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# Long-range dependence and the rank of decompositions

Céline Lévy-Leduc and Murad S. Taqqu

**ABSTRACT.** We review and compare different methodologies for studying the asymptotic behavior of partial sums of nonlinear functionals of the following type  $\sum_{i=1}^N h(X_i)$  in the long-range dependence setting. Here  $(X_i)_{i \geq 1}$  is either a stationary mean-zero Gaussian process or a linear process. The methodologies, we consider, are based on different decompositions of the function  $h$ . This includes the decomposition of [Sur82] and of [HH97] in the case of linear processes. The so-called “rank” of these decompositions plays an essential role. We show that all ranks coincide when the function  $h$  is a polynomial.

## 1. Introduction

We focus here on long-range dependence and on its impact on central, or more precisely, non-central limit theorems. Long-range dependence, also called “long-memory” or “strong dependence”, occurs in a stationary time series when the covariances of that series tend to zero like a power function but so slowly that their sums diverge. Such a behavior is often observed in economics, telecommunications and hydrology and was of great interest to Benoît Mandelbrot. Many of his articles on the subject have been collected in his book [Man02].

The notion of long-range dependence is closely related to self-similarity. Self-similarity refers to invariance in distribution under a suitable change of scale. More precisely, the process  $(Z(t), t \geq 0)$  is self-similar with parameter  $H$  if  $(Z(at), t \geq 0)$  has the same finite-dimensional distributions as  $(a^H Z(t), t \geq 0)$ , for all non negative  $a$ . For instance, *Brownian motion* is self-similar with parameter  $H = 1/2$ . In such an example the increments  $Z(t+1) - Z(t)$  are stationary and independent over disjoint intervals. But now consider standard *fractional Brownian motion*. It is self-similar with parameter  $0 < H < 1$ , satisfies  $\mathbb{E}[Z(t)] = 0$ ,  $\mathbb{E}[Z(t)^2] = 1$  and has stationary increments. This last fact, together with self-similarity, implies that its covariance function equals

$$(1.1) \quad \mathbb{E}[Z(t_1)Z(t_2)] = \{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\}/2, \quad t_1, t_2 \geq 0.$$

Observe that if  $H = 1/2$ , then (1.1) reduces to  $\mathbb{E}[Z(t_1)Z(t_2)] = \min(t_1, t_2)$ , for  $t_1, t_2 \geq 0$ , which is the covariance of Brownian motion. If  $H \neq 1/2$ , however, the increments of standard fractional Brownian motion, while stationary, are not independent anymore. In fact, when  $1/2 < H < 1$ , they have the long-range dependence property. Indeed,

$$\mathbb{E}[\{Z(t+1) - Z(t)\}\{Z(s+t+1) - Z(s+t)\}] \sim H(2H-1)s^{2H-2},$$

as  $s$  tends to infinity. Since  $H > 1/2$ , the sum of these covariances diverges.

We will consider Gaussian processes converging to fractional Brownian motion as well as linear processes which may be non-Gaussian. A linear process  $(X_i)_{i \in \mathbb{Z}}$  is defined as

$$(1.2) \quad X_i = \sum_{j \geq 1} a_j \varepsilon_{i-j}, \quad i \in \mathbb{Z},$$

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Here the  $\varepsilon_i$ 's are "innovations". These are zero-mean independent and identically (i.i.d.) random variables with at least finite second moments. The coefficients  $a_j$  in (1.2) are such that  $\sum_{j \geq 1} a_j^2 < \infty$ , which ensures that  $\mathbb{E}(X_i^2) < \infty$ . By choosing the  $a_j$ 's judiciously, we can construct linear processes  $(X_i)$  with long-range dependence such that their partial sums  $\sum_{i=1}^{\lfloor Nt \rfloor} X_i$ ,  $0 \leq t \leq 1$ , suitably normalized, converge to fractional Brownian motion as  $N$  tends to infinity.

In this paper, we focus on processes  $(h(X_i))_{i \geq 1}$  which are *non-linear* functionals of Gaussian or linear processes with long-range dependence and are interested in the asymptotic behavior of their partial sums  $\sum_{i=1}^N h(X_i)$ . We will study this behavior in various settings. Each setting involves a decomposition and a notion of "rank". Our goal is to highlight the connections between the different methodologies.

Because  $h$  is in general a non-linear function, the limits are typically non-Gaussian. The limits are called *Hermite process*  $\{Z_D^{(m)}(t), t \geq 0\}$  which are defined in terms of multiple Wiener-Itô integrals as follows:

$$(1.3) \quad Z_D^{(m)}(t) = a_{m,D} \int_{x_1 < \dots < x_m} \left\{ \int_0^t \prod_{j=1}^m (s - x_j)_+^{-(1+D)/2} ds \right\} dB(x_1) \dots dB(x_m),$$

where  $0 < D < 1/m$ ,  $a_{m,D}$  is a constant,  $\{B(x)\}_{x \in \mathbb{R}}$  denotes the standard Brownian motion, and

$$(u)_+ = \begin{cases} u, & \text{if } u \geq 0; \\ 0, & \text{if } u < 0, \end{cases}$$

is the "positive part" function. Physically,  $Z_D^{(m)}(t)$  is an aggregation of products of independent Gaussian noises with power weights. The multiple integrals are well-defined because

$$\int_{x_1 < \dots < x_m} \left| \int_0^t \prod_{j=1}^m (s - x_j)_+^{-(1+D)/2} ds \right|^2 dx_1 \dots dx_m < \infty.$$

The representation (1.3) is called a *time-domain representation*.

There are other equivalent ones. There is the *spectral representation* of  $Z_D^{(m)}(t)$ , namely

$$b_{m,D} \int_{\lambda_1 < \dots < \lambda_m} \frac{e^{i(\lambda_1 + \dots + \lambda_m)t} - 1}{i(\lambda_1 + \dots + \lambda_m)} \prod_{j=1}^m |\lambda_j|^{-(1-D)/2} d\tilde{B}(\lambda_1) \dots d\tilde{B}(\lambda_m), \quad t \geq 0,$$

where  $b_{m,D}$  is a constant and  $\{\tilde{B}(\lambda)\}_{\lambda \in \mathbb{R}}$  denotes a complex Brownian motion. There is the *positive half-time representation*

$$c_{m,D} \int_{0 < x_1 < \dots < x_m < t} \left\{ \int_0^t \prod_{j=1}^m x_j^{-(1-D)/2} (1 - sx_j)^{-(1+D)/2} \right\} dB(x_1) \dots dB(x_m), \quad t \geq 0,$$

where  $c_{m,D}$  is a constant and  $\{B(x)\}_{x \in \mathbb{R}}$  is a standard Brownian motion. There is finally the *finite interval representation*

$$d_{m,D} \int_{0 < x_1 < \dots < x_m < t} \left\{ \prod_{j=1}^m x_j^{-(1-D)/2} \int_0^t s^{m(1-D)/2} \prod_{j=1}^m (s - x_j)_+^{-(1+D)/2} ds \right\} dB(x_1) \dots dB(x_m),$$

where  $d_{m,D}$  is a constant and  $\{B(x)\}_{x \in \mathbb{R}}$  is a standard Brownian motion. For more details see [PT10, Theorem 1.1].

The Hermite processes have interesting properties: they have mean zero, finite moments of all order, have stationary increments and are self-similar in the sense that for all positive  $a$ ,

$$\{Z_D^{(m)}(at)\}_{t \geq 0} \stackrel{d}{=} \{a^H Z_D^{(m)}(t)\}_{t \geq 0},$$

where

$$H = 1 - \frac{mD}{2},$$

and hence depends on  $D$  and  $m$ .

The paper is organized as follows. In Section 2, we state conditions for  $\sum_{i=1}^{\lfloor Nt \rfloor} h(X_i)$ , suitably normalized, to converge to  $Z_D^{(m)}(t)$  when  $(X_i)_{i \geq 1}$  is a stationary mean-zero long-range dependent Gaussian process. In Section 3, we consider the case where  $(X_i)_{i \geq 1}$  is not a Gaussian process anymore but is a long-range dependent linear process and  $h$  is an entire function as done in [Sur82]. In Section 4,  $(X_i)_{i \geq 1}$  is still assumed to be a long-range dependent linear process but the regularity assumptions on  $h$  are somewhat alleviated as done in [HH97]. In Section 5, we consider the particular case where  $h$  is a polynomial and show that all ranks coincide. Finally, in Section 6, we illustrate the methods by providing sketches of proofs.

## 2. The Gaussian case

We suppose here that, the underlying process  $(X_i)_{i \geq 1}$  satisfies the following assumption:

- (A1)  $(X_i)_{i \geq 1}$  is a stationary mean-zero Gaussian process with covariances  $\rho(k) = \mathbb{E}(X_1 X_{k+1})$  satisfying:

$$\rho(0) = 1 \text{ and } \rho(k) = k^{-D} L(k), \quad 0 < D < 1,$$

where  $L$  is slowly varying at infinity and is positive for large  $k$ .

Recall that a slowly varying function  $L(x)$ ,  $x > 0$  is such that  $L(xt)/L(x) \rightarrow 1$ , as  $x \rightarrow \infty$  for any  $t > 0$ . Constants and logarithms are examples of slowly varying functions.

If  $h$  is such that  $\mathbb{E}[h(X_1)^2] < \infty$ , the idea is to use the expansion of  $h$  in the basis of Hermite polynomials, that is:

$$h(x) = \sum_{k \geq 0} (J(k)/k!) H_k(x),$$

where  $H_k$  is the  $k$ th Hermite polynomial with leading coefficient equal to 1, that is  $H_0(x) = 1$ ,  $H_1(x) = x$ ,  $H_2(x) = x^2 - 1$ ,  $H_3(x) = x^3 - 3x, \dots$  and

$$J(k) = \mathbb{E}[h(X_1)H_k(X_1)].$$

DEFINITION 2.1. We shall say that  $h$  is of *Hermite rank*  $m \geq 1$  if  $m$  is the smallest positive integer such that

$$(2.1) \quad J(m) = \mathbb{E}[h(X)H_m(X)] \neq 0,$$

where  $X$  is a standard Gaussian random variable. The corresponding *rank coefficient* is  $J(m)$ .

Suppose that not only  $(X_i)_{i \geq 1}$  but also  $h(X_i)_{i \geq 1}$  is long-range dependent which happens if  $0 < D < 1/m$ . Then by the reduction theorem of [Taq75], the leading term of  $\sum_{i=1}^N h(X_i)$  properly normalized is the first term of the decomposition of  $h$  in the Hermite polynomials basis, namely

$$\frac{J(m)}{m!} \sum_{i=1}^N H_m(X_i).$$

Moreover, this leading term, properly normalized, converges in distribution to  $Z_D^{(m)}(1)$ , where  $\{Z_D^{(m)}(t), t \geq 0\}$  is the *Hermite process* of order  $m$ , evaluated at time  $t = 1$ . For this last step, one needs to show that

$$(2.2) \quad \sum_{i=1}^N \frac{H_m(X_i)}{m!} / \text{Var} \left( \sum_{i=1}^N H_m(X_i)/m! \right)^{1/2}$$

converges in distribution to  $Z_D^{(m)}(1)$ , as  $N \rightarrow \infty$ . To gain some insight, note that not only the limit  $Z_D^{(m)}(1)$  is represented by a multiple Wiener-Itô integral of order  $m$  (see (1.3)), but also the summands  $H_m(X_i)$ . This is because the Gaussian sequence  $X_i$ ,  $i \geq 1$ , can be expressed as

$$X_i = \int_{\mathbb{R}} \psi_i(x) dB(x), \quad \int_{\mathbb{R}} \psi_i(x) \psi_j(x) dx = \mathbb{E}(X_i X_j), \quad i, j \geq 1,$$

with  $\mathbb{E}(X_i^2) = 1$ ,  $i \geq 1$ . Then one has (see Proposition 8.1.2 of [PT11] or Theorem 9.6.9 in [Kuo06]),

$$(2.3) \quad \frac{H_m(X_i)}{m!} = \int_{x_1 < x_2 < \dots < x_m} \psi_i(x_1) \dots \psi_i(x_m) dB(x_1) \dots dB(x_m),$$

sometimes written

$$H_m(X_i) = \int_{\mathbb{R}^m}' \psi_i(x_1) \dots \psi_i(x_m) dB(x_1) \dots dB(x_m),$$

where the prime indicates that one does not integrate over the diagonals.

Here is the precise result (see [Taq75], [DM79] and [Taq79].)

**THEOREM 2.2.** *Assume that  $h$  is such that  $\mathbb{E}[h(X_1)^2] < \infty$  and that  $m$  is the smallest integer greater than 1 such that  $J(m) = \mathbb{E}[h(X_1)H_m(X_1)] \neq 0$ , where  $H_m$  denotes the  $m$ th Hermite polynomial. Assume also that Assumption (A1) holds with  $D$  in  $(0, 1/m)$ . Then*

$$(2.4) \quad \frac{1}{\sigma_{N,m}} \sum_{i=1}^N \left\{ h(X_i) - \mathbb{E}[h(X_i)] \right\} \xrightarrow{d} J(m) Z_D^{(m)}(1), \quad N \rightarrow \infty,$$

where

$$(2.5) \quad \sigma_{N,m}^2 = \text{Var} \left( \sum_{i=1}^N H_m(X_i)/m! \right) \sim \frac{2N^{2-mD} L^m(N)}{m!(1-mD)(2-mD)}, \quad N \rightarrow \infty,$$

and  $\{Z_D^{(m)}(t)\}_{t \in \mathbb{R}}$  is the Hermite process of order  $m$  and parameter  $D$  defined by (1.3) where

$$a_{m,D} = \left[ m!(1-mD/2)(1-mD) \left\{ \int_0^\infty (x+x^2)^{-(1+D)/2} dx \right\}^{-m} \right]^{1/2},$$

ensures that  $\mathbb{E}[Z_D^{(m)}(1)^2] = 1$ .

A sketch of proof of Theorem 2.2 is given in Section 6.

### 3. The linear case: Surgailis approach

Suppose now that  $(X_i)$  is a linear process. Thus, replace Assumption (A1) by the following assumption.

**(A2)**  $(X_i)$  is defined by

$$(3.1) \quad X_i = \sum_{j \geq 1} a_j \varepsilon_{i-j}, \quad i \in \mathbb{Z},$$

where the innovations  $\varepsilon_i$ 's are zero-mean i.i.d. random variables having at least finite second moments and the  $a_j$ 's are such that  $\sum_{j \geq 1} a_j^2 < \infty$ . The  $a_j$ 's are assumed to be such that

$$(3.2) \quad a_j = j^{-\beta} L(j),$$

where  $\beta \in (1/2, 1)$  and  $L$  is a slowly varying function at infinity.

Note that under Assumption (A2),

$$\mathbb{E}(X_1 X_{k+1}) \sim CL(k)^2 k^{1-2\beta}, \quad \text{as } k \rightarrow \infty,$$

where  $C$  is a positive constant.

In this situation, [Sur82] proposes a methodology for studying the limiting behavior of  $\sum_{i=1}^N h(X_i)$  in the case where  $h$  is an entire function, that is,  $h(z) = \sum_{k \geq 0} c_k z^k$ ,  $z \in \mathbb{C}$ . The idea is to prove that, this time, the leading term is

$$\frac{\mathbb{E}[h^{(m)}(X_1)]}{m!} \sum_{i=1}^N X_i^m,$$

where the rank  $m$  is the exponent rank of  $h$ .

DEFINITION 3.1. We shall say that  $h$  is of *exponent rank*  $m \geq 1$  if  $m$  is the smallest integer such that

$$(3.3) \quad \mathbb{E}[h^{(m)}(X_1)] \neq 0 ,$$

where  $h^{(m)}$  denotes the  $m$ th derivative of  $h$ . The corresponding *rank coefficient* is  $\mathbb{E}[h^{(m)}(X_1)]$ .

In view of the definition (3.1) of a linear process, the idea is to show that  $\sum_{i=1}^N h(X_i)$  has the same asymptotic behavior as

$$\frac{\mathbb{E}[h^{(m)}(X_1)]}{m!} \sum_{i=1}^N X_i^m = \frac{\mathbb{E}[h^{(m)}(X_1)]}{m!} \sum_{i=1}^N \left( \sum_{j \geq 1} a_j \varepsilon_{i-j} \right)^m ,$$

which in turns has the same asymptotic behavior as  $\mathbb{E}[h^{(m)}(X_1)]Y_{N,m}$ , where

$$(3.4) \quad Y_{N,m} = \sum_{n=1}^N \sum_{1 \leq j_1 < j_2 < \dots < j_m} \prod_{s=1}^m a_{j_s} \varepsilon_{n-j_s} .$$

One has then finally to prove that  $Y_{N,m}/[\text{Var}(Y_{N,m})]^{1/2}$  converges to the Wiener-Itô multiple integral defined in (1.3). In view of (2.3), this is not too different from focusing on (2.2). Observe, however, that in (3.4) we are dealing with a discrete convolution and that the  $\varepsilon$ 's are not assumed normal.

Here is the precise result of [Sur82].

THEOREM 3.2. *Let  $h$  be an entire function defined by  $h(z) = \sum_{k \geq 0} c_k z^k$ ,  $z \in \mathbb{C}$ , such that*

$$\sum_{k,j \geq 0} |c_k| |c_j| (k!j!)^2 2^{2(k+j)} \bar{\mu}_{k+j} < \infty ,$$

where  $\bar{\mu}_k = \mathbb{E}[|\varepsilon_1|^k]$ ,  $k \geq 0$  and let  $m$  be the smallest integer larger than 1 such that

$$\mathbb{E}[h^{(m)}(X_1)] \neq 0 .$$

Then, under Assumption (A2), with  $0 < D = 2\beta - 1 < 1/m$ ,  $\beta$  being defined in (3.2),

$$s_{N,m}^{-1} \sum_{n=1}^N \{h(X_n) - \mathbb{E}[h(X_n)]\}$$

has the same limit in distribution, as  $N$  tends to infinity, as

$$\mathbb{E}[h^{(m)}(X_1)] \frac{Y_{N,m}}{s_{N,m}} ,$$

where  $Y_{N,m}$  is defined in (3.4) and  $s_{N,m}^2 = \text{Var}(Y_{N,m})$ . Moreover, that limit is

$$\mathbb{E}[h^{(m)}(X_1)] Z_D^{(m)}(1) ,$$

where  $\{Z_D^{(m)}(t)\}_{t \in \mathbb{R}}$  is the Hermite process of order  $m$  and parameter  $D$  defined in (1.3).

A sketch of proof of Theorem 3.2 is given in Section 6. A related approach, focusing on Appell polynomials, can be found in [AT87].

#### 4. The linear case: Ho and Hsing approach

We need to introduce first some notation. Let  $F$  be the distribution of the linear process  $X_n = \sum_{i \geq 1} a_i \varepsilon_{n-i}$  and  $F_j$  the distribution of

$$X_{n,j} = \sum_{1 \leq i \leq j} a_i \varepsilon_{n-i} ,$$

for  $j \geq 1$ , with the convention:  $X_{n,0} = 0$ . Let

$$(4.1) \quad h_j(x) = \int_{\mathbb{R}} h(x+y) dF_j(y) , \quad h_\infty(x) = \int_{\mathbb{R}} h(x+y) dF(y) ,$$

and

$$(4.2) \quad h_j^{(r)}(x) = \frac{d^r}{dx^r} \int_{\mathbb{R}} h(x+y) dF_j(y), \quad h_\infty^{(r)}(x) = \frac{d^r}{dx^r} \int_{\mathbb{R}} h(x+y) dF(y).$$

If the  $r$ th derivative  $h_j^{(r)}$  of  $h_j$  exists, define

$$h_{j,\lambda}^{(r)}(x) = \sup_{|y| \leq \lambda} |h_j^{(r)}(x+y)|, \quad \lambda \geq 0.$$

We shall say that  $h$  satisfies the *Condition*  $C(r, j, \lambda)$  if

- (1)  $h_j^{(r)}(x)$  exists for all  $x$  and  $h_j^{(r)}$  is continuous.
- (2) For all  $x \in \mathbb{R}$ ,

$$\sup_{I \subset \{1, 2, \dots\}} \mathbb{E}[\{h_{j,\lambda}^{(r)}(x + \sum_{i \in I} a_i \varepsilon_i)\}^4] < \infty,$$

where the supremum is taken over all subsets  $I$  of  $\{1, 2, \dots\}$ .

Let us comment on Condition  $C(r, j, \lambda)$ . It is satisfied if the  $r$ th derivative of  $h$  is bounded and continuous, in which case one can take any  $j$ . Moreover, if  $h$  is any polynomial, then  $C(r, j, \lambda)$  holds provided that  $\varepsilon_i$  has finite moments of sufficiently high order.

The novelty here is that  $C(r, j, \lambda)$  can hold without  $h$  being smooth. An important example is the *indicator function*. If  $h(x) = \mathbb{1}_{\{x \leq u\}}$ , for some fixed  $u$ , let us prove that  $h$  satisfies  $C(r, 1, \lambda)$  for all positive  $\lambda$  as soon as the probability density function  $g$  of  $\varepsilon_1$  has a continuous and integrable  $r$ th derivative.

Since  $X_{n,1} = a_1 \varepsilon_{n-1}$ , we have

$$h_1(x) = \int_{\mathbb{R}} h(x+y) dF_1(y) = \int_{\mathbb{R}} h(x + a_1 y_1) g(y_1) dy_1 = a_1^{-1} \int_{\mathbb{R}} h(z) g\left(\frac{z-x}{a_1}\right) dz.$$

Note that,

$$\frac{\partial^r}{\partial x^r} \left\{ h(z) g\left(\frac{z-x}{a_1}\right) \right\} = \frac{(-1)^r}{a_1^r} h(z) g^{(r)}\left(\frac{z-x}{a_1}\right).$$

Since, by assumption,  $\int_{\mathbb{R}} |g^{(r)}(y)| dy < \infty$ , we get

$$h_1^{(r)}(x) = \frac{(-1)^r}{a_1^{r+1}} \int_{\mathbb{R}} h(z) g^{(r)}\left(\frac{z-x}{a_1}\right) dz = \frac{(-1)^r}{a_1^r} \int_{\mathbb{R}} h(x + a_1 y_1) g^{(r)}(y_1) dy_1.$$

Moreover,  $h_1^{(r)}$  is a continuous function since  $g^{(r)}$  is assumed to be a continuous function. This gives (1) of  $C(r, 1, \lambda)$ . Let us now check (2) of  $C(r, 1, \lambda)$ . For all subset  $I$  of  $\{1, 2, \dots\}$ , we have

$$\begin{aligned} & \mathbb{E} \left[ \left\{ \sup_{|y| \leq \lambda} \left| h_1^{(r)} \left( x + \sum_{i \in I} a_i \varepsilon_i + y \right) \right| \right\}^4 \right] \\ &= \frac{1}{a_1^{4r}} \mathbb{E} \left[ \left\{ \sup_{|y| \leq \lambda} \left| \int_{\mathbb{R}} h \left( x + \sum_{i \in I} a_i \varepsilon_i + y + a_1 y_1 \right) g^{(r)}(y_1) dy_1 \right| \right\}^4 \right], \end{aligned}$$

which is bounded by  $a_1^{-4r} (\int_{\mathbb{R}} |g^{(r)}(y_1)| dy_1)^4$ , which is finite since  $\int_{\mathbb{R}} |g^{(r)}(y)| dy < \infty$  and thus ensures that  $h(x) = \mathbb{1}_{\{x \leq u\}}$  satisfies Condition  $C(r, 1, \lambda)$ .

Observe that the indicator function  $h(x) = \mathbb{1}_{\{x \leq u\}}$  is allowed in the Gaussian case but not in the situation considered by [Sur82]. As we have just seen, it is allowed in the methodology proposed by [HH97].

The idea here is to use a mixingale decomposition as explained in Section 6 and to prove that the leading term is once again  $Y_{N,m}$ , defined in (3.4), where here  $m$  is the power rank of  $h$  defined as follows.

**DEFINITION 4.1.** We shall say that  $h$  is of *power rank*  $m \geq 1$  if it is the smallest integer such that

$$(4.3) \quad h_\infty^{(m)}(0) \neq 0.$$

The corresponding *rank coefficient* is  $h_\infty^{(m)}(0)$ .

The idea is once again to prove that  $Y_{n,m}$  properly normalized converges to the Wiener-Itô multiple integral defined in (1.3). Here is the precise result due to [HH97].

**THEOREM 4.2.** *Assume that Assumption (A2) holds and that  $h$  is of power rank  $m$ , that is,  $m$  is the smallest positive integer such that  $h_\infty^{(m)}(0) \neq 0$ . Assume also that for some  $j$  and  $\lambda$ , condition  $C(r, j, \lambda)$  holds for  $r = 0, \dots, m+2$  and  $\mathbb{E}[h(X_1)^2] < \infty$ . If*

$$0 < D = (2\beta - 1) < 1/m,$$

$\beta$  being defined in (3.2), and  $\mathbb{E}[|\varepsilon_1|^{2m \vee 8}] < \infty$  then

$$s_{N,m}^{-1} \sum_{i=1}^N \{h(X_i) - \mathbb{E}[h(X_i)]\}$$

has the same limit in distribution, as  $N$  tends to infinity, as

$$h_\infty^{(m)}(0) \frac{Y_{N,m}}{s_{N,m}},$$

where  $Y_{N,m}$  is defined in (3.4) and  $s_{N,m}^2 = \text{Var}(Y_{N,m})$ . Moreover, that limit is  $h_\infty^{(m)}(0) Z_D^{(m)}(1)$ , where  $(Z_D^{(m)}(t))_{t \in \mathbb{R}}$  is defined in (1.3).

A sketch of proof of Theorem 4.2 is given in Section 6.

**REMARK 4.3.** The second parts of Theorems 3.2 and 4.2 are similar. This shows that the key is to reduce the original  $\sum_{n=1}^N h(X_n)$  to  $Y_{N,m}$ . It is  $Y_{N,m}$  which converges to the limit after suitable normalization.

## 5. Application to the polynomial case

In the case where  $h$  is a polynomial, we prove in the following proposition that the three definitions of ranks introduced previously coincide.

**PROPOSITION 5.1.** *If  $h$  is a polynomial defined by  $h(x) = \sum_{k=0}^K c_k x^k$  then the three rank coefficients*

$$J(m), \mathbb{E}[h^{(m)}(X_1)] \text{ and } h_\infty^{(m)}(0)$$

defined in (2.1), (3.3) and (4.3) respectively, are identical and equal to

$$(5.1) \quad \sum_{k=m}^K c_k k(k-1) \dots (k-m+1) \mathbb{E}[X_1^{k-m}],$$

where  $m$  is the corresponding rank.

**PROOF OF PROPOSITION 5.1.** Let

$$h(X) = \sum_{k=0}^K c_k X^k$$

denote the polynomial.

Suppose first that  $X$  is a standard Gaussian random variable. The  $m$ th coefficient of the expansion in Hermite polynomials of  $h$  is given by

$$J(m) = \sum_{k=0}^K c_k \mathbb{E}[X^k H_m(X)].$$

We first show that  $J(m)$  is equal to (5.1). Using the relation between powers and Hermite polynomials [Kuo06, p. 159], we have, for all  $x \in \mathbb{R}$  and  $k \in \mathbb{N}$ ,

$$(5.2) \quad x^k = \sum_{\ell=0}^{\lfloor k/2 \rfloor} \frac{k!}{(k-2\ell)! \ell! 2^\ell} H_{k-2\ell}(x),$$



where  $[y]$  denotes the integer part of  $y$ . Thus, by orthogonality of the  $H_k$ 's in  $L^2$  equipped with the  $\mathcal{N}(0, 1)$  Gaussian measure,

$$\begin{aligned} \mathbb{E}[X^k H_m(X)] &= \sum_{\ell=0}^{[k/2]} \frac{k!}{(k-2\ell)! \ell! 2^\ell} \mathbb{E}[H_{k-2\ell}(X) H_m(X)] \\ &= \frac{k!}{m! \{(k-m)/2\}! 2^{(k-m)/2}} m! \\ &= \frac{k!}{(k-m)! \{(k-m)/2\}! 2^{(k-m)/2}} \\ &= k(k-1) \dots (k-m+1) \mathbb{E}[X^{k-m}], \end{aligned}$$

when  $k \geq m$  and  $k-m$  is even, otherwise,  $\mathbb{E}[X^k H_m(X)] = 0$ . We used here the fact that when  $X$  is a standard Gaussian random variable,  $\mathbb{E}[X^{2p+1}] = 0$  and  $\mathbb{E}[X^{2p}] = (2p)!/(p!2^p)$ . Now  $c_k \mathbb{E}[X^k H_m(X)]$  vanishes if either  $c_k = 0$  or if  $\mathbb{E}[X^{k-m}] = 0$ . The Hermite rank  $m$  is thus equal to the smallest  $k$  such that both  $c_k \neq 0$  and  $\mathbb{E}[X^{k-m}] \neq 0$ . The corresponding coefficient  $J(m)$  is thus given by (5.1).

We now suppose that  $(X_i)_{i \geq 1}$  is a linear process and consider  $\mathbb{E}[h^{(m)}(X_1)]$  defined in (3.3). Observe that

$$h^{(m)}(x) = \sum_{k=m}^K c_k k(k-1) \dots (k-m+1) x^{k-m},$$

for all  $x$  in  $\mathbb{R}$ . Thus,

$$(5.3) \quad \mathbb{E}[h^{(m)}(X_1)] = \sum_{k=m}^K c_k k(k-1) \dots (k-m+1) \mathbb{E}[X_1^{k-m}],$$

which is the same as (5.1).

We finally consider  $h_\infty^{(m)}(0)$  defined in (4.3). Since

$$h_\infty(x) = \sum_{k=0}^K c_k \mathbb{E}[(x + X_1)^k] = \sum_{k=0}^K c_k \sum_{j=0}^k \binom{k}{j} x^j \mathbb{E}[X_1^{k-j}],$$

we get

$$(5.4) \quad h_\infty^{(m)}(0) = \sum_{k=0}^K c_k \binom{k}{m} m! \mathbb{E}[X_1^{k-m}] = \sum_{k=m}^K c_k k(k-1) \dots (k-m+1) \mathbb{E}[X_1^{k-m}],$$

which is the same as (5.1) and (5.3).  $\square$

To understand the significance of the proposition, let  $h(X) = \sum_{k=1}^K c_k X^k$  be a polynomial and set for convenience

$$C(k, j) = k(k-1) \dots (k-j+1), \quad k \geq j.$$

As noted, the rank of  $h(X)$  is  $m$  if

$$(5.5) \quad \sum_{k=j}^K c_k C(k, j) \mathbb{E}[X^{k-j}] = 0, \quad j = 1, \dots, m-1,$$

and

$$e_m = \sum_{k=m}^K c_k C(k, m) \mathbb{E}[X^{k-m}] \neq 0.$$

In this case,  $\sum_{i=1}^N h(X_i)$  behaves asymptotically like  $e_m Y_{N,m}$ , where  $Y_{N,m}$  is defined in (3.4) and where, here,  $e_m$  stands for any of the three rank coefficients  $J(m)$ ,  $\mathbb{E}[h^{(m)}(X_1)]$  and  $h_\infty^{(m)}(0)$ .

EXAMPLE 5.2. Suppose  $h(X) = c_k X^k$ ,  $c_k \neq 0$ ,  $k \geq 1$ . If  $k$  is odd, then  $\mathbb{E}[X^{k-1}] \neq 0$  and the rank is  $m = 1$ . The corresponding rank coefficient is  $c_k C(k, 1) \mathbb{E}[X^{k-1}]$ . If  $k \geq 2$  is even, then  $\mathbb{E}[X^{k-1}] = 0$ ,  $\mathbb{E}[X^{k-2}] \neq 0$  and thus the corresponding rank is  $m = 2$  and the rank coefficient is  $c_k C(k, 2) \mathbb{E}[X^{k-2}]$ .

EXAMPLE 5.3. Suppose  $h(X) = c_k X^k + c_{k+1} X^{k+1}$ ,  $c_k \neq 0$ ,  $c_{k+1} \neq 0$ ,  $k \geq 1$ . Since

$$c_k C(k, 1) \mathbb{E}[X^{k-1}] + c_{k+1} C(k+1, 1) \mathbb{E}[X^{k+1-1}] \neq 0,$$

the rank is  $m = 1$  and the rank coefficient is  $c_{k+1} C(k+1, 1) \mathbb{E}[X^k]$  if  $k$  is even and  $c_k C(k, 1) \mathbb{E}[X^{k-1}]$  if  $k$  is odd.

EXAMPLE 5.4. Suppose  $h(X) = X^3 - 3X$ . For  $j = 1$ , (5.5) equals

$$C(3, 1) \mathbb{E}[X^2] - 3C(1, 1) = C(3, 1) - 3C(1, 1) = 3 - 3 = 0,$$

and for  $j = 2$ , it equals  $C(3, 2) \mathbb{E}[X^{3-2}] = 0$ . For  $j = 3$ , it equals

$$C(3, 3) \mathbb{E}[X^{3-3}] = C(3, 3) = 3 \times 2 = 6.$$

Hence the rank is  $m = 3$  and the rank coefficient is 6. One can arrive to this conclusion immediately by supposing  $X \sim \mathcal{N}(0, 1)$  and noting that

$$J(j) = \mathbb{E}[h(X)H_j(X)] = \mathbb{E}[(X^3 - 3X)H_j(X)] = \mathbb{E}[H_3(X)H_j(X)]$$

equals 0 if  $j \neq 3$  and equals  $3! = 6$  if  $j = 3$ .

## 6. Sketches of proofs of Theorems 2.2, 3.2 and 4.2

The proof for each theorem involves a decomposition of the function  $h$  and has two parts. The first involves showing that the only contribution to the limit is due to the term of the decomposition with index  $m$ , where  $m$  is the rank. The second part consists in showing that the term with index  $m$  converges in distribution to  $Z_D^{(m)}(1)$ . In fact convergence also holds for the finite-dimensional distributions as well as in function space. We will focus here on the first part of each proof.

SKETCH OF PROOF OF THEOREM 2.2. The first part of the proof consists in showing that the remainder  $h^*(x) = h(x) - \frac{J(m)}{m!} H_m(x)$  is negligible, namely that

$$N^{-2+mD} L^{-m}(N) \text{Var} \left( \sum_{i=1}^N h^*(X_i) \right) \rightarrow 0, \text{ as } N \rightarrow \infty.$$

This implies that  $N^{-1+mD/2} L^{-m/2}(N) \sum_{i=1}^N h^*(X_i) = o_P(1)$  and that the convergence in distribution of

$$N^{-1+mD/2} L^{-m/2}(N) \sum_{i=1}^N h(X_i)$$

reduces to the convergence in distribution of

$$N^{-1+mD/2} L^{-m/2}(N) \sum_{i=1}^N (J(m)/m!) H_m(X_i).$$

Using the Mehler formula, namely

$$\mathbb{E}[H_p(X_i)H_q(X_j)] = \delta_{p,q} \frac{J(p)^2}{p!} (\rho(i-j))^p,$$

where  $\delta_{p,q}$  is 1 if  $p = q$  and 0 if  $p \neq q$ , we get that

$$\begin{aligned} \text{Var} \left( \sum_{i=1}^N h^*(X_i) \right) &\leq \sum_{i=1}^N \sum_{j=1}^N \left( \sum_{p \geq 1} \frac{J(p)^2}{p!} \right) |\rho(i-j)|^{m+1} \\ &= \mathbb{E}[h^2(X_1)] \sum_{i=1}^N \sum_{j=1}^N |\rho(i-j)|^{m+1} \\ &\leq N \mathbb{E}[h^2(X_1)] \sum_{|k| < N} |\rho(k)|^{m+1}. \end{aligned}$$

Taking into account the normalization, we have

$$(6.1) \quad N^{-2+mD} L^{-m}(N) \text{Var} \left( \sum_{i=1}^N h^*(X_i) \right) \leq N^{-1+mD} L^{-m}(N) \mathbb{E}[h^2(X_1)] \sum_{|k| < N} |\rho(k)|^{m+1}.$$

There are two possibilities: either the series  $\sum_{k \in \mathbb{Z}} |\rho(k)|^{m+1}$  is convergent and in this case the expression (6.1) tends to zero since  $D < 1/m$  or the series  $\sum_{k \in \mathbb{Z}} |\rho(k)|^{m+1}$  is divergent. In the latter case,  $\sum_{|k| < N} |\rho(k)|^{m+1}$  is of order  $N^{-(m+1)D+1}$  and thus again, the expression of (6.1) tends to zero since  $D > 0$ .

To conclude the proof, it remains to show that, as  $N \rightarrow \infty$ ,

$$\frac{J(m)}{\sigma_{N,m}} \sum_{i=1}^N H_m(X_i)/m! \xrightarrow{d} J(m) Z_D^{(m)}(1),$$

where  $\sigma_{N,m}$  is defined in (2.5). This is done in [DM79], [Taq79] and [Maj81].  $\square$

SKETCH OF PROOF OF THEOREM 3.2. Let  $(X_j)$  be defined in (3.1). The idea of the proof consists of decomposing

$$(6.2) \quad (X_j)^k = \sum_{p_1, p_2, \dots, p_k \geq 1} a_{p_1} a_{p_2} \dots a_{p_k} \varepsilon_{j-p_1} \varepsilon_{j-p_2} \dots \varepsilon_{j-p_k}$$

in terms of the cardinality  $|\{p_1, \dots, p_k\}|$  of the set  $\{p_1, \dots, p_k\}$ . When  $|\{p_1, \dots, p_k\}| = k$ , the term  $a_{p_1} a_{p_2} \dots a_{p_k} \varepsilon_{j-p_1} \varepsilon_{j-p_2} \dots \varepsilon_{j-p_k}$  is not modified. When  $|\{p_1, \dots, p_k\}| < k$  and for instance equal to  $k-1$  with  $p_1 = p_2$ , it is split in two parts:

$$(6.3a) \quad a_{p_1} a_{p_2} \dots a_{p_k} (\varepsilon_{j-p_1} \varepsilon_{j-p_2} \dots \varepsilon_{j-p_k}) = a_{p_1}^2 a_{p_3} \dots a_{p_k} (\varepsilon_{p_1-j}^2 \varepsilon_{j-p_3} \dots \varepsilon_{j-p_k})$$

$$(6.3b) \quad = a_{p_1}^2 a_{p_3} \dots a_{p_k} (\mu_2 \varepsilon_{j-p_3} \dots \varepsilon_{j-p_k}) + a_{p_1}^2 a_{p_3} \dots a_{p_k} (\eta_{p_1}(2) \varepsilon_{j-p_3} \dots \varepsilon_{j-p_k}),$$

where  $\mu_\ell = \mathbb{E}(\varepsilon_1^\ell)$  and  $\eta_p(\ell) = \varepsilon_p^\ell - \mu_\ell$ . One then shows that the term with  $\eta_{p_1}(2)$  is negligible.

Let us focus on the first term in 6.3b and consider the general case. The idea is to replace  $(X_j)^k$  by

$$(6.4) \quad (Z_j)^{*k} = \sum_{\ell=0}^k \binom{k}{\ell} \sum_{(p)_\ell} a_{p_1} \varepsilon_{j-p_1} \dots a_{p_\ell} \varepsilon_{j-p_\ell} \sum_{(V)(k-\ell)} \sum_{(q)_r} a_{q_1}^{v_1} \mu_{v_1} \dots a_{q_r}^{v_r} \mu_{v_r},$$

where the summation over  $(p)_\ell$  corresponds to the summation over the sets  $\{p_1, \dots, p_\ell\}$  of cardinality  $|\{p_1, \dots, p_\ell\}| = \ell$ , that is, over  $p_1, \dots, p_\ell$  which take different values. In the perspective of (6.3b), one should take the sets  $\{q_1, \dots, q_r\}$  and  $\{p_1, \dots, p_\ell\}$  disjoint, but we shall not impose this restriction in (6.4). The sum  $\sum_{(V)(k-\ell)}$  is taken over all partitions of the set  $\{1, \dots, k-\ell\}$  of cardinality  $v_1, \dots, v_r$  such that  $v_i \geq 2$ , for all  $i$ . By convention, this sum equals 1 for  $(V)(0)$ . In contrast to  $(X_j)^k$ , the notation  $(Z_j)^{*k}$  is a shorthand for the r.h.s of (6.4) and does not mean  $Z_j$  to the power  $k$ . The difference between  $(X_j)^k$  and  $(Z_j)^{*k}$  is that when there is an  $\varepsilon_{p-j}^\ell$  with

$\ell > 1$  in (6.2), it is replaced by  $\mu_\ell = \mathbb{E}(\varepsilon_1^\ell)$  in (6.4). Observe also that the summands of  $(Z_j)^{*k}$  with  $\ell > 0$  have zero mean. This ensures that

$$\mathbb{E}[(X_j)^k] = \mathbb{E}[(Z_j)^{*k}] = \sum_{(V)(k)} \sum_{(q)_r} a_{q_1}^{v_1} \mu_{v_1} \dots a_{q_r}^{v_r} \mu_{v_r} ,$$

for  $k \geq 0$  and hence for  $0 \leq s \leq k$ ,

$$(6.5) \quad \mathbb{E}[(X_j)^{k-s}] = \mathbb{E}[(Z_j)^{*k-s}] = \sum_{(V)(k-s)} \sum_{(q)_r} a_{q_1}^{v_1} \mu_{v_1} \dots a_{q_r}^{v_r} \mu_{v_r} .$$

Let us now define formally  $h^*(Z_j)$  as  $\sum_{s \geq 0} c_s (Z_j)^{*s}$ , where  $(Z_j)^{*s}$  is given in (6.4) and prove that

$$(6.6) \quad h^*(Z_j) = \sum_{s \geq 0} c_s (Z_j)^{*s} = \sum_{s \geq 0} (e_s / s!) \sum_{(p)_s} a_{p_1} \varepsilon_{j-p_1} \dots a_{p_s} \varepsilon_{j-p_s} ,$$

where here

$$e_s = \mathbb{E}[h^{(s)}(X_j)] .$$

Observe that by (5.3) and (6.5),

$$\begin{aligned} \frac{e_s}{s!} &= \frac{\mathbb{E}[h^{(s)}(X_j)]}{s!} = \frac{1}{s!} \sum_{k \geq s} c_k k(k-1) \dots (k-s+1) \mathbb{E}[(X_j)^{k-s}] \\ &= \sum_{k \geq s} c_k \binom{k}{s} \mathbb{E}[(X_j)^{k-s}] = \sum_{k \geq s} c_k \binom{k}{s} \mathbb{E}[(Z_j)^{*k-s}] . \end{aligned}$$

Using this and again (6.5), we note that the last term of (6.6) can be expressed as

$$\begin{aligned} \sum_{s \geq 0} \frac{e_s}{s!} \sum_{(p)_s} a_{p_1} \varepsilon_{j-p_1} \dots a_{p_s} \varepsilon_{j-p_s} &= \sum_{s \geq 0} \left( \sum_{k \geq s} c_k \binom{k}{s} \mathbb{E}[(Z_j)^{*k-s}] \right) \sum_{(p)_s} a_{p_1} \varepsilon_{j-p_1} \dots a_{p_s} \varepsilon_{j-p_s} \\ &= \sum_{s \geq 0} \left( \sum_{k \geq s} c_k \binom{k}{s} \sum_{(V)(k-s)} \sum_{(q)_r} a_{q_1}^{v_1} \mu_{v_1} \dots a_{q_r}^{v_r} \mu_{v_r} \right) \sum_{(p)_s} a_{p_1} \varepsilon_{j-p_1} \dots a_{p_s} \varepsilon_{j-p_s} \\ &= \sum_{k \geq 0} c_k \left( \sum_{s=0}^k \binom{k}{s} \sum_{(p)_s} a_{p_1} \varepsilon_{j-p_1} \dots a_{p_s} \varepsilon_{j-p_s} \sum_{(V)(k-s)} \sum_{(q)_r} a_{q_1}^{v_1} \mu_{v_1} \dots a_{q_r}^{v_r} \mu_{v_r} \right) = \sum_{k \geq 0} c_k (Z_j)^{*k} , \end{aligned}$$

by (6.4), hence proving (6.6).

The proof of **[Sur82]** consists in showing that  $\sum_{j=1}^N h(X_j)$  can be replaced by  $\sum_{j=1}^N h^*(Z_j)$  and that the leading term in  $\sum_{j=1}^N h^*(Z_j)$  is the term corresponding to  $s = m$  in  $\sum_{j=1}^N h(Z_j)$ , that is by (6.6) and (3.4),

$$e_m Y_{N,m} = (e_m / m!) \sum_{j=1}^N \sum_{(p)_m} a_{p_1} \varepsilon_{j-p_1} \dots a_{p_m} \varepsilon_{j-p_m} .$$

More precisely, it is proved in Lemma 2 and 3 of **[Sur82]** that

$$\mathbb{E} \left[ \left\{ \sum_{j=1}^N \left( h(X_j) - h^*(Z_j) \right) \right\}^2 \right] \leq CN ,$$

where  $C$  is a positive constant and that

$$\begin{aligned} \mathbb{E} \left[ \left\{ \sum_{j=1}^N \sum_{\ell \geq m+1} \sum_{(p)_\ell} (e_\ell / \ell!) a_{p_1} \varepsilon_{j-p_1} \dots a_{p_\ell} \varepsilon_{j-p_\ell} \right\}^2 \right] &= \mathbb{E} \left[ \left\{ \sum_{\ell \geq m+1} e_\ell Y_{N,\ell} \right\}^2 \right] \\ &= o(L(N) 2^m N^{2-mD}) , \end{aligned}$$

as  $N$  tends to infinity.

To conclude, it remains to show that, as  $N \rightarrow \infty$ ,

$$e_m \frac{Y_{N,m}}{s_{N,m}} \xrightarrow{d} e_m Z_D^{(m)}(1),$$

where  $s_{N,m}^2 = \text{Var}(Y_{N,m})$ . This is done in Lemma 5 of [Sur82]. The basic idea is to express  $Y_{N,m}$  as a discrete multiple stochastic integral and use the fact that  $N^{-1/2} \sum_{k=1}^N \varepsilon_k$  converges to a normal distribution.  $\square$

SKETCH OF PROOF OF THEOREM 4.2. The idea is to condition on the  $\sigma$ -fields

$$\mathcal{F}_k = \sigma(\varepsilon_i; i < k),$$

using the telescoping expression

$$(6.7) \quad h(X_n) - \mathbb{E}[h(X_n)] = \sum_{j \geq 1} \{ \mathbb{E}[h(X_n) | \mathcal{F}_{n-j+1}] - \mathbb{E}[h(X_n) | \mathcal{F}_{n-j}] \},$$

since the extreme summands are such that  $\mathbb{E}[h(X_n) | \mathcal{F}_n] = h(X_n)$  and  $\mathbb{E}[h(X_n) | \mathcal{F}_{-\infty}] = \mathbb{E}[h(X_n)]$ . Now write

$$\mathbb{E}[h(X_n) | \mathcal{F}_{n-j}] = \mathbb{E}[h(X_{n,j} + \tilde{X}_{n,j}) | \mathcal{F}_{n-j}],$$

with

$$X_{n,j} = \sum_{i=1}^j a_i \varepsilon_{n-i} \quad \text{and} \quad \tilde{X}_{n,j} = \sum_{i>j} a_i \varepsilon_{n-i}.$$

Since  $\tilde{X}_{n,j}$  is  $\mathcal{F}_{n-j}$ -measurable and  $X_{n,j}$  is independent of  $\mathcal{F}_{n-j}$ ,

$$(6.8) \quad \mathbb{E}[h(X_n) | \mathcal{F}_{n-j}] = \int_{\mathbb{R}} h(x + \tilde{X}_{n,j}) dF_j(x) = h_j(\tilde{X}_{n,j}),$$

where  $F_j$  is the distribution of  $X_{n,j}$  and  $h_j(y) = \mathbb{E}[h(X_{n,j} + y)]$ . Using (6.7) and (6.8), we get that

$$\begin{aligned} & \sum_{n=1}^N \{ h(X_n) - \mathbb{E}[h(X_n)] \} = \sum_{n=1}^N \sum_{j_1 \geq 1} [h_{j_1-1}(\tilde{X}_{n,j_1-1}) - h_{j_1}(\tilde{X}_{n,j_1})] \\ &= \sum_{n=1}^N \sum_{j_1 \geq 1} (\tilde{X}_{n,j_1-1} - \tilde{X}_{n,j_1}) h_{j_1}^{(1)}(\tilde{X}_{n,j_1}) \\ &+ \left[ h_{j_1-1}(\tilde{X}_{n,j_1-1}) - h_{j_1}(\tilde{X}_{n,j_1}) - (\tilde{X}_{n,j_1-1} - \tilde{X}_{n,j_1}) h_{j_1}^{(1)}(\tilde{X}_{n,j_1}) \right] \\ &= \sum_{n=1}^N \sum_{j_1 \geq 1} a_{j_1} \varepsilon_{n-j_1} h_{j_1}^{(1)}(\tilde{X}_{n,j_1}) \\ (6.9) \quad &+ \left[ h_{j_1-1}(\tilde{X}_{n,j_1-1}) - h_{j_1}(\tilde{X}_{n,j_1}) - a_{j_1} \varepsilon_{n-j_1} h_{j_1}^{(1)}(\tilde{X}_{n,j_1}) \right] \end{aligned}$$

$$(6.10) \quad \approx \sum_{n=1}^N \sum_{j_1 \geq 1} a_{j_1} \varepsilon_{n-j_1} h_{j_1}^{(1)}(\tilde{X}_{n,j_1}),$$

after proving that the terms in brackets can be neglected. We have introduced the  $\varepsilon$ 's. We need now to introduce  $h_\infty^{(1)}(0), \dots, h_\infty^{(m)}(0)$ . To do so, we express the summands in the remaining term (6.10) as

$$\begin{aligned} (6.11) \quad a_{j_1} \varepsilon_{n-j_1} h_{j_1}^{(1)}(\tilde{X}_{n,j_1}) &= a_{j_1} \varepsilon_{n-j_1} h_\infty^{(1)}(0) + a_{j_1} \varepsilon_{n-j_1} [h_{j_1}^{(1)}(\tilde{X}_{n,j_1}) - h_\infty^{(1)}(0)] \\ &= a_{j_1} \varepsilon_{n-j_1} h_\infty^{(1)}(0) + a_{j_1} \varepsilon_{n-j_1} \sum_{j_2 \geq j_1+1} [h_{j_2-1}^{(1)}(\tilde{X}_{n,j_2-1}) - h_{j_2}^{(1)}(\tilde{X}_{n,j_2})]. \end{aligned}$$

Focusing on the term in brackets, we write as before (see (6.9)),

$$(6.12) \quad \begin{aligned} & h_{j_2-1}^{(1)}(\tilde{X}_{n,j_2-1}) - h_{j_2}^{(1)}(\tilde{X}_{n,j_2}) \\ &= a_{j_2} \varepsilon_{n-j_2} h_{j_2}^{(2)}(\tilde{X}_{n,j_2}) + \left[ h_{j_2-1}^{(1)}(\tilde{X}_{n,j_2-1}) - h_{j_2}^{(1)}(\tilde{X}_{n,j_2}) - a_{j_2} \varepsilon_{n-j_2} h_{j_2}^{(2)}(\tilde{X}_{n,j_2}) \right] \\ &\approx a_{j_2} \varepsilon_{n-j_2} h_{j_2}^{(2)}(\tilde{X}_{n,j_2}) = a_{j_2} \varepsilon_{n-j_2} h_{\infty}^{(2)}(0) + a_{j_2} \varepsilon_{n-j_2} [h_{j_2}^{(2)}(\tilde{X}_{n,j_2}) - h_{\infty}^{(2)}(0)] , \end{aligned}$$

where in that last equality, we proceeded as in (6.11). Relations (6.11) and (6.12) yield

$$\begin{aligned} a_{j_1} \varepsilon_{n-j_1} h_{j_1}^{(1)}(\tilde{X}_{n,j_1}) &= a_{j_1} \varepsilon_{n-j_1} h_{\infty}^{(1)}(0) \\ &+ a_{j_1} \varepsilon_{n-j_1} \sum_{j_2 \geq j_1+1} \left\{ a_{j_2} \varepsilon_{n-j_2} h_{\infty}^{(2)}(0) + a_{j_2} \varepsilon_{n-j_2} [h_{j_2}^{(2)}(\tilde{X}_{n,j_2}) - h_{\infty}^{(2)}(0)] \right\} . \end{aligned}$$

Thus, we get

$$(6.13) \quad \begin{aligned} \sum_{n=1}^N \{h(X_n) - \mathbb{E}[h(X_n)]\} &= Y_{N,1} h_{\infty}^{(1)}(0) + Y_{N,2} h_{\infty}^{(2)}(0) \\ &+ \sum_{n=1}^N \sum_{j_2 > j_1 \geq 1} a_{j_1} \varepsilon_{n-j_1} a_{j_2} \varepsilon_{n-j_2} [h_{j_2}^{(2)}(\tilde{X}_{n,j_2}) - h_{\infty}^{(2)}(0)] , \end{aligned}$$

where

$$Y_{N,r} = \sum_{n=1}^N \sum_{1 \leq j_1 < j_2 < \dots < j_r} \prod_{s=1}^r a_{j_s} \varepsilon_{n-j_s} .$$

Iterating, we get

$$(6.14) \quad \sum_{n=1}^N \{h(X_n) - \mathbb{E}[h(X_n)]\} = Y_{N,1} h_{\infty}^{(1)}(0) + Y_{N,2} h_{\infty}^{(2)}(0) + \dots + Y_{N,m} h_{\infty}^{(m)}(0) + R_N ,$$

where  $R_N$  can be shown to be a negligible remainder term. Hence, the first term of the expansion (6.14) is given by  $Y_{N,m} h_{\infty}^{(m)}(0)$ , where  $m$  is the power rank, namely the first  $k$  such that  $h_{\infty}^{(k)}(0) \neq 0$ . This is the same  $Y_{N,m}$  as in (3.4). One concludes by applying the last part of Theorem 3.2.  $\square$

## 7. Conclusion

We considered a stationary sequence  $(X_i)$  which is either

- Gaussian with long-range dependence
- a linear process with long-range dependence,

and focused on the convergence of

$$\sum_{i=1}^N (h(X_i) - \mathbb{E}[h(X_i)])$$

properly normalized to that of a Hermite process  $Z(t)$ , at  $t = 1$ , when  $(h(X_i))_{i \geq 1}$  itself is long-range dependent. We

- described the type of functions  $h$  considered in the literature,
- showed that their notions of rank coincide for  $h$  polynomial,
- and indicated heuristically why we expect the limit to be a Hermite process.

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