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# On Powers of Polynomials 

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#### Abstract

The aim of this short note is to show that, under certain conditions, when the coefficients of a power of a polynomial over a field lie in a subfield, then the coefficients of the polynomial itself lie in the subfield.


Let $K$ be a field and $k$ a subfield of $K$. If $P(X) \in K[X], m \in N^{*}$ and $P^{m}(X) \in k[X]$, then in general we cannot say that $P(X) \in k[X]$. For example, if $P(X)=i X \in \mathbb{C}[X]$, then $P^{2}(X)=-X^{2} \in \mathbb{Q}[X]$, or if $P(X)=\sqrt{2}+\frac{1}{\sqrt{2}} X \in \mathbb{R}[X]$, then $P^{2}(X)=2+2 X+\frac{1}{2} X^{2} \in \mathbb{Q}[X]$. In both cases the square of the polynomial $P$ lies in a subfield of a field and the polynomial itself does not lie in this subfield. In this note we will show that this cannot occur if the coefficients of the polynomial $P(X)$ satisfy a certain simple condition. To fix our ideas we will initially suppose that $K=\mathbb{C}$ and $k=\mathbb{Q}$. Of course, if $m=1$ there is nothing to prove, so we will suppose that this is not the case.

Let us write $P(X)=\sum_{i=0}^{n} a_{i} X^{i}$, with the $a_{i} \in \mathbb{C}$. Using the multinomial theorem we obtain

$$
P^{m}(X)=\sum_{k_{0}+k_{1}+\cdots+k_{n}=m}\binom{m}{k_{0}, k_{1}, \ldots, k_{n}} \prod_{0 \leq t \leq n}\left(a_{t} X^{t}\right)^{k_{t}}
$$

where

$$
\binom{m}{k_{0}, k_{1}, \ldots, k_{n}}=\frac{m!}{k_{0}!k_{1}!\cdots k_{n}!}
$$

We will write $Q(X)=P^{m}(X)=\sum_{j=0}^{m n} b_{j} X^{j}$ and at first suppose that $a_{0} \neq 0$. Then the coefficient $b_{s}$ of $X^{s} \in Q(X)$ may be written

$$
b_{s}=\sum a_{0}^{k_{0}} a_{1}^{k_{1}} \cdots a_{n}^{k_{n}}\binom{m}{k_{0}, k_{1}, \ldots, k_{n}}
$$

where

$$
\begin{align*}
k_{1}+2 k_{2}+\cdots+n k_{n} & =s  \tag{1}\\
k_{0}+k_{1}+k_{2}+\cdots+k_{n} & =m . \tag{2}
\end{align*}
$$

If $s=0$, then from the above equations we obtain $k_{0}=m$ and $k_{i}=0$ for $i \neq 0$, thus $b_{0}=a_{0}^{m}$.
If $s=1$, then, from equation (1), $k_{1}=1$ and $k_{i}=0$ if $i>1$. Now, using equation (2), we obtain $k_{0}=m-1$ and so $b_{1}=m a_{0}^{m-1} a_{1}$.

Now let us consider the case $s=2$. From (1) $k_{i}=0$ if $i>2$ and for $\left(k_{1}, k_{2}\right)$ we have $(0,1)$ or $(2,0)$. From (2) the corresponding values of $k_{0}$ are $m-1$ and $m-2$. Hence we have

$$
b_{2}=m a_{0}^{m-1} a_{2}+a_{0}^{m-2} a_{1}^{2}\binom{m}{m-2,2} .
$$

Continuing, we now consider the case $s=3$. From (1) we see that $k_{i}=0$ if $i>3$ and for $\left(k_{1}, k_{2}, k_{3}\right)$ we have $(0,0,1),(1,1,0)$ or $(3,0,0)$. Using (2) we obtain the corresponding values of $k_{0}$, namely $m-1$, $m-2$ or $m-3$. Therefore

$$
b_{3}=m a_{0}^{m-1} a_{3}+a_{0}^{m-2} a_{1} a_{2}\binom{m}{m-2,1,1}+a_{0}^{m-3} a_{1}^{3}\binom{m}{m-3,3} .
$$

We now turn to the general case. One possibility for the $k_{i}$ 's is $k_{0}=m-1, k_{s}=1$ and $k_{i}=0$ for $i \neq 0, s$. This gives us the term $m a_{0}^{m-1} a_{s}$. All other possibilities have $k_{s}=0$ which from (1) and (2) above gives us

$$
\begin{align*}
k_{1}+2 k_{2}+\cdots+(s-1) k_{s-1} & =s  \tag{3}\\
k_{0}+k_{1}+k_{2}+\cdots+k_{s-1} & =m \tag{4}
\end{align*}
$$

We thus obtain

$$
b_{s}=m a_{0}^{m-1} a_{s}+\sum a_{0}^{k_{0}} a_{1}^{k_{1}} \cdots a_{s-1}^{k_{s-1}}\binom{m}{k_{0}, k_{1}, \ldots, k_{s-1}}
$$

where the sum is over the $s$-tuples $\left(k_{0}, k_{1}, \ldots, k_{s-1}\right)$ satisifying equations (3) and (4) above. (In passing, notice that in all the terms in the sum the power of $a_{0}$ is strictly less than $m-1$.)

Lemma 1 If $b_{j} \in \mathbb{Q}$ for all $j$, then $m a_{0}^{m-1} a_{i} \in \mathbb{Q}$ for all $i \geq 1$.
PROOF We use an induction argument. As $b_{1}=m a_{0}^{m-1} a_{1}$, the statement is true for $s=1$. We now suppose that the statement is true up to a given $s$ and consider the case $s+1$. We have

$$
b_{s+1}=m a_{0}^{m-1} a_{s+1}+\sum a_{0}^{k_{0}} a_{1}^{k_{1}} \cdots a_{s}^{k_{s}}\binom{m}{k_{0}, k_{1}, \ldots, k_{s}}
$$

where

$$
\begin{align*}
k_{1}+2 k_{2}+\cdots+s k_{s} & =s+1  \tag{5}\\
k_{0}+k_{1}+k_{2}+\cdots+k_{s} & =m . \tag{6}
\end{align*}
$$

We can write

$$
\begin{aligned}
a_{0}^{k_{0}} a_{1}^{k_{1}} \cdots a_{s}^{k_{s}} & =\frac{a_{0}^{k_{0}}\left(m a_{0}^{m-1} a_{1}\right)^{k_{1}} \cdots\left(m a_{0}^{m-1} a_{s}\right)^{k_{s}}}{m^{k_{1}+\cdots+k_{s}} a_{0}^{(m-1)\left(k_{1}+\cdots+k_{s}\right)}} \\
& =\frac{\left(m a_{0}^{m-1} a_{1}\right)^{k_{1}} \cdots\left(m a_{0}^{m-1} a_{s}\right)^{k_{s}}}{m^{k_{1}+\cdots+k_{s}} a_{0}^{(m-1)\left(k_{1}+\cdots+k_{s}-1\right)-k_{0}}} \\
& =\frac{\left(m a_{0}^{m-1} a_{1}\right)^{k_{1}} \cdots\left(m a_{0}^{m-1} a_{s}\right)^{k_{s}}}{m^{k_{1}+\cdots+k_{s}} a_{0}^{m\left(k_{1}+\cdots+k_{s}-1\right)}} .
\end{aligned}
$$

As $a_{0}^{m}=b_{0} \in \mathbb{Q}$ and $m a_{0}^{m-1} a_{i} \in \mathbb{Q}$, for $i=1, \ldots, s$, by hypothesis, we have $a_{0}^{k_{0}} a_{1}^{k_{1}} \cdots a_{s}^{k_{s}} \in \mathbb{Q}$. However, $b_{s+1} \in \mathbb{Q}$, so $m a_{0}^{m-1} a_{s+1} \in \mathbb{Q}$. This finishes the induction step.

We can now prove our first result on powers of polynomials.
Theorem 1 Let $P(X) \in \mathbb{C}[X]$ and $m \in \mathbb{N}^{*}$ and suppose that $Q(X)=P^{m}(X) \in \mathbb{Q}[X]$. If there is a coefficient $a_{s}$ of $P(X)$ such that $a_{s} \in \mathbb{Q}^{*}$, then $P(X) \in \mathbb{Q}(X]$.

PROOF Let us first suppose that $a_{0} \neq 0$. We will show that $a_{0} \in \mathbb{Q}^{*}$. If $s=0$, then there is nothing to prove, so let us suppose that $s \neq 0$. From what we have just seen, if $P(X)=\sum_{i=0}^{n} a_{i} X^{i}$, then $a_{0}^{m} \in \mathbb{Q}$ and $m a_{0}^{m-1} a_{i} \in \mathbb{Q}$ for $i=1, \ldots, n$. Now

$$
\frac{a_{0}}{m a_{s}}=\frac{a_{0}^{m}}{m a_{0}^{m-1} a_{s}} \in \mathbb{Q} \Longrightarrow a_{0} \in \mathbb{Q}^{*}
$$

because $a_{s} \in \mathbb{Q}^{*}$. Now suppose that $i \neq 0, s$. Then

$$
m a_{0}^{m-1} a_{i} \in \mathbb{Q}, a_{0} \in \mathbb{Q}^{*} \Longrightarrow a_{i} \in \mathbb{Q}
$$

Thus $P(X) \in \mathbb{Q}[X]$.
To finish we consider the case where the coefficient $a_{0}=0$. If $a_{u}$ is the first non-zero coefficient, then we may write $P(X)=X^{u} P_{1}(X)$, with $P_{1}(X)=\sum_{i=0}^{n-u} c_{i} X^{i}$ and $c_{i}=a_{i+u}$. Then $P^{m}(X)=X^{u m} P_{1}^{m}(X)$ and applying the above argument to $P_{1}(X)$ gives us the desired result.

It is not difficult to generalize the result we have proved to a more general situation.
Theorem 2 Let $k$ be a subfield of a field $K$ and $m \in \mathbb{N}^{*}$. If the polynomial $P(X) \in K[X]$ is such that $Q(X)=P^{m}(X) \in k[X]$ and there is a coefficient of $P(X)$, which belongs to $k^{*}$, then $P(X) \in k[X]$.

PROOF If $k$ is of characteristic 0 , or of characteristic $p>0$ and $m$ is not a multiple of $p$, then we may use an argument analogous to that used in the preceding theorem. Suppose now that $k$ has characteristic $p>0$ and $m=p^{\alpha} m^{\prime}$, with $\alpha \geq 1$ and $\left(p, m^{\prime}\right)=1$. First we notice that

$$
P^{m}(X)=\left(P^{p^{\alpha}}(X)\right)^{m^{\prime}}=P^{m^{\prime}}\left(X^{p^{\alpha}}\right) .
$$

Hence $P^{m^{\prime}}\left(X^{p^{\alpha}}\right) \in k[X]$. However, if $P(X)=\sum_{i=0}^{n} a_{i} X^{i}$, then

$$
P\left(X^{p^{\alpha}}\right)=\sum_{i=0}^{n} a_{i} X^{p^{\alpha} i}=\sum_{j=0}^{n p^{\alpha}} c_{j} X^{j}
$$

with

$$
c_{j}=\left\{\begin{array}{ll}
a_{i} & \text { for } j=p^{\alpha} i, i=0, \ldots, n \\
0 & \text { otherwise }
\end{array} .\right.
$$

If we write $P_{1}(X)=\sum_{j=0}^{n p^{\alpha}} c_{j} X^{j}$, then $P_{1}^{m^{\prime}}(X) \in k[X]$ and one of the coefficients $c_{j}$ belongs to $k^{*}$; also, $m^{\prime}$ is not a multiple of $p$. We thus deduce that $P_{1}(X) \in k[X]$. This completes the proof.

Corollary 1 Let $k$ be a subfield of a field $K$ and $m \in \mathbb{N}^{*}$. If the polynomial $P(X) \in K[X]$ is monic and such that $Q(X)=P^{m}(X) \in k[X]$, then $P(X) \in k[X]$.

We might be tempted to generalize Corollary 1 to products of distinct polynomials, i.e., if $P_{1}(X), \ldots, P_{s}(X) \in$ $\mathbb{C}[X]$, each having a coefficient in $\mathbb{Q}$, and $P_{1}(X) \cdots P_{s}(X) \in \mathbb{Q}[X]$, then $P_{i}(X) \in \mathbb{Q}[X]$ for all $i$. However, it is easy to find a counterexample; for instance, if $P_{1}(X)=-i+X$ and $P_{2}(X)=i+X$, then $P_{1}(X) P_{2}(X)=1+X^{2} \in \mathbb{Z}[X] \subset \mathbb{Q}[X]$.

