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# SAMPLING, INTERPOLATION AND RIESZ BASES IN SMALL FOCK SPACES

A. BARANOV, A. DUMONT, A. HARTMANN, K. KELLAY

**ABSTRACT.** We give a complete description of Riesz bases of reproducing kernels in small Fock spaces. This characterization is in the spirit of the well known Kadets–Ingham  $1/4$  theorem for Paley–Wiener spaces. Contrary to the situation in Paley–Wiener spaces, a link can be established between Riesz bases in the Hilbert case and corresponding complete interpolating sequences in small Fock spaces with associated uniform norm. These results allow to show that if a sequence has a density strictly different from the critical one then either it can be completed or reduced to a complete interpolating sequence. In particular, this allows to give necessary and sufficient conditions for interpolation or sampling in terms of densities.

**Résumé.** Nous donnons une description complète des bases de Riesz de noyaux reproduisants dans les espaces de Fock pondérés par un poids radial à croissance lente. Cette caractérisation est de même nature que celle du Théorème  $1/4$  de Kadets–Ingham pour les espaces de Paley–Wiener. Contrairement à la situation dans les espaces de Paley–Wiener, on peut établir un lien entre les bases de Riesz dans le cadre Hilbertien et les suites d’interpolation complètes dans les petits espaces de Fock munis de la norme uniforme associée. Ces résultats permettent de montrer que si une suite a une densité strictement différente de la densité critique alors elle peut être soit complétée soit réduite à une suite d’interpolation complète. En particulier, cela nous permet de donner des conditions nécessaires et suffisantes en termes de densité pour qu’une suite soit d’interpolation ou d’échantillonnage.

## 1. INTRODUCTION AND MAIN RESULTS.

Interpolation and sampling problems are well studied objects. Complete results for corresponding sequences are known for broad classes of spaces of analytic functions. We refer the reader to the monograph by Seip for an account on these problems [25]. Two prominent examples here are the Fock spaces and the Bergman spaces. For these, interpolating and sampling sequences have been studied by Seip in the classical situation. More general weights have been discussed in the 1990s by Berndtsson and Ortega-Cerdà [4], Lyubarskii and Seip [15], later by Marco, Massaneda and Ortega-Cerdà [17] for weights whose Laplacian is a doubling measure, and in [6] for radial weights growing more rapidly

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than polynomially. More recently, a series of results was obtained for "small" (i.e., with slowly growing weights) versions of these spaces. In this case the geometric properties of sampling/interpolating sequences change significantly. Seip showed that in small Bergman spaces, locally, interpolating sequences look like interpolating sequences in Hardy spaces [26]. In small Fock spaces, Borichev and Lyubarskii recently exhibited Riesz bases of reproducing kernels [7]. In this paper we will investigate further the situation of small Fock spaces  $\mathcal{F}_\varphi^p$  for the weight  $\varphi(z) = \alpha(\log^+ r)^2$ . We focus on the Hilbert situation  $p = 2$  and on the case  $p = \infty$ . As it turns out, no density characterization can be expected for interpolation or sampling. There are actually sequences which are simultaneously interpolating and sampling, also called *complete interpolating sequences*. Note that complete interpolating sequences necessarily have critical density (as defined below). We also provide sequences with critical density which are neither interpolating nor sampling for  $p = 2, \infty$ .

The central result of this paper is a characterization of complete interpolating sequences when  $p = 2$  which is in the spirit of the famous 1/4 Kadets–Ingham theorem in the Paley–Wiener space [12, 14] and its more general version due to Avdonin [1]. A different characterization using in particular a Muckenhoupt type condition and based on different techniques is discussed by Belov, Mengestie and Seip for more general spaces in [3]. The novelty of our approach is the use of a rather elementary Hilbert space result, namely Bari’s Theorem, adapted to our situation. Moreover, we will show that, surprisingly, the same characterization applies to complete interpolating sequences in  $\mathcal{F}_\varphi^\infty$ , the  $L^\infty$ -counterpart of the Fock space  $\mathcal{F}_\varphi^2$ . From our characterization, we will also deduce sufficient density conditions for interpolation and sampling.

We also would like to emphasize the connection between these spaces and the de Branges spaces [8]. The complete interpolating sequence introduced by Borichev and Lyubarskii in [7] defines a generating function  $G$  which, when  $p = 2$ , allows to identify the Fock spaces we are interested in with the de Branges space  $\mathcal{H}(G)$ . Consequently, the measure  $dx/|G(x)|^2$  is a sampling measure for our Fock spaces, which answers a question raised in [19, Section 4.3 p.142]. Note that in [18], the authors consider sampling and interpolation in the class of de Branges spaces for which the phase function defines a doubling measure. Our space corresponds to the situation when the phase function is locally but not globally doubling so that their results apparently do not apply here. Still it can be observed that these authors obtain a similar kind of density characterization as ours when  $p = 2$  (at least for real sequences they consider).

**1.1. Definition of small Fock spaces.** We now introduce the necessary notation. Let  $\varphi(z) = \alpha(\log^+ |z|)^2$ ,  $\alpha > 0$ , which is a subharmonic radial function with  $\varphi(r) \nearrow +\infty$ ,  $r \rightarrow +\infty$ , and define the associated Fock space,

$$\mathcal{F}_\varphi^2 = \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_{\varphi,2}^2 := \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)} dm(z) < \infty \right\}.$$

In [7] Borichev and Lyubarski have shown the existence of *complete interpolating* sequences in  $\mathcal{F}_\varphi^2$  (i.e., simultaneously interpolating and sampling for  $\mathcal{F}_\varphi^2$ , see precise definitions below).

The sequence they introduced will be the reference sequence for our considerations:

$$\Gamma = \Gamma_\alpha = \{e^{\frac{n+1}{2\alpha}} e^{i\theta_n}\}_{n \geq 0}, \quad \theta_n \in \mathbb{R}. \quad (1)$$

In order to define sampling and interpolating sequences for  $\mathcal{F}_\varphi^2$ , we consider first  $\mathbf{k}_z$ , the reproducing kernel of  $\mathcal{F}_\varphi^2$ :

$$\langle f, \mathbf{k}_z \rangle_{\mathcal{F}_\varphi^2} = f(z), \quad f \in \mathcal{F}_\varphi^2, \quad z \in \mathbb{C}.$$

According to [7, Lemma 2.7], the kernel admits the following estimate:

$$\|\mathbf{k}_z\|_{\varphi,2}^2 = \mathbf{k}_z(z) \asymp \frac{e^{2\varphi(z)}}{1 + |z|^2}, \quad z \in \mathbb{C}. \quad (2)$$

The sequence  $\Lambda \subset \mathbb{C}$  is called *sampling* for  $\mathcal{F}_\varphi^2$  if

$$\|f\|_{\varphi,2}^2 \asymp \|f\|_{\varphi,2,\Lambda}^2 := \sum_{\lambda \in \Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)}, \quad f \in \mathcal{F}_\varphi^2,$$

and *interpolating* if for every  $v = (v_\lambda)_{\lambda \in \Lambda} \in \ell_{\varphi,\Lambda}^2$ , i.e.,  $\|v\|_{\varphi,2,\Lambda} < \infty$ , there exists  $f \in \mathcal{F}_\varphi^2$  such that

$$v = f|_\Lambda.$$

Let  $\mathbb{k}_\lambda = \mathbf{k}_\lambda / \|\mathbf{k}_\lambda\|_{\varphi,2}$  be the normalized reproducing kernel at  $\lambda$ . Let  $\Lambda \subset \mathbb{C}$ . We say that  $\{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$  is a *Riesz sequence* in  $\mathcal{F}_\varphi^2$  if for some  $C > 0$  and for each finite sequence  $\{a_\lambda\}$ , we have

$$\frac{1}{C} \sum_{\lambda \in \Lambda} |a_\lambda|^2 \leq \left\| \sum_{\lambda \in \Lambda} a_\lambda \mathbb{k}_\lambda \right\|_{2,\varphi}^2 \leq C \sum_{\lambda \in \Lambda} |a_\lambda|^2,$$

and a *Riesz basis* if it is also complete. It is well known that  $\Lambda$  is interpolating if and only if  $\{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$  is a Riesz sequence, and  $\Lambda$  is complete interpolating if and only if  $\{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$  is a Riesz basis in  $\mathcal{F}_\varphi^2$ .

It should be mentioned that our case  $\varphi(r) = \alpha(\log^+ r)^2$  corresponds to the critical growth of the weight for which Riesz bases of reproducing kernels exist. Recall that, by the results of Seip and Seip–Wallstén [22, 27] there are no complete interpolating sequences for the classical Fock space (i.e., for  $\varphi(r) = r^2$ ). For more general (in particular, rapidly growing) weights the same was shown in [6, 17, 15] (see also [25] as a general source). Some examples of slowly growing weights such that no complete interpolating sequences exist were given in [13]. Finally, Borichev and Lyubarskii [7, Theorem 2.5] have shown that, under some regularity conditions, if  $(\log^+ r)^2 \ll \varphi(r) \ll r^2$ , then the corresponding Fock space does not possess a Riesz basis of reproducing kernels.

On the contrary, if  $\varphi(r) = \alpha(\log^+ r)^\beta$ ,  $1 < \beta \leq 2$ , then in the space  $\mathcal{F}_\varphi^2$  there exist Riesz bases of reproducing kernels corresponding to real points [7]. Note that if a Fock type space contains a Riesz basis  $\{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$  with  $\Lambda \subset \mathbb{R}$ , then it coincides with a de Branges space with equivalence of norms (see, e.g., [3, Section 2.3]). All de Branges spaces which are Fock spaces were explicitly described in [2].

**1.2. Description of complete interpolating sequences in  $\mathcal{F}_\varphi^2$ .** Our central result is a characterization of complete interpolating sequences in terms of their deviation from the sequence  $\Gamma = \Gamma_\alpha$  defined in (1).

Before stating the theorem we need to define separation as in [17, 7]. Set

$$\rho(z) = (\Delta\varphi(z))^{-1/2},$$

where  $\Delta\varphi(r) = \varphi''(r) + \varphi'(r)/r$ ,  $r > 0$ . We associate with  $\rho$  a “distance” (a semi-metric):

$$d_\rho(z, w) = \frac{|z - w|}{1 + \min(\rho(z), \rho(w))}, \quad z, w \in \mathbb{C}.$$

Note that when  $z, w$  are in a fixed disk, this distance is comparable to Euclidean distance.

The sequence  $\Lambda \subset \mathbb{C}$  is said to be  $d_\rho$ -separated if there is  $d_\Lambda > 0$  such that

$$\inf\{d_\rho(z, w) : z, w \in \Lambda, z \neq w\} \geq d_\Lambda.$$

In the specific situation  $\varphi(r) = \alpha(\log^+ r)^2$ , we have  $\rho(r) = r/\sqrt{2\alpha}$ ,  $r \geq 1$ . Hence

$$d_\rho(z, w) = \sqrt{2\alpha} \frac{|z - w|}{\sqrt{2\alpha} + \min(|z|, |w|)}.$$

In particular, for  $\beta < 1$  and  $0 \neq \lambda \in \mathbb{C}$  the ball corresponding to this distance is given by

$$D_\rho(\lambda, \beta) := \{z \in \mathbb{C} : d_\rho(z, \lambda) < \beta\}.$$

When  $\beta$  is small,  $D_\rho(\lambda, \beta)$  is comparable to a Euclidean disk  $D(\lambda, q|\lambda|)$  with a suitable constant  $q$  depending on  $\beta$ . From this we deduce that  $\Lambda$  is  $d_\rho$ -separated if and only if there exists  $c > 0$  such that the Euclidean disks  $D(\lambda, c|\lambda|)$ ,  $\lambda \in \Lambda$ , are disjoint.

**Theorem 1.1.** *Let  $\alpha > 0$ ,  $\varphi(r) = \alpha(\log^+ r)^2$ , let  $\Gamma = \{\gamma_n\}_{n \geq 0} = \{e^{\frac{n+1}{2\alpha}}\}_{n \geq 0}$  and let  $\Lambda = \{\lambda_n\}_{n \geq 0}$  with  $\lambda_n = \gamma_n e^{\delta_n} e^{i\theta_n}$ ,  $|\lambda_n| \leq |\lambda_{n+1}|$ ,  $\theta_n \in \mathbb{R}$ . Then  $\{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$  is a Riesz basis for  $\mathcal{F}_\varphi^2$  if and only if the following three conditions hold.*

- (a)  $\Lambda$  is  $d_\rho$ -separated,
- (b)  $(\delta_n) \in \ell^\infty$ ,
- (c) there exists  $N \geq 1$  and  $\delta > 0$  such that

$$\sup_n \frac{1}{N} \left| \sum_{k=n+1}^{n+N} \delta_k \right| \leq \delta < \frac{1}{4\alpha}.$$

For  $N = 1$ , condition (c) of Theorem 1.1 becomes  $\sup_n |\delta_n| < 1/4$ , which resembles a well-known stability result for complete interpolating sequences in the Paley–Wiener space (equivalently, Riesz bases of exponentials) – the famous 1/4–Theorem due to Ingham and Kadets (see, e.g., [12, 20]). Also, for arbitrary  $N$ , Avdonin considers the sufficiency of that condition in the Paley–Wiener space [1]. However, there is an essential difference since complete interpolating sequences in the Paley–Wiener space can not be described in terms of perturbations and more subtle characteristics (e.g., the Muckenhoupt condition) appear [12], while in the case of spaces of very slow growth such a characterization turns out to be possible. As already mentioned, very close results were obtained by Belov, Mengestie and Seip in [3] where the boundedness and invertibility problem of a discrete Hilbert transform

on lacunary sequences was solved. Though Theorem 1.1 is not formally covered by the results stated in [3] (see Remark 3.1), it seems that one can obtain our characterization using the methods of that paper. However, as mentioned earlier, our proof being based on Bari's theorem is essentially elementary. Note also that (c) implies (b) but we prefer to state condition (b) separately.

As an immediate consequence of Theorem 1.1, the reader should note that if a sequence  $\{\lambda_n\}$  is complete interpolating then any  $d_\rho$ -separated sequence of the form  $\{\lambda_n e^{i\theta_n}\}$  will be complete interpolating (as is the case for  $\Gamma_\alpha$ ).

**1.3. Weighted Fock spaces with uniform norm.** We also deal with the case  $p = \infty$ . The corresponding weighted Fock space is defined by

$$\mathcal{F}_\varphi^\infty = \{f \in \text{Hol}(\mathbb{C}) : \|f\|_{\varphi, \infty} := \sup_{z \in \mathbb{C}} |f(z)| e^{-\varphi(z)} < \infty\}.$$

A sequence  $\Lambda \subset \mathbb{C}$  is called *sampling* for  $\mathcal{F}_\varphi^\infty$ , if there exists  $L > 0$  such that

$$\|f\|_{\varphi, \infty} \leq L \|f\|_{\varphi, \infty, \Lambda} := L \sup_{\lambda \in \Lambda} |f(\lambda)| e^{-\varphi(\lambda)}, \quad f \in \mathcal{F}_\varphi^\infty.$$

A sequence  $\Lambda$  is called *interpolating* for  $\mathcal{F}_\varphi^\infty$  if for every sequence  $v = (v_\lambda)_{\lambda \in \Lambda}$  in  $\ell_{\varphi, \Lambda}^\infty$ , i.e., such that  $\|v\|_{\varphi, \infty, \Lambda} < \infty$ , there is a function  $f \in \mathcal{F}_\varphi^\infty$  such that  $v = f|_\Lambda$ .

It turns out that there exist complete interpolating sequences for  $\mathcal{F}_\varphi^\infty$  (i.e., as in  $\mathcal{F}_\varphi^2$ , simultaneously interpolating and sampling sequences for  $\mathcal{F}_\varphi^\infty$ ), which differs in this respect from most other known spaces of entire functions (e.g., complete interpolating sequences exist for the Paley–Wiener space  $PW_a^p$  for  $1 < p < \infty$  [16], but not for  $p = \infty$ ).

Our second main result is as follows:

**Theorem 1.2.** *Let  $\varphi(r) = \alpha(\log^+ r)^2$ ,  $\alpha > 0$ . Then a sequence  $\Lambda$  is a complete interpolating sequence for  $\mathcal{F}_\varphi^\infty$  if and only if for some (any)  $\lambda \in \Lambda$  the sequence  $\Lambda \setminus \{\lambda\}$  is a complete interpolating sequence for  $\mathcal{F}_\varphi^2$ .*

Thus, any complete interpolating sequence for  $\mathcal{F}_\varphi^\infty$  is a small (in the sense of conditions (a)–(c) of Theorem 1.1) perturbation of the sequence  $\tilde{\Gamma} = \Gamma \cup \{1\}$ .

We derive from Theorems 1.1 and 1.2 several density conditions for interpolation and sampling in  $\mathcal{F}_\varphi^p$ ,  $p = 2, \infty$ . Before stating these results, we need some more notation.

Let  $\mathcal{A}(r, R)$  be the annulus centered at the origin with inner and outer radii  $r$  and  $R$ :  $\mathcal{A}(r, R) := \{z \in \mathbb{C} : r \leq |z| < R\}$ . For a  $d_\rho$ -separated sequence  $\Lambda$  we define the lower and upper densities respectively by

$$D^-(\Lambda) = \liminf_{R \rightarrow +\infty} \inf_{r > 0} \frac{\text{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R}$$

and

$$D^+(\Lambda) = \limsup_{R \rightarrow +\infty} \sup_{r > 0} \frac{\text{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R}.$$

These densities do not change when we remove or add a finite number of points to  $\Lambda$ .

**1.4. Density criteria for sampling and interpolation.** As an application of our result on Riesz bases, Theorem 1.1 (as well as Theorem 1.2) we can show that each set with  $D^+(\Lambda) < 2\alpha$  (respectively  $D^-(\Lambda) > 2\alpha$ ) can be completed (reduced) to a complete interpolating sequence. More precisely:

**Theorem 1.3.** *Let  $\varphi(r) = \alpha(\log^+ r)^2$ , let  $p = 2, \infty$ , and let  $\Lambda$  be a  $d_\rho$ -separated sequence. Then*

- (i) *if  $D^+(\Lambda) < 2\alpha$ , then  $\Lambda$  is a subset of some complete interpolating sequence in  $\mathcal{F}_\varphi^p$ ;*
- (ii) *if  $D^-(\Lambda) > 2\alpha$ , then  $\Lambda$  contains a complete interpolating sequence in  $\mathcal{F}_\varphi^p$ .*

This result, together with a classical comparison method by Ramanathan-Steger [17, 21], allows us to deduce our density results.

**Theorem 1.4.** *(Sampling,  $p = \infty$ ) Let  $\varphi(r) = \alpha(\log^+ r)^2$ ,  $\alpha > 0$ . Then*

- (i) *every  $d_\rho$ -separated sequence  $\Lambda$  with  $D^-(\Lambda) > 2\alpha$ , is a set of sampling for  $\mathcal{F}_\varphi^\infty$ ;*
- (ii) *if the sequence  $\Lambda$  is a set of sampling for  $\mathcal{F}_\varphi^\infty$  then it contains a  $d_\rho$ -separated subsequence  $\tilde{\Lambda}$  with  $D^-(\tilde{\Lambda}) \geq 2\alpha$ .*

**Theorem 1.5.** *(Sampling,  $p = 2$ ) Let  $\varphi(r) = \alpha(\log^+ r)^2$ ,  $\alpha > 0$ . Then*

- (i) *every  $d_\rho$ -separated sequence  $\Lambda$  with  $D^-(\Lambda) > 2\alpha$ , is a set of sampling for  $\mathcal{F}_\varphi^2$ ;*
- (ii) *if the sequence  $\Lambda$  is a set of sampling for  $\mathcal{F}_\varphi^2$ , then it is a finite union of  $d_\rho$ -separated subsequences and  $\Lambda$  contains a  $d_\rho$ -separated sequence  $\tilde{\Lambda}$  such that  $D^-(\tilde{\Lambda}) \geq 2\alpha$ .*

**Theorem 1.6.** *(Interpolation,  $p = 2, \infty$ ) Let  $\varphi(r) = \alpha(\log^+ r)^2$ ,  $\alpha > 0$ . Then*

- (i) *every  $d_\rho$ -separated sequence  $\Lambda$  with  $D^+(\Lambda) < 2\alpha$  is a set of interpolation for  $\mathcal{F}_\varphi^p$ ,  $p = 2, \infty$ ;*
- (ii) *if the sequence  $\Lambda$  is a set of interpolation for  $\mathcal{F}_\varphi^p$ ,  $p = 2, \infty$ , then it is a  $d_\rho$ -separated sequence with  $D^+(\Lambda) \leq 2\alpha$ .*

In the case when the density is critical, i.e.,  $D^+(\Lambda) = D^-(\Lambda) = 2\alpha$ , any of the following situations may occur: a system may be complete interpolating, either complete or interpolating, and, finally, neither complete nor interpolating (see Section 6 for the corresponding examples). Thus, there are no density characterizations for sampling or interpolating sequences for  $p = 2, \infty$ .

The paper is organized as follows. In the next section, we present some elementary results on sampling and interpolation in our spaces. Some of them follow from a Bernstein type inequality that we will also give in this section. Furthermore, we show that the lower density of a zero sequence has to be less than or equal to the critical density. Sections 3 and 4 are devoted to the proof of our main results (Theorems 1.1 and 1.2), from which we deduce the density results on sampling and interpolation in Section 5. Examples of sequences of critical density which are neither sampling nor interpolating are discussed in Section 6. There we will also show that it is not possible to switch from an interpolating sequence to a sampling sequence by adding one point without one of the sequences being complete interpolating. We finish this section by stating two open problems.

A final word on notation:  $A \lesssim B$  means that there is a constant  $C$  independent of the relevant variables such that  $A \leq CB$ . We write  $A \asymp B$  if both  $A \lesssim B$  and  $B \lesssim A$ .

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## 2. PRELIMINARY RESULTS

2.1.  $d_\rho$ -separated sequences. Recall that  $\varphi(r) = \alpha(\log^+ r)^2$ , and

$$d_\rho(z, w) = \sqrt{2\alpha} \frac{|z - w|}{\sqrt{2\alpha} + \min(|z|, |w|)}.$$

A central tool in our discussion is the following Bernstein type result whose proof can be found in [6, Lemma 4.1].

**Lemma 2.1.** *Let  $f$  be a holomorphic function.*

- (i) *If  $\|f\|_{\varphi, \infty} = 1$ , then for every  $c > 0$  there exists  $0 < \beta < 1$  such that whenever  $|f(z_0)|e^{-\varphi(z_0)} \geq c$  for some  $z_0 \in \mathbb{C}$ , then for every  $z \in D_\rho(z_0, \beta)$  we have*

$$|f(z)|e^{-\varphi(z)} \geq \frac{c}{2}e^{-\alpha\pi^2}.$$

- (ii) *There is  $\beta_0 > 0$ , such that if  $0 < \beta < \beta_0$ , then for  $z \in \mathbb{C}$  with  $d_\rho(z, z_0) \leq \beta$  we have*

$$\left| |f(z)|e^{-\varphi(z)} - |f(z_0)|e^{-\varphi(z_0)} \right| \lesssim d_\rho(z, z_0) \max_{D_\rho(z_0, \beta)} |f|e^{-\varphi}.$$

- (iii)  $|f(z)|e^{-\varphi(z)} \lesssim \frac{1}{|z|^2} \int_{D_\rho(z, \beta)} |f(w)|e^{-\varphi(w)} dm(w).$

From Lemma 2.1 we can deduce the following immediate corollaries (proofs can be found for instance in [11, Chapter 5] or [28, Chapter 4]).

**Corollary 2.2.** *If  $\Lambda$  is sampling for  $\mathcal{F}_\varphi^p$ ,  $p = 2, \infty$ , then there exists a  $d_\rho$ -separated sequence  $\tilde{\Lambda} \subset \Lambda$  which is sampling for  $\mathcal{F}_\varphi^p$ .*

**Corollary 2.3.** *Every set of interpolation for  $\mathcal{F}_\varphi^p$ ,  $p = 2, \infty$ , is  $d_\rho$ -separated.*

We also need a uniform control of the sampling constant for small perturbations.

**Corollary 2.4.** *Let  $\Lambda = \{\lambda_n\}$  be a separated sampling sequence. Then there is a  $\delta > 0$  and  $C > 0$  depending only on  $\delta$  such that for every  $\tilde{\Lambda} = \{\tilde{\lambda}_n\}$  with  $d_\rho(\lambda_n, \tilde{\lambda}_n) \leq \delta$ , we have*

$$\frac{1}{C} \|f\|_{2, \varphi} \lesssim \|f|_{\tilde{\Lambda}}\|_{2, \varphi, \tilde{\Lambda}}.$$



*Proof.* First note that when  $d_\rho(\lambda_n, \tilde{\lambda}_n) \leq \delta$ , then  $1 + |\lambda_n|^2 \asymp 1 + |\tilde{\lambda}_n|^2$  (constants are uniform for  $\delta \leq \delta_0$ ). Now

$$\begin{aligned} \|f\|_{2,\varphi}^2 &\asymp \sum_n (1 + |\lambda_n|^2) |f(\lambda_n)|^2 e^{-2\varphi(\lambda_n)} \\ &\leq \sum_n (1 + |\lambda_n|^2) |f(\tilde{\lambda}_n)|^2 e^{-2\varphi(\tilde{\lambda}_n)} + \sum_n (1 + |\lambda_n|^2) \left| |f(\tilde{\lambda}_n)|^2 e^{-2\varphi(\tilde{\lambda}_n)} - |f(\lambda_n)|^2 e^{-2\varphi(\lambda_n)} \right| \\ &\lesssim \sum_n (1 + |\tilde{\lambda}_n|^2) |f(\tilde{\lambda}_n)|^2 e^{-2\varphi(\tilde{\lambda}_n)} + \sum_n (1 + |\lambda_n|^2) d_\rho(\lambda_n, \tilde{\lambda}_n) \max_{z \in D_\rho(\lambda_n, \delta)} |f(z)|^2 e^{-2\varphi(z)}, \end{aligned}$$

where we have used Lemma 2.1 (constants only depend on  $\delta$ ). Let  $\delta_0$  be such that the disks  $D(\lambda_n, \delta_0|\lambda_n|)$  are disjoint. Then, when  $\delta > 0$  is sufficiently small and  $z \in D(\lambda_n, \delta|\lambda_n|)$ , every disk  $D(z, \delta_0|z|/2)$  is contained in  $D(\lambda_n, \delta_0|\lambda_n|)$ . Now, by Lemma 2.1, for every  $z \in D(\lambda_n, \delta|\lambda_n|)$ ,

$$|f(z)|^2 e^{-2\varphi(z)} \lesssim \frac{1}{\delta_0^2 |z|^2} \int_{D(z, \delta_0|z|/2)} |f(w)|^2 e^{-2\varphi(w)} dm(w) \leq \frac{1}{\delta_0^2 |z|^2} \int_{D(\lambda_n, \delta_0|\lambda_n|)} |f(w)|^2 e^{-2\varphi(w)} dm(w).$$

Since  $d_\rho(z, \lambda_n) \leq 2\delta$ , we have  $1 + |\lambda_n|^2 \asymp 1 + |z|^2$ , and hence

$$\|f\|_{2,\varphi}^2 \lesssim \sum_n (1 + |\tilde{\lambda}_n|^2) |f(\tilde{\lambda}_n)|^2 e^{-2\varphi(\tilde{\lambda}_n)} + \delta \sum_n \int_{D_\rho(\lambda_n, 2\delta)} |f(w)|^2 e^{-2\varphi(w)} dm(w),$$

where the constants only depend on  $\delta$ . It remains to choose  $\delta$  sufficiently small.  $\square$

Recall that  $\Gamma = \Gamma_\alpha = \{e^{\frac{n+1}{2\alpha}} e^{i\theta_n}\}_{n \geq 0}$ ,  $\theta_n \in \mathbb{R}$ , is a complete interpolating sequence for  $\mathcal{F}_\varphi^2$ . We will need the following simple estimates from [7].

**Lemma 2.5.** [7, Lemma 2.6]. *Let  $\varphi(r) = \alpha(\log^+ r)^2$  and  $\Gamma = \Gamma_\alpha$ . The product*

$$G(z) = \prod_{\gamma \in \Gamma} \left(1 - \frac{z}{\gamma}\right)$$

*converges uniformly on compact sets in  $\mathbb{C}$  and satisfies*

$$|G(z)| \asymp e^{\varphi(z)} \frac{\text{dist}(z, \Gamma)}{1 + |z|^{3/2}}, \quad z \in \mathbb{C},$$

*where the constants are independent of the choice of  $\theta_n$ . Here  $\text{dist}(z, \Gamma)$  denotes the Euclidean distance between  $z$  and  $\Gamma$ . Also*

$$|G'(\gamma)| \asymp \frac{e^{\varphi(\gamma)}}{1 + |\gamma|^{3/2}}, \quad \gamma \in \Gamma.$$

**Lemma 2.6.** *Let  $\Lambda \subset \mathbb{C}$ . Then*

$$\|f\|_{2,\varphi,\Lambda} \leq c(\Lambda) \|f\|_{2,\varphi}, \quad f \in \mathcal{F}_\varphi^2, \quad (3)$$

*if and only if  $\Lambda$  is a finite union of  $d_\rho$ -separated subsets.*

*Proof.* If  $\Lambda$  is a finite union of  $d_\rho$ -separated subsets, then (3) follows from Lemma 2.1 (iii). In the opposite direction, let  $\Gamma$  and  $G$  be as in Lemma 2.5, and put

$$G_\gamma(z) = \frac{G(z)}{(z - \gamma)G'(\gamma)} \frac{e^{\varphi(\gamma)}}{\gamma}.$$

Then

$$|G_\gamma(z)| \asymp \frac{|\gamma|^{1/2}}{1 + |z|^{3/2}} e^{\varphi(z)} \frac{\text{dist}(z, \Gamma)}{|z - \gamma|}.$$

The function  $G_\gamma$  belongs to  $\mathcal{F}_\varphi^2$  and  $\sup_\gamma \|G_\gamma\|_{2,\varphi} \lesssim 1$  (see [7, Proof of Theorem 2.5]). Hence, by Lemma 2.5, we have

$$\begin{aligned} 1 \gtrsim \|G_\gamma\|_{2,\varphi} &\gtrsim \|G_\gamma\|_{2,\varphi,\Lambda} \geq \sum_{\lambda \in \Lambda \cap D_\rho(\gamma, \beta)} |G_\gamma(\lambda)|^2 e^{-2\varphi(\lambda)} (1 + |\lambda|^2) \\ &\geq c \text{Card}(\Lambda \cap D_\rho(\gamma, \beta)) \end{aligned}$$

(note that  $\text{dist}(\lambda, \Gamma) \asymp |\lambda - \gamma|$  since  $\Gamma$  is separated and we can choose  $\beta$  such that  $D_\rho(\gamma, \beta)$  stays sufficiently far from  $\Gamma \setminus \{\gamma\}$ ). So

$$\sup_{\gamma \in \Gamma} \text{Card}(\Lambda \cap D_\rho(\gamma, \beta)) < \infty$$

for arbitrary real numbers  $\theta_n$  (we can pick  $\beta$  such that that  $D_\rho(\gamma, \beta)$  covers  $\mathbb{C}$  when  $\gamma$  runs through  $\Gamma$  and  $\theta_n$  through  $\mathbb{R}$ ). Hence  $\Lambda$  is a finite union of  $d_\rho$ -separated sequences.  $\square$

**Lemma 2.7.** *If  $\Lambda$  is  $d_\rho$ -separated and sampling for  $\mathcal{F}_{(1+\varepsilon)\varphi}^\infty$  for some  $\varepsilon > 0$ , then  $\Lambda$  is sampling for  $\mathcal{F}_\varphi^2$ .*

*Proof.* We are going to use the same Beurling duality argument as in [17, Theorem 36] (see also [5, pp. 348-358], [23, pp.36-37]). Let

$$\mathcal{F}_{(1+\varepsilon)\varphi}^{\infty,0} = \{f \in \mathcal{F}_\varphi^\infty : \lim_{|z| \rightarrow \infty} |f(z)| e^{-(1+\varepsilon)\varphi(z)} = 0\}.$$

By the sampling property, the operator family  $T_z : \{f(\lambda)\}_{\lambda \in \Lambda} \mapsto e^{-(1+\varepsilon)\varphi(z)} f(z)$ ,  $z \in \mathbb{C}$ , is uniformly bounded from  $\{f|_\Lambda : f \in \mathcal{F}_{(1+\varepsilon)\varphi}^{\infty,0}\} \subset c_0$  to  $\mathbb{C}$ . Hence, by duality, there exists a family  $(g(z, \lambda))_{\lambda \in \Lambda}$  such that

$$e^{-(1+\varepsilon)\varphi(z)} f(z) = \sum_{\lambda \in \Lambda} e^{-(1+\varepsilon)\varphi(\lambda)} f(\lambda) g(z, \lambda), \quad f \in \mathcal{F}_{(1+\varepsilon)\varphi}^{\infty,0}.$$

and  $\sup_z \sum_\lambda |g(z, \lambda)| < \infty$ . Let now  $\Gamma = \Gamma_{\varepsilon\alpha} = \{e^{\frac{n+1}{2\alpha\varepsilon}} e^{i\theta_n}\}$ , and consider the function  $G \in \mathcal{F}_{\varepsilon\varphi}^\infty$  of Lemma 2.5 vanishing on  $\Gamma$ . When  $e^{\frac{n+1}{2\alpha\varepsilon}} \leq |z| \leq e^{\frac{n+2}{2\alpha\varepsilon}}$ , let  $\gamma_z = e^{\frac{n+2}{2\alpha\varepsilon}} e^{i\theta_n}$ , so that

$$e^{-1/(2\alpha\varepsilon)} \leq |z/\gamma_z| \leq 1.$$

Set

$$P_z(w) = \frac{G(w)}{(w - \gamma_z)G'(\gamma_z)} \frac{w^2}{z^2}.$$

For  $w \in \mathbb{C}$  we have

$$|P_z(w)| \asymp e^{\varepsilon(\varphi(w) - \varphi(\gamma_z))} \frac{|w|^{1/2} \operatorname{dist}(w, \Gamma)}{|z|^{1/2} |w - \gamma_z|} \lesssim e^{\varepsilon(\varphi(w) - \varphi(z))} \frac{|w|^{1/2} \operatorname{dist}(w, \Gamma)}{|z|^{1/2} |w - \gamma_z|}. \quad (4)$$

Given  $f \in \mathcal{F}_\varphi^2$ , by Lemma 2.1 (iii) and (4) we have  $w \mapsto f(w)P_z(w) \in \mathcal{F}_{(1+\varepsilon)\varphi}^{\infty, 0}$  and hence

$$e^{-(1+\varepsilon)\varphi(z)} f(z)P_z(z) = \sum_{\lambda \in \Lambda} e^{-(1+\varepsilon)\varphi(\lambda)} f(\lambda)P_z(\lambda)g(z, \lambda).$$

Since  $|P_z(z)| \asymp 1$ , again by (4) we obtain that

$$|f(z)|e^{-\varphi(z)} \lesssim \sum_{\lambda \in \Lambda} |f(\lambda)|e^{-\varphi(\lambda)} \frac{|\lambda|^{1/2} \operatorname{dist}(\lambda, \Gamma)}{|z|^{1/2} |\lambda - \gamma_z|} |g(z, \lambda)|.$$

Since  $\sum_\lambda |g(z, \lambda)| < \infty$ , Hölder's inequality and (2) give us that

$$\begin{aligned} |f(z)|^2 e^{-2\varphi(z)} &\lesssim \left( \sum_{\lambda \in \Lambda} |f(\lambda)|^2 e^{-2\varphi(\lambda)} \frac{|\lambda| \operatorname{dist}(\lambda, \Gamma)^2}{|z| |\lambda - \gamma_z|^2} \right) \left( \sum_{\lambda \in \Lambda} |g(z, \lambda)|^2 \right) \\ &\lesssim \left( \sum_{\lambda \in \Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)} \frac{1}{|\lambda z|} \frac{\operatorname{dist}(\lambda, \Gamma)^2}{|\lambda - \gamma_z|^2} \right) \left( \sum_{\lambda \in \Lambda} |g(z, \lambda)| \right)^2 \\ &\lesssim \sum_{\lambda \in \Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)} \frac{1}{|\lambda z|} \frac{\operatorname{dist}(\lambda, \Gamma)^2}{|\lambda - \gamma_z|^2}. \end{aligned} \quad (5)$$

It remains to verify that

$$I(\lambda) = \frac{1}{|\lambda|} \int_{\mathbb{C}} \frac{\operatorname{dist}(\lambda, \Gamma)^2}{|\lambda - \gamma_z|^2 |z|} dm(z) < \infty \quad (6)$$

uniformly in  $\lambda$ , since by (5), we then obtain the sampling inequality

$$\|f\|_{2, \varphi}^2 \lesssim \sup_{\lambda} I(\lambda) \|f\|_{2, \varphi, \Lambda}^2.$$

We will now show (6). Since  $\operatorname{dist}(\lambda, \Gamma) \leq |\lambda - \gamma_z|$  and  $|\gamma_z| \asymp |z|$ , we have

$$\frac{1}{|\lambda|} \int_{z: |\gamma_z| < 2|\lambda|} \frac{\operatorname{dist}(\lambda, \Gamma)^2}{|\lambda - \gamma_z|^2 |z|} dm(z) \lesssim \frac{1}{|\lambda|} \int_{|z| \lesssim |\lambda|} \frac{1}{|z|} dm(z) \lesssim 1.$$

If  $|\gamma_z| \geq 2|\lambda|$ , then  $|\lambda - \gamma_z| \geq |\gamma_z|/2 \asymp |z|$  and  $\operatorname{dist}(\lambda, \Gamma) \lesssim |\lambda|$ , so that

$$\int_{z: 2|\lambda| \leq |\gamma_z|} \frac{1}{|\lambda| |z|} \frac{\operatorname{dist}(\lambda, \Gamma)^2}{|\lambda - \gamma_z|^2} dm(z) \lesssim |\lambda| \int_{|\lambda| \lesssim |z|} \frac{1}{|z|^3} dm(z) \lesssim 1.$$

Hence (6) is established and the proof is complete.  $\square$

**2.2. De Branges spaces.** We will identify the Fock space  $\mathcal{F}_\varphi^2$  with a de Branges space. Let  $G$  be the generating function associated with the sequence  $\Gamma$  defined by (1) with  $\theta_n = -\pi/2$ . Then  $G$  is a Hermite–Biehler function, which means that  $|G(z)| > |G(\bar{z})|$ ,  $z \in \mathbb{C}^+$  (where  $\mathbb{C}^+$  is the upper half-plane). Recall that the de Branges space associated with  $G$  (see [8]) is given by

$$\mathcal{H}(G) := \{f \text{ entire} : f/G \in H^2(\mathbb{C}^+) \text{ and } f^*/G \in H^2(\mathbb{C}^+)\},$$

where  $f^*(z) = \overline{f(\bar{z})}$  and  $H^2(\mathbb{C}^+)$  stands for the standard Hardy space. The space  $\mathcal{H}(G)$  is normed by

$$\|f\|_{\mathcal{H}(G)}^2 := \int_{\mathbb{R}} \left| \frac{f(x)}{G(x)} \right|^2 dx, \quad f \in \mathcal{H}(G).$$

We have the following result.

**Proposition 2.8.** *The space  $\mathcal{F}_\varphi^2$  is a de Branges space:  $\mathcal{F}_\varphi^2 = \mathcal{H}(G)$ .*

*Proof.* We know that the normalized reproducing kernels  $\{\mathbf{k}_\gamma\}_{\gamma \in \Gamma} = \{\mathbf{k}_\gamma / \|\mathbf{k}_\gamma\|_{\varphi,2}\}_{\gamma \in \Gamma}$  form a Riesz basis in  $\mathcal{F}_\varphi^2$ . Then the biorthogonal family

$$\frac{\|\mathbf{k}_\gamma\|_{\varphi,2}}{G'(\gamma)} \cdot \frac{G(z)}{z - \gamma}, \quad \gamma \in \Gamma,$$

is a Riesz basis in  $\mathcal{F}_\varphi^2$ . By formula (2) and the estimate for  $|G'(\gamma)|$  in Lemma 2.5 we conclude that the above biorthogonal system is of the form  $a_\gamma |\gamma|^{1/2} \cdot G(z)/(z - \gamma)$ , where  $|a_\gamma| \asymp 1$ . Hence, any function  $f$  in the space  $\mathcal{F}_\varphi^2$  can be written as

$$f(z) = \sum_{\gamma \in \Gamma} c_\gamma |\gamma|^{1/2} \cdot \frac{G(z)}{z - \gamma}, \quad (7)$$

where  $\{c_\gamma\} \in \ell^2$  and  $\|f\|_{\varphi,2} \asymp \|\{c_\gamma\}\|_{\ell^2}$ . Writing for simplicity  $\gamma_n = -iy_n$ , we have

$$\frac{f(z)}{G(z)} = \sum_n \frac{c_n y_n^{1/2}}{z + iy_n},$$

and the series converges in the Hardy space  $H^2 = H^2(\mathbb{C}^+)$ , since  $\gamma$  satisfies the Carleson condition and so is an  $H^2$ -interpolating sequence. Analogously, if we put  $\Theta = G^*/G$ , we get

$$\frac{f^*(z)}{G(z)} = \sum_n \bar{c}_n y_n^{1/2} \frac{\Theta(z)}{z - iy_n},$$

again the series converges in  $H^2$ , since  $\Theta$  is an interpolating Blaschke product (with zeros  $iy_n$ ). We conclude that  $f/G$  and  $f^*/G$  are in  $H^2$ . Conversely, any function in  $\mathcal{H}(G)$  can be written as a series of the form (7), since the functions  $(\operatorname{Im} \gamma)^{1/2} G(z)/(z - \gamma)$  form a Riesz basis in  $\mathcal{H}(G)$  whenever the zero set of  $G$  is an interpolating sequence.  $\square$

In [3] a class of spaces  $\mathcal{H}(\Gamma, \mu)$  (Cauchy transforms of discrete measures) was introduced. Note that (7) means that  $f/G \in \mathcal{H}(\Gamma, \mu)$  where  $\mu = \sum_{\gamma \in \Gamma} |\gamma| \delta_\gamma$ , where  $\delta_\gamma$  is the Dirac

measure at  $\gamma$ .

It is also possible to have the comparison with the integral over the positive or negative rays.

**Corollary 2.9.** *Let  $G$  be the generating function of the sequence (1) with  $\theta_n = -\pi/2$ . Then the measure  $dx/|G(x)|^2$  is sampling on  $\mathbb{R}_+$ ,  $\mathbb{R}_-$  or  $\mathbb{R}$  for  $\mathcal{F}_\varphi^2$ : for every  $f \in \mathcal{F}_\varphi^2$ ,*

$$\|f\|_{\varphi,2}^2 \asymp \int_{\mathbb{R}_+} \left| \frac{f(x)}{G(x)} \right|^2 dx \asymp \int_{\mathbb{R}_-} \left| \frac{f(x)}{G(x)} \right|^2 dx \asymp \int_{\mathbb{R}} \left| \frac{f(x)}{G(x)} \right|^2 dx.$$

This answers a question appearing in equation (4.3.1) in [19].

*Proof.* Since  $\|f\|_{\varphi,2} \asymp \int_{\mathbb{R}} \left| \frac{f(x)}{G(x)} \right|^2 dx$ , it is sufficient to prove that

$$\int_{\mathbb{R}} \left| \frac{f(x)}{G(x)} \right|^2 dx \lesssim \int_{\mathbb{R}_+} \left| \frac{f(x)}{G(x)} \right|^2 dx.$$

The sequence  $\Gamma$  given by (1) (with  $\theta_n = 0$ ) is sampling, and hence, according to Corollary 2.4 there exists  $\delta > 0$  and  $C$  depending only on  $\delta$  such that for every perturbation  $\tilde{\Gamma}$  of  $\Gamma$  with  $d_\rho(\gamma_n, \tilde{\gamma}_n) \leq \delta$  we have

$$\frac{1}{C} \|f\|_{\varphi,2} \leq \|f\|_{\varphi,2,\tilde{\Gamma}}.$$

Consider now the sequence  $\Lambda = \{\lambda_n\}_{n \geq 0}$  such that  $\lambda_n > \gamma_n$  and  $d_\rho(\gamma_n, \lambda_n) = \delta$ . We set  $I_n = [\gamma_n, \lambda_n)$ ,  $n \geq 0$ . Clearly, the intervals  $I_n$  are disjoint if  $\delta$  is sufficiently small. By the mean value theorem, there exists  $x_n \in I_n$  such that

$$\int_{I_n} \left| \frac{f(x)}{G(x)} \right|^2 dx = |I_n| \times \left| \frac{f(x_n)}{G(x_n)} \right|^2.$$

It is also clear that  $|I_n| \asymp 1 + x_n$ . Since  $x_n \in \mathbb{R}^+$ , using the estimate in Lemma 2.5 and taking into account that  $\text{dist}(x_n, -i\Gamma) \asymp 1 + x_n$ , we get

$$\begin{aligned} \int_{\mathbb{R}_+} \left| \frac{f(x)}{G(x)} \right|^2 dx &\geq \sum_{n \geq 0} \int_{I_n} \left| \frac{f(x)}{G(x)} \right|^2 dx \asymp \sum_{n \geq 0} (1 + x_n) \left| \frac{f(x_n)}{G(x_n)} \right|^2 \\ &\asymp \sum_{n \geq 0} |f(x_n)|^2 (1 + x_n^2) e^{-2\varphi(x_n)} \asymp \sum_{n \geq 0} \frac{|f(x_n)|^2}{\mathbf{k}_{x_n}(x_n)}. \end{aligned}$$

Since  $d_\rho(x_n, \gamma_n) \leq \delta$ , the sequence  $\tilde{\Gamma} := \{x_n\}_{n \geq 0}$  is sampling for  $\mathcal{F}_\varphi^2$ , and we get

$$\int_{\mathbb{R}_+} \left| \frac{f(x)}{G(x)} \right|^2 dx \gtrsim \|f\|_{\varphi,2,\tilde{\Gamma}} \geq \frac{1}{C} \|f\|_{\varphi,2}.$$

□

### 2.3. Density results.

**Lemma 2.10.** *If  $\Lambda$  is  $d_\rho$ -separated then  $D^-(\Lambda) \leq D^+(\Lambda) < \infty$ .*

*Proof.* As already mentioned in the beginning of Section 2, when  $\Lambda$  is  $d_\rho$ -separated, then there exists  $c$  such that the Euclidean disks  $D(\lambda, c|\lambda|)$ ,  $\lambda \in \Lambda$ , are disjoint. A standard argument, based for instance on the consideration of the Euclidean area of  $\mathcal{A}(x, Rx)$  and that of the disks  $D(\lambda, c|\lambda|)$ ,  $\lambda \in \Lambda \cap \mathcal{A}(x, Rx)$ , shows that this implies in particular that for fixed  $\eta > 0$  every annulus  $\mathcal{A}(x, \eta x)$  contains a uniformly bounded number of points of  $\Lambda$  (this number depends on  $\eta$ ):

$$\text{Card}(\Lambda \cap \mathcal{A}(x, \eta x)) \leq M, \quad x > 0.$$

Suppose now that  $R > R_0$  and  $r > r_0$  with  $R_0, r_0$  big enough. Let  $N$  be the least integer such that  $r\eta^N \geq rR$  so that  $N \asymp \log R / \log \eta$ . Then

$$\text{Card}(\Lambda \cap \mathcal{A}(r, Rr)) = \sum_{\lambda \in \Lambda \cap \mathcal{A}(r, Rr)} 1 \leq \sum_{n=1}^N \text{Card}(\Lambda \cap \mathcal{A}(r\eta^{n-1}, r\eta^n)) \lesssim \frac{M}{\log \eta} \log R.$$

□

**Proposition 2.11.** *If  $\Lambda$  is a zero sequence for  $\mathcal{F}_\varphi^\infty$  then*

$$\liminf_{R \rightarrow +\infty} \frac{\text{Card}(\Lambda \cap D(0, R))}{\log R} \leq 2\alpha. \quad (8)$$

*Proof.* Suppose there is a function  $g$  that vanishes on  $\Lambda$  with

$$\liminf_{R \rightarrow +\infty} \frac{\text{Card}(\Lambda \cap D(0, R))}{\log R} > 2\alpha. \quad (9)$$

Assuming  $g(0) \neq 0$  (otherwise divide by a suitable power of  $z$  which does not change the other zeros of  $g$  and gives a function still in  $\mathcal{F}_\varphi^\infty$ ), Jensen's formula yields for every  $R > 0$ ,

$$\sum_{\lambda \in \Lambda: |\lambda| < R} \log \frac{R}{|\lambda|} = \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\theta})| d\theta - \log |g(0)| \leq \alpha(\log R)^2 + C.$$

Denote now by  $n_g(R)$  the number of zeros of  $g$  in  $D(0, R)$ . Then

$$\sum_{\lambda \in \Lambda: |\lambda| < R} \log \frac{R}{|\lambda|} = \int_0^R \frac{n_g(t)}{t} dt.$$

From (9) we deduce that for  $\varepsilon > 0$  small enough there exists  $R_0 > 0$  such that for every  $R \geq R_0$ ,

$$n_g(R) = \text{Card}(\Lambda \cap D(0, R)) \geq 2\alpha(1 + \varepsilon) \log R.$$

Then for every  $R \geq R_0$ ,

$$\sum_{\lambda \in \Lambda: |\lambda| < R} \log \frac{R}{|\lambda|} \geq \int_{R_0}^R \frac{n_g(t)}{t} dt \geq 2\alpha(1 + \varepsilon) \int_{R_0}^R \frac{\log t}{t} dt \geq 2\alpha(1 + \varepsilon) \left( \frac{(\log R)^2}{2} - \frac{(\log R_0)^2}{2} \right).$$

It follows that

$$\alpha(1 + \varepsilon) \left( (\log R)^2 - (\log R_0)^2 \right) \leq \alpha(\log R)^2 + C$$

which is impossible when  $R$  is big.  $\square$

We may deduce the following corollary.

**Corollary 2.12.** *If  $\Lambda$  satisfies (8) then  $D^-(\Lambda) \leq 2\alpha$ .*

*Proof.* By contraposition, suppose that  $D^-(\Lambda) > 2\alpha$ . Then there are  $R_0$  and  $r_0$  such that for every  $R > R_0$  and  $r > r_0$  we have

$$\frac{\text{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R} \geq 2(1 + \varepsilon)\alpha,$$

for a suitable fixed  $\varepsilon$ . Set  $\eta = \max(R_0, r_0)$  and let  $x = \eta^{N+\kappa} \in [\eta^N, \eta^{N+1})$  ( $\kappa \in [0, 1)$ ) be big, then

$$\begin{aligned} \frac{\text{Card}(\Lambda \cap D(0, x))}{\log x} &\geq \frac{\sum_{k=1}^{N-1} \text{Card}(\Lambda \cap \mathcal{A}(\eta^k, \eta^{k+1}))}{\log x} \\ &\geq \frac{2(1 + \varepsilon)\alpha(N-1) \log \eta}{(N + \kappa) \log \eta} \rightarrow 2(1 + \varepsilon)\alpha, \quad N \rightarrow \infty, \end{aligned}$$

i.e.,  $\Lambda$  does not satisfy (8).  $\square$

The two preceding results together team up in:

**Corollary 2.13.** *If  $\Lambda$  is a zero sequence for  $\mathcal{F}_\varphi^\infty$ , then  $D^-(\Lambda) \leq 2\alpha$ .*

### 3. PROOF OF THE RESULT ON RIESZ BASES

We recall Bari's Theorem [20, p.132] which will be used in the proof of our Theorem 1.1. Let  $\mathcal{H}$  be a Hilbert space  $\mathcal{H}$  and let  $\mathcal{X} = \{x_n : n \in \mathbb{N}\}$  be a family of elements of  $\mathcal{H}$ . The family  $\mathcal{X}$  is called *minimal* if  $x_n \notin \bigvee \{x_i, i \neq n\}$ , and is called *complete* if  $\mathcal{H} = \bigvee \{x_i, i \in \mathbb{N}\}$ .

Note that  $\mathcal{X}$  is minimal if and only if  $\mathcal{X}$  admits a biorthogonal family  $\mathcal{X}'$ . Let  $x \in \mathcal{X}$  and let  $\mathcal{X}' = \{x'_n : n \in \mathbb{N}\}$  be the corresponding biorthogonal system. We set

$$\mathcal{J}_{\mathcal{X}}x := \{(x, x'_n) : n \in \mathbb{N}\}.$$

**Bari's Theorem.** *Let  $\mathcal{X} = \{x_n : n \in \mathbb{N}\}$  be a minimal family in  $\mathcal{H}$  and let  $\mathcal{X}'$  be the corresponding biorthogonal system. Then the following statements are equivalent*

- (1)  $\mathcal{X}$  is a Riesz basis in  $\mathcal{H}$ ;
- (2)  $\mathcal{X}$  is complete in  $\mathcal{H}$  and  $\mathcal{J}_{\mathcal{X}}\mathcal{H} \subset \ell^2$ ,  $\mathcal{J}_{\mathcal{X}'}\mathcal{H} \subset \ell^2$ .

**Proof of Theorem 1.1.** “ $\Leftarrow$ ”: As in the introduction, let  $\mathbf{k}_\lambda$  be the reproducing kernel of  $\mathcal{F}_\varphi^2$  and let  $\mathbb{k}_\lambda = \mathbf{k}_\lambda / \|\mathbf{k}_\lambda\|_{\varphi,2}$  be the normalized kernel at  $\lambda$ . Let  $F$  be an entire function with simple zeros at each  $\lambda \in \Lambda$ ,

$$F(z) := \prod_{n \geq 0} \left(1 - \frac{z}{\lambda_n}\right), \quad z \in \mathbb{C},$$

and set

$$g_\lambda(z) = \frac{F(z)}{F'(\lambda)(z - \lambda)} \|\mathbf{k}_\lambda\|_{\varphi,2}, \quad z \in \mathbb{C}.$$

If the functions  $g_\lambda$  are in  $\mathcal{F}_\varphi^2$ , then the family  $\{g_\lambda\}_{\lambda \in \Lambda}$  is biorthogonal to  $\mathcal{K}_\Lambda := \{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$ . We use Bari's Theorem, to show that  $\mathcal{K}_\Lambda$  is Riesz basis it suffices to prove

- (i)  $F/(\cdot - \lambda) \in \mathcal{F}_\varphi^2$  for  $\lambda \in \Lambda$ ;
- (ii)  $\mathcal{K}_\Lambda$  is complete :  $\bigvee \{\mathbb{k}_\lambda, \lambda \in \Lambda\} = \mathcal{F}_\varphi^2$ ;
- (iii)  $\sum_{\lambda \in \Lambda} |\langle f, \mathbb{k}_\lambda \rangle|^2 \lesssim \|f\|_{\varphi,2}^2$ ;
- (iv)  $\sum_{\lambda \in \Lambda} |\langle f, g_\lambda \rangle|^2 \lesssim \|f\|_{\varphi,2}^2$ .

To prove (i), let  $|z| = e^t$  with  $|\lambda_{n-1}| \leq |z| \leq |\lambda_n|$  and suppose that  $\text{dist}(z, \Lambda) = |z - \lambda_{n-1}|$ . Let  $m \in \mathbb{N}$  be such that

$$\frac{m}{2\alpha} - \frac{1}{4\alpha} \leq t < \frac{m}{2\alpha} + \frac{1}{4\alpha}.$$

Then, since  $|\lambda_n| = \gamma_n e^{\delta_n} \asymp \gamma_m$  by the above, and  $\delta_m$  is uniformly bounded, we have that  $|m - n|$  is uniformly bounded in  $|z|$ , and  $\log |\lambda_s| - t$  is bounded uniformly in  $z$  and  $s$  between  $m$  and  $n$ . We use that for  $d_\rho$ -separated sequences the behavior of the function  $F$  is essentially given by the first  $n$  terms. We have

$$\begin{aligned} \log |F(z)| &= \sum_{0 \leq k \leq n-2} \log \frac{|z|}{|\lambda_k|} + \log \left|1 - \frac{z}{\lambda_{n-1}}\right| + O(1) \\ &= \sum_{0 \leq k \leq n-1} \left(t - \frac{k+1}{2\alpha}\right) + \log \text{dist}(z, \Lambda) - t - \sum_{0 \leq k \leq n-1} \delta_k + O(1) \\ &= \sum_{0 \leq k \leq m-1} \left(t - \frac{k+1}{2\alpha}\right) + \log \text{dist}(z, \Lambda) - t - \sum_{0 \leq k \leq m-1} \delta_k + O(1) \\ &= mt - \frac{m(m+1)}{4\alpha} - \sum_{0 \leq k \leq m-1} \delta_k + \log \text{dist}(z, \Lambda) - t + O(1) \\ &= \alpha t^2 - \frac{3}{2}t + \log \text{dist}(z, \Lambda) - \sum_{0 \leq k \leq m-1} \delta_k + O(1), \quad t \rightarrow \infty. \end{aligned} \tag{10}$$



Next, if  $m = lN + r$ ,  $0 \leq r < N$ , then

$$\left| \sum_{k=0}^{m-1} \delta_k \right| \leq \sum_{j=0}^{l-1} \left| \sum_{i=0}^{N-1} \delta_{jN+i} \right| + \left| \sum_{i=0}^{r-1} \delta_{lN+i} \right| \leq lN\delta + O(1) = 2\alpha\delta t + O(1). \quad (11)$$

Therefore, for some  $\eta > 0$ ,

$$e^{\varphi(z)} \frac{\text{dist}(z, \Lambda)}{(1 + |z|)^{2-\eta}} \lesssim |F(z)| \lesssim e^{\varphi(z)} \frac{\text{dist}(z, \Lambda)}{(1 + |z|)^{1+\eta}}, \quad z \in \mathbb{C}. \quad (12)$$

This proves (i).

Next we pass to property (iii). By assumption,  $\Lambda$  is  $d_\rho$ -separated, and so by Lemma 2.1 we have

$$\sum_{\lambda \in \Lambda} |\langle f, \mathbb{k}_\lambda \rangle|^2 = \sum_{\lambda \in \Lambda} \frac{|f(\lambda)|^2}{\|\mathbf{k}_\lambda\|_{\varphi,2}^2} \lesssim \|f\|_{\varphi,2}^2.$$

Property (iii) also follows from [3, Theorem 3.1].

Let us turn to (ii). By Lemma 2.5 we have

$$|G(z)| \asymp e^{\varphi(z)} \frac{\text{dist}(z, \Lambda)}{1 + |z|^{3/2}}, \quad z \in \mathbb{C}.$$

If  $f \in \mathcal{F}_\varphi^2$ , then by Lemma 2.1 (iii),  $|f(z)| = o(e^{\varphi(z)}/(1 + |z|))$  and so

$$|f(z)/G(z)| = o\left(\frac{1 + \sqrt{|z|}}{\text{dist}(z, \Gamma)}\right), \quad z \in \mathbb{C}. \quad (13)$$

Let  $|z| = e^t$  and let  $n$  be such that

$$\frac{n}{2\alpha} - \frac{1}{4\alpha} \leq t < \frac{n}{2\alpha} + \frac{1}{4\alpha}.$$

If we denote by  $k(n)$  the integer such that the point  $\lambda_{k(n)} \in \Lambda$  is the closest to  $z$ , then, by condition (b),  $|k(n) - n|$  is uniformly bounded. Hence, keeping in mind that  $\Lambda$  is separated and  $|\lambda_m| \asymp |z|$ ,  $|m - n| \leq |k(n) - n|$ , we have

$$|F(z)| \asymp \prod_{k=1}^{k(n)} \left| 1 - \frac{z}{\lambda_k} \right| \asymp \frac{\text{dist}(z, \Lambda)}{|z|} \prod_{k=1}^{n-1} \frac{|z|}{|\lambda_k|}. \quad (14)$$

Thus,

$$\frac{|F(z)|}{|G(z)|} \asymp \frac{\prod_{k=0}^{n-1} |z/\lambda_k|}{\prod_{k=0}^{n-1} |z/\gamma_k|} \cdot \frac{\text{dist}(z, \Lambda)}{\text{dist}(z, \Gamma)} \asymp \exp\left(-\sum_{k=0}^{n-1} \delta_k\right) \frac{\text{dist}(z, \Lambda)}{\text{dist}(z, \Gamma)}.$$

As in (11), and recalling that  $|m - n|$  is uniformly bounded, we have  $|\sum_{k=0}^{n-1} \delta_k| \leq 2\alpha\delta t + O(1)$ , where  $2\alpha\delta < 1/2$ . Therefore,

$$\frac{1}{|z|^{2\alpha\delta}} \frac{\text{dist}(z, \Lambda)}{\text{dist}(z, \Gamma)} \lesssim \frac{|F(z)|}{|G(z)|}, \quad z \notin \Gamma.$$

If  $z$  is  $d_\rho$ -far from  $\Lambda$ , we have  $\text{dist}(z, \Lambda)/|z|^{2\alpha\delta} \asymp |z|^{1-2\alpha\delta} \gg \sqrt{|z|}$  so that with (13) in mind we see that  $F \notin \mathcal{F}_\varphi^2$ . Now, if  $\mathcal{K}_\Lambda$  is not complete, then there exists a nonzero  $f \in \mathcal{F}_\varphi^2$  vanishing on  $\Lambda$ . So  $f = FS$  for some entire function  $S$ , and

$$|S(z)| = \left| \frac{f(z)}{F(z)} \right| = \left| \frac{f(z)G(z)}{G(z)F(z)} \right| = o\left( \frac{1 + |z|^{1/2+2\alpha\delta}}{\text{dist}(z, \Lambda)} \right).$$

Now, for every  $R > 0$ , we can find a closed contour  $C_R$  surrounding  $D(0, R)$  which is  $d_\rho$ -separated from  $\Lambda$  (and not meeting  $\Gamma$ ). We have  $\max_{z \in C_R} |S(z)| = o(1)$ ,  $R \rightarrow \infty$ , so that  $\sup_{|z| \leq R} |S(z)| = o(R)$ , and hence  $S$  vanishes identically, which is impossible. Statement (ii) is proved.

It remains to show (iv). Let us estimate  $F(\gamma_m)$  and  $F'(\lambda_n)$ . As above, we only need to consider the terms  $k \leq n-1$ . By (14),

$$|F(\gamma_m)| \asymp \frac{\text{dist}(\gamma_m, \Lambda)}{\gamma_m} \prod_{0 \leq k \leq m-1} \frac{\gamma_m}{|\lambda_k|} \quad (15)$$

and

$$|F'(\lambda_n)| \asymp \frac{1}{|\lambda_n|} \prod_{0 \leq k \leq n-1} \left| \frac{\lambda_n}{\lambda_k} \right|. \quad (16)$$

Since the family  $\{\mathbf{k}_\gamma\}_{\gamma \in \Gamma}$  is a Riesz basis, we can write

$$f = \sum_{m \geq 0} a_m \mathbf{k}_{\gamma_m}, \quad (a_m)_{m \geq 0} \in \ell^2,$$

and the sum in (iv) becomes

$$\sum_{n \geq 0} \left| \sum_{m \geq 0} a_m \underbrace{\frac{F(\gamma_m)}{F'(\lambda_n)(\gamma_m - \lambda_n)} \cdot \frac{\|\mathbf{k}_{\lambda_n}\|_{\varphi, 2}}{\|\mathbf{k}_{\gamma_m}\|_{\varphi, 2}}}_{A_{n,m}} \right|^2.$$

It remains to check that the matrix  $[A_{n,m}]$  defines a bounded operator in  $\ell^2$ .

It follows from (15), (16) and (2) that

$$\begin{aligned} |A_{n,m}| &\asymp \frac{\text{dist}(\gamma_m, \Lambda)}{|\gamma_m - \lambda_n|} \cdot \frac{\gamma_m^m}{|\lambda_n|^n} \cdot \left( \prod_{0 \leq k \leq n-1} |\lambda_k| \right) \cdot \left( \prod_{0 \leq k \leq m-1} |\lambda_k| \right)^{-1} \cdot e^{\alpha(\log |\lambda_n|)^2 - \alpha(\log \gamma_m)^2} \\ &= \frac{\text{dist}(\gamma_m, \Lambda)}{|\gamma_m - \lambda_n|} e^{c(n,m)}, \end{aligned}$$

where

$$\begin{aligned} c(n, m) &= \frac{m(m+1)}{2\alpha} - \left(\frac{n+1}{2\alpha} + \delta_n\right)n - \sum_{k=0}^{m-1} \left(\frac{k+1}{2\alpha} + \delta_k\right) \\ &\quad - \sum_{k=0}^{n-1} \left(\frac{k+1}{2\alpha} + \delta_k\right) + \alpha \left(\frac{n+1}{2\alpha} + \delta_n\right)^2 - \frac{(m+1)^2}{4\alpha} \\ &= -\frac{m-n}{4\alpha} + \sum_{k=0}^{n-1} \delta_k - \sum_{k=0}^{m-1} \delta_k + O(1). \end{aligned}$$

By condition (b) there exists  $M$  such that  $|\gamma_m - \lambda_n| \asymp |\lambda_n|$  when  $n > m + M$ ,  $|\gamma_m - \lambda_n| \asymp |\gamma_m|$  when  $m > n + N$ , and  $|\gamma_m| \asymp |\lambda_n|$  for  $|m - n| \leq M$ .

• Let  $|m - n| \leq M$ . Then it is clear that  $|A_{n,n}| \lesssim 1$  with a bound independent of  $n$  and  $m$ .

• If  $m > n + M$ , then

$$|A_{n,m}| \asymp \frac{\text{dist}(\gamma_m, \Lambda)}{\gamma_m} \exp\left(-\frac{m-n}{4\alpha} + \sum_{k=n}^{m-1} \delta_k + O(1)\right).$$

• If  $n > m + M$ , then  $|\gamma_m - \lambda_n| \asymp |\lambda_n|$  and so

$$\begin{aligned} |A_{n,m}| &\asymp \frac{\text{dist}(\gamma_m, \Lambda)}{\gamma_m} \cdot \frac{\gamma_m}{|\lambda_n|} \cdot \exp\left(\frac{n-m}{4\alpha} + \sum_{k=m}^{n-1} \delta_k + O(1)\right) \\ &\asymp \frac{\text{dist}(\gamma_m, \Lambda)}{\gamma_m} \cdot \exp\left(-\frac{n-m}{4\alpha} + \sum_{k=m}^{n-1} \delta_k + O(1)\right), \end{aligned}$$

since  $\gamma_m/|\lambda_n| \asymp \exp\left(\frac{m-n}{2\alpha}\right)$ .

Thus,

$$|A_{n,m}| \asymp \frac{\text{dist}(\gamma_m, \Lambda)}{\gamma_m} \exp\left(-\frac{|m-n|}{4\alpha} + \sum_{k=m}^{n-1} \delta_k + O(1)\right), \quad (17)$$

where the sum is taken with negative sign if  $m > n$ . It follows from (c) that  $|A_{n,m}| \lesssim \exp(-\delta|n-m|)$  for some  $\delta > 0$ , and so the matrix  $[A_{n,m}]$  defines a bounded operator in  $\ell^2$ . Statement (iv) is proved.

“ $\implies$ ”:

(a) By Corollary 2.3,  $\Lambda$  is  $d_\rho$ -separated.

(b) Suppose that  $(\delta_n) \notin \ell^\infty$ . Then there exists an infinite subsequence of indices  $\mathcal{N} = \{n_k\}$  such that for each  $k$  there exists  $m_k$  such that  $d_\rho(\lambda_{n_k}, \gamma_{m_k}) \lesssim 1$ , but  $|n_k - m_k| \rightarrow \infty$  as  $k \rightarrow \infty$ . Passing possibly to another subsequence (also denoted by  $\lambda_{n_k}$  to not overcharge notation) we can suppose that this subsequence is in an angle. Now, since  $\mathcal{F}_\varphi^2$  is rotation invariant,  $\{e^{i\theta}\lambda_n\}$  is also a complete interpolating sequence for any  $\theta \in \mathbb{R}$  for which the

subsequence  $(\lambda_{n_k})_k$  is  $d_\rho$  separated from  $\Gamma$ . Thus, we may assume without loss of generality that

$$d_\rho(\lambda_{n_k}, \Gamma) \asymp 1,$$

that is,  $|\lambda_{n_k} - \gamma| \geq c|\lambda_{n_k}|$ ,  $\gamma \in \Gamma$  (with constants independent of  $k$ ) and also that

$$d_\rho(\gamma_{m_k}, \Lambda) \asymp 1.$$

To simplify the notations we write  $n$  and  $m$  in place of  $n_k$  and  $m_k$ , Now let us consider

$$A_{n,m} = \langle g_{\lambda_n}, \mathbb{k}_{\gamma_m} \rangle = \frac{F(\gamma_m)}{F'(\lambda_n)(\gamma_m - \lambda_n)} \cdot \frac{\|\mathbf{k}_{\lambda_n}\|_{\varphi,2}}{\|\mathbf{k}_{\gamma_m}\|_{\varphi,2}}.$$

Observe that we have assumed conditions (a) and (c) which ensure that  $g_\lambda \in \mathcal{F}_\varphi^2$  (cf. proof of (i) in Bari's theorem in the beginning of this section). As in (15)–(16) and taking into account that  $\text{dist}(\gamma_m, \Lambda) \asymp \gamma_m$ ,

$$|F(\gamma_m)| \asymp \frac{\gamma_m^n}{|\lambda_0 \lambda_1 \dots \lambda_{n-1}|}.$$

Analogously,

$$|F'(\lambda_n)| \asymp \frac{|\lambda_n|^{n-1}}{|\lambda_0 \lambda_1 \dots \lambda_{n-1}|}.$$

Thus, using the estimate (2) for the norm of the reproducing kernel as well as the estimates  $\gamma_m \asymp |\lambda_n| \asymp |\gamma_m - \lambda_n|$ , we get

$$|A_{n,m}| \asymp \frac{\gamma_m^n}{|\lambda_n|^n} e^{\alpha(\log^2 |\lambda_n| - \log^2 \gamma_m)}.$$

Note that  $\log \gamma_m = \frac{m+1}{2\alpha}$  and  $\log |\lambda_n| = \frac{n+1}{2\alpha} + \delta_n$ . Let us write  $|\lambda_n| = \gamma_m e^{\delta_{nm}}$  with  $|\delta_{nm}| \lesssim 1$ . Hence  $\exp(\frac{n+1}{2\alpha} + \delta_n) = \exp(\frac{m+1}{2\alpha} + \delta_{nm})$  and so

$$\frac{n-m}{2\alpha} = -\delta_n + \delta_{nm}. \quad (18)$$

Without loss of generality, let  $\delta_n \rightarrow +\infty$ . Moreover, we may assume that  $\delta_{nm} > 1$ . Otherwise, we may replace  $m$  by  $m - m_0$  for some sufficiently large  $m_0$  that can be chosen to be independent of  $k$ . (If  $\delta_n \rightarrow -\infty$  then we may assume that  $\delta_{nm} > 1$ . Otherwise, we may replace  $m$  by  $m + m_0$  for some sufficiently large  $m_0$ )

Then by (18) we have

$$\begin{aligned} \log |A_{n,m}| &= n \frac{m+1}{2\alpha} - n \left( \frac{n+1}{2\alpha} + \delta_n \right) - \frac{(m+1)^2}{4\alpha} + \alpha \left( \frac{n+1}{2\alpha} + \delta_n \right)^2 + O(1) \\ &= -\frac{(n-m)^2}{4\alpha} + \frac{n-m}{2\alpha} + \delta_n + \alpha \delta_n^2 + O(1) \\ &= 2\alpha \delta_n \delta_{nm} + O(1) \end{aligned}$$

Thus,  $|A_{n,m}| \gtrsim e^{2\alpha \delta_n \delta_{nm}}$ . Repeating this estimate for each  $k$  (note that all asymptotic estimates  $\asymp$  and  $\gtrsim$  hold uniformly with respect to  $k$ ) we conclude that  $\lim_{k \rightarrow \infty} |A_{n_k, m_k}| = \infty$ . Since  $A_{m,n} = (g_{\lambda_n}, \mathbb{k}_{\gamma_m})$ , we conclude that  $\|g_{\lambda_{n_k}}\|_{\varphi,2} \rightarrow \infty$ ,  $k \rightarrow \infty$ , and so  $\{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$  is not

uniformly minimal (in particular, it is not a Riesz basis).

(c) Given  $N$ , set

$$\sup_n \frac{1}{N} \left| \sum_{k=n+1}^{n+N} \delta_k \right| = \frac{1}{4\alpha} + \varepsilon_N. \quad (19)$$

Since we already have shown that  $(\delta_n)$  is bounded, the sequence  $(\varepsilon_N)$  is bounded. If for some  $N$  we have  $\varepsilon_N < 0$ , then (c) is proved. Assuming the converse, we have  $\varepsilon_N \geq 0$ .

Replacing, if necessary,  $\Lambda$  by  $e^{i\theta}\Lambda$  we may assume that  $\text{dist}(\gamma_m, \Lambda) \asymp \gamma_m$ .

Suppose first that there is a subsequence  $(N_l)$  such that  $\varepsilon_{N_l}$  is bounded from below by  $\varepsilon > 0$ . Then there exists  $n_l$  such that

$$\frac{1}{N_l} \left| \sum_{k=n_l+1}^{n_l+N_l} \delta_k \right| \geq \frac{1}{4\alpha} + \varepsilon_{N_l}/2 \geq \frac{1}{4\alpha} + \varepsilon/2.$$

It follows from (17) that the sum  $|A_{n_l, n_l+N_l}| + |A_{n_l+N_l, n_l}|$  is unbounded and so *a fortiori* the matrix  $[A_{n,m}]$ . Suppose now that  $(\varepsilon_N)$  is a sequence of positive numbers tending to zero. Then for every  $N$  there exists  $n_N$  such that

$$\left| \sum_{k=n_N+1}^{n_N+N} \delta_k \right| \geq N \left( \frac{1}{4\alpha} + \varepsilon_N \right) - 1$$

By definition

$$\left| \sum_{k=n_N+1}^{n_N+K} \delta_k \right| \leq K \left( \frac{1}{4\alpha} + \varepsilon_K \right)$$

for every  $1 \leq K \leq N$ . Then

$$\left| \sum_{k=n_N+K+1}^{n_N+N} \delta_k \right| \geq \frac{N-K}{4\alpha} + N\varepsilon_N - K\varepsilon_K - 1. \quad (20)$$

Two cases may occur.

Case (1): If  $(N\varepsilon_N)$  has a subsequence which tends to  $+\infty$ , then for fixed  $K$  and  $N$  in this subsequence, the expression  $N\varepsilon_N - K\varepsilon_K$  is positive and unbounded. Thus, again by (17), the sum of matrix entries  $|A_{M+K, M+N}| + |A_{M+N, M+K}|$  is unbounded and so the matrix  $[A_{n,m}]$  can not define a bounded operator in  $\ell^2$ .

Case (2): Assume that the sequence  $(N\varepsilon_N)$  is bounded. If there exist arbitrarily large  $N$  such that  $\sum_{k=n_N+1}^{n_N+N} \delta_k > 0$  then for these values of  $N$  and every  $1 \leq K \leq N$  we have by (20)

$$\sum_{k=n_N+K+1}^{n_N+N} \delta_k \geq \frac{N-K}{4\alpha} + N\varepsilon_N - K\varepsilon_K - 1.$$

Hence, by (17),

$$|A_{n_N+N, n_N+K+1}| \asymp \exp \left( \sum_{k=n_N+K+1}^{n_N+N} \delta_k - \frac{N-K}{4\alpha} \right) \asymp 1.$$

Analogously, if there exist arbitrarily large  $N$  such that  $\sum_{k=n_N+1}^{n_N+N} \delta_k < 0$ , then by (20) for  $1 \leq K \leq N$  we have

$$\sum_{k=n_N+1}^{n_N+N-K} \delta_k \leq -\frac{N-K}{4\alpha} + O(1),$$

and so, by (17),

$$|A_{n_N+1, n_N+N-K}| \asymp \exp \left( -\frac{N-K}{4\alpha} - \sum_{k=n_N+1}^{n_N+N-K} \delta_k \right) \asymp 1.$$

In each of these two situations we conclude that the matrix  $[A_{n,m}]$  has an increasing number of entries in one line which are bounded away from zero, and, thus, cannot define a bounded operator.  $\square$

**Remark 3.1.** Now we compare in more detail Theorem 1.1 with the results of Belov, Mengestie and Seip [3]. By Proposition 2.8 (see formula (7)) any function  $f \in \mathcal{F}_\varphi^2$  may be represented as

$$f(z) = G(z) \sum_{\gamma \in \Gamma} \frac{c_\gamma |\gamma|^{1/2}}{z - \gamma},$$

where  $(c_\gamma)_{\gamma \in \Gamma} \in \ell^2$  and  $\|f\|_{\varphi, 2} \asymp \|(c_\gamma)\|_{\ell^2}$ . Thus, the space  $\mathcal{F}_\varphi^2$  is a special case of the spaces  $\mathcal{H}(\Gamma, v)$  introduced in [3, Section 2] and which consist of entire functions of the form  $f(z) = G(z) \sum_{\gamma \in \Gamma} \frac{c_\gamma v_\gamma^{1/2}}{z - \gamma}$ . The space  $\mathcal{F}_\varphi^2$  corresponds to the choice of weights  $v_n = |\gamma_n|$  (we write  $v_n$  in place of  $v_{\gamma_n}$ ).

In [3, Theorems 5.1, 5.2] a description of Riesz bases in the spaces  $\mathcal{H}(\Gamma, v)$  was obtained under two additional restrictions:

$$\text{either } v_n = o\left(\sum_{k < n} v_k\right) \quad \text{or} \quad \frac{v_n}{|\gamma_n|^2} = o\left(\sum_{k > n} \frac{v_k}{|\gamma_k|^2}\right) \quad (21)$$

as  $n \rightarrow \infty$ . The criteria in [3] are also of the perturbative nature and are very close to our result. However, for the choice of the weight  $v_n = |\gamma_n|$  neither of conditions in (21) is satisfied. Thus, the results stated in [3] do not cover Theorem 1.1. It seems very probable that Theorem 1.1 can be proved using the powerful methods developed in [3] (in particular, Theorem 1.3), however our feeling is that such a proof will be not be shorter and more transparent than the one presented in this section.

4. COMPLETE INTERPOLATING SEQUENCES IN  $\mathcal{F}_\varphi^\infty$ 

In this section we prove Theorem 1.2. The proof is in many ways similar to the proof of Theorem 1.1. Put

$$\tilde{\Gamma} = \Gamma \cup \{\tilde{\gamma}\} = \{\gamma_n\}_{n \geq 0} \cup \{\tilde{\gamma}\},$$

where  $\Gamma = \Gamma_\alpha = \{e^{\frac{n+1}{2\alpha}} e^{i\theta_n}\}_{n \geq 0}$  is our reference sequence (1) and  $\tilde{\gamma} \notin \Gamma$ .

**Proposition 4.1.** *The sequence  $\tilde{\Gamma}$  is a complete interpolating sequence for  $\mathcal{F}_\varphi^\infty$  for any  $\tilde{\gamma} \in \mathbb{C} \setminus \Gamma$ .*

*Proof.* Let  $\tilde{G}(z) = \prod_{\gamma \in \tilde{\Gamma}} (1 - z/\gamma)$ , that is  $\tilde{G}(z) = (1 - z/\tilde{\gamma})G(z)$  where  $G$  is the associated function of Lemma 2.5 vanishing exactly on  $\Gamma$  (without loss of generality assume that  $\tilde{\gamma} \neq 0$ ). For every sequence  $v = (v_\gamma)_{\gamma \in \tilde{\Gamma}}$  with  $\|v\|_{\infty, \varphi, \tilde{\Gamma}} < \infty$ , we construct the corresponding interpolating function and estimate its norm:

$$f_v(z) = \sum_{\gamma \in \tilde{\Gamma}} v_\gamma \frac{\tilde{G}(z)}{\tilde{G}'(\gamma)(z - \gamma)}.$$

By the estimates of Lemma 2.5 we have, for any  $z \in \mathbb{C}$ ,

$$\begin{aligned} |f_v(z)| &\lesssim \sum_{\gamma \in \tilde{\Gamma}} |v_\gamma| e^{\varphi(z)} \frac{\text{dist}(z, \tilde{\Gamma})}{1 + |z|^{3/2}} \cdot \frac{|z|}{\gamma} \cdot \frac{1 + |\gamma|^{3/2}}{e^{\varphi(\gamma)} |z - \gamma|} \\ &\lesssim e^{\varphi(z)} \|v\|_{\infty, \varphi, \tilde{\Gamma}} \sum_{\gamma \in \tilde{\Gamma}} \frac{\text{dist}(z, \tilde{\Gamma})}{|z - \gamma|} \frac{1 + \gamma^{1/2}}{1 + |z|^{1/2}} \lesssim \|v\|_{\infty, \varphi, \tilde{\Gamma}} e^{\varphi(z)}, \end{aligned}$$

and we deduce that  $\tilde{\Gamma}$  is an interpolating sequence for which we can construct a linear operator of interpolation.

We now show that  $\tilde{\Gamma}$  is a uniqueness sequence. For this, let  $f \in \mathcal{F}_\varphi^\infty$  vanish on  $\tilde{\Gamma}$ . Consider the holomorphic function  $g = f/\tilde{G}$ . Then, again by Lemma 2.5,

$$|g(z)| \lesssim \frac{1 + |z|^{1/2}}{\text{dist}(z, \tilde{\Gamma})}$$

and, by the maximum modulus principle,  $g = 0$ .

As a conclusion the sequence  $\tilde{\Gamma}$  is an interpolating sequence which is also a uniqueness sequence and thus a sampling sequence.  $\square$

*Proof of Theorem 1.2.* Let  $\Lambda = (\lambda_n)_{n \geq 0}$  be a sequence of complex numbers tending to infinity, with  $|\lambda_n| \leq |\lambda_{n+1}|$ . As before, we write  $\lambda_n = \gamma_n e^{\delta_n} e^{i\theta_n}$ ,  $\theta_n \in \mathbb{R}$ ,  $n \geq 0$ . Now let  $\tilde{\Lambda} = \Lambda \cup \{\tilde{\lambda}\}$ . We can also write  $\tilde{\Lambda} = (\lambda_n)_{n \geq -1}$  with  $\lambda_{-1} = \tilde{\lambda}$ , and analogously for  $\tilde{\Gamma}$ .

To prove Theorem 1.2 we need to show that  $\tilde{\Lambda}$  is a complete interpolating sequence for  $\mathcal{F}_\varphi^\infty$  if and only if  $\Lambda$  is complete interpolating for  $\mathcal{F}_\varphi^2$ , that is,  $\Lambda$  satisfies the conditions (a)–(c) of Theorem 1.1.

$\Lambda$  is complete interpolating for  $\mathcal{F}_\varphi^2 \implies \tilde{\Lambda}$  is a complete interpolating sequence for  $\mathcal{F}_\varphi^\infty$ .

By Theorem 1.1,  $\Lambda$  satisfies the conditions (a)–(c). Hence, the infinite product  $\tilde{F}(z) = \prod_{\lambda \in \tilde{\Lambda}} (1 - z/\lambda)$  converges uniformly on compact sets and, by (12), there exists  $\eta > 0$  such that

$$e^{\varphi(z)} \frac{\text{dist}(z, \tilde{\Lambda})}{(1 + |z|)^{1-\eta}} \lesssim |\tilde{F}(z)| \lesssim e^{\varphi(z)} \frac{\text{dist}(z, \tilde{\Lambda})}{(1 + |z|)^\eta}, \quad z \in \mathbb{C}.$$

Thus,  $\tilde{F} \notin \mathcal{F}_\varphi^\infty$ , while  $\tilde{F}/(\cdot - \lambda) \in \mathcal{F}_\varphi^\infty$ ,  $\lambda \in \tilde{\Lambda}$ . Also, if  $\tilde{F}g \in \mathcal{F}_\varphi^\infty$ , then  $|g(z)| \lesssim |z|^{1-\eta}/\text{dist}(z, \tilde{\Lambda})$ , whence  $g \equiv 0$ . Thus,  $\tilde{\Lambda}$  is a uniqueness set for  $\mathcal{F}_\varphi^\infty$ .

It remains to show that  $\tilde{\Lambda}$  is an interpolating sequence for  $\mathcal{F}_\varphi^\infty$ . Let  $(v_n)_{n \geq 0}$  be a finite sequence and put

$$f_v(z) = \sum_n v_n \frac{\tilde{F}(z)}{\tilde{F}'(\lambda_n)(z - \lambda_n)}.$$

Since we already know that  $\tilde{\Gamma}$  is a sampling sequence for  $\mathcal{F}_\varphi^\infty$ , we have

$$\begin{aligned} \|f_v\|_{\varphi, \infty} &\asymp \sup_{\gamma_m \in \tilde{\Gamma}} e^{-\varphi(\gamma_m)} \left| \sum_n v_n \frac{\tilde{F}(\gamma_m)}{\tilde{F}'(\lambda_n)(\gamma_m - \lambda_n)} \right| \\ &= \sup_{\gamma_m \in \tilde{\Gamma}} \left| \sum_n v_n e^{-\varphi(\lambda_n)} B_{n,m} \right|, \end{aligned} \quad (22)$$

where

$$B_{n,m} = e^{\varphi(\lambda_n) - \varphi(\gamma_m)} \frac{\tilde{F}(\gamma_m)}{\tilde{F}'(\lambda_n)(\gamma_m - \lambda_n)}.$$

Note that  $|\tilde{F}(\gamma_m)| \asymp \gamma_m |F(\gamma_m)|$  and  $|\tilde{F}'(\lambda_n)| \asymp |\lambda_n F'(\lambda_n)|$ , and so

$$|B_{n,m}| = e^{\varphi(\lambda_n) - \varphi(\gamma_m)} |A_{n,m}| \cdot \frac{\gamma_m}{|\lambda_n|} \cdot \frac{\|\mathbf{k}_{\gamma_m}\|_{\varphi, 2}}{\|\mathbf{k}_{\lambda_n}\|_{\varphi, 2}} \asymp |A_{n,m}|,$$

where  $A_{n,m}$  are defined in the previous section. By (17) and (c),  $|A_{n,m}| \lesssim \exp(-\delta|n - m|)$  for some  $\delta > 0$ . Hence,

$$\|f_v\|_{\varphi, \infty} \leq \|v\|_{\varphi, \infty, \tilde{\Lambda}} \sup_{\gamma_m \in \tilde{\Gamma}} \sum_{\lambda_n \in \tilde{\Lambda}} |B_{n,m}| \lesssim \|v\|_{\varphi, \infty, \tilde{\Lambda}}.$$

By a normal family argument, we can find an interpolating function  $f_v$  for any sequence  $v$  with  $\|v\|_{\varphi, \infty, \tilde{\Lambda}} < \infty$ .

$\tilde{\Lambda}$  is a complete interpolating sequence for  $\mathcal{F}_\varphi^\infty \implies \Lambda$  is complete interpolating for  $\mathcal{F}_\varphi^2$ .

We need to show that the sequence  $\Lambda$  satisfies the conditions (a)–(c). By Corollary 2.3,  $\Lambda$  is  $d_\rho$ -separated. Also, if the sequence  $(\delta_n)$  is unbounded, then, as in the proof of necessity part of Theorem 1.1, there exists a subsequence  $|A_{n_k, m_k}| \rightarrow \infty$ . Since  $|A_{n_k, m_k}| \asymp |B_{n_k, m_k}|$ , it



follows from (22) that the interpolation operator is unbounded (choose the data  $v_\lambda = e^{\varphi(\lambda)}$  for  $\lambda = \lambda_{n_k}$  and  $v_\lambda = 0$  otherwise).

It remains to prove (c). As in Section 3 define  $\varepsilon_N$  by (19). If the sequence  $N\varepsilon_N$  is unbounded then again there exists a subsequence  $|A_{n_k, m_k}| \rightarrow \infty$ . Finally, if the sequence  $N\varepsilon_N$  is bounded, then, analogously to the proof of (c) in Section 3 we can show that there exist arbitrary large  $N$ ,  $m_N$  and  $n_N$  such that for  $1 \leq K \leq N$  we have  $|B_{n_N+K, m_N}| \asymp |A_{n_N+K, m_N}| \gtrsim 1$ . It is clear from (22) that in this case the interpolation operator is unbounded (choose the data  $v_n e^{-\varphi(\lambda_n)} \in \ell^\infty$  such that  $v_n e^{-\varphi(\lambda_n)} B_{n, m_N} \asymp |B_{n, m_N}|$  for  $n_N + 1 \leq n \leq n_N + N$ ).  $\square$

## 5. PROOF OF THE DENSITY RESULTS

**5.1. Sufficient conditions.** First we deduce the sufficient conditions of Theorems 1.4–1.6 from Theorems 1.1 and 1.2.

These conditions follow immediately from Theorem 1.3 which we recall and prove here.

**Theorem.** *Let  $\varphi(r) = \alpha(\log^+ r)^2$ , let  $p = 2, \infty$ , and let  $\Lambda$  be a  $d_\rho$ -separated sequence. Then*

- (i) *if  $D^+(\Lambda) < 2\alpha$ , then  $\Lambda$  is a subset of some complete interpolating sequence in  $\mathcal{F}_\varphi^p$ ;*
- (ii) *if  $D^-(\Lambda) > 2\alpha$ , then  $\Lambda$  contains a complete interpolating sequence in  $\mathcal{F}_\varphi^p$ .*

We give a proof for the case  $p = 2$ ; the proof for the case  $p = \infty$  is completely analogous. To simplify the notations, we choose  $\alpha = 1/2$ . Recall that with our choice of  $\alpha$  the set  $\Gamma = \{\gamma_n\} = \{e^n\}_{n \in \mathbb{N}}$  becomes a complete interpolating sequence for the space.

*Proof of (i).* It follows from the condition  $D^+(\Lambda) < 1$  that for sufficiently large  $M > 0$ , every annulus

$$A_m = \{z : e^{Mm+\frac{1}{2}} < |z| < e^{M(m+1)+\frac{1}{2}}\}, \quad m \geq 0,$$

contains at most  $M - 1$  points from  $\Lambda$ . Fix such  $M$ . Furthermore, there exists an  $\eta > 0$  such that each  $A_m$  contains an annulus  $B_m$  of width  $\eta$  which contains no points of  $\Lambda$ .

Our goal is to add some sequence  $\Lambda'$  to  $\Lambda$  so that the new sequence  $\Lambda \cup \Lambda'$  could be written as  $\gamma_n e^{\delta_n} e^{i\theta_n}$  and for some  $N$  we would have

$$\sup_n \frac{1}{N} \left| \sum_{k=n}^{n+N} \delta_k \right| \leq \delta < \frac{1}{4\alpha} = \frac{1}{2}. \quad (23)$$

Let us denote the points from  $\Lambda \cap A_m$  by  $\lambda_1^m, \dots, \lambda_{l_m}^m$  (we of course assume that  $\lambda_l$  are ordered so that the modulus is nondecreasing), and let us associate with each of them some point from  $\Gamma \cap A_m$ . E.g., let us write

$$\lambda_l^m = e^{Mm+l} e^{\delta_{Mm+l}} e^{i\theta_{Mm+l}}, \quad 1 \leq l \leq l_m.$$

In each annulus  $A_m$  we still have at least one point from  $\Gamma \cap A_m$  to which nothing is associated.

We now take a large number  $N$  (the choice will be specified later) and consider the groups of the annuli  $A_m$ , namely put

$$\tilde{A}_k = \bigcup_{m=kN+1}^{kN+N} A_m, \quad k \geq 0.$$

Now in the whole group of annuli  $\tilde{A}_k$  there are at least  $N$  free points of  $\Gamma$  to which we need to assign some element of the sequence  $\Lambda'$  that we want to construct. We will do this in such a way that for any  $k$  we have

$$\left| \sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n \right| \leq CM, \quad (24)$$

for some absolute constant  $C$  whence for sufficiently large  $N$ , (23) will be satisfied. Thus, from now on,  $k$  will be fixed.

We use an idea from the paper [24] by Seip. The points of  $\Lambda' \cap \tilde{A}_m$  will be chosen within the annuli  $B_m$  (of the width  $\eta$ ). Note that we can even put all missing points in one annulus  $B_m$ , if we want, and still have  $\rho$ -separation, but, of course, the separation constant will depend on  $\eta$ ,  $M$  and  $N$  and may be rather small. Let us consider all possible sequences  $\Lambda' \subset \bigcup_{m=kN+1}^{kN+N} B_m$  with separation constants uniformly bounded away from zero, and let us write the elements of  $\Lambda \cup \Lambda'$  as  $\gamma_n e^{\delta_n} e^{i\theta_n}$ . Note that for any  $m$  and  $n = Mm+1, \dots, Mm+l_m$  the values  $\delta_n$  are already fixed. Moreover, since for these  $n$  the corresponding  $\lambda$ -s are in the same annulus  $A_m$  we have  $|\delta_n| \leq M$ , whence

$$-M^2 N \leq \sum_{m=kN+1}^{kN+N} \sum_{n=Mm+1}^{Mm+l_m} \delta_n \leq M^2 N.$$

Now assume that we chose all the points of  $\Lambda'$  in the annulus  $B_{kN+1}$  (the smallest of all  $B_m$  in our group). Then for

$$kNM + jM + l_{kN+j} + 1 < n \leq kNM + (j+1)M, \quad 2 \leq j \leq N-1,$$

we have

$$\delta_n \leq -(j-1)M,$$

whence (using the fact that we have at least  $N$  free indices in each  $A_m$ )

$$\sum_{j=2}^{N-1} \sum_{n=kNM+jM+l_{kN+j}+1}^{kNM+(j+1)M} \delta_n \leq - \sum_{j=2}^{N-1} (j-1)MN \leq -\frac{MN^2}{3},$$

when  $N$  is sufficiently large. Thus, with this choice of  $\Lambda'$  we have

$$\sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n \leq -\frac{MN^2}{3} + O(M^2N) < 0,$$

if  $N \gg M$ .

Analogously, if we choose all the points of  $\Lambda'$  in the annulus  $B_{kN+N}$  (the largest of all  $B_m$  in our group), we will have  $\delta_n \geq (N - 2 - j)M$  for  $kNM + jM + l_{kN+j} + 1 < n \leq kNM + (j + 1)M$ ,  $0 \leq j \leq N - 3$ , whence

$$\sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n \geq \frac{MN^2}{3} - O(M^2N) > 0.$$

Finally, note that if two choices of  $\Lambda'$  coincide up to one point which is in some  $B_m$  for one choice and which is in  $B_{m+1}$  for the other choice, then the corresponding sums

$$\sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n$$

considered for these two choices of  $\Lambda'$  will differ by at most  $2M$ . Since the two configurations of  $\Lambda'$  described above may be obtained from the other by changing only one point and moving it to a neighboring annulus  $B_n$ , we conclude that there exists some intermediate choice of  $\Lambda'$  with the property (24) (with  $C = 2$ ).  $\square$

*Proof of (ii).* The idea is the same and so we may omit some details. Let  $M, N$  be as above, but now we assume that each  $A_m$  contains at least  $M + 1$  points for some fixed  $M$ . Let us assume that  $N \gg M$  and choose  $j_0 \in \mathbb{N}$  so that  $3j_0M < N \leq 3(j_0 + 1)M$ .

For  $j_0 \leq j \leq N - j_0$  and  $kNM + jM + 1 \leq n \leq kNM + (j + 1)M$  we choose in an arbitrary way  $\lambda_n \in \Lambda \cap A_j$  and write them as

$$\gamma_n e^{\delta_n} e^{i\theta_n}.$$

Then  $|\delta_n| \leq M$  and

$$-(N - 2)M^2 \leq \sum_{j=j_0}^{N-j_0} \sum_{n=kNM+jM+1}^{kNM+(j+1)M} \delta_n \leq (N - 2)M^2.$$

Note that we did not assign any point from  $\Lambda$  to  $n$ -s in the first and in the last interval, namely, for  $kNM + 1 \leq n \leq kNM + j_0M$  and for  $kNM + (N - j_0)M + 1 \leq n \leq kNM + NM$ .

Recall that we still have  $N$  free points of  $\Lambda$  in each  $A_m$ . Now consider two choices of  $\lambda_n$  for these values of  $n$ . For the first choice let us assign some points  $\lambda_n \in \Lambda \cap A_j$  to  $kNM + jM + 1 \leq n \leq kNM + (j + 1)M$  and  $N - j_0 \leq j \leq N - 1$ . However, for  $kNM + 1 \leq n \leq kNM + j_0M$  let us choose  $j_0M$  points  $\lambda_n$  in  $\cup_{j>2N/3} A_j$ . This is possible, since we have at least  $N/3 > j_0M$  free points from  $\Lambda$  in  $\cup_{j>2N/3} A_j$ . Then, for  $kNM + 1 \leq n \leq kNM + j_0M$ , we have

$$\delta_n \leq -\frac{MN}{3},$$

and hence,

$$\begin{aligned} \sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n &= \sum_{j=0}^{j_0-1} \sum_{n=kNM+jM+1}^{kNM+(j+1)M} \delta_n + \sum_{j=j_0}^{N-1} \sum_{n=kNM+jM+1}^{kNM+(j+1)M} \delta_n \\ &\leq -\frac{j_0 M^2 N}{3} + O(M^2 N) < 0. \end{aligned}$$

Analogously, choosing the points  $\lambda_n \in \Lambda \cap A_j$  for  $kNM+jM+1 \leq n \leq kNM+(j+1)M$  and  $0 \leq j \leq j_0-1$ , and taking  $\lambda_n$  in  $\cup_{j < N/3} A_j$  for  $kNM+(N-j_0)M+1 \leq n \leq kNM+NM$ , we see that the corresponding sum of  $\delta_n$  is positive.

The proof is completed as in (i): each configuration  $\Lambda_n$  may be obtained from the other by changing exactly on point at each step, and, moreover, these points can be chosen at the distance (with respect to the logarithm) at most  $2M$ . Thus, the corresponding sum will be at most  $4M$  for some choice of  $\{\lambda_n\} \subset \Lambda$ .  $\square$

**5.2. Necessary conditions for sampling/interpolation,  $p = 2, \infty$ , Theorems 1.4, 1.5 and 1.6.** To obtain the necessary conditions for the sequence to be sampling/interpolating, we use the technique developed by Ramanathan and Steger [21, 17]. We follow the scheme of proof proposed in [17, Lemma 40] and concentrate mainly on the places where the proofs differ.

**Lemma 5.1.** *Let  $\varepsilon > 0$ . Assume  $\Lambda$  to be interpolating for  $\mathcal{F}_{(1-\varepsilon)\varphi}^p$ ,  $p = 2, \infty$ , and  $\mathcal{S}$  to be sampling for  $\mathcal{F}_\varphi^2$  and  $d_\rho$ -separated. Then for small  $\delta > 0$  we have for sufficiently big  $R$ ,*

$$(1 - \delta^2) \text{Card}(\Lambda \cap \mathcal{A}(x, Rx)) \leq \text{Card}(\mathcal{S} \cap \mathcal{A}(\delta x, Rx/\delta)).$$

*Proof.* Let  $p = 2$ . Since  $\Lambda$  is interpolating for  $\mathcal{F}_{(1-\varepsilon)\varphi}^2$ , for every  $\lambda \in \Lambda$  there exists  $f_\lambda \in \mathcal{F}_{(1-\varepsilon)\varphi}^2$ , such that  $f_\lambda(\lambda) = 1$ ,  $f_\lambda|_{\Lambda \setminus \{\lambda\}} = 0$  and  $\|f_\lambda\|_{(1-\varepsilon)\varphi, 2} \lesssim |\lambda| e^{-(1-\varepsilon)\varphi(\lambda)}$ . By (2),

$$|f_\lambda(z)| = |\langle f_\lambda, \mathbf{k}_z \rangle| \lesssim e^{(1-\varepsilon)(\varphi(z) - \varphi(\lambda))} |\lambda| / (1 + |z|). \quad (25)$$

Let  $G$  be the function from Lemma 2.5 associated to  $\Gamma := \Gamma_{\varepsilon\alpha} = \{e^{\frac{n+1}{2\varepsilon\alpha}} e^{i\theta_n} : n \geq 0\}$ , and let  $\gamma_\lambda \in \Gamma$  be a point such that  $\text{dist}(\lambda, \Gamma) = |\lambda - \gamma_\lambda|$ . Then  $|\lambda| \asymp \gamma_\lambda$ . With an appropriate choice of  $\theta_n$  we can assume that  $d_\rho(\Lambda, \Gamma) > 0$ . Define

$$\kappa(z, \lambda) := \begin{cases} f_\lambda(z) \frac{G(z)}{z - \gamma_\lambda} \frac{\lambda - \gamma_\lambda}{G(\lambda)} \frac{z}{\lambda} \|\mathbf{k}_\lambda\|_{\varphi, 2} & \text{if } z \in \mathbb{C} \setminus \{\gamma_\lambda\}, \\ G'(\lambda) \frac{\lambda - \gamma_\lambda}{G(\lambda)} \|\mathbf{k}_\lambda\|_{\varphi, 2} & \text{if } z = \gamma_\lambda. \end{cases}$$

By construction,  $\kappa(\cdot, \lambda) \in \mathcal{F}_\varphi^2$ ,  $\lambda \in \Lambda$ , and the system  $\{\kappa(\cdot, \lambda)\}_{\lambda \in \Lambda}$  is biorthogonal to  $\{\mathbf{k}_\lambda\}_{\lambda \in \Lambda}$ . Moreover  $\|\kappa(\cdot, \lambda)\|_{\varphi, 2}$  is uniformly bounded. To verify this, it suffices to estimate this norm on the Borichev–Lyubarskii sampling sequence  $\Gamma_\alpha$  with  $\text{dist}(\gamma, \Gamma_{\varepsilon\alpha}) \asymp |\gamma|$ ,  $\gamma \in$

$\Gamma_\alpha$ , using Lemma 2.5 and (25):

$$\begin{aligned}
\|\kappa(\cdot, \lambda)\|_{\varphi, 2}^2 &\asymp \sum_{\gamma \in \Gamma_\alpha} |\kappa(\gamma, \lambda)|^2 e^{-2\varphi(\gamma)} (1 + |\gamma|^2) \\
&\asymp \sum_{\gamma \in \Gamma_\alpha} |f_\lambda(\gamma)|^2 \frac{e^{2\varepsilon\varphi(\gamma)} \text{dist}^2(\gamma, \Gamma_{\varepsilon\alpha}) |\lambda - \gamma\lambda|^2 (1 + |\lambda|^3)}{(1 + |\gamma|^3) |\gamma - \gamma\lambda|^2 e^{2\varepsilon\varphi(\lambda)} \text{dist}^2(\lambda, \Gamma_{\varepsilon\alpha})} \times \left| \frac{\gamma}{\lambda} \right|^2 \frac{e^{2\varphi(\lambda)}}{1 + |\lambda|^2} e^{-2\varphi(\gamma)} (1 + |\gamma|^2) \\
&\lesssim \sum_{\gamma \in \Gamma_\alpha} \frac{\text{dist}^2(\gamma, \Gamma_{\varepsilon\alpha})}{|\gamma - \gamma\lambda|^2} \frac{1 + |\lambda|}{1 + |\gamma|} \lesssim \sum_{\gamma \in \Gamma_\alpha, |\gamma| \leq |\lambda|} \frac{1 + |\gamma|}{1 + |\lambda|} + \sum_{\gamma \in \Gamma_\alpha, |\gamma| > |\lambda|} \frac{1 + |\lambda|}{1 + |\gamma|}. \tag{26}
\end{aligned}$$

Both sums are majorized by the sum of a geometric progression, and, hence, are uniformly bounded.

Let now  $\{\tilde{k}(\cdot, s)_{s \in \mathcal{S}}\}$  be the dual frame (see, for example, [9]) for  $\{\mathbf{k}_s / \|\mathbf{k}_s\|_{\varphi, 2}\}_{s \in \mathcal{S}}$  in  $\mathcal{F}_\varphi^2$ . Consider the following finite dimensional subspaces of  $\mathcal{F}_\varphi^2$ :

$$W_{\mathcal{S}} = \{\tilde{k}(\cdot, s) : s \in \mathcal{S} \cap \mathcal{A}(\delta x, Rx/\delta)\} \quad \text{and} \quad W_\Lambda = \{\kappa(\cdot, \lambda) : \lambda \in \Lambda \cap \mathcal{A}(x, Rx)\}.$$

We define  $P_{\mathcal{S}}$  and  $P_\Lambda$  as the orthogonal projections of  $\mathcal{F}_\varphi^2$  onto  $W_{\mathcal{S}}$  and  $W_\Lambda$ , respectively. Consider the operator  $T = P_\Lambda P_{\mathcal{S}}$  defined from  $W_\Lambda$  to  $W_\Lambda$ . Clearly

$$\text{tr}(T) \leq \text{rank} P_{\mathcal{S}} \leq \text{Card}(\mathcal{S} \cap \mathcal{A}(\delta x, Rx/\delta)).$$

Since  $(P_\Lambda \mathbf{k}_\lambda)_\lambda$  is biorthogonal to  $(\kappa(\cdot, \mu))_{\mu \in \Lambda \cap \mathcal{A}(x, Rx)}$  in  $W_\Lambda$ , we have

$$\text{tr}(T) = \sum_{\lambda \in \Lambda \cap \mathcal{A}(x, Rx)} \langle T\kappa(\cdot, \lambda), P_\Lambda \mathbf{k}_\lambda \rangle.$$

Hence, since  $\langle T\kappa(\cdot, \lambda), P_\Lambda \mathbf{k}_\lambda \rangle = \langle \kappa(\cdot, \lambda) + (P_{\mathcal{S}} - Id)\kappa(\cdot, \lambda), P_\Lambda \mathbf{k}_\lambda \rangle$ , we also get

$$\text{tr}(T) \geq \text{Card}(\Lambda \cap \mathcal{A}(x, Rx)) [1 - \sup_\lambda \|P_{\mathcal{S}}(\kappa(\cdot, \lambda)) - \kappa(\cdot, \lambda)\|_{\varphi, 2}]$$

It remains to verify that  $\|P_{\mathcal{S}}(\kappa(\cdot, \lambda)) - \kappa(\cdot, \lambda)\|_{\varphi, 2}$  are small for sufficiently small  $\delta$  independently of  $\lambda$ . By Lemma 2.5 we have

$$\frac{|\kappa(s, \lambda)|^2}{\|\mathbf{k}_s\|_{\varphi, 2}^2} \lesssim \frac{|\lambda|}{1 + |s|} \frac{\text{dist}(s, \Gamma_{\varepsilon\alpha})^2}{|s - \gamma\lambda|^2}. \tag{27}$$

Since  $\mathcal{S}$  is sampling and  $d_\rho$ -separated, and since  $\lambda \in \mathcal{A}(x, Rx)$ , we have, using also  $|\lambda| \asymp |\gamma\lambda|$ ,

$$\begin{aligned}
\|P_{\mathcal{S}}(\kappa(\cdot, \lambda)) - \kappa(\cdot, \lambda)\|_{\varphi, 2} &\lesssim \sum_{s \notin \mathcal{A}(\delta x, Rx/\delta)} \frac{|\kappa(s, \lambda)|^2}{\|\mathbf{k}_s\|_{\varphi, 2}^2} \\
&\lesssim \sum_{s \notin \mathcal{A}(\delta x, Rx/\delta)} \frac{|\gamma\lambda|}{1 + |s|} \frac{\text{dist}(s, \Gamma_{\varepsilon\alpha})^2}{|s - \gamma\lambda|^2} \\
&\lesssim \sum_{|s| \leq |\gamma\lambda|, s \notin \mathcal{A}(\delta x, Rx/\delta)} \frac{|s|}{|\gamma\lambda|} + \sum_{|s| \geq |\gamma\lambda|, s \notin \mathcal{A}(\delta x, Rx/\delta)} \frac{|\gamma\lambda|}{|s|}.
\end{aligned}$$

For  $R$  big enough, we can suppose  $\gamma_\lambda \in \mathcal{A}(x, Rx)$ . Since  $\mathcal{S}$  is separated, each annulus  $\mathcal{A}_k := \mathcal{A}(2^k, 2^{k+1})$  contains a uniformly bounded number of points of  $\mathcal{S}$ . Let  $M$  be an upper bound of these numbers. Let  $N, K$  be such that  $2^N \leq \delta x \leq 2^{N+1}$  and  $2^K \leq Rx/\delta \leq 2^{K+1}$ . Then an estimate analogous to the above yields:

$$\begin{aligned} \sum_{|s| \leq |\gamma_\lambda|, s \notin \mathcal{A}(\delta x, Rx/\delta)} \frac{|s|}{|\gamma_\lambda|} &\lesssim \frac{1}{|\gamma_\lambda|} \sum_{k=1}^N \sum_{s \in \mathcal{A}_k} |s| \lesssim \frac{M2^N}{|\gamma_\lambda|} \lesssim \delta, \\ \sum_{|s| \geq |\gamma_\lambda|, s \notin \mathcal{A}(\delta x, Rx/\delta)} \frac{|\gamma_\lambda|}{|s|} &\lesssim |\gamma_\lambda| \sum_{k \geq K} \sum_{s \in \mathcal{A}_k} \frac{1}{|s|} \lesssim \frac{M|\gamma_\lambda|}{2^K} \lesssim \delta, \end{aligned}$$

and we are done.

Now let  $p = \infty$ . Since  $\Lambda$  is interpolating for  $\mathcal{F}_{(1-\varepsilon)\varphi}^\infty$ , then for every  $\lambda \in \Lambda$ , there exists  $f_\lambda \in \mathcal{F}_{(1-\varepsilon)\varphi}^\infty$  such that  $f_\lambda(\lambda) = 1$ ,  $f_\lambda|\_{\Lambda \setminus \{\lambda\}} = 0$  and  $\|f_\lambda\|_{(1-\varepsilon)\varphi, \infty} \lesssim e^{-(1-\varepsilon)\varphi(\lambda)}$ , i.e.,  $|f_\lambda(z)| \lesssim e^{(1-\varepsilon)(\varphi(z) - \varphi(\lambda))}$ . Set

$$\kappa(z, \lambda) = f_\lambda(z) \frac{G(z)}{z - \gamma_\lambda} \frac{\lambda - \gamma_\lambda}{G(\lambda)} \|\mathbf{k}_\lambda\|_{\varphi, 2}, \quad z \in \mathbb{C}.$$

(Observe that we do not need the factor  $z/\lambda$  in this case.) As in (26) it can be shown that  $\kappa(\cdot, \lambda)$  is in  $\mathcal{F}_\varphi^2$  with uniformly bounded norms. Again by Lemma 2.5 we have (27) and applying the same arguments as above (note that  $\mathcal{S}$  is again assumed sampling in  $\mathcal{F}_\varphi^2$ ), we get our result.  $\square$

- Let us first deduce from this lemma the necessary condition for sampling in Theorem 1.5 ( $p = 2$ ). Suppose that  $\Lambda$  is a sampling sequence for  $\mathcal{F}_\varphi^2$ . By Corollary 2.2 and Lemma 2.6,  $\Lambda$  is a finite union of  $d_\rho$ -separated subsets and contains a  $d_\rho$ -separated subset  $\Lambda^*$  which is also a sampling set for  $\mathcal{F}_\varphi^2$ . Let  $\Gamma = \Gamma_{(1-\varepsilon)\alpha} = \{e^{\frac{n+1}{2(1-\varepsilon)\alpha}} e^{i\theta_n} : n \geq 0\}$  for some  $\varepsilon > 0$ . We have  $D^-(\Gamma) = D^+(\Gamma) = 2(1-\varepsilon)\alpha$ . By [7, Theorem 2.8],  $\Gamma$  is an interpolating sequence for  $\mathcal{F}_{(1-\varepsilon)\varphi}^2$  and the comparison Lemma 5.1 gives  $D^-(\Lambda^*) \geq 2\alpha$ .

- The necessary condition for sampling in Theorem 1.4 ( $p = \infty$ ) follows from Corollary 2.2, Lemma 2.7, and the necessary condition for sampling in Theorem 1.5.

- Next we consider the necessary condition for interpolation in Theorem 1.6. Consider  $\Gamma = \Gamma_{(1+\varepsilon)\alpha} = \{e^{\frac{n+1}{2(1+\varepsilon)\alpha}} e^{i\theta_n} : n \geq 0\}$  for some  $\varepsilon > 0$ . We have  $D^-(\Gamma) = D^+(\Gamma) = 2(1+\varepsilon)\alpha$  and by [7, Theorem 2.8],  $\Gamma$  is a sampling sequence for  $\mathcal{F}_{(1+\varepsilon)\varphi}^2$ . If  $\Lambda$  is an interpolating sequence for  $\mathcal{F}_\varphi^p$ ,  $p = 2, \infty$ , then by Lemma 5.1, comparing the densities between interpolating and sampling sequences, we obtain  $D^+(\Lambda) \leq 2\alpha$ .

## 6. EXAMPLES AND PROBLEMS

**6.1. Examples.** In this section we give explicit examples of  $d_\rho$ -separated sequences of critical density which are uniqueness sets for  $\mathcal{F}_\varphi^p$  but are neither sampling nor interpolating.

We will also show that it is not possible to switch from a sampling sequence to an interpolating sequence by removing a point without one being already complete interpolating.

For the first of these examples we will need a two-sided version of the function from Lemma 2.5, which we estimate using essentially the same argument. As before,  $\varphi(z) = a(\log^+ |z|)^2$ ,  $a > 0$ .

**Lemma 6.1.** *Let  $a > 0$  and consider  $\Gamma^\pm = \{-e^{\frac{n}{a}}e^{i\theta_n}, e^{\frac{n}{a}}e^{i\theta_n}\}_{n \geq 1} =: \{\gamma_n\}_{n \in \mathbb{Z} \setminus \{0\}}$ , where  $\theta_n$  are arbitrary real numbers. Then the infinite product*

$$E(z) = \prod_{\gamma \in \Gamma^\pm} \left(1 - \frac{z}{\gamma}\right)$$

converges on every compact subset of  $\mathbb{C}$ ,

$$|E(z)| \asymp e^{\varphi(z)} \frac{\text{dist}(z, \Gamma^\pm)}{1 + |z|^2}, \quad z \in \mathbb{C}, \quad (28)$$

and

$$|E'(\gamma)| \asymp \frac{e^{\varphi(\gamma)}}{|\gamma|^2}, \quad \gamma \in \Gamma^\pm.$$

*Proof.* For every  $t > 0$  there exists a unique  $m$  such that

$$\frac{1}{a} \left(m - \frac{1}{2}\right) \leq t < \frac{1}{a} \left(m + \frac{1}{2}\right).$$

We consider  $z$  with  $|z| = e^t$ . Then

$$\begin{aligned} \log |E(z)| &= \sum_{1 \leq |k| \leq m} \log \left|1 - \frac{z}{\gamma_k}\right| + O(1) \\ &= \sum_{1 \leq |k| \leq m-1} \left(t - \frac{|k|}{a}\right) + \log \text{dist}(z, \Gamma) - t + O(1) \\ &= 2(m-1)t - \frac{2(m-1)m}{a} + \log \text{dist}(z, \Gamma^\pm) - t + O(1) \\ &= at^2 - 2t + \log \text{dist}(z, \Gamma^\pm) + O(1), \end{aligned}$$

which gives the first estimate. The second estimate can be immediately deduced from the first.  $\square$

**Example 6.2.** Let  $\Gamma^\pm$  be the sequence from Lemma 6.1. In view of Theorem 1.1, this sequence cannot be complete interpolating for  $\mathcal{F}_\varphi^2$  since the points  $\gamma_n$ ,  $n < 0$ , are too far from the reference sequence. Clearly, the upper and lower densities of this sequence are equal to  $2\alpha$ . Let  $E$  be the function of Lemma 6.1. Hence the functions  $g_n := E/E'(\gamma_n)(\cdot - \gamma_n)$  are in  $\mathcal{F}_\varphi^\infty$  and  $\mathcal{F}_\varphi^2$ .

Let us prove that it is neither sampling nor interpolating for  $\mathcal{F}_\varphi^2$ .

Suppose first that it is sampling. Then for every finite sequence  $v = (v_n)_{n \in \mathbb{Z} \setminus \{0\}} \in \ell_{\varphi, \Gamma^\pm}^2$  the function  $f_v = \sum_n v_n g_n$  would interpolate  $v$  on  $\Gamma^\pm$ , and by the sampling property of  $\Gamma^\pm$ ,

$$\|f_v\|_{\varphi, 2}^2 \asymp \|f_v|_{\Gamma^\pm}\|_{\varphi, 2, \Gamma^\pm}^2 = \sum_n |v_n|^2 e^{-2\varphi(\gamma_n)} (1 + |\gamma_n|^2) = \|v\|_{\varphi, 2, \Gamma^\pm}^2.$$

In other words, the interpolation operator  $v \mapsto f_v$  would be continuous from  $\ell_{\varphi, \Gamma}^2$  to  $\mathcal{F}_\varphi^2$ , and the sequence would be interpolating. Since it was supposed sampling it would thus be complete interpolating, and we would get a contradiction.

Suppose next that it is interpolating. Since the function  $E$  vanishes on  $\Gamma^\pm$  and satisfies (28),  $\Gamma^\pm$  is a uniqueness sequence (see, for example, [10, Theorem 3]). Since it is also interpolating, it is complete interpolating, and again we obtain a contradiction.

**Example 6.3.** Let  $\Gamma_2 := \Gamma^\pm \cup \{-1, 1\} = \{\pm e^{n/\alpha}\}_{n \geq 0} = \{\gamma_n\}_{n \in \mathbb{Z}}$  where  $\Gamma^\pm$  is again the sequence from Lemma 6.1 and  $\gamma_{-n} = -\gamma_n$ . Then  $\Gamma_2$  is neither sampling nor interpolating for  $\mathcal{F}_\varphi^\infty$ . Clearly, by (28),  $\Gamma_2$  is a uniqueness set for  $\mathcal{F}_\varphi^\infty$ .

Set  $E_2(z) = (z^2 - 1)E(z)$ . Then, in view of (28),  $\Gamma_2$  is a uniqueness sequence for  $\mathcal{F}_\varphi^\infty$  and  $g_\gamma = E_2/E_2'(\gamma)(\cdot - \gamma)$  with  $\gamma \in \Gamma_2$  defines a function in  $\mathcal{F}_\varphi^\infty$ . Next, let the family of functions  $(f_n)_{n \geq 0} \in \mathcal{F}_\varphi^\infty$  be given by

$$f_n(z) = \sum_{|\gamma| \leq \gamma_n} \varepsilon_\gamma g_\gamma(z), \quad z \in \mathbb{C},$$

where  $\varepsilon_\gamma := e^{-i \arg g_\gamma(z_n)} e^{\varphi(\gamma)}$  and  $z_n = i\gamma_{n+1}$ . We have

$$f_n(z_n) = \sum_{|\gamma| \leq \gamma_n} \varepsilon_\gamma g_\gamma(z_n) = \sum_{|\gamma| \leq \gamma_n} e^{\varphi(\gamma)} |g_\gamma(z_n)| \asymp e^{\varphi(z_n)} \sum_{|\gamma| \leq \gamma_n} \frac{\text{dist}(z_n, \Gamma)}{|z_n - \gamma|} \asymp n e^{\varphi(z_n)}.$$

As a consequence,  $\|f_n\|_{\varphi, \infty} \rightarrow \infty$ , while  $\|f_n|_{\Gamma_2}\|_{\varphi, \infty, \Gamma_2} = 1$ . Hence the sequence is not sampling, and since it is uniqueness it can neither be interpolating (otherwise it would be complete interpolating and hence sampling).

Next we give an example of a one-sided sequence of critical density which is neither sampling nor interpolating for  $\mathcal{F}_\varphi^2$ . Let  $0 < \delta < \frac{1}{2a}$  and let

$$\Lambda = \{\lambda_n\}_{n \geq 0} = \{e^{\frac{n+1}{2a} + \delta}\}_{n \geq 0}.$$

Then, by the estimates analogous to Lemma 6.1, the generating function  $F$  of the sequence  $\Lambda$  satisfies

$$|F(z)| \asymp e^{\varphi(z)} \frac{\text{dist}(z, \Lambda)}{1 + |z|^{3/2 + 2a\delta}}, \quad z \in \mathbb{C}. \quad (29)$$

It is clear that  $F \in \mathcal{F}_\varphi^2$  if and only if  $\delta > \frac{1}{4a}$ . Moreover, by Theorem 1.1,  $\Lambda$  is a complete interpolating sequence for  $\mathcal{F}_\varphi^2$  if  $0 < \delta < \frac{1}{4a}$  and  $\Lambda \cup \{1\}$  is a complete interpolating sequence if  $\frac{1}{4a} < \delta < \frac{1}{2a}$ .



**Example 6.4.** If  $\delta = \frac{1}{4a}$ , then  $\Lambda$  is neither sampling nor interpolating for  $\mathcal{F}_\varphi^2$ . First, it follows from (29) that  $\Lambda$  is a uniqueness set for  $\mathcal{F}_\varphi^2$ . Indeed, if  $SF \in \mathcal{F}_\varphi^2$  for an entire function  $S$ , then  $S$  should be constant, but  $F \notin \mathcal{F}_\varphi^2$  (a detailed proof can be found in [10, Theorem 3]). Hence,  $\Lambda$  is not an interpolating sequence (otherwise, it would be complete interpolating which is not true).

Assume now that  $\Lambda$  is sampling for  $\mathcal{F}_\varphi^2$ . Then, by the stability result of Corollary 2.4, the sequence

$$\tilde{\Lambda} = \{e^{\frac{n+1}{2a} + \frac{1}{4a} + \varepsilon}\}_{n \geq 0}$$

also is sampling for sufficiently small  $\varepsilon > 0$ . However, it follows from (29) that  $\tilde{\Lambda}$  is the zero set of some nontrivial function in  $\mathcal{F}_\varphi^2$ , a contradiction.

As mentioned in the introduction we finish this section showing that it is not possible to switch from a sampling sequence to an interpolating sequence by removing a point without one being already complete interpolating.

**Proposition 6.5.** *Suppose  $\Lambda$  is an interpolating sequence for  $\mathcal{F}_\varphi^\infty$  and  $\Lambda \cup \{\lambda\}$  is not interpolating for a  $\lambda \notin \Lambda$ . Then  $\Lambda$  is complete interpolating for  $\mathcal{F}_\varphi^\infty$ .*

*Proof.* The assertion of the theorem is clearly true when  $\Lambda$  is complete interpolating.

Suppose now that  $\Lambda$  is an interpolating sequence which is not a uniqueness set. Then there exists  $h$  vanishing on  $\Lambda$  and such that  $h(\lambda) = 1$  which implies that  $\Lambda \cup \{\lambda\}$  is interpolating.  $\square$

**Lemma 6.6.** *If  $\Lambda$  is sampling for  $\mathcal{F}_\varphi^\infty$  and  $\Lambda \setminus \{\lambda\}$  is a zero set for  $\mathcal{F}_\varphi^\infty$ , then  $\Lambda \setminus \{\lambda\}$  is interpolating.*

*Proof.* By assumption there exists a function  $f \in \mathcal{F}_\varphi^\infty$  vanishing on  $\Lambda \setminus \{\lambda\}$ , and  $f(\lambda) = 1$ . For  $\mu \in \Lambda \setminus \{\lambda\}$  define the entire function

$$g_\mu(z) = \begin{cases} f(z) \frac{z - \lambda}{z - \mu} & \text{if } z \neq \mu \\ f'(\mu)(\mu - \lambda) & \text{if } z = \mu, \end{cases}$$

which is in  $\mathcal{F}_\varphi^\infty$ . Clearly  $g_\mu$  vanishes on  $\{\lambda\} \cup \Lambda \setminus \{\mu\}$ .

Pick now a finite sequence  $(v_\mu)_{\mu \neq \lambda}$  and set

$$f_v(z) = \sum_{\mu \neq \lambda} v_\mu \frac{g_\mu(z)}{g_\mu(\mu)}.$$

By construction  $f_v \in \mathcal{F}_\varphi^\infty$  as a finite sum of functions in  $\mathcal{F}_\varphi^\infty$ , and  $f_v$  interpolates  $v_\mu$  in  $\mu \neq \lambda$ . Let us estimate the norm of  $f_v$ . Observe that  $f_v(\lambda) = 0$  since  $g_\mu(\lambda) = 0$  for every  $\mu \neq \lambda$ . Using the fact that  $\Lambda$  is sampling we have

$$\|f_v\|_{\varphi, \infty} \asymp \sup_{\mu \in \Lambda} |f_v(\mu)| e^{-\varphi(\mu)} = \sup_{\mu \in \Lambda \setminus \{\lambda\}} |f_v(\mu)| e^{-\varphi(\mu)} = \sup_{\mu \in \Lambda \setminus \{\lambda\}} |v_\mu| e^{-\varphi(\mu)} = \|v\|_{\varphi, \infty, \Lambda \setminus \{\lambda\}}.$$

Hence we can define a bounded interpolation operator  $v \mapsto f_v$  and  $\Lambda \setminus \{\lambda\}$  is an interpolating sequence.  $\square$

**Corollary 6.7.** *If  $\Lambda$  is sampling for  $\mathcal{F}_\varphi^\infty$  and  $\Lambda \setminus \{\lambda\}$  is a zero set for  $\mathcal{F}_\varphi^\infty$ , then  $\Lambda$  is complete interpolating.*

*Proof.* We already know from the preceding lemma that  $\Lambda \setminus \{\lambda\}$  is interpolating. Two cases may occur. Either the sequence  $\Lambda$  is interpolating, in which case nothing has to be proved, or  $\Lambda$  is not interpolating. In the latter case,  $\Lambda \setminus \{\lambda\}$  is interpolating and  $\Lambda$  is not, so that from Proposition 6.5 we conclude that  $\Lambda \setminus \{\lambda\}$  is complete interpolating which is impossible since  $\Lambda \setminus \{\lambda\}$  is a zero set.  $\square$

**Corollary 6.8.** *If a sequence  $\Lambda$  is interpolating for  $\mathcal{F}_\varphi^\infty$  and  $\Lambda \cup \{\lambda\}$  is sampling for  $\mathcal{F}_\varphi^\infty$ , then either  $\Lambda$  or  $\Lambda \cup \{\lambda\}$  is complete interpolating.*

The reader might have noticed that these last results work in a quite general setting. In particular, if there are no complete interpolating sequences in such a general space, then it is not possible to switch from an interpolating sequence to a sampling sequence by adding a sole point.

**6.2. Problems.** In Theorems 1.1 and 1.2 we described complete interpolating sequences for  $\mathcal{F}_\varphi^p$ ,  $p = 2, \infty$ . The case of other values of  $p$  remains open. For  $\varphi(r) = \alpha(\log^+ r)^2$ ,  $1 \leq p < \infty$ , let

$$\mathcal{F}_\varphi^p := \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_{\varphi,p}^p := \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} dm(z) < \infty \right\}.$$

The sequence  $\Lambda \subset \mathbb{C}$  is called *sampling* for  $\mathcal{F}_\varphi^p$  if

$$\|f\|_{\varphi,p}^p \asymp \|f\|_{\varphi,p,\Lambda}^p := \sum_{\lambda \in \Lambda} |f(\lambda)|^p e^{-p\alpha(\log^+ |\lambda|)^2} (1 + |\lambda|^2), \quad f \in \mathcal{F}_\varphi^2,$$

and *interpolating* if for every  $v = (v_\lambda)_{\lambda \in \Lambda} \in \ell_{\varphi,\Lambda}^p$ , i.e.,  $\|v\|_{\varphi,p,\Lambda} < \infty$ , there exists  $f \in \mathcal{F}_\varphi^p$  such that  $v = f|_\Lambda$ .

**Problem 1.** *Describe complete interpolating sequences for  $\mathcal{F}_\varphi^p$ ,  $p \neq 2, \infty$ .*

We believe the sequence  $\Gamma = \{e^{\frac{n+1}{2\alpha}}\}_{n \geq 0}$  is still complete and interpolating for  $1 < p < \infty$  and all complete interpolating sequences can be described in terms of their perturbations from the sequence  $\Gamma$ . Namely, one may suggest that  $\Lambda$  is a complete interpolating sequence in  $\mathcal{F}_\varphi^p$  if and only if conditions (a)–(c) from Theorem 1.1 hold with one change :  $1/4\alpha$  in (c) should be replaced by  $1/(2p'\alpha)$ , where  $p' = \max(p, q)$  and  $q$  is the conjugate exponent,  $1/p + 1/q = 1$  (compare with [16, Theorem 2] for the Paley–Wiener space  $PW_\pi^p$ ). However, the case  $p \neq 2$  requires probably new techniques.

As we mentioned in the Introduction, in the case  $\varphi(r) = \alpha(\log^+ r)^\beta$ ,  $1 < \beta < 2$ , there also exist Riesz bases of reproducing kernels [7]. We believe that the methods of our paper (as well as those of [3]) apply to this situation also though the estimates become much more technical.

**Problem 2.** *Describe complete interpolation sequences for  $\mathcal{F}_\varphi^2$  where  $\varphi(r) = \alpha(\log^+ r)^\beta$ .*

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