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FROBENIUS AND NON LOGARITHMIC RAMIFICATION

Stéphanie Reglade

Abstract: *A ℓ -extension is said logarithmically unramified if it is locally cyclotomic. The purpose of this article is to explain the construction of the logarithmic Frobenius, which plays the role usually played by the classical Frobenius, but in the context of the logarithmic ramification. The interesting point is that usual and logarithmic Frobenius coincide when usual and logarithmic ramification are the same.*

Key words: class field theory, ℓ -adic class field theory.

AMS Classification: 11R37

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Introduction:

The notion of logarithmic ramification was developed by Jaulent in [2]. The starting point of this article is the following fact: let L/K be an extension of number fields, let \mathfrak{p} be a prime of K , \mathfrak{p} is logarithmically unramified in L if $L_{\mathfrak{p}} \subseteq \hat{K}_{\mathfrak{p}}^c$, where $\hat{K}_{\mathfrak{p}}^c$ is the $\hat{\mathbb{Z}}$ -cyclotomic extension of $K_{\mathfrak{p}}$. Consequently the decomposition sub-group is cyclic. Thus we may naturally wonder if there exists, in this decomposition sub-group, an element which is going to play the role usually played by the classical Frobenius ?

Following Neukirch's abstract theory [1, chap.II], we first build the logarithmic local symbol. Neukirch's context starts with an abstract Galois theory and an abstract profinite group G . The key points of this formal theory are two fundamental morphisms (the degree map and the valuation) and the class field axiom (cohomological condition on the G -module A we work with).

In this article, the local object we study is the ℓ -adification of the multiplicative group of a local field defined by Jaulent in [2]. It is endowed with the logarithmic valuation (we recall its construction). We first define the degree map. We then prove that Neukirch's abstract theory applies. This allows us to define the logarithmic local symbol. In particular we get an explicit expression for the logarithmic local symbol on the ℓ -adification of \mathbb{Q}_p :

Proposition. *Let ζ be a root of unity of order a ℓ -th power, and $a \in \mathcal{R}_{\mathbb{Q}_p} = \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathbb{Q}_p^\times$. The logarithmic local symbol is:*

$$(a, (\mathbb{Q}_p(\zeta)/\mathbb{Q}_p)_\ell)(\zeta) = \zeta^{n_p}$$

with

$$n_p = \begin{cases} p^{v_p(a)} & \text{for } p \neq \ell \text{ and } p \neq \infty \\ (1 + \ell)^{-\tilde{v}_\ell(a)} & \text{for } p = \ell \\ \text{sgn}(a) & \text{for } p = \infty \end{cases}$$

where $(\mathbb{Q}(\zeta)/\mathbb{Q})_\ell$ denotes the projection on the ℓ -Sylow sub-group of $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

Thus we study the global case. Finally, after choosing the logarithmic uniformizer $\tilde{\pi}_\mathfrak{p}$, we obtain the explicit construction of the logarithmic Frobenius:

Definition. *Let L/K be an abelian ℓ -extension of number fields. Let \mathfrak{p} be a prime of K logarithmically unramified in L . The logarithmic Frobenius attached to \mathfrak{p} is:*

$$\left(\frac{\widetilde{L/K}}{\mathfrak{p}} \right) = ([\tilde{\pi}_\mathfrak{p}], L/K)$$

where $[\tilde{\pi}_\mathfrak{p}]$ is the image of the uniformizing element $\tilde{\pi}_\mathfrak{p}$, through the logarithmic global symbol defined on the ℓ -adic idele group $\mathcal{I}_K = \prod \mathcal{R}_{K_\mathfrak{p}}$.

We are now able to extend this map by multiplicativity. By this way, we obtain the logarithmic Artin map:

Definition. *Let L/K be a finite and abelian ℓ -extension and \mathfrak{p} a prime of K logarithmically unramified in L . Let $D\ell_K$ be the group of logarithmic divisors of K , $\tilde{\mathfrak{f}}_{L/K}$ the logarithmic global conductor of L/K and $D\ell_K^{\tilde{\mathfrak{f}}_{L/K}}$ the sub-module of logarithmic divisors prime to the conductor $\tilde{\mathfrak{f}}_{L/K}$. We define the logarithmic Artin map on $D\ell_K^{\tilde{\mathfrak{f}}_{L/K}}$ as follows:*

$$\left(\frac{\widetilde{L/K}}{\mathfrak{p}} \right) : \mathfrak{p} \in D\ell_K^{\tilde{\mathfrak{f}}_{L/K}} \mapsto \left(\frac{\widetilde{L/K}}{\mathfrak{p}} \right) \in \text{Gal}(L/K)$$

We study the properties of this map and give an expression of its kernel. We then focus on the quadratic case and generalize it to the case of a ℓ -extension.

Notations

Let ℓ be a fixed prime number. Let's introduce the notations.

For a local field $K_{\mathfrak{p}}$ with maximal ideal \mathfrak{p} and uniformizer $\pi_{\mathfrak{p}}$, we let:

- $\mathcal{R}_{K_{\mathfrak{p}}} = \varprojlim_k K_{\mathfrak{p}}^{\times} / K_{\mathfrak{p}}^{\times \ell^k}$: the ℓ -adification of the multiplicative group of a local field
- $\mathcal{U}_{K_{\mathfrak{p}}} = \varprojlim_k U_{\mathfrak{p}} / U_{\mathfrak{p}}^{\ell^k}$: the ℓ -adification of the group of units $U_{\mathfrak{p}}$ of $K_{\mathfrak{p}}$
- $U_{\mathfrak{p}}^1$: the group of principal units of $K_{\mathfrak{p}}$
- $\mu_{\mathfrak{p}}^0$: the subgroup of $U_{\mathfrak{p}}$, whose order is finite and prime to \mathfrak{p}
- $\mu_{\mathfrak{p}}$: the ℓ -Sylow subgroup of $\mu_{\mathfrak{p}}^0$

For a number field K we define:

- $\mathcal{R}_K = \mathbb{Z}_{\ell} \otimes_{\mathbb{Z}} K^{\times}$: the ℓ -adic group of principal ideles
- $\mathcal{J}_K = \prod_{\mathfrak{p} \in Pl_K}^{res} \mathcal{R}_{K_{\mathfrak{p}}}$: the ℓ -adic idele group
- $\mathcal{U}_K = \prod_{\mathfrak{p} \in Pl_K} \mathcal{U}_{K_{\mathfrak{p}}}$: the subgroup of units
- $\mathcal{C}_K = \mathcal{J}_K / \mathcal{R}_K$: the ℓ -adic idele class group

In the logarithmic context, we denote:

- $\hat{\mathbb{Q}}_{\mathfrak{p}}^c$: the cyclotomic $\widehat{\mathbb{Z}}$ -extension of $\mathbb{Q}_{\mathfrak{p}}$
- $\mathbb{Q}_{\mathfrak{p}}^c$: the cyclotomic \mathbb{Z}_{ℓ} -extension of $\mathbb{Q}_{\mathfrak{p}}$
- $\tilde{v}_{\mathfrak{p}}$: the logarithmic valuation attached to \mathfrak{p} sur $\mathcal{R}_{K_{\mathfrak{p}}}$
- $\tilde{\mathcal{U}}_{K_{\mathfrak{p}}} = \text{Ker}(\tilde{v}_{\mathfrak{p}})$: the sub-group of local logarithmic units
- $\tilde{\mathcal{U}}_K = \prod_{\mathfrak{p} \in Pl_K} \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}$: the sub-group of logarithmic units
- $\tilde{e}_{\mathfrak{p}} = [K_{\mathfrak{p}} : \hat{\mathbb{Q}}_{\mathfrak{p}}^c \cap K_{\mathfrak{p}}]$: the logarithmic absolute ramification index of \mathfrak{p}
- $\tilde{f}_{\mathfrak{p}} = [\hat{\mathbb{Q}}_{\mathfrak{p}}^c \cap K_{\mathfrak{p}} : \mathbb{Q}_{\mathfrak{p}}]$: the logarithmic absolute inertia degree of \mathfrak{p}
- $\tilde{e}_{L_{\mathfrak{p}}/K_{\mathfrak{p}}} = [L_{\mathfrak{p}} : \hat{K}_{\mathfrak{p}}^c \cap L_{\mathfrak{p}}]$: the relative logarithmic ramification index of \mathfrak{p}
- $\tilde{f}_{L_{\mathfrak{p}}/K_{\mathfrak{p}}} = [\hat{K}_{\mathfrak{p}}^c \cap L_{\mathfrak{p}} : K_{\mathfrak{p}}]$: the logarithmic relative inertia degree of \mathfrak{p}
- $\mathcal{J}_K^{(m)} = \prod_{\mathfrak{p} \nmid m} \mathcal{R}_{K_{\mathfrak{p}}} \prod_{\mathfrak{p} | m} \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^{v_{\mathfrak{p}}(m)}$
- $\mathcal{J}_K^m = \prod_{\mathfrak{p} \nmid m} \mathcal{R}_{K_{\mathfrak{p}}} \prod_{\mathfrak{p} | m} \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}$
- $\tilde{\mathcal{U}}_K^{(m)} = \prod_{\mathfrak{p} \nmid m} \mathcal{U}_{K_{\mathfrak{p}}} \prod_{\mathfrak{p} | m} \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^{v_{\mathfrak{p}}(m)}$
- $\mathcal{R}_K^{(m)} = \mathcal{R}_K \cap \mathcal{J}_K^{(m)}$

$$\tilde{div} : \alpha = (\alpha_{\mathfrak{p}}) \in \mathcal{J}_K^m \mapsto \tilde{div}(\alpha) = \prod \mathfrak{p}^{\tilde{v}_{\mathfrak{p}}(\alpha_{\mathfrak{p}})} \in D\ell_K^m$$

$D\ell_K^m = \tilde{div}(\mathcal{J}_K^m)$: logarithmic divisors prime to m

$\mathcal{P}\ell_K^{(m)} = \tilde{div}(\mathcal{R}_K^{(m)})$: principal logarithmic divisors attached to m

$\tilde{f}_{L/K}$: the logarithmic global conductor L/K

$\tilde{f}_{\mathfrak{p}}$: the logarithmic local conductor attached to \mathfrak{p}

$(\frac{\widehat{L/K}}{\mathfrak{p}})$: the logarithmic Frobenius attached to \mathfrak{p}

$\mathcal{A}\ell_{L/K}$ the logarithmic Artin group of L/K

1 Recall: Logarithmic ramification

We first recall the notion of logarithmic ramification developed by Jaulent in [2].

Let K, L be number fields, p a prime number, \mathfrak{p} a prime of K above p and \mathfrak{P} a prime of L lying above \mathfrak{p} . Let's denote $\widehat{\mathbb{Q}}_p^c$ the $\widehat{\mathbb{Z}}$ -cyclotomic extension of \mathbb{Q}_p , i.e. the compositum of all \mathbb{Z}_q -cyclotomic extensions of \mathbb{Q}_p for all primes q .

Definition 1. *Absolute and relative indexes : [2, definition 1.3]*

i) *the absolute and relative logarithmic ramification index of \mathfrak{p} are respectively:*

$$\tilde{e}_{\mathfrak{p}} = [K_{\mathfrak{p}} : \widehat{\mathbb{Q}}_p^c \cap K_{\mathfrak{p}}] \quad \tilde{e}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} = [L_{\mathfrak{P}} : \widehat{K}_{\mathfrak{p}}^c \cap L_{\mathfrak{P}}]$$

ii) *the absolute and relative logarithmic inertia degree of \mathfrak{p} are respectively:*

$$\tilde{f}_{\mathfrak{p}} = [\widehat{\mathbb{Q}}_p^c \cap K_{\mathfrak{p}} : \mathbb{Q}_p] \quad \tilde{f}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} = [\widehat{K}_{\mathfrak{p}}^c \cap L_{\mathfrak{P}} : K_{\mathfrak{p}}]$$

iii) *K/\mathbb{Q} is said logarithmically unramified at \mathfrak{p} if $\tilde{e}_{\mathfrak{p}} = 1$, which means $K_{\mathfrak{p}} \subseteq \widehat{\mathbb{Q}}_p^c$.*

iv) *L/K is said logarithmically unramified at \mathfrak{p} if $\tilde{e}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} = 1$, which implies $L_{\mathfrak{P}} \subseteq \widehat{K}_{\mathfrak{p}}^c$.*

v) *These indexes satisfy the relations: $\tilde{e}_{\mathfrak{P}} = \tilde{e}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} \cdot \tilde{e}_{\mathfrak{p}}$ et $\tilde{f}_{\mathfrak{P}} = \tilde{f}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} \cdot \tilde{f}_{\mathfrak{p}}$*

According to the diagramm: [7, 1.1.3]

$$\begin{array}{ccccc}
 & & & & L_{\mathfrak{P}} \\
 & & & \nearrow \tilde{f}_{\mathfrak{P}} & \downarrow \tilde{e}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} \\
 & & L_{\mathfrak{P}} \cap \widehat{\mathbb{Q}}_p^c & \xrightarrow{\tilde{e}_{\mathfrak{p}}} & \widehat{K}_{\mathfrak{p}}^c \cap L_{\mathfrak{P}} \\
 & \nearrow \tilde{e}_{\mathfrak{P}} & \downarrow & & \downarrow \tilde{f}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} \\
 \mathbb{Q}_p & \xrightarrow{\tilde{f}_{\mathfrak{p}}} & K_{\mathfrak{p}} \cap \widehat{\mathbb{Q}}_p^c & \xrightarrow{\tilde{e}_{\mathfrak{p}}} & K_{\mathfrak{p}}
 \end{array}$$

Proposition 1.0.1. [2, theorem 1.4] *With the previous notations, classical and logarithmic indexes are linked by this formula:*

$$\tilde{e}_{\mathfrak{p}} \cdot \tilde{f}_{\mathfrak{p}} = e_{\mathfrak{p}} \cdot f_{\mathfrak{p}} = [K_{\mathfrak{p}} : \mathbb{Q}_p]$$

Fundamental remark: [2, p. 4] Assume K/\mathbb{Q} is a finite ℓ -extension such that $[K : \mathbb{Q}] = \ell^n$ with $n \geq 1$. Then $\widehat{\mathbb{Q}}_p^c/\mathbb{Q}_p^c$ only contains sub-extensions of order prime to ℓ .

In particular the degree of $[\widehat{\mathbb{Q}}_p^c \cap K_{\mathfrak{p}} : \mathbb{Q}_p^c \cap K_{\mathfrak{p}}]$ is prime to ℓ and as it also divides ℓ^n , we deduce $[\widehat{\mathbb{Q}}_p^c \cap K_{\mathfrak{p}} : \mathbb{Q}_p^c \cap K_{\mathfrak{p}}] = 1$. The equality of fields $\widehat{\mathbb{Q}}_p^c \cap K_{\mathfrak{p}} = \mathbb{Q}_p^c \cap K_{\mathfrak{p}}$ implies this equivalence:

$$\tilde{e}_{\mathfrak{p}} = 1 \Leftrightarrow K_{\mathfrak{p}} \subseteq \mathbb{Q}_p^c \quad \text{and} \quad \tilde{f}_{\mathfrak{p}} = 1 \Leftrightarrow \mathbb{Q}_p^c \cap K_{\mathfrak{p}} = \mathbb{Q}_p^c.$$

As $\mathbb{Q}_p^c/\mathbb{Q}_p$ is a Galois extension, the previous condition means that the extensions $K_{\mathfrak{p}}$ et \mathbb{Q}_p^c are linearly separated on \mathbb{Q}_p .

As we work with ℓ -extensions, we may replace in the previous definitions $\widehat{\mathbb{Q}}_p^c$ by \mathbb{Q}_p^c .

2 The local case

2.1 The degree map

Let $K_{\mathfrak{p}}^{ab}$ be the maximal abelian pro- ℓ -extension of $K_{\mathfrak{p}}$. On the Galois group $\text{Gal}(K_{\mathfrak{p}}^{ab}(\zeta_{\ell^\infty})/K_{\mathfrak{p}})$, the Teichmüller's character ω is defined as the character of the action on roots of unity. It is defined for pro- ℓ -extensions as the restriction to the pro- ℓ -part of the whole character. We thus define the degree map:

$$\begin{aligned} \text{deg} : G = \text{Gal}(K_{\mathfrak{p}}^{ab}/K_{\mathfrak{p}}) &\rightarrow \mathbb{Z}_{\ell} \\ \phi &\mapsto \omega(\phi) \end{aligned}$$

where $K_{\mathfrak{p}}^{ab}$ is the maximal abelian pro- ℓ -extension of $K_{\mathfrak{p}}$.

We then follow Neukirch's abstract construction. If $L_{\mathfrak{p}}$ is a finite ℓ -extension of $K_{\mathfrak{p}}$, the logarithmic ramification index and the logarithmic inertia degree appear naturally:

$$\tilde{f}_{L_{\mathfrak{p}}/K_{\mathfrak{p}}} = [L_{\mathfrak{p}} \cap K_{\mathfrak{p}}^c : K_{\mathfrak{p}}] \quad \tilde{e}_{L_{\mathfrak{p}}/K_{\mathfrak{p}}} = [L_{\mathfrak{p}} : L_{\mathfrak{p}} \cap K_{\mathfrak{p}}^c]$$

Those definitions coincide with those given by Jaulent [2].

Remark: We replace here in the definitions $\hat{K}_{\mathfrak{p}}^c$ by $K_{\mathfrak{p}}^c$ because we work with ℓ -extensions.

2.2 The G -module

We introduce the ℓ -adification of the multiplicative group of a local field: $\mathcal{R}_{L_{\mathfrak{p}}} = \varprojlim_k L_{\mathfrak{p}}^{\times} / L_{\mathfrak{p}}^{\times \ell^k}$ defined by Jaulent [3, def.1.2]. The G -module we study is the same as before [5, §2.2]: $A = \varinjlim \mathcal{R}_{L_{\mathfrak{p}}}$, where $L_{\mathfrak{p}}$ runs through all finite sub-extensions of $K_{\mathfrak{p}}$. It can be canonically identified to: $A = \bigcup_{[L_{\mathfrak{p}}:K_{\mathfrak{p}}] < \infty} \mathcal{R}_{L_{\mathfrak{p}}}$. If $L_{\mathfrak{p}}$ is a finite extension of $K_{\mathfrak{p}}$, $A_{L_{\mathfrak{p}}} = \mathcal{R}_{L_{\mathfrak{p}}}$ is the $\text{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}})$ -module we are going to work with.

2.3 Logarithmic valuation and ℓ -adic degree

Definition 2. Let K be a finite extension of \mathbb{Q} , p a prime number. Let's denote $\widehat{\mathbb{Q}}_p^c$ the $\widehat{\mathbb{Z}}$ -cyclotomic extension of \mathbb{Q}_p and let \mathfrak{p} be a prime of K above p .

$$i) \text{ The } \ell\text{-adic degree of } p \text{ is: } \text{deg}_{\ell}(p) = \begin{cases} \text{Log}_{I_w}(p) & \text{if } p \neq \ell \\ \text{Log}_{I_w}(1 + \ell) & \text{if } p = \ell \end{cases}$$

$$ii) \text{ The } \ell\text{-adic degree of } \mathfrak{p} \text{ is: } \text{deg}(\mathfrak{p}) = \tilde{f}_{\mathfrak{p}} \cdot \text{deg}_{\ell}(p)$$

iii) Let $v_{\mathfrak{p}}$ be the usual normalized valuation on $K_{\mathfrak{p}}$, the absolute principal ℓ -adic valuation, defined on $K_{\mathfrak{p}}^{\times}$ is:

$$\text{if } \mathfrak{p} \nmid \ell \text{ then } |x_{\mathfrak{p}}| = \langle N_{\mathfrak{p}}^{-v_{\mathfrak{p}}(x)} \rangle$$

$$\text{if } \mathfrak{p} | \ell \text{ then } |x_{\mathfrak{p}}| = \langle N_{K_{\mathfrak{p}}/\mathbb{Q}_{\ell}}(x) N_{\mathfrak{p}}^{-v_{\mathfrak{p}}(x)} \rangle$$

where $N_{\mathfrak{p}}$ is the absolute norm of \mathfrak{p} and $u \rightarrow \langle u \rangle$ is the canonical surjection from $\mathbb{Z}_{\ell}^{\times}$ to the group of principal units.

iv) the logarithmic valuation attached to \mathfrak{p} is: $\tilde{v}_{\mathfrak{p}}(x) = -\text{Log}_{I_w}(N_{K_{\mathfrak{p}}/\mathbb{Q}_{\ell}}(x)) / \text{deg}_{\ell}(\mathfrak{p})$, defined on $\mathcal{R}_{K_{\mathfrak{p}}}$ valued to \mathbb{Z}_{ℓ} , [1][Prop.1.2]

Remark: This definition of the logarithmic valuation is here different from the one given by Jaulent [2, définition 1.1]. Indeed the ℓ -adic degree of ℓ , $\deg_\ell(\ell)$, is equal to ℓ initially and to $\text{Log}_{Iw}(1 + \ell)$ in our case. This is motivated by the fact that we want an explicit expression for the logarithmic uniformizers.

Proposition 2.3.1. [2, proposition 1.2]

Let p be a prime number, \mathbb{Q}_p^c the $\hat{\mathbb{Z}}$ -cyclotomic extension of \mathbb{Q}_p , $K_{\mathfrak{p}}/\mathbb{Q}_p$ a finite extension of degree $d_{\mathfrak{p}}$ and $|\cdot|_{\mathfrak{p}}$ the principal absolute ℓ -adic valuation on $K_{\mathfrak{p}}^\times$. Then:

- i) for all $x \in K_{\mathfrak{p}}^\times$ the expression $h_{\mathfrak{p}}(x) = -\text{Log}(|x|_{\mathfrak{p}}/d_{\mathfrak{p}} \cdot \deg_\ell(p))$ does not depend on the choice of the extension of \mathbb{Q}_p containing x ;
- ii) the restriction $h_{\mathfrak{p}}$ of h_p to the multiplicative group of $K_{\mathfrak{p}}^\times$, yields a \mathbb{Z}_ℓ -morphism from $\mathcal{R}_{K_{\mathfrak{p}}}$ to \mathbb{Q}_ℓ , whose kernel is $\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}$ and whose image is the \mathbb{Z}_ℓ -lattice $1/\tilde{e}_{\mathfrak{p}} \cdot \mathbb{Z}_\ell$
- iii) the logarithmic valuation satisfies: $\tilde{v}_{\mathfrak{p}} = \tilde{e}_{\mathfrak{p}} \cdot h_{\mathfrak{p}}$.
- iv) let \mathfrak{p} be a prime of K which is not above ℓ , classical and logarithmic valuations are proportional:

$$\tilde{v}_{\mathfrak{p}} = \frac{f_{\mathfrak{p}}}{f_{\mathfrak{p}}} \cdot v_{\mathfrak{p}} = \frac{\tilde{e}_{\mathfrak{p}}}{e_{\mathfrak{p}}} \cdot v_{\mathfrak{p}}$$

Proposition 2.3.2. The logarithmic valuation $\tilde{v}_{\mathfrak{p}}$ checks two properties:

- i) $\tilde{v}_{\mathfrak{p}}(\mathcal{R}_{K_{\mathfrak{p}}}) = Z$ with $\mathbb{Z} \subset Z$ et $Z/n \cdot Z \simeq \mathbb{Z}/n \cdot \mathbb{Z}$ for all n ;
- ii) $\tilde{v}_{\mathfrak{p}}(N_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} \mathcal{R}_{L_{\mathfrak{q}}}) = \tilde{f}_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} Z$ for all finite extension $L_{\mathfrak{q}}$ of $K_{\mathfrak{p}}$.

Thus the logarithmic valuation is henselian with respect to the degree map, according to Neukirch's definition.

Proof. For the first criterium (i) we use the definition of the logarithmic valuation :

$$\tilde{v}_{\mathfrak{p}}(\mathcal{R}_{K_{\mathfrak{p}}}) = \mathbb{Z}_\ell ; \text{ ainsi } Z = \mathbb{Z}_\ell \text{ et } Z/n \cdot Z \simeq \mathbb{Z}/n \cdot \mathbb{Z} \text{ for all } n.$$

For the second criterium (ii) we use this diagramm [2, proposition 1.1] :

$$\begin{array}{ccccc} K_{\mathfrak{p}}^\times & \xrightarrow{\text{extension}} & L_{\mathfrak{q}}^\times & \xrightarrow{\text{Norm}} & K_{\mathfrak{p}}^\times \\ \downarrow & & & & \downarrow h_{\mathfrak{p}} \\ \mathbb{Q}_\ell & \longrightarrow & \mathbb{Q}_\ell & \xrightarrow{[L_{\mathfrak{q}}:K_{\mathfrak{p}}]} & \mathbb{Q}_\ell \end{array}$$

It follows $h_{\mathfrak{p}}(N_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} \mathcal{R}_{L_{\mathfrak{q}}}) = [L_{\mathfrak{q}} : K_{\mathfrak{p}}] h_{\mathfrak{p}}$. From, $\tilde{v}_{\mathfrak{p}}(N_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} \mathcal{R}_{L_{\mathfrak{q}}}) = \tilde{e}_{\mathfrak{p}} h_{\mathfrak{p}}(N_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} \mathcal{R}_{L_{\mathfrak{q}}})$ we deduce $\tilde{v}_{\mathfrak{p}}(N_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} \mathcal{R}_{L_{\mathfrak{q}}}) = [L_{\mathfrak{q}} : K_{\mathfrak{p}}] \tilde{e}_{\mathfrak{p}} h_{\mathfrak{p}}(\mathcal{R}_{L_{\mathfrak{q}}})$ thus $\tilde{v}_{\mathfrak{p}}(N_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} \mathcal{R}_{L_{\mathfrak{q}}}) = [L_{\mathfrak{q}} : K_{\mathfrak{p}}] \tilde{e}_{\mathfrak{p}}/\tilde{e}_{\mathfrak{q}} \mathbb{Z}_\ell$ by [2, proposition 1.2], then $h_{\mathfrak{q}}(\mathcal{R}_{L_{\mathfrak{q}}}) = 1/\tilde{e}_{\mathfrak{q}} \mathbb{Z}_\ell$. According to [proposition 5.1.1], we get:

$$[L_{\mathfrak{q}} : K_{\mathfrak{p}}] = \tilde{f}_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} \tilde{e}_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} ; \text{ so } [L_{\mathfrak{q}} : K_{\mathfrak{p}}] \tilde{e}_{\mathfrak{p}}/\tilde{e}_{\mathfrak{q}} = \frac{\tilde{f}_{\mathfrak{q}} \tilde{e}_{\mathfrak{q}} \tilde{e}_{\mathfrak{p}}}{f_{\mathfrak{p}} \tilde{e}_{\mathfrak{p}} \tilde{e}_{\mathfrak{q}}} \text{ Thus } \tilde{v}_{\mathfrak{p}}(N_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} \mathcal{R}_{L_{\mathfrak{q}}}) = \frac{\tilde{f}_{\mathfrak{q}}}{f_{\mathfrak{p}}} \mathbb{Z}_\ell$$

and finally $\tilde{v}_{\mathfrak{p}}(N_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} \mathcal{R}_{L_{\mathfrak{q}}}) = \tilde{f}_{L_{\mathfrak{q}}/K_{\mathfrak{p}}} Z$. □

2.4 The logarithmic local symbol

Theorem 2.4.1. *(deg, \tilde{v}_p) is a class field theory and \mathcal{R}_{K_p} satisfies the class field axiom [5, theorem 2.5.1]. Thus for all finite and abelian ℓ -extension $L_{\mathfrak{F}}$ of K_p (finite extension of \mathbb{Q}_p) we have this isomorphism :*

$$\text{Gal}(L_{\mathfrak{F}}/K_p) \simeq \mathcal{R}_{K_p}/N_{L_{\mathfrak{F}}/K_p} \mathcal{R}_{L_{\mathfrak{F}}}$$

Definition 3. *It allows us to define a surjective homomorphism: the logarithmic local symbol*

$$(\cdot, L_{\mathfrak{F}}/K_p) : \mathcal{R}_{K_p} \longrightarrow \text{Gal}(L_{\mathfrak{F}}/K_p)$$

Moreover, for all archimedean place p of \mathbb{Q} , we have [2, i), p. 4]

$$\mathcal{R}_{\mathbb{Q}_p} = \begin{cases} \mu_p \cdot p^{\mathbb{Z}_\ell} & \text{for } p \neq \ell \\ (1 + \ell)^{\mathbb{Z}_\ell} \cdot \ell^{\mathbb{Z}_\ell} & \text{for } p = \ell \end{cases}$$

Thus we have a decomposition of the shape : $\mathcal{R}_{\mathbb{Q}_p} \simeq \tilde{\mathcal{U}}_{\mathbb{Q}_p} \cdot \tilde{\pi}_p^{\mathbb{Z}_\ell}$.

Proposition 2.4.1. *We have an explicit expression for the logarithmic local symbol. Let ζ be a root of unity of ℓ -th power and $a \in \mathcal{R}_{\mathbb{Q}_p}$. The logarithmic local symbol is:*

$$(a, (\mathbb{Q}_p(\zeta)/\mathbb{Q}_p))_\ell = \zeta^{n_p}$$

with

$$n_p = \begin{cases} p^{v_p(a)} & \text{for } p \neq \ell \text{ and } p \neq \infty \\ (1 + \ell)^{-\tilde{v}_\ell(a)} & \text{for } p = \ell \\ \text{sgn}(a) & \text{for } p = \infty \end{cases}$$

where $(\mathbb{Q}(\zeta)/\mathbb{Q})_\ell$ denotes the projection on the ℓ -Sylow sub-group of $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

Remark: If p is real and $\ell = 2$ then we have $\mathcal{R}_{\mathbb{Q}_\infty} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}}$, it is trivial in any other cases, [3, Prop.1.2] : $\text{sgn}(a)$ is ± 1 in the first case and 1 in the other cases.

Proof. Let ζ be a ℓ^m -th root of unity, with $\ell^m \neq 2$. We take $a \in \mathcal{R}_{\mathbb{Q}_p}$ and write $a = u_p \cdot p^{v_p(a)}$ for $p \neq \ell$, where v_p is the usual normalized valuation of \mathbb{Q}_p , which coincide in this particular case with the logarithmic valuation. For $p = \ell$, we write $a = \ell^{v_\ell(a)} \cdot (1 + \ell)^{\tilde{v}_\ell(a)}$. For $p \neq \ell$ and $p \neq \infty$ the extension $\mathbb{Q}_p(\zeta)/\mathbb{Q}_p$ is an unramified extension. The fundamental principle [1, Th. 2.6, p. 25] states that the local symbol associates the Frobenius elements to the uniformizing elements; we have already seen that $(p, (\mathbb{Q}_p(\zeta)/\mathbb{Q}_p))_\ell$ is the usual Frobenius automorphism $\phi_p : \zeta \longrightarrow \zeta^p$. Moreover this diagramm is commutative:

$$\begin{array}{ccc} K_p^\times & \xrightarrow{(\cdot; \text{Gal}(L_{\mathfrak{F}}/K_p))} & \text{Gal}(L_{\mathfrak{F}}/K_p) \\ \downarrow & & \downarrow \\ \mathcal{R}_{K_p} & \xrightarrow{(\cdot; \text{Gal}(L_{\mathfrak{F}}/K_p))_\ell} & \text{Gal}(L_{\mathfrak{F}}/K_p)_\ell \end{array}$$

where the symbol on the top is the usual local symbol and the symbol on the bottom is the ℓ -adic local symbol. That is why we deduce:

$$(a, (\mathbb{Q}_p(\zeta)/\mathbb{Q}_p)_\ell) \zeta = \zeta^{n_p}$$

with

$$n_p = \begin{cases} p^{v_p(a)} & \text{for } p \neq \ell \text{ and } p \neq \infty \\ \text{sgn}(a) & \text{for } p = \infty \end{cases}$$

But we want to have $[a, (\mathbb{Q}(\zeta)/\mathbb{Q})_\ell] = 1$ for $a \in \mathcal{R}_\mathbb{Q}$ in order to be able to define the valuation in the logarithmic global context:

$$[a, (\mathbb{Q}(\zeta)/\mathbb{Q})_\ell]\zeta = \prod_{\mathfrak{p}} (a, (\mathbb{Q}_{\mathfrak{p}}(\zeta)/\mathbb{Q}_{\mathfrak{p}})_\ell)\zeta = \zeta^\alpha.$$

Thus, by the product formula, taking $n_\ell = (1 + \ell)^{-v_\ell(a)}$, we get : $\alpha = \prod_p n_p = \text{sgn}(a) \cdot \prod_{p \neq \infty} p^{v_p(a)} \cdot \ell^{-v_\ell(a)} \cdot (1 + \ell)^{-\tilde{v}_\ell(a)} = a \cdot a^{-1} = 1$, as wanted. \square

3 The global case

3.1 G and the G -module

Let G be the Galois group of the maximal abelian pro- ℓ -extension of \mathbb{Q} . We introduce the ℓ -adic idele class group for a given number field K refer to Jaulent [3, definition 1.4]. Like previously [5, §3.4] the G -module is the union of all ℓ -adic idele class groups \mathcal{C}_K , where K runs through all finite extensions of \mathbb{Q} : $\bigcup_{[K:\mathbb{Q}] < \infty} \mathcal{C}_K$ and \mathcal{C}_L is our $\text{Gal}(L/K)$ -module.

3.2 The degree map

We fix an isomorphism: $\text{Gal}(\tilde{\mathbb{Q}}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}$. This allows to define:

$$\begin{aligned} \widetilde{\text{deg}} & : G = \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) & \rightarrow & \hat{\mathbb{Z}} \\ & \phi & \mapsto & \phi|_{\tilde{\mathbb{Q}}} \end{aligned}$$

where \mathbb{Q}^{ab} is the maximal abelian pro- ℓ -extension of \mathbb{Q} . Let K/\mathbb{Q} be a finite extension, we define: $f_K = [K \cap \tilde{\mathbb{Q}} : \mathbb{Q}]$ and we get, by analogy with the abstract case [1, ch.2], a surjective homomorphism $\widetilde{\text{deg}}_K = \frac{1}{f_K} \cdot \widetilde{\text{deg}}$ such that $\widetilde{\text{deg}}_K : G_K \rightarrow \hat{\mathbb{Z}}$ defines the $\hat{\mathbb{Z}}$ -extension \tilde{K} of K .

3.3 The valuation

Due to the property, $\forall a \in \mathcal{R}_K$, $[a, \tilde{K}/K] = 1$. we put this definition:

Definition 4. We define the valuation $\tilde{v}_K : \mathcal{C}_K \rightarrow \hat{\mathbb{Z}}$ as follows:

$$\mathcal{C}_K \xrightarrow{[\cdot, \tilde{K}/K]} G(\tilde{K}/K) \xrightarrow{\widetilde{\text{deg}}_K} \hat{\mathbb{Z}}$$

Lemma 3.3.1. This valuation \tilde{v}_K is henselian with respect to the degree map $\widetilde{\text{deg}}$.

Proof. Arguments are the same as [5, lemma 3.6.4] replacing $\mathcal{U}_{K_{\mathfrak{p}}}$ by $\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}$. \square

Theorem 3.3.1. $(\widetilde{\text{deg}}, \tilde{v}_K)$ is a class field theory and \mathcal{C}_K [4, theorem 3.3.1] satisfies the class field axiom. Thus for all finite and abelian ℓ -extensions L of K , we get an isomorphism:

$$\text{Gal}(L/K) \simeq \mathcal{I}_K / N_{L/K}(\mathcal{I}_L) \mathcal{R}_K$$

Definition 5. This allows to define a surjective homomorphism, called the global logarithmic symbol:

$$(\cdot, L/K) : \mathcal{J}_K \longrightarrow \text{Gal}(L/K)$$

4 The logarithmic Frobenius

4.1 Logarithmic uniformizing elements on \mathbb{Q}_p

Logarithmic uniformizing elements on $\mathcal{R}_{\mathbb{Q}_p}$:

If $p \nmid \ell$: the classical uniformizing element p is also a uniformizing element for the logarithmic valuation.

If $p = \ell$: due to the expression of $\mathcal{R}_{\mathbb{Q}_\ell}$, we have

$$\mathcal{R}_{\mathbb{Q}_\ell} = \mathcal{U}_{\mathbb{Q}_\ell} \tilde{\mathcal{U}}_{\mathbb{Q}_\ell}$$

with $\tilde{\mathcal{U}}_{\mathbb{Q}_\ell} \simeq \ell^{\mathbb{Z}_\ell}$ and $\mathcal{U}_{\mathbb{Q}_\ell} \simeq 1 + \ell\mathbb{Z}_\ell$.

We consider a logarithmic uniformizing element $\tilde{\ell}$ such that

$$\tilde{v}_\ell(\tilde{\ell}) = 1 \text{ et } \tilde{\ell} \in \mathcal{U}_{\mathbb{Q}_\ell}.$$

thus we obtain the decomposition:

$$\mathcal{R}_{\mathbb{Q}_\ell} = \tilde{\ell}^{\mathbb{Z}_\ell} \ell^{\mathbb{Z}_\ell}.$$

On \mathbb{Q}_ℓ **this is the choice of the denominator** in the expression of the logarithmic valuation, i.e. the ℓ -adic degree of ℓ , which **enforces** $\tilde{\ell}$. The condition $\text{Log}(\tilde{\ell}) = \text{deg}_\ell(\ell)$, which comes from $\tilde{v}_\ell(\tilde{\ell}) = 1$, defines $\tilde{\ell}$ up to a logarithmic unit. But we have $\tilde{\mathcal{U}}_{\mathbb{Q}_\ell} \cap \mathcal{U}_{\mathbb{Q}_\ell} = 1$, thus $\tilde{\ell}$ is determined by the choice of the denominator.

For instance, if we choose the ℓ -adic degree of ℓ as equal to $\text{Log}(1 + \ell)$, we get: $\tilde{\ell} = 1 + \ell$.

4.2 Logarithmic uniformizing elements on \mathcal{R}_{K_p}

If $\mathfrak{p} \nmid \ell$, the classical uniformizing element $\pi_{\mathfrak{p}}$ is also a uniformizer for the logarithmic valuation, in this case both valuations are proportional: $\pi_{\mathfrak{p}} = \tilde{\pi}_{\mathfrak{p}}$.

If $\mathfrak{p} | \ell$, then we have in this case $\mathcal{R}_{K_p} \simeq U_{\mathfrak{p}}^1 \pi_{\mathfrak{p}}^{\mathbb{Z}_\ell}$. By definition a uniformizing element is an element $\tilde{\pi}_{\mathfrak{p}}$ of \mathcal{R}_{K_p} such that :

$$\text{Log}_{Iw}(N_{K_p/\mathbb{Q}_\ell}(\tilde{\pi}_{\mathfrak{p}})) = \tilde{f}_{\mathfrak{p}} \text{deg}_\ell(\ell) = \text{Log}_{Iw}(\tilde{\ell}^{\tilde{f}_{\mathfrak{p}}}).$$

This defines $\tilde{\pi}_{\mathfrak{p}}$ up to a logarithmic unit.

4.3 The logarithmic conductor

In usual local class field theory, we have the decreasing filtration of the group of units U_{K_p} of K_p^\times :

$$U_{K_p}^{(n)} = 1 + \mathfrak{p}^n$$

where \mathfrak{p} is the maximal ideal of the ring of integers.

$(U_{K_p}^{(n)})_n$ are a system of neighborhoods of 1 in K_p^\times . Thanks to this filtration, we define the local and the global conductor attached to an extension.

Let's give a decreasing filtration of the group of logarithmic units $(\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^n)_{n \in \mathbb{N}}$ with $\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^0 = \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}$.

Definition 6. We are now able to define a logarithmic local and global conductor, as follows:

- i) Let $L_{\mathfrak{p}}/K_{\mathfrak{p}}$ be an abelian ℓ -extension and n the smallest integer such that $\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^n \subseteq N_{L_{\mathfrak{p}}/K_{\mathfrak{p}}}(\mathcal{R}_{L_{\mathfrak{p}}})$ then the ideal:

$$\tilde{\mathfrak{f}}_{\mathfrak{p}} = \mathfrak{p}^n$$

defines the logarithmic local conductor attached to this extension.

- ii) Let L/K be a finite and abelian ℓ -extension, the global logarithmic conductor is:

$$\tilde{\mathfrak{f}}_{L/K} = \prod_{\mathfrak{p}} \tilde{\mathfrak{f}}_{\mathfrak{p}}$$

Proposition 4.3.1. The \mathfrak{p} -conductor $\tilde{\mathfrak{f}}_{\mathfrak{p}}$ is trivial if and only if the extension L/K is logarithmically unramified at \mathfrak{p} . The logarithmic global conductor $\tilde{\mathfrak{f}}_{L/K}$ contains all the primes of K which are logarithmically unramified in L and only those. Besides, if M is between K and L then $\tilde{\mathfrak{f}}_{M/K}$ divides $\tilde{\mathfrak{f}}_{L/K}$.

Proof. If \mathfrak{p} is a prime of K logarithmically unramified in L , we have $\tilde{e}_{L_{\mathfrak{p}}/K_{\mathfrak{p}}} = 1$, and due to [1]Prop.2.2.p.22 :

$$H^0(\text{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}}), \tilde{\mathcal{U}}_{L_{\mathfrak{p}}}) = 1;$$

it follows:

$$\tilde{\mathcal{U}}_{K_{\mathfrak{p}}} = N_{L_{\mathfrak{p}}/K_{\mathfrak{p}}}(\tilde{\mathcal{U}}_{L_{\mathfrak{p}}}),$$

which means

$$\tilde{\mathcal{U}}_{K_{\mathfrak{p}}} \subseteq N_{L_{\mathfrak{p}}/K_{\mathfrak{p}}}(\mathcal{R}_{L_{\mathfrak{p}}}).$$

Conversely, let's assume that $\tilde{\mathfrak{f}}_{\mathfrak{p}}$ is trivial, we deduce $\tilde{\mathcal{U}}_{K_{\mathfrak{p}}} \subseteq N_{L_{\mathfrak{p}}/K_{\mathfrak{p}}}(\mathcal{R}_{L_{\mathfrak{p}}})$. Let $n = [\mathcal{R}_{K_{\mathfrak{p}}} : N_{L_{\mathfrak{p}}/K_{\mathfrak{p}}}(\mathcal{R}_{L_{\mathfrak{p}}})]$; from $\tilde{\pi}_{\mathfrak{p}}^n \in N_{L_{\mathfrak{p}}/K_{\mathfrak{p}}}(\mathcal{R}_{L_{\mathfrak{p}}})$ we get $(\tilde{\pi}_{\mathfrak{p}}^n)\tilde{\mathcal{U}}_{K_{\mathfrak{p}}} \subseteq N_{L_{\mathfrak{p}}/K_{\mathfrak{p}}}(\mathcal{R}_{L_{\mathfrak{p}}})$. Thus it follows $L_{\mathfrak{p}} \subseteq M$, where M is the class field of $(\tilde{\pi}_{\mathfrak{p}}^n)\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}$ i.e. $N_{M/K_{\mathfrak{p}}}(\mathcal{R}_M) = (\tilde{\pi}_{\mathfrak{p}}^n) \cdot \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}$. But $\tilde{\mathcal{U}}_{K_{\mathfrak{p}}} \subseteq N_{M/K_{\mathfrak{p}}}(\mathcal{R}_M)$ which means that the \mathfrak{p} -conductor is trivial. Applying the first part, we deduce that $M/K_{\mathfrak{p}}$ is logarithmically unramified. By ℓ -adic class field theory, we obtain: $\text{Gal}(M/K_{\mathfrak{p}}) \simeq (\tilde{\pi}_{\mathfrak{p}})\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}/(\tilde{\pi}_{\mathfrak{p}}^n)\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}$. Consequently, M is the ℓ -extension of degree n logarithmically unramified; and from $L_{\mathfrak{p}} \subseteq M$, we conclude that \mathfrak{p} is logarithmically unramified in L . \square

Examples

- 1) For \mathbb{Q}_{ℓ} , we have $\tilde{\mathcal{U}}_{\mathbb{Q}_{\ell}} \simeq \ell^{\mathbb{Z}_{\ell}} \simeq \mathbb{Z}_{\ell}$, that is why we get a filtration of $\tilde{\mathcal{U}}_{\mathbb{Q}_{\ell}}$ by raising in $\tilde{\mathcal{U}}_{\mathbb{Q}_{\ell}}$ the canonical filtration \mathbb{Z}_{ℓ} by $\ell^n \mathbb{Z}_{\ell}$, for $n \in \mathbb{N}$.
- 2) Given a local field $K_{\mathfrak{p}}$, we get a filtration of the logarithmic units $\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}$ as follows:

- if $\mathfrak{p}|\ell$ by raising the local norm $N_{K_{\mathfrak{p}}/\mathbb{Q}_{\ell}}$ the filtration of units on $\mathbb{Q}_{\ell} : \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^n = \{x \in \mathcal{R}_{K_{\mathfrak{p}}} \mid N_{K_{\mathfrak{p}}/\mathbb{Q}_{\ell}}(x) \in (\ell)^{\ell^n \mathbb{Z}_{\ell}}\}$. So, we have $\bigcap_n \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^n = \{x \in \mathcal{R}_{K_{\mathfrak{p}}} \mid N_{K_{\mathfrak{p}}/\mathbb{Q}_{\ell}}(x) = 1\}$. But by class field theory, the compositum $K_{\mathfrak{p}}\mathbb{Q}_{\ell}^{ab}$ is fixed by the kernel of the norm. Thus the decreasing sequence $(\tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^n)_n$ is not exhaustive.
- if $p \nmid \ell$, as $\tilde{\mathcal{U}}_{K_{\mathfrak{p}}} \simeq \mu_{\mathfrak{p}}$ the natural filtration is $0 \subset \mu_{\mathfrak{p}}$.

4.4 The logarithmic Artin map

Let K be a number field, the ℓ -group of logarithmic divisors of K is [2, Def.2.1]:

$$\mathcal{D}l_K = \mathcal{J}_K / \tilde{\mathcal{U}}_K \simeq \bigoplus_{\mathfrak{p}} \mathbb{Z}_{\ell} \mathfrak{p}$$

through the logarithmic valuations $(\tilde{v}_{\mathfrak{p}})_{\mathfrak{p}}$ it can be identified to a free \mathbb{Z}_{ℓ} -module built on finite primes of K .

Definition 7. Let L/K be a finite and abelian ℓ -extension, \mathfrak{p} a prime of K logarithmically unramified in L , $\mathcal{D}l_K$ be the group of logarithmic divisors of K , $\tilde{\mathfrak{f}}_{L/K}$ be the global logarithmic conductor L/K and $\mathcal{D}l_K^{\tilde{\mathfrak{f}}_{L/K}}$ the sub-group of logarithmic divisors prime to $\tilde{\mathfrak{f}}_{L/K}$. We define the logarithmic Frobenius of a prime \mathfrak{p} logarithmically unramified, as follows:

$$\left(\frac{\widetilde{L/K}}{\mathfrak{p}} \right) = ([\tilde{\pi}_{\mathfrak{p}}], L/K)$$

where $\tilde{\pi}_{\mathfrak{p}}$ is the logarithmic uniformising element, defined in proposition 3.7.3 and $[\tilde{\pi}_{\mathfrak{p}}]$ the image of $\tilde{\pi}_{\mathfrak{p}}$ in \mathcal{J}_K .

We extend this map by multiplicativity:

$$\begin{aligned} \left(\frac{\widetilde{L/K}}{\cdot} \right) : \mathcal{D}l_K^{\tilde{\mathfrak{f}}_{L/K}} &\rightarrow \text{Gal}(L/K) \\ \mathfrak{p} &\mapsto \left(\frac{\widetilde{L/K}}{\mathfrak{p}} \right) \end{aligned}$$

Few remarks:

- 1) The previous application is extended by multiplicativity to $\mathcal{D}l_K^{\tilde{\mathfrak{f}}_{L/K}}$ as by hypothesis L/K is a ℓ -extension.
- 2) This map is a surjective \mathbb{Z}_{ℓ} -morphism, due to the surjectivity of the global symbol.
- 3) The motivation of this definition is the fact that in abstract class field theory [1, Prop.3.4], if $a \in A_K$, $(\cdot, L/K)$ the norm residue symbol of L/K satisfies:

$$(a, \tilde{K}/K) = \phi_K^{v_K(a)}$$

where ϕ_K is the generic Frobenius of \tilde{K}/K .

- 4) When $\mathfrak{p} \nmid \ell$, the usual valuation and the logarithmic one are proportional: consequently $\tilde{\pi}_{\mathfrak{p}} = \pi_{\mathfrak{p}}$, the \mathbb{Z}_{ℓ} -unramified extension and the \mathbb{Z}_{ℓ} -cyclotomic one are the same. Thus the classical Artin symbol and the logarithmic one coincide on ideals.

Proposition 4.4.1. *Properties of the logarithmic Artin map*

Let L/K be a finite and abelian ℓ -extension, $\tilde{f}_{L/K}$ the logarithmic global conductor. Then we have:

- i) If M is between L and K , the restriction of $(\widetilde{\frac{L/K}{\mathfrak{a}}})$ to M is $(\widetilde{\frac{M/K}{\mathfrak{a}}})$, for all $\mathfrak{a} \in \mathcal{D}\ell_{K/\tilde{f}_{L/K}}^{\tilde{f}_{L/K}}$
- ii) Let L and L' be abelian ℓ -extensions, let's consider $\text{Gal}(LL'/K)$ as a sub-group of $\text{Gal}(L/K) \times \text{Gal}(L'/K)$, through the map $\sigma \in \text{Gal}(LL'/K) \rightarrow (\sigma|_L, \sigma|_{L'}) \in \text{Gal}(L/K) \times \text{Gal}(L'/K)$, then $(\widetilde{\frac{LL'/K}{\mathfrak{a}}})$ is $((\widetilde{\frac{L/K}{\mathfrak{a}}}), (\widetilde{\frac{L'/K}{\mathfrak{a}}}))$ for all $\mathfrak{a} \in \mathcal{D}\ell_{LL'/K}^{\tilde{f}_{LL'/K}}$
- iii) Let K' be any sub-extension of K , and let's consider $\text{Gal}(LK'/K')$ as isomorphic, by restriction to K , to a sub-group of $\text{Gal}(L/K)$. The restriction of $(\widetilde{\frac{LK'/K'}{\mathfrak{a}'}})$ to K is $(\widetilde{\frac{K}{N_{K'/K}\mathfrak{a}'}})$ for all $\mathfrak{a}' \in \mathcal{D}\ell_{K'}^{\tilde{f}_{L/K'}}$
- iv) In particular, if M is a field between L and K , we have $(\widetilde{\frac{L/M}{\mathfrak{U}}}) = (\widetilde{\frac{L/K}{N_M\mathfrak{U}}})$, for all $\mathfrak{U} \in \mathcal{D}\ell_M^{\tilde{f}_{L/K}}$.

Proof. i) It suffices to check the property on each prime \mathfrak{p} . By the functoriality property of the reciprocity map, we have: $\text{res}_M \circ \phi_{L/K}(\tilde{\pi}_{\mathfrak{p}}) = \phi_{M/K}(\tilde{\pi}_{\mathfrak{p}})$.

ii) is a consequence of i)

iii) We have to prove the equality between $(\widetilde{\frac{LK'/K'}{\mathfrak{a}'}})|_K = \phi_{LK'/K'}(\mathfrak{a}')$ and $(\widetilde{\frac{L}{N_{K'}\mathfrak{a}'}}) = \phi_{L/K}(N_{K'}(\mathfrak{a}'))$ for all $\mathfrak{a}' \in \mathcal{D}\ell_{K'}^{\tilde{f}_{L/K'}}$. By the functoriality property of the reciprocity map, we obtain this commutative diagramm:

$$\begin{array}{ccc} \text{Gal}(L'/K') & \xrightarrow{r_{L'/K'}} & \mathcal{J}_{K'}/N_{L'/K'}(\mathcal{J}_{L'})\mathcal{R}_{K'} \\ \downarrow & & \downarrow N_{K'/K} \\ \text{Gal}(L/K) & \xrightarrow{r_{L/K}} & \mathcal{J}_K/N_{L/K}(\mathcal{J}_L)\mathcal{R}_K \end{array}$$

where $r_{L/K}$ is the reciprocity map for global ℓ -adic class field theory. Consequently we have the following diagramm for the logarithmic global symbol:

$$\begin{array}{ccc} \mathcal{J}_{K'} & \xrightarrow{(\cdot, L'/K')} & \text{Gal}(LK'/K') \\ N_{K'/K} \downarrow & & \downarrow \text{res} \\ \mathcal{J}_K & \xrightarrow{(\cdot, L/K)} & \text{Gal}(L/K) \end{array}$$

and we deduce the property from $\text{res} \circ (\cdot, LK'/K') = (\cdot, L/K) \circ N_{K'/K}$.

iv) is a particular case of iii) taking $K' = L$

□

Definition 8. The kernel $\mathcal{A}_{L/K}$ of the previous application $(\widetilde{\frac{L/K}{\cdot}})$ is called the Artin logarithmic sub-module. For every modulus \mathfrak{m} divisible by the global logarithmic conductor of L/K , we put $\mathcal{A}_{L/K, \mathfrak{m}} = \mathcal{A}_{L/K} \cap \mathcal{D}\ell_K^{\mathfrak{m}}$.

Definition 9. Let L/K be an abelian ℓ -extension, \mathfrak{m} a modulus dividing the logarithmic global conductor, we define:

$$\mathcal{J}_K^{(\mathfrak{m})} = \prod_{\mathfrak{p}|\mathfrak{m}} \mathcal{R}_{K_{\mathfrak{p}}} \prod_{\mathfrak{p}|\mathfrak{m}} \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^{v_{\mathfrak{p}}(\mathfrak{m})}$$

$$\mathcal{R}_K^{(\mathfrak{m})} = \mathcal{R}_K \cap \mathcal{J}_K^{(\mathfrak{m})}$$

Theorem 4.4.1. Let L/K be a finite and abelian ℓ -extension, we have :

$$\mathcal{A}_{L/K} = \text{Gal}(L/K) \simeq \mathcal{P}_K^{(\tilde{\mathfrak{f}}_{L/K})} \cdot N_{L/K}(\mathcal{D}_L^{\tilde{\mathfrak{f}}_{L/K}})$$

where $\mathcal{P}_K^{(\tilde{\mathfrak{f}}_{L/K})}$ is the sub-module of logarithmic principal divisors, image of the elements of $\mathcal{R}_K^{(\tilde{\mathfrak{f}}_{L/K})}$.

Proof. Let's denote here the logarithmic global conductor of L/K by $\tilde{\mathfrak{f}}$. By global ℓ -adic class field theory, we know:

$$\text{Gal}(L/K) \simeq \mathcal{J}_K / N_{L/K}(\mathcal{J}_L) \mathcal{R}_K.$$

By the ℓ -adic approximation lemma [4, II.2], we know that the morphism of semi-localization $\mathcal{R}_K \rightarrow \prod_{\mathfrak{p} \in S} \mathcal{R}_{K_{\mathfrak{p}}}$ is surjective for all finite set of primes S .

It follows:

$$\mathcal{J}_K = \mathcal{J}_K^{\tilde{\mathfrak{f}}} \mathcal{R}_K.$$

Thus we get:

$$\text{Gal}(L/K) \simeq \mathcal{J}_K^{\tilde{\mathfrak{f}}} / N_{L/K}(\mathcal{J}_L^{\tilde{\mathfrak{f}}}) \mathcal{R}_K^{\tilde{\mathfrak{f}}}.$$

But the elements of $\tilde{\mathcal{U}}_K^{(\tilde{\mathfrak{f}})} = \prod_{\mathfrak{p}|\tilde{\mathfrak{f}}} \tilde{\mathcal{U}}_{K_{\mathfrak{p}}} \prod_{\mathfrak{p}|\tilde{\mathfrak{f}}} \tilde{\mathcal{U}}_{K_{\mathfrak{p}}}^{v_{\mathfrak{p}}(\tilde{\mathfrak{f}})}$ are norms : if $\mathfrak{p} \nmid \tilde{\mathfrak{f}}$, \mathfrak{p} is logarithmically unramified and units are norms, moreover if $\mathfrak{p}|\tilde{\mathfrak{f}}$, the definition of the conductor implies that those elements are norms. Due to this remark, we deduce:

$$\text{Gal}(L/K) \simeq \mathcal{J}_K^{\tilde{\mathfrak{f}}} / N_{L/K}(\mathcal{J}_L^{\tilde{\mathfrak{f}}}) \tilde{\mathcal{U}}_K^{(\tilde{\mathfrak{f}})} \mathcal{R}_K^{\tilde{\mathfrak{f}}}.$$

But by the next lemma, we have: $\tilde{\mathcal{U}}_K^{(\tilde{\mathfrak{f}})} \mathcal{R}_K^{\tilde{\mathfrak{f}}} = \tilde{\mathcal{U}}_K \mathcal{R}_K^{(\tilde{\mathfrak{f}})}$. Finally we obtain:

$$\text{Gal}(L/K) \simeq \mathcal{D}_K^{\tilde{\mathfrak{f}}} / \mathcal{P}_K^{(\tilde{\mathfrak{f}})} \cdot N_{L/K}(\mathcal{D}_L^{\tilde{\mathfrak{f}}}).$$

□

Lemma 4.4.1. With the same notations, we get:

$$\tilde{\mathcal{U}}_K^{(\tilde{\mathfrak{f}})} \mathcal{R}_K^{\tilde{\mathfrak{f}}} = \tilde{\mathcal{U}}_K \mathcal{R}_K^{(\tilde{\mathfrak{f}})}.$$

Proof. Let's take $\alpha \in \tilde{\mathcal{U}}_K^{(\tilde{f})} \tilde{\mathcal{R}}_K^{\tilde{f}}$ and write $\alpha = ur$ with $u \in \tilde{\mathcal{U}}_K^{(\tilde{f})}$ and $r \in \tilde{\mathcal{R}}_K^{\tilde{f}}$. This last condition implies that for all primes \mathfrak{p} dividing the conductor, the local component $r_{\mathfrak{p}}$ is a logarithmic unit. The approximation lemma gives a principal idele β , whose local components for the primes dividing the conductor are $r_{\mathfrak{p}}$. Finally we get $\alpha\beta^{-1} \in \tilde{\mathcal{U}}_K \tilde{\mathcal{R}}_K^{(\tilde{f})}$: this is the first inclusion.

Conversely, let's consider $\alpha \in \tilde{\mathcal{U}}_K \tilde{\mathcal{R}}_K^{(\tilde{f})}$ and write $\alpha = ur$ with now $u \in \tilde{\mathcal{U}}_K$ and $r \in \tilde{\mathcal{R}}_K^{(\tilde{f})}$. As $u \in \tilde{\mathcal{U}}_K$, the local component $u_{\mathfrak{p}}$ is a logarithmic unit and in particular for $\mathfrak{p}|\tilde{f}$. Due to the approximation lemma we get a principal idele β , whose local components for the primes $\mathfrak{p}|\tilde{f}$ are $u_{\mathfrak{p}}$. Finally $\alpha\beta^{-1} \in \tilde{\mathcal{U}}_K^{(\tilde{f})} \tilde{\mathcal{R}}_K^{\tilde{f}}$. The equality follows. \square

Remark : Let's notice the analogy between the expression of the classical Takagi's group and the logarithmic Artin sub-module.

Theorem 4.4.2. *Let L/K be a finite and abelian ℓ -extension, \mathfrak{m} a modulus of K divisible by $\tilde{f}_{L/K}$, then $(\frac{L/K}{\cdot})$ restricted to $\mathcal{D}\ell_K^{\mathfrak{m}}$ is surjective and leads to an isomorphism between $\mathcal{D}\ell_K^{\mathfrak{m}}/\mathcal{A}_{L/K,\mathfrak{m}}$ and $\text{Gal}(L/K)$.*

4.5 Example: the quadratic case

To better understand the differences between the classical and the logarithmic case, let's focus on a quadratic extension: $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. This is a ℓ -extension for $\ell = 2$, thus classical ramification and logarithmic ramification only differ for the primes above ℓ (cf [2]).

1) Let's consider first a prime p which is not above 2 :

- either the prime p is ramified in the classical and the logarithmic sense: the classical Frobenius and the logarithmic one are not defined.
- or the prime p is unramified both in the classical and the logarithmic sense: we have to consider two cases
 - either p is inert and logarithmically inert: the decomposition sub-group is then isomorphic to the Galois group of the quadratic extension. It contains the identity and the usual Frobenius which coincides with the logarithmic one.
 - or p is completely splitted in the classical and the logarithmic sense : the decomposition sub-group is trivial, and so are the classical Frobenius and the logarithmic one.

2) Let's study the case of 2.

2 is logarithmically unramified, means by definition:

$$[\mathbb{Q}_2(\sqrt{d}) \cap \widehat{\mathbb{Q}}_2^c : \mathbb{Q}_2(\sqrt{d})] = 1 \Leftrightarrow \mathbb{Q}_2(\sqrt{d}) \subseteq \mathbb{Q}_2^c$$

with $\widehat{\mathbb{Q}}_2^c$ the $\widehat{\mathbb{Z}}$ -cyclotomic extension of \mathbb{Q}_2 and \mathbb{Q}_2^c the \mathbb{Z}_2 -cyclotomic extension of \mathbb{Q} . $\widehat{\mathbb{Q}}_2^c$ is just the compositum of all \mathbb{Z}_q -cyclotomic extensions for all primes q , those extensions are linearly separated and their Galois group is isomorphic to \mathbb{Z}_p . The previous equivalence is due to the fact that \mathbb{Z}_2 is the only one which has a quotient isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

The \mathbb{Z}_2 -cyclotomic extension of \mathbb{Q} is cyclic, thus we may focus on the first step of the tower: $\mathbb{Q}_2(\sqrt{d}) \subseteq \mathbb{Q}_2(\sqrt{2})$. So we have two cases:

$$\mathbb{Q}_2(\sqrt{d}) = \mathbb{Q}_2 \quad \text{or} \quad \mathbb{Q}_2(\sqrt{d}) = \mathbb{Q}_2(\sqrt{2})$$

thus

$$d \in \mathbb{Q}_2^{\times 2} \text{ i.e. } d \equiv 1 \pmod{8} \quad \text{or} \quad d/2 \in \mathbb{Q}_2^{\times 2} \text{ i.e. } d/2 \equiv 1 \pmod{8}$$

as the squares of \mathbb{Q}_2 are of the shape $2^n \cdot u$ where u is a 2-adic unit congruent to 1 mod 8 and n is a rational even number .

- If $d \equiv 1 \pmod{8}$, $\mathbb{Q}_2(\sqrt{d}) = \mathbb{Q}_2$, the local extension is trivial, the same is true for the decomposition subgroup. In this case, 2 is completely splitted in the classical and in the logarithmic sense.
- If $d \equiv 2 \pmod{16}$, 2 is ramified in the classical sense but logarithmically inert. Its decomposition sub-group contains two elements : the trivial one and the other one which is the logarithmic Frobenius. $\mathbb{Q}_2(\sqrt{d})/\mathbb{Q}_2$ is the maximal real sub-extension of $\mathbb{Q}_2(\zeta_8)/\mathbb{Q}_2$, which is fixed by $\langle -1 \rangle$. Moreover $\text{Gal}(\mathbb{Q}_2(\zeta_8)/\mathbb{Q}_2) \simeq (\mathbb{Z}/8\mathbb{Z})^\times$, thus we may explicit the action of the logarithmic Frobenius of 2 : $(\frac{K/\mathbb{Q}}{2})(\zeta_8) = \zeta_8^3$.

4.6 Generalization: from quadratic to ℓ -extensions

Proposition 4.6.1. *Let K/\mathbb{Q} be an abelian ℓ -extension with $\ell \neq 2$ and p a prime of \mathbb{Q} .*

- *For $p \neq \ell$, classical and logarithmic ramification coincide. If p is unramified, the classical Frobenius and the logarithmic one are the same.*
- *For $p = \ell$ we have several cases : (let \mathfrak{p} be a prime of K above ℓ)*
 - *either $K_{\mathfrak{p}} = \mathbb{Q}_\ell$: ℓ is then completely splitted in the classical and the logarithmic sense. The decomposition sub-group is trivial, classical and logarithmic Frobenius are equal and both trivial.*
 - *and $\mathbb{Q}_\ell \neq K_{\mathfrak{p}} \subset \mathbb{Q}_\ell^{nr}$: ℓ is classically unramified and logarithmically ramified, only the classical Frobenius exists*
 - *or $\mathbb{Q}_\ell \neq K_{\mathfrak{p}} \subset \mathbb{Q}_\ell^c$: ℓ is logarithmically unramified but classically ramified, only the logarithmic Frobenius exists*
 - *in any other case, ℓ is both classically and logarithmically ramified.*

Proof. If $\ell \neq 2$, the maximal abelian pro- ℓ -extension \mathbb{Q}_ℓ^{ab} of \mathbb{Q}_ℓ is the compositum of \mathbb{Z}_ℓ -unramified extension \mathbb{Q}_ℓ^{nr} and the cyclotomic one \mathbb{Q}_ℓ^c .

Let's assume $p \neq \ell$, then the classical and the logarithmic uniformizing elements are equal, the local symbol coincide : thus the classical and the logarithmic Frobenius are the same.

Let's assume $p = \ell$ and that ℓ is logarithmically unramified in K . As we work with ℓ -extensions, we obtain:

$$K_{\mathfrak{p}} \subseteq \mathbb{Q}_\ell^c,$$

where \mathfrak{p} is a prime of K above ℓ . As K/\mathbb{Q} is a finite extension, we may concentrate on the first steps of the \mathbb{Z}_ℓ -cyclotomic extension : there exists an integer n such that

$$K_{\mathfrak{p}} \subseteq B_{n+1},$$

where B_{n+1} is the maximal real subextension of $\mathbb{Q}_\ell(\zeta_{\ell^{n+1}})$ with $\zeta_{\ell^{n+1}}$ a ℓ^{n+1} -root of unity and $[B_{n+1} : \mathbb{Q}_\ell] = \ell^n$. This yields to:

$$K_{\mathfrak{p}} = \mathbb{Q}_\ell \text{ or } K_{\mathfrak{p}} = B_{n+1}.$$

In the first case, the local extension is trivial: \mathfrak{p} is completely splitted in the classical and in the logarithmic sense. In the second case, ℓ is ramified in the classical sense but logarithmically unramified.

□

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