e_*} & \mathbf{A}_{(2s_*)h_*} & \mathbf{A}_{(2s_*)s_*} & \mathbf{A}_{(2s_*)(2s_*)} \end{pmatrix}$$

where the diagonal block matrices are of sizes $e_* \times e_*$, $h_* \times h_*$, $s_* \times s_*$, and $s_* \times s_*$, respectively. When $\mathbf{A}\overline{\mathbf{A}} = \mathbf{I}$, for $\tilde{\mathbf{A}}\overline{\tilde{\mathbf{A}}} = \mathbf{I}$ we need $\boldsymbol{\gamma}^{-1} + \mathbf{A} + \boldsymbol{\gamma}^{-1}\overline{\mathbf{A}}\overline{\boldsymbol{\gamma}} = 0$. Recall that $\gamma_1^2, \ldots, \gamma_{e_*+h_*}^2$ are real and distinct. It is easy to see that $\mathbf{A}_{e_*h_*} = 0$, $\mathbf{A}_{h_*e_*} = 0$, and $\mathbf{A}_{e_*e_*}, \mathbf{A}_{h_*h_*}$ are diagonal. Also,

$$\mathbf{A}_{e_*e_*} + \overline{\mathbf{A}}_{e_*e_*} = -\boldsymbol{\gamma}_{e_*}^{-1}, \quad \mathbf{A}_{h_*h_*} + \overline{\mathbf{A}}_{h_*h_*} = -\boldsymbol{\gamma}_{h_*}^{-1}. \tag{11.9}$$

In block matrices, we obtain

$$\boldsymbol{\gamma}_j^{-1}\overline{\mathbf{A}}_{j(2s_*)}\overline{\tilde{\boldsymbol{\gamma}}}_{s_*} = -\mathbf{A}_{js_*}, \qquad \tilde{\boldsymbol{\gamma}}_{s_*}^{-1}\overline{\mathbf{A}}_{(2s_*)j}\overline{\boldsymbol{\gamma}}_j = -\mathbf{A}_{s_*j}; \tag{11.10}$$

$$\boldsymbol{\gamma}_j^{-1}\overline{\mathbf{A}}_{js_*}\overline{\boldsymbol{\gamma}}_{s_*} = -\mathbf{A}_{j(2s_*)}, \qquad \boldsymbol{\gamma}_{s_*}^{-1}\overline{\mathbf{A}}_{s_*j}\boldsymbol{\gamma}_j = -\mathbf{A}_{(2s_*)j}; \tag{11.11}$$

$$\tilde{\boldsymbol{\gamma}}_{s_*}^{-1}\overline{\mathbf{A}}_{(2s_*)(2s_*)}\overline{\tilde{\boldsymbol{\gamma}}}_{s_*} = -\mathbf{A}_{s_*s_*}, \qquad \tilde{\boldsymbol{\gamma}}_{s_*}^{-1}\overline{\mathbf{A}}_{(2s_*)s_*}\overline{\boldsymbol{\gamma}}_{s_*} = -\mathbf{A}_{s_*(2s_*)} - \tilde{\boldsymbol{\gamma}}_{s_*}^{-1}, \tag{11.12}$$

$$\boldsymbol{\gamma}_{s_*}^{-1}\overline{\mathbf{A}}_{s_*(2s_*)}\overline{\tilde{\boldsymbol{\gamma}}}_{s_*} = -\mathbf{A}_{(2s_*)s_*} - \boldsymbol{\gamma}_{s_*}^{-1}, \qquad \boldsymbol{\gamma}_{s_*}^{-1}\overline{\mathbf{A}}_{s_*s_*}\overline{\boldsymbol{\gamma}}_{s_*} = -\mathbf{A}_{(2s_*)(2s_*)}. \tag{11.13}$$

In the first 4 equations, we have $j = e_*, h_*$. Note that the last two equations are of the form (11.12).

By Lemma 11.2, we know that $\gamma_e^2, \gamma_h^2$, and $\gamma_s\overline{\gamma}_{s+s_*}$ are distinct. Thus, $\mathbf{A}_{js_*} = \mathbf{A}_{j(2s)_*} = \mathbf{0}$ and $\mathbf{A}_{s_*j} = \mathbf{A}_{(2s_*)j} = \mathbf{0}$ for $j = e_*, h_*$. Since $\gamma_s\overline{\gamma}_{s+s_*}$ is different from all $\gamma_{s+s_*}\overline{\gamma}_s$, then $\mathbf{A}_{s_*s_*} = \mathbf{A}_{(2s_*)(2s_*)} = \mathbf{0}$ while $\mathbf{A}_{s_*(2s_*)}$, $\mathbf{A}_{(2s_*)s_*}$ are diagonal. Now $\mathbf{A}\overline{\mathbf{A}} = \mathbf{I}$ implies that

$$\mathbf{A}_{e_*e_*}\overline{\mathbf{A}}_{e_*e_*} = \mathbf{I}, \quad \mathbf{A}_{h_*h_*}\overline{\mathbf{A}}_{h_*h_*} = \mathbf{I}, \quad \mathbf{A}_{s_*(2s_*)}\overline{\mathbf{A}}_{(2s_*)s_*} = \mathbf{I}. \tag{11.14}$$

Combining the first identities in (11.9) and (11.14), we know that the diagonal $e$th element $a_e$ of $\mathbf{A}_{e_*e_*}$ must satisfy

$$2a_e = \gamma_e^{-1}, \quad a_e^2 = 1.$$

Since $0 < \gamma_e < 1/2$, there is no such solution $a_e$ if $e_* > 0$. We have verified (i).

Note that $\gamma_h^{-1} = \lambda_h + \overline{\lambda}_h$ with $|\gamma_h| = 1$. For the hyperbolic components, by the second identities in (11.9) and (11.14), one set of solutions is given by

$$\mathbf{A}_{h_*h_*} = -\boldsymbol{\lambda}_{h_*} \quad \tilde{\mathbf{A}}_{h_*h_*} = -\boldsymbol{\lambda}_{h_*}^{-1}.$$

For the complex components, we use $\mathbf{A}_{s_*(2s_*)}\overline{\mathbf{A}}_{(2s_*)s_*} = \mathbf{I}$ and multiply both sides of the second identity in (11.12) by $\mathbf{A}_{s_*(2s_*)}$. The $(s - s_*)$th diagonal element $a_s$ must satisfy

$$a_s(a_s + \tilde{\gamma}_s^{-1}) + \tilde{\gamma}_s^{-1}\overline{\gamma}_s = 0.$$

By the last identity in (11.5), we get

$$a_s^2 + (1 - \overline{\mu}_s)a_s + \overline{\mu}_s = 0.$$

Hence, $a_s = -1$ or $a_s = -\overline{\mu}_s$. We get one set of solutions

$$\begin{aligned} \mathbf{A}_{(2s_*)s_*} &= -\mathbf{I}, & \mathbf{A}_{s_*(2s_*)} &= -\mathbf{I}, \\ \tilde{\mathbf{A}}_{(2s_*)s_*} &= -\boldsymbol{\mu}_{s_*}^{-1}, & \tilde{\mathbf{A}}_{s_*(2s_*)} &= -\overline{\boldsymbol{\mu}}_{s_*}. \end{aligned}$$

There are exactly $2^{h_*+s_*-1}$ solutions for $\mathbf{A}, \tilde{\mathbf{A}}$ since we can only determine the pairs

$$\{\mathbf{A}_{h_*h_*}, \tilde{\mathbf{A}}_{h_*h_*}\}, \quad \{\mathbf{A}_{s_*(2s_*)}, \mathbf{A}_{s_*(2s_*)}\}.$$

Note that

$$\tilde{\mathbf{A}}^{-1}\mathbf{A} = \operatorname{diag}\nu, \tag{11.15}$$

$$\nu = \mu_\epsilon = (\mu_1^{\epsilon_1}, \dots, \mu_p^{\epsilon_p}), \quad \epsilon_j^2 = 1, \quad \nu_{s+s_*} = \overline{\nu}_s^{-1}, \tag{11.16}$$

where there are $2^{h_*+s_*-1}$ distinct combinations. Thus, we get exactly $2^{h_*+s_*-1}$ pairs $\{K_\epsilon^1, K_\epsilon^2\}$ of asymptotic linear submanifolds indexed by $\epsilon = (\epsilon_1, \dots, e_{h_s+s_*})$ with $\epsilon_j^2 = 1$ for the product quadric. We may restrict to $\epsilon_1 = 1$. The attached formal submanifolds associated to these linear asymptotic submanifolds are unique and it is given by

$$\begin{aligned} z_{p+h} &= (1-4\gamma_h^2)z_h^2, \\ z_{p+s} &= (1-2\gamma_{s+s_*})^2 z_{s+s_*}^2, \\ z_{p+s+s_*} &= (1-2\gamma_s)^2 z_s^2. \end{aligned}$$

This finishes the proof of (ii).

(iii). Let us continue the computation for the perturbations. We have determined linear parts of antiholomorphic involutions $\rho_i$. We expand components of $R(z')$ as

$$R_j(z') = \sum_{k=2}^{\infty} R_{j;k}(z'), \quad 1 \le j \le p.$$

Here $R_{j;k}$ are homogeneous terms of degree $k$. We expand $\tilde{R}_j$ analogously. Suppose that terms of order up to $k-1$ in $R_j, \tilde{R}_j$ have been determined. For the hyperbolic components, we need to solve the equations

$$4\sqrt{1-4\gamma_h^2}z_h(R_{h;k}(z') + \tilde{R}_{h;k}(z')) = \cdots, \tag{11.17}$$

where the right-hand side has been determined. Indeed, let us compute the $(k+1)$-jet of (11.8). We obtain

$$(1-2\gamma_j\lambda_j)^2 z_j^2 + 2(1-2\gamma_j\lambda_j)z_j R_{j;k} = (1-2\gamma_j\lambda_j^{-1})^2 z_j^2 + 2(1-2\gamma_j\lambda_j^{-1})z_j\tilde{R}_{j;k} + \mathcal{R}$$

where $\mathcal{R}$ is polynomial that depends on $\tilde{R}_{j;l}, R_{j;l}$, $l < k$. Since $(1-2\gamma_j\lambda_j) = -(1-2\gamma_j\lambda_j^{-1})$, we obtain (11.17).

When $p > 1$, the system of equations (11.17) cannot be solved even formally, unless the right-hand side is divisible by $z_h$. When $p = 1$, the equation (11.17) is clearly solvable. In fact, under the non-resonant condition on $\mu_1$, the formal anti-holomorphic involutions $\{\rho_1, \rho_2\}$ can be uniquely determined.

Let us keep the above notation and compute for the example stated in (iii). We need to solve

$$\begin{aligned} (z_1 + 2\gamma_1\tilde{w}_1)^2 + (z_2 + 2\gamma_2\tilde{w}_2)^3 &= (z_1 + 2\gamma_1 w_1)^2 + (z_2 + 2\gamma_2 w_2)^3, \\ (z_2 + 2\gamma_2\tilde{w}_2)^2 &= (z_2 + 2\gamma_2 w_2)^2. \end{aligned}$$

Again $\tilde{w}_2 - w_2$ cannot be identically zero. Thus $\tilde{w}_2 = -w_2 - \gamma_2^{-1} z_2$. Then we need to solve

$$(z_1 + 2\gamma_1 \tilde{w}_1)^2 = (z_1 + 2\gamma_1 w_1)^2 + 2(z_2 + 2\gamma_2 w_2)^3.$$

By (ii), we know that $w_1 = \lambda_1 z_1 + R_1(z')$ and $w_2 = \lambda_2 z_2 + R_2(z')$ with $R_i(z') = O(2)$. Also $\tilde{w}_1 = \overline{\lambda}_1 z_1 + \tilde{R}_1(z')$ and $\tilde{w}_2 = \overline{\lambda}_2 z_2 + \tilde{R}_2(z')$. Comparing the cubic terms implies that $z_1$ must divide $2(1 + 2\gamma_2\lambda_2)^3 z_2^3$, which is a contradiction.

(iv) For a general $M$, following Klingenberg [Kli85] we reformulate the problem by considering the following equations

$$\begin{aligned} h(z') &= q(z', \overline{\rho}_i(z')) + H(z', \overline{\rho}_i(z')), \quad i = 1, 2, \\ h^*(\overline{\rho}_i(z')) &= \overline{q}(\overline{\rho}_i(z'), z') + \overline{H}(\overline{\rho}_i(z'), z'), \quad i = 1, 2. \end{aligned}$$

Here $h, h^*, \overline{\rho}_i$ are unknowns. Initially, we only require that $\overline{\rho}_1, \overline{\rho}_2$ be arbitrary biholomorphic maps, except their linear parts match with $z' \to Az'$ and $z' \to \tilde{A}z'$. This will ensure that the solutions $\overline{\rho}_i$ are unique and they are involutions.

As demonstrated in (iii), in general there is no formal submanifold attached to $M$. We now assume that $M$ admits the maximum number of deck transformation. By Lemma 2.8 and Proposition 2.10 we know that in suitable holomorphic coordinates, $M$ is given by

$$\begin{aligned} z_{p+j} = \Big( &\sum_h b_{jh}(z_h + 2\gamma_h \overline{z}_h) + \sum_s b_{js}(z_s + 2\gamma_s \overline{z}_{s+s_*}) \\ &+ \sum_s b_{j(s+s_*)}(z_{s+s_*} + 2\gamma_{s+s_*} \overline{z}_s) + E_j(z, \overline{z}) \Big)^2, \quad 1 \le j \le p. \end{aligned}$$

Here $(b_{jk})$ is invertible and $E_j(z, \overline{z}) = O(2)$. This special form, which has not played significant roles until now, will allow us removing the obstruction to formal solutions $\rho_i$.

For the proof of our result, we will restrict $(b_{jk})$ to be the identity matrix. Let $M$ be defined by

$$\begin{aligned} z_{p+h} &= (z_h + 2\gamma_h \overline{z}_h + E_j(z', \overline{z}'))^2, \\ z_{p+s} &= (z_s + 2\gamma_s \overline{z}_{s+s_*} + E_{p+s}(z', \overline{z}'))^2, \\ z_{p+s+s_*} &= (z_{s+s_*} + 2\gamma_{s+s_*} \overline{z}_s + E_{p+s+s_*}(z', \overline{z}'))^2. \end{aligned}$$

We fix linear parts of $\rho_i$ such that

$$\rho_1(z') = \overline{A}\overline{z}' + \overline{R}(\overline{z}'), \quad \rho_2(z') = \overline{\tilde{A}}\overline{z}' + \overline{\tilde{R}}(\overline{z}').$$

For $i = 1, 2$ we then need to solve $w, \tilde{w}$ from

$$z_h + 2\gamma_h \overline{\rho}_{ih} + E_h(z', \overline{\rho}_i) = (-1)^i f_h, \tag{11.18}$$

$$z_s + 2\gamma_s \overline{\rho}_{is+s_*} + E_{p+s}(z', \overline{\rho}_i) = (-1)^i f_s, \tag{11.19}$$

$$z_{s+s_*} + 2\gamma_{s+s_*} \overline{\rho}_{is} + E_{p+s+s_*}(z', \overline{\rho}_i) = (-1)^i f_{s+s_*}, \tag{11.20}$$

$$2\gamma_h z_h + \overline{\rho}_{ih} + \overline{E}_h(\overline{\rho}_i, z') = (-1)^i f_h^*(\overline{\rho}_i), \tag{11.21}$$

$$2\overline{\gamma}_s z_{s+s_*} + \overline{\rho}_{is} + \overline{E}_{p+s}(\overline{\rho}_i, z') = (-1)^i f_s^*(\overline{\rho}_i), \tag{11.22}$$

$$2\overline{\gamma}_{s+s_*} z_s + \overline{\rho}_{is+s_*} + \overline{E}_{p+s+s_*}(\overline{\rho}_i, z') = (-1)^i f_{s+s_*}^*(\overline{\rho}_i). \tag{11.23}$$

Suppose that we have already determined terms of $R_j, \tilde{R}_j, f_j, f_j^*$ of order $< k$. We have

$$\overline{\rho}_1(z') = \mathbf{A}z' + R(z'), \quad \overline{\rho}_1^{-1}(z') = \mathbf{A}^{-1}z' - \mathbf{A}^{-1}R'(\mathbf{A}^{-1}z'),$$

where the terms in $R' - R$ of order $k$ depend only on terms of $R$ of order $< k$. Recall that $\tilde{\mathbf{A}}^{-1}\mathbf{A} = \operatorname{diag}\nu$ is given by (11.15). For terms of order $k$, by eliminating $f_j, f_j^*$, we therefore need to solve

$$R_{jQ} + \tilde{R}_{jQ} = \cdots \tag{11.24}$$

where the dots denote terms which have been determined. We compose from right in the last 3 identities for $i = 1$ (resp. $i = 2$) by $\overline{\rho}_1^{-1}$ (resp $\overline{\rho}_2^{-1}$). From the new identities, we obtain

$$\mathbf{A}^{-1}R(\mathbf{A}^{-1}z') + \tilde{\mathbf{A}}^{-1}\tilde{R}(\tilde{\mathbf{A}}^{-1}z') = \cdots .$$

Recall that the linear part of $\tilde{\mathbf{A}}^{-1}\mathbf{A}$ is $\operatorname{diag}\nu$ with $\nu := \nu_\epsilon$. Thus we need to solve (11.24) and

$$\nu_j^{-1}R_{j,Q} + \nu^Q\tilde{R}_{j,Q} = \cdots .$$

The equation admits a unique solution as

$$\nu^Q \neq \nu_j^{-1}, \quad Q \in \mathbf{N}^p, \quad |Q| > 1, \quad 1 \leq j \leq p. \tag{11.25}$$

This shows that $R_{j,Q}, \tilde{R}_{j,Q}$ are uniquely determined.

To verify that $\rho_i$ are involutions, we compose by $\overline{\rho}_i^{-1}$ from right in (11.18)-(11.20), and we apply complex conjugate to the coefficients of the new identities. This results in (11.21)-(11.23) in which $(\overline{\rho}_i, f_j^*)$ are replaced by $(\overline{(\overline{\rho}_i)^{-1}}, \overline{f}_i)$. We can also start with (11.21)-(11.23) and apply the same procedure to get (11.18)-(11.20), in which $(\overline{\rho}_i, f_i)$ are replaced by $(\overline{(\overline{\rho}_i)^{-1}}, \overline{f}_i^*)$. By the uniqueness of the solutions, we conclude that $\overline{(\overline{\rho}_i)^{-1}} = \overline{\rho}_i$ as both sides have the same linear part. We now have $\overline{(\overline{\rho}_i)^{-1}(\overline{z}')} = \overline{\rho_i(z')}$. Hence $\overline{z}' = \overline{\rho}_i(\rho_i(z')) = \overline{\rho_i^2(z')}$. This shows that each $\rho_i$ is an involution. □

We remark that given complex numbers

$$\mu_1, \ldots, \mu_{h_*}, \quad \mu_{h_*+1}, \ldots, \mu_{h_*+s_*}, \quad \mu_{h_*+s_*+s} = \overline{\mu}_{h_*+s}^{-1}$$

with $|\mu_h| = 1$. Let $\nu = \mu_\epsilon$ be given by (11.6). The set of $\nu$ that violate (11.7) is contained in the union of the sets defined by $\nu^Q = \nu_j^{-1}$. Here $Q \in \mathbf{N}^p$, $|Q| > 1$ and $1 \leq j \leq p$. For each $Q, j$, the above equations define an algebraic set of codimension at least 1 in the space $(S^1)^{h_*} \times \mathbf{C}^{s_*}$.

We now can prove the following theorem.

**Theorem 11.4.** *Let $M$ be a higher order perturbation of a product quadric. Assume that in $(\xi, \eta)$ coordinates, its associated $\sigma$ has a linear part given by the diagonal matrix with diagonal entries $\mu_1, \ldots, \mu_p$, $\mu_1^{-1}, \ldots, \mu_p^{-1}$. Let $\nu = \nu_e$ be of the form (11.16) and satisfy (11.25). Then $M$ admits a unique pair of asymptotic submanifold $\{K_1^\epsilon, K_2^\epsilon\}$ such that the complexification of $K_1^\epsilon$ in $\mathcal{M}$ is an invariant formal submanifold $\mathcal{H}_\epsilon$ of $\sigma$ that is tangent to*

$$\mathcal{H}_\epsilon = \Big(\bigcap_{e_j=1, 1\leq j\leq p} \{\xi_j = 0\}\Big) \cap \Big(\bigcap_{e_i=-1, 1\leq i\leq p} \{\eta_i = 0\}\Big). \tag{11.26}$$

*Furthermore, the complexification of $K_2^\epsilon$ equals $\tau_1\mathcal{H}_\epsilon$.*

*Proof.* We will follow Klingenberg's approach for $p = 1$, by using the deck transformations. Here we assume that $M$ admits the maximum number of deck transformations. Suppose that $K$ is an attached formal complex submanifold which intersects with $M$ at two totally real formal submanifolds $K_1, K_2$. We first embed $K_1 \cup K_2$ into $\mathcal{M}$ as $M$ is embedded into $\mathcal{M}$. Let $\mathcal{K}_i$ be the complexification of $K_i$ in $\mathcal{M}$. Since $\rho$ fixes $K_i$ pointwise, then $\rho\mathcal{K}_i = \mathcal{K}_i$.

We want to show that $\tau_1(\mathcal{K}_1) = \mathcal{K}_2$; thus $\mathcal{K}_i$ is invariant under $\sigma$. Recall that $\mathcal{K}_i$ is defined by

$$\overline{\rho}_i(z') = w'. \tag{11.27}$$

On $\mathcal{K}_1$, by (11.18) and (11.20) we have $\tilde{L}(z', w') + E(z', w') = -f(z')$. The latter defines a complex submanifold of dimension $p$. Thus it must be $\mathcal{K}_1$. On $\mathcal{M}$,

$$(\tilde{L}_j(z', w') + E_j(z', w'))^2 = z_{p+j}$$

are invariant by $\tau_1$. Thus each $\tilde{L}_j(z', w') + E_j(z', w')$ is either invariant or skew-invariant by $\tau_1$. Computing the linear part, we conclude that they are all skew-invariant by $\tau_1$. Hence $\tau_1(\mathcal{K}_1)$ is defined by

$$\tilde{L}(z', w') + E(z', w') = f(z'),$$

which is the defining equations for $\mathcal{K}_2$.

Finally, if $\mathcal{K}_1$ is convergent, then (11.27) implies that $\overline{\rho}_1$ is convergent. Hence $K_1$, the fixed point set of $\rho_1$, is convergent. □

We now study the convergence of attached formal submanifolds. Let us first recall a theorem of Pöschel [Pös86]. Let $\nu$ and $\epsilon$ be as in (11.16). Define

$$\omega_\nu(k) = \min_{1<|P|\leq 2^k, P\in\mathbf{N}^p} \min_{1\leq i\leq p} \left\{ |\nu^P - \nu_i|, |\nu^P - \nu_i^{-1}| \right\}.$$

Suppose that

$$-\sum \frac{\log \omega_\nu(2^k)}{2^k} < \infty. \tag{11.28}$$

Then the unique invariant formal submanifold of $\sigma$ that is tangent to the $\mathcal{H}_\epsilon$ defined by (11.26) is convergent.

We now obtain a consequence of Theorem 11.4 and Pöschel's theorem.

**Theorem 11.5.** *Let $M$ be a higher order perturbation of a product quadric. Suppose that $M$ admits the maximum number of deck transformations. Assume that the CR singularity of $M$ has no elliptic components. Let $\nu = \mu_\epsilon$ be given by* (11.16)*. Assume that $\nu = (\mu_1^{\epsilon_1}, \dots, \mu_p^{\epsilon_p})$ satisfy* (11.28)*. Then $M$ admits an attached complex submanifold.*

Since the eigenvalues of $\sigma$ are special, we verify that the condition (11.28) can be satisfied.

Let us first prove Proposition 1.11, by considering the case when the complex tangent has pure complex type. Then condition (11.28) always holds if $\nu_1, \dots, \mu_p$ satisfy the weaker non-resonance condition (11.25). Indeed, in this case, the eigenvalues of $\sigma$ are

$$\mu_s, \quad \mu_{s_*+s} = \overline{\mu}_s^{-1}, \quad \mu_{p+s} = \mu_s^{-1}, \quad \mu_{p+s_*+s} = \overline{\mu}_s.$$

Recall that $1 \leq s \leq s_*$ and $p = 2s_*$. We may assume that $|\mu_s| > 1$. We take

$$\nu_s = \mu_s, \quad \nu_{s+s_*} = \overline{\mu}_s. \tag{11.29}$$

Assume that

$$\nu^Q - \nu_j \neq 0, \quad Q \in \mathbf{N}^p, \quad |Q| > 1, \quad 1 \leq j \leq p.$$

Under the condition (11.29), we can find a positive integer $r$ such that

$$\min\{|\nu_1|^r, \dots, |\nu_p|^r\} > \max\{|\nu_1|, \dots, |\nu_p|\}.$$

It is easy to see that $|\mu^P - \nu_j| \geq c$ for some positive constant and all $Q \in \mathbf{N}^p$ with $|Q| > 1$. Hence (11.28) holds. We have proved Proposition 1.11.

We now consider the general case by showing that the set of $\{\mu_h, \mu_{s+s_*}, \overline{\mu}_{s+s_*}^{-1}\}$ that satisfy (11.28) for some choice of $\nu$ has the full measure. Without loss of generality, we may assume that $|\mu_{h_*+s}| > 1$. Thus we list the eigenvalues of $\sigma$ as

$$\boldsymbol{\mu}_{h_*}, \quad \boldsymbol{\mu}_{s_*}, \quad \tilde{\boldsymbol{\mu}}_{s_*}, \quad \overline{\boldsymbol{\mu}_{h_*}}, \quad \overline{\tilde{\boldsymbol{\mu}}_{s_*}}, \quad \overline{\boldsymbol{\mu}_{s_*}}$$

with

$$\boldsymbol{\mu}_{h_*} = (\mu_1, \dots, \mu_{h_*}), \quad \boldsymbol{\mu}_{s_*} = (\mu_{h_*+1}, \dots, \mu_{h_*+s_*}), \quad \tilde{\boldsymbol{\mu}}_{s_*} = (\overline{\mu}_{h_*+1}^{-1}, \dots, \overline{\mu}_{h_*+s_*}^{-1}).$$

We take $(\nu_1, \dots, \nu_p) = (\boldsymbol{\mu}_{h_*}, \boldsymbol{\mu}_{s_*}, \overline{\boldsymbol{\mu}_{s_*}})$. We first note that there are only finitely many $R_1, \dots, R_d \in \mathbf{N}^{2s_*}$ such that

$$|(\nu_{h_*+1}, \dots, \nu_p)^{R_i}| < 2C_*, \quad C_* := \max\{|\nu_1|, \dots, |\nu_p|\}.$$

Denote by $\{b_1, \dots, b_{4s_*d}\}$ the set of numbers:

$$\nu_j(\nu_{h_*+1}, \dots, \nu_p)^{-R_i}, \quad \nu_j^{-1}(\nu_{h_*+1}, \dots, \nu_p)^{-R_i}$$

with $h_* < j \leq p$ and $1 \leq i \leq d$. Let $\mathcal{S}_m$ be the set of $\boldsymbol{\mu}_{h_*} \in (S^1)^{h_*}$ satisfying the Siegel condition

$$\min_{i,j}\left\{\left|\boldsymbol{\mu}_{h_*}^P - b_j\right|, |\boldsymbol{\mu}_{h_*}^P - \mu_h|, |\boldsymbol{\mu}_{h_*}^P - \mu_h^{-1}|\right\} \geq \frac{C}{(1+|P|)^m}, \tag{11.30}$$

for $P \in \mathbf{N}^{h_*}$ and $|P| > 2$. One can verify that, $\cup_{m=2}^{\infty}\mathcal{S}_m$ has the full measure on $(S^1)^{h_*}$ for a fixed set of $\{b_j\}$.

We take any $\mu_{h_*+1}, \dots, \mu_p$ such that

$$(|\mu_{h_*+1}|, \dots, |\mu_p|)^Q \neq |\mu_j|, \quad \forall Q \in \mathbf{N}^{p-*}, \quad |Q| > 1.$$

We then take $(\mu_1, \dots, \mu_{h_*}) \in (S^1)^{h_*}$ satisfying (11.30). We have

$$|\nu^P - \nu_j| \geq \frac{C}{(1+|P|)^m}, \quad P \in \mathbf{N}^{h_*}, |P| > 2. \tag{11.31}$$

To verify it, we write $P = (P', P'')$ with $P' \in \mathbf{N}^{h_*}$. If $P'' \neq R_i$ for $1 \leq i \leq d$, we have $|\nu^P - \nu_j| \geq C_*$, which satisfies (11.31). Suppose that $P'' = R_i$. Then for $j > h_*$ we have

$$|\nu^P - \nu_j| \geq |(\nu_{h_*+1}, \dots, \nu_p)^{R_i}| \min_i\{|(\mu_1, \dots, \mu_{h_*})^{P'} - b_i|\} \geq \frac{C'}{(1+|P|)^m}.$$

Suppose that $1 \leq j \leq h_*$. If $P'' \neq 0$, we have

$$|\nu^P - \nu_j| \geq \min(|\mu_{h_*+1}|, \dots, |\mu_p|) - 1.$$

This gives us (11.31). Suppose now that $P'' = 0$. Then we get (11.31) immediately. We have verified (11.31) for all cases.

We have proved that the non-resonant product quadric has a unique attached complex submanifold. Let us first show that the unique complex submanifolds attached to the product quadric may split into two attached submanifolds after a perturbation. In fact, a stronger result hods; it could split into a divergent attached submanifold and a convergent one simultaneously.

**Proposition 11.6.** *There is a non-resonant 4-dimensional real analytic submanifolds $M$ that has pure complex type and admits a convergent attached submanifold and a divergent one too.*

*Proof.* By Proposition 1.11, it suffice to show the existence of a divergent attached submanifold. The proof is an application of small divisors, as shown in previous divergent result. However, the proof is much simple. We will be brief.

Consider

$$M\colon z_3 = (z_1 + 2\gamma_1\overline{z}_2 + a(z_1z_2))^2, \quad z_4 = (z_2 + 2\gamma_2\overline{z}_1)^2.$$

Here $a$ is holomorphic in $z_1z_2$ and $a(0) = a'(0) = 0$. By (11.18) for $i = 1, 2$, we eliminate $f_1$ to obtain

$$2\gamma_1 R_1 + 2\gamma_1\tilde{R}_1 + a(z_1z_2) = \cdots,$$
$$R_1 \circ \overline{\rho}_1 \circ \overline{\rho}_2 + R_2 + \overline{a}((z_1z_2) \circ \overline{\rho}_1(z_1, z_2)) = \cdots.$$

Here the right-hand sides depend on coefficients of lower orders. Thus for $Q = (k, k)$, we have

$$R_{1,kk} = \frac{a_k - (\mu_1\overline{\mu}_1^{-1})^k\overline{a}_k + e_{kk}}{(\mu_s\overline{\mu}_s^{-1})^k - 1}.$$

Here $e_{kk}$ depends only on coefficients of $a_j$ with $j < k$. We will choose $a_k$ as follows. If $|e_{kk}| > 1$, we choose $a_k = 0$. If $|e_{kk}| \leq 1$, we choose an $a_k$ such that $|a_k| = 1$ and $|a_k - (\mu_1\overline{\mu}_1^{-1})^k\overline{a}_k| = 2$. In both cases, we obtain

$$|R_{1,kk}| \geq \frac{1}{|(\mu_1\overline{\mu}_1^{-1})^k - 1|}.$$

We can find $\mu_1$ such that $0 < |(\mu_1\overline{\mu}_1^{-1})^k - 1| \leq \frac{1}{k!}$ for a sequence of integer $k = k_j \to \infty$. Furthermore, $\mu_1\overline{\mu}_1^{-1}$ is not a root of unity and $|\mu_1| \neq 1$. This shows that $R_1$ is divergent. □

**Remark 11.7.** It is plausible that there are $2^{h_*+s_*-1}$ attached formal complex submanifolds to a generic $M$ that is a higher order perturbation of non-resonant product quadric and has the maximum number of deck transformations.

To study the existence of convergence of all attached formal manifolds, we use the following theorem in [Sto13] to conclude simultaneous convergence of all attached formal submanifolds. In fact the conclusion is much more stronger. Here we recall the technique of linearization of $\sigma$ on the resonant ideal, i.e. the ideal generated by $\xi_1\eta_1, \ldots, \xi_p\eta_p$.

For the convenience of the reader, we state the result only for the family $F = \{F_1, \ldots, F_l\}$, where $F$ is $\{\sigma_1, \ldots, \sigma_p\}$, or a single mapping $\sigma$. Recall that the linear part $D = \{D_i\colon 1 \leq i \leq l\}$ of $F$ is $\{S_1, \ldots, S_p\}$ or $S$. The matrix of $D_i$ is diagonal, which is denoted by $\operatorname{diag}\mu_i$ for $\mu_i = (\mu_{i,1}, \ldots, \mu_{i,n})$. Let $\mathcal{I}$ be a monomial ideal on $\mathbf{C}^n$. Define

$$\omega_k(D, \mathcal{I}) = \inf\left\{\max_{1\leq i\leq l}|\mu_i^Q - \mu_{i,j}| \neq 0\colon |\ 2 \leq |Q| \leq 2^k, 1 \leq j \leq n, Q \in \mathbf{N}^n, x^Q \notin \mathcal{I}\right\}$$

where $\mu_i^Q := \mu_{i,1}^{q_1} \cdots \mu_{i,n}^{q_n}$. Let $\{\omega_k(D)\}_{k\geq 1}$ be the sequence of positive numbers defined by

$$\omega_k(D) = \inf\left\{\max_{1\leq i\leq l} |\mu_i^Q - \mu_{i,j}| \neq 0 \colon \mid 2 \leq |Q| \leq 2^k, 1 \leq j \leq n, Q \in \mathbf{N}^n\right\}.$$

According to [Sto13], we say that the family $D$ is *diophantine* (reps. on $\mathcal{I}$), if

$$-\sum \frac{\log w_k(D)}{2^k} < \infty, \quad (resp. -\sum \frac{\log w_k(D,\mathcal{I})}{2^k} < \infty). \tag{11.32}$$

When $D$ is reduced to one element and $\mathcal{I} = \{0\}$, this condition is Brjuno condition [Brj71, Rüs02]. Let $\mathcal{C}^D$ denote the centralizer of the family $D$.

We now state the following theorem proved in [Sto13].

**Theorem 11.8.** *Let $\mathcal{I}$ be a monomial ideal on $\mathbf{C}^n$. Let $F$ be the above family of holomorphic mappings. Assume that the family $D$ is diophantine on $\mathcal{I}$. Suppose that there is a formal mapping $\Phi$ satisfies the following:*

*(i) $\Phi$ is tangent to the identity and has a zero projection on $\mathcal{C}_D \cup \hat{\mathcal{I}}^n$, i.e. $\Phi = (\Phi_1, \ldots, \Phi_n)$ satisfy that $\Phi_{j,Q} x^Q e_j = 0$ if $x^Q \in \mathcal{I}$ or $x^Q e_j \in \mathcal{C}_D$.*

*(ii) $\Phi^{-1} F_i \Phi = D_i$ modulo $\hat{\mathcal{I}}^n$ for all $i$.*

*Then $\Phi$ is convergent.*

We apply the above theorem to $\Phi$ in $\mathcal{C}^c(S, ResI)$ and $\sigma$ which arises from a real analytic submanifold which is a higher order perturbation of a non-resonant product quadric. Note that $\mathcal{C}_S$ is contained in the resonant ideal and the condition on the projection (ii) of the above theorem is satisfied by the unique normalized map that linearizes $\sigma$ on $\hat{\mathcal{I}}^n$.

As a corollary of the above theorem, we have the following result.

**Corollary 11.9.** *Let $ResI$ be the resonant ideal of $S$. Assume that $\sigma$ satisfies the diophantine on $\mathcal{I}$. Then $\sigma$ is holomorphically linearizable on $\mathcal{I}$. In particular, if $\{\mu_1, \ldots, \mu_p\}$ is non-resonant in $\mathbf{Z}^p$, i.e. $\mu^Q \neq 1$ for all $Q \in \mathbf{Z}^p$ with $|Q| > 0$, then in suitable holomorphic coordinates, the $\sigma$ is linear and diagonal on the $(\xi_{i_1}, \ldots, \xi_{i_s}, \eta_{i_{s+1}}, \ldots, \eta_{i_p})$-subspace for any partition $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$.*

As a consequence of Corollary 11.9 and Theorem 11.4, we obtain immediately Theorem 1.12, which we restate here in a stronger form.

**Theorem 11.10.** *Let $M$ be a third order perturbation of a product quadric. Suppose that $M$ admits the maximum number of deck transformations and is non resonant. Suppose that $M$ has no elliptic component and that the eigenvalues of $\sigma$ satisfy diophantine condition* (11.32)*, then all attached formal submanifolds are convergent. Moreover, and the restrictions of $\sigma$ on these invariant submanifolds are simultaneously linearizable by a single change of holomorphic coordinates of the ambient space.*

As mentioned earlier, the eigenvalues of $\sigma$ are special. Let us verify that the set of $\mu$ that satisfy the diophantine condition (11.32) has the full measure. Recall that the resonant ideal is generated by $\xi_1\eta_1, \ldots, \xi_p\eta_p$. Suppose that $\xi^P\eta^Q$ is not in the ideal. Then $p_j q_j = 0$ and $|p_i - q_j| = p_j + q_j$. We need to consider non-zero small divisors of the form

$$\boldsymbol{\mu}_{h_*}^P \boldsymbol{\mu}_{s_*}^Q \overline{\boldsymbol{\mu}}_{s_*}^R - \mu_j, \quad 1 \leq j \leq p$$

for $(P,Q,R)\in\mathbf{Z}^p$. Let $m$ be a positive number such that

$$\big||\boldsymbol{\mu}_{s_*}^Q\overline{\boldsymbol{\mu}}_{s_*}^R|-|\mu_j|\big|\ge\frac{1}{2}\min_j\{|\mu_j|,|\mu_j|^{-1}\},\quad |Q+R|\ge m.$$

Let us define

$$\mu_s=r_s\nu_s,\quad r_s=|\mu_s|,\quad \mu_h=\nu_h,\quad r_h=1.$$

Then we can write

$$\mu^{P-Q}-\mu_j=\mu_j^{-1}\left(r_j^{-1}\nu_j^{-1}\prod_s r_s^{p_s-p_s'-q_s+q_s'}\cdot\prod\nu_h^{p_h-q_h}\prod\nu_s^{p_s+p_s'-q_s-q_s'}-1\right).$$

Here $P=(p_h,p_s,p_s')$ and $Q=(q_h,q_s,q_s')$. We set

$$\prod_s r_s^{p_s-p_s'-q_s+q_s'}=r^{R'},\quad \nu_j^{-1}\prod\nu_h^{p_h-q_h}\prod\nu_s^{p_s+p_s'-q_s-q_s'}=\nu_j^{-1}\nu^R.$$

Note that $|R'|\le|P|+|Q|$ and $|R|\le|P|+|Q|$. In view of

$$|\rho e^{i\theta}-1|^2=(r-1)^2+r\sin^2(\theta/2)\ge C\max\{|r-1|^2,|e^{i\theta}-1|\}$$

we obtain

$$|\mu^{P-Q}-\mu_j|\ge Cr_j^{-1}\max\left\{|r_j^{-1}r^{R'}-1|,|\nu_j^{-1}\nu^R-1|\right\}.$$

Now one can see that the set of $\mu=\{\mu_h,\mu_{h_*+s},\overline{\mu}_{h_*+s}^{-1}\}$ that satisfies the diophantine condition (11.32) has the full measure.

Finally, we indicate a consequence of $\sigma$ being linear on the zero set of the resonant ideal. In this case the solutions $\{\overline{\rho}_1,\overline{\rho}_2\}$ to (11.18)-(11.23) are linear and there are $2^{h_*+s_*-1}$ pairs $\{\overline{\rho}_{j1},\overline{\rho}_{j2}\}$ of solutions. Now (11.18)-(11.23) imply that

$$E(z',\overline{\rho}_1(z'))=-E(z',\overline{\rho}_2(z')),\quad \overline{E}(\overline{\rho}_1(z'),z')=-\overline{E}(\overline{\rho}_2(z'),z').$$

The complex submanifold associated to $\{\rho_{i1},\rho_{i2}\}$ then has the form

$$K_j\colon z_{p+i}=(L_i(z',\overline{\rho}_{j1}(z'))+E_i(z',\overline{\rho}_{j1}(z')))^2,\quad 1\le i\le p.$$

Of course, there are additional hidden symmetries in $E$ for $\sigma$ to preserve the resonant ideal. On the other hand, $E$ can be quite general as shown by the algebraic example (Example 5.6).

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