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# Semi-concave singularities and the Hamilton-Jacobi equation

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## Abstract

We study the Cauchy problem for the Hamilton-Jacobi equation with a semi-concave initial condition. We prove an inequality between the two types of weak solutions emanating from such an initial condition (the variational and the viscosity solution). We also give conditions for an explicit semi-concave function to be a viscosity solution. These conditions generalize the entropy inequality characterizing piecewise smooth solutions of scalar conservation laws in dimension one.

## 1 Introduction

We consider the Cauchy problem for the Hamilton-Jacobi equation

$$\partial_t u(t, x) + H(t, x, \partial_x u(t, x)) = 0, \quad (\text{HJ})$$

of an unknown function  $u(t, x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ . It will also be useful to consider the associated Hamiltonian system

$$\dot{q}(t) = -\partial_p H(t, q(t), p(t)) \quad , \quad \dot{p}(t) = \partial_q H(t, q(t), p(t)). \quad (\text{HS})$$

We will most of the time assume :

**Hypothesis 1.** *The Hamiltonian  $H(t, x, p) : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is  $C^2$  and there exists a constant  $A$  such that*

$$|d^2 H(t, x, p)| \leq A, \quad |dH(t, x, p)| \leq A(1 + |p|), \quad |H(t, x, p)| \leq A(1 + |p|)^2$$

for each  $(t, x, p)$ .

In particular, the Hamiltonian system is complete. No convexity assumption is made on  $H$ .

We focus our attention on the case where the initial condition  $u_0$  is semi-concave and Lipschitz, given as the infimum of an equi-Lipschitz family  $F_0$  of  $C^2$  functions with uniformly equi-bounded second derivatives, which means that there exists a constant  $B$  such that  $|d^2 f_0(x)| \leq B$  for each  $x \in \mathbb{R}^d, f_0 \in F_0$ . See Section 3 for more details on semi-concave functions. The general theory of Hamilton Jacobi equations allows to define two solutions for the Cauchy problem with the Lipschitz initial condition  $u_0$  at time 0:

The variational solution  $g(t, x) = G_0^t u_0(x)$ , and the viscosity solution  $v(t, x) = V_0^t u_0(x)$ , see Section 2 for more details. One of our goals in the present work is to compare these two solutions and the following natural third candidate.

The theory of characteristics implies that there exists a constant  $T(B) > 0$ , which depends on  $B$  (and on  $A$ ), such that, to each function  $f_0 \in F_0$  is associated a  $C^2$  solution  $f : ]-T(B), T(B)[ \times \mathbb{R}^d \rightarrow \mathbb{R}$  of (HJ) satisfying  $f(0, \cdot) = f_0$ , more details in Section 2. We denote by  $F$  this family of  $C^2$  solutions. Their infimum is a natural candidate to be a solution of our Cauchy problem on  $[0, T(B)[ \times \mathbb{R}^d$ , although it depends on the family  $F_0$ , and not just on the function  $u_0$ .

**Theorem 1.** *The solutions  $v$  and  $g$  are semi-concave on  $[0, T[ \times \mathbb{R}^d$  for each  $T \in ]0, T(B)[$ , and*

$$v \leq g \leq \inf_{f \in F} f$$

on  $[0, T(B)[ \times \mathbb{R}^d$ .

This theorem is proved in section 4, where a sufficient condition for the equality  $g = \inf f$  is also given. We discuss this condition here under the additional assumption that  $F$  is closed for the  $C_{loc}^1$  topology (this is a very minor restriction since one can always replace  $F$  by its closure). In this case, we denote by  $\partial_x F(t, x)$  the set  $\{\partial_x \underline{f}(t, x), \underline{f} \in F, \underline{f}(t, x) = \min f(t, x)\}$ .

**Addendum 1.** If  $\partial_x F(0, x)$  is convex for each  $x$ , then the equality  $g = \inf f$  holds in Theorem 1.

If  $\partial_x F(t, x)$  is convex for each  $(t, x)$ , then the equality  $v = g = \inf f$  holds.

The sufficient condition for the equality  $v = \inf f$  mentioned above is actually too demanding, for example it is usually not satisfied in the context of the Hopf formula for concave solutions, see below. We now propose more reasonable sufficient conditions inspired by the famous entropy inequalities which characterize piecewise smooth entropy solutions of conservation laws, see [8].

We consider a semi-concave function  $u(t, x) : ]0, T[ \times \mathbb{R}^d \rightarrow \mathbb{R}$  which solves (HJ) at each point of differentiability. It can be for example the variational solution  $g$  or the function  $\inf f$  in Theorem 1. We call such a function a semi-concave solution. We denote by  $D^e u(t, x)$  the set of extremal points of the super-differential  $Du(t, x)$  of  $u$ , and by  $D_x^e u(t, x)$  the projection of  $D^e u(t, x)$  on the spacial directions. We also denote by  $D_x u(t, x)$  the projection on the spacial directions of  $Du(t, x)$ . Note that  $D_x^e u(t, x)$  is bigger than the set  $D^e u_t(x)$  of extremal super-differentials of the function  $u_t = u(t, \cdot)$ . We then denote by  $\check{H}_{t,x}^u$  the greatest convex function of  $\mathbb{R}^d$  which is smaller than or equal to  $H_{t,x}$  on  $D_x^e u(t, x)$ . Similarly, we denote by  $\hat{H}_{t,x}^u$  the smallest concave function of  $\mathbb{R}^d$  which is greater than or equal to  $H_{t,x}$  on  $D_x^e u(t, x)$ . These functions take the value  $+\infty$  (resp  $-\infty$ ) outside of  $D_x u(t, x)$ . The following result holds for all continuous Hamiltonians (not necessarily satisfying Hypothesis 1) :

**Theorem 2.** Let  $u(t, x) : ]0, T[ \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a semi-concave solution of (HJ). If

$$H_{t,x} \leq \check{H}_{t,x}^u \quad \text{on} \quad D_x u(t, x) \quad (1)$$

for each  $(t, x) \in ]0, T[ \times \mathbb{R}^d$ , then  $u$  is a viscosity solution of (HJ). Conversely, if  $u$  is a viscosity solution of (HJ) then the inequality

$$H_{t,x} \leq \hat{H}_{t,x}^u \quad \text{on} \quad D_x u(t, x) \quad (2)$$

holds for each  $(t, x)$ .

Conditions of the same kind were introduced in [10] in a different setting. In the case where  $u$  is the minimum of two functions  $f^-$  and  $f^+$ , conditions (1) and (2) are equivalent, and they can be expressed as

$$H(t, x, sp^- + (1-s)p^+) \leq sH(t, x, p^-) + (1-s)H(t, x, p^+) \quad \forall s \in [0, 1], \quad (3)$$

where  $p^\pm = \partial_x f^\pm(t, x)$ . Theorem 2 then reads

**Corollary 1.** Let  $f^-, f^+ : ]0, T[ \times \mathbb{R}^d \rightarrow \mathbb{R}$  be two  $C^2$  solutions of the Hamilton-Jacobi equation with bounded second derivative and bounded derivative. The function  $u := \min(f^-, f^+)$  is a viscosity solution of (HJ) if and only if the entropy condition (3) is satisfied at each point  $(t, x) \in ]0, T[ \times \mathbb{R}^d$ .

In the case  $d = 1$  this corollary is the counterpart of a standard result concerning the conservation law

$$\partial_t p(t, x) + \partial_x (H(t, x, p(t, x))) = 0, \quad (\text{CL})$$

which, formally, is the equation solved by the differential  $p = \partial_x u$  of solutions of (HJ). For such equations, there is a theory of entropy solutions, which is the counterpart of the theory of viscosity solutions. Let us consider a solution  $p(t, x)$  of (CL) which is composed of two smooth branches of solutions  $p^-(t, x)$  and  $p^+(t, x)$  on both sides of a discontinuity  $\chi(t)$ , with  $p(t, x) = p^-(t, x)$  for  $x \leq \chi(t)$  and  $p = p^+$  for  $x \geq \chi(t)$ . Assuming in addition that  $p^- \geq p^+$  (a condition satisfied by the derivative of a semi-concave singularity), it is known that the function  $p(t, x)$  is an entropy solution of (CL) if and only if the entropy condition (3) holds at  $(t, \chi(t))$  for each  $t$ , see [8]. Corollary 1 is the transposition of this celebrated result for viscosity solutions of the Hamilton-Jacobi equation.

All the results presented in this note have obvious counterparts for semi-convex initial conditions. The reason why we preferred to work with semi-concave solutions is that they play a special role in the case of convex Hamiltonians (meaning that  $\partial_p^2 H > 0$ ). As is well-known, viscosity solutions are variational in this case, forming a single notion of weak solution ( $v = g$ ). This solution is given by an explicit expression, the so-called Lax-Oleinik semi-group, and it has the property of being locally semi-concave whatever the initial condition, see [4] for example. In this convex case, the generalized entropy condition (1) (hence (2)) always holds, hence each semi-concave solution is a viscosity solution (this property is well-known, see [4] for example). As a consequence, equality always holds in Theorem 1. This could also be proved easily with the Lax-Oleinik formula.

Still in the case of a convex Hamiltonian, semi-convex solutions are also of interest. The first conclusion of Theorem 1 (the solution emanating from a semi-convex initial data is semi-convex) was first proved in [2]. This solution is then locally  $C^{1,1}$  (because a function which is both semi-convex and semi-concave is  $C^{1,1}$ ). This is reflected in Theorem 2 as follows: The necessary condition to be a semi-convex viscosity solution reads  $H_{t,x} \geq \tilde{H}_{t,x}^u$ . In view of the assumption of strict convexity of  $H$ , this can hold only if the sub-differential  $D_x u(t, x)$  is reduced to a point, which implies that the function  $u$  is  $C^1$ .

All these considerations indicate that Theorem 2 is not useful in the convex case. To illustrate its possible usefulness, let us use it to recover the formula of Hopf for concave solutions of integrable Hamiltonians. We suppose in this discussion that  $H = H(p)$  does not explicitly depend on  $t$  and  $x$ , and consider a concave initial condition  $u_0$ , that we also assume Lipschitz for simplicity (this assumption is removed in Section 6). We write  $u_0$  as an infimum of affine functions with the formula  $u_0(x) = \inf_p (p \cdot x - u_0^*(p))$ , where  $u_0^*(p)$  is the (concave version of the) Legendre transform  $u_0^*(p) := \inf_x (p \cdot x - u_0(x))$ . This dual  $u_0^*$  takes the value  $-\infty$ , outside of a bounded domain denoted by  $P$ . The function  $f(t, x) = px - u_0^*(p) - tH(p)$  is a  $C^2$  solution emanating from the affine initial condition  $f_0(x) = px - u_0^*(p)$ .

**Corollary 2.** *If  $H(p)$  is a continuous Hamiltonian, and  $u_0$  is a Lipschitz and concave initial condition, then the function*

$$u(t, x) := \min_{p \in P} (px - u_0^*(p) - tH(p))$$

*is a viscosity solution on  $[0, \infty[ \times \mathbb{R}^d$ . If, in addition,  $H$  satisfies Hypothesis 1, then*

$$v(t, x) = g(t, x) = \min_{p \in P} (px - u_0^*(p) - tH(p)).$$

That the right hand side in this expression is actually a viscosity solution is well-known, even in broader contexts, see for example [9, 7].

The paper is organized as follows: In Section 2, we quickly recall some basic facts on the various notions of solutions of the Cauchy problem. In Section 3, we settle some notations and elementary properties on semi-concave functions seen as infima of  $C^2$  functions. We then prove Theorem 1 and its addendum in Section 4, using a Proposition proved in [2] in the case of a convex Hamiltonian. We prove Theorem 2 in Section 5, in the general setting of a continuous Hamiltonian. We return to the Hopf formula and prove Corollary 2 in Section 6.

## 2 The Cauchy Problem

We give here a very brief survey on the Cauchy problem associated to the Hamilton-Jacobi equation (HJ).

### 2.1 Classical solutions

The theory of characteristics links classical solutions of (HJ) with the Hamiltonian system (HS). We give here a brief account on the results, see for example [4, 1] for more details.

For each  $C^2$  initial condition  $f_s$  with bounded second derivative, there exist a time  $T > 0$  and a  $C^2$  solution  $f : ]s - T, s + T[ \times \mathbb{R}^d \rightarrow \mathbb{R}$  of (HJ) satisfying  $f(s, \cdot) = f_s$ . It is necessary to be more quantitative.

There exists a non-decreasing semi-group  $Q^t(r)$  on  $[0, \infty]$ , such that the time  $T(r) := \sup\{t \geq 0, Q^t(r) < \infty\}$  is positive for each  $r \in [0, \infty[$  and such that:

For each  $C^2$  initial data  $f_s$  satisfying  $\|d^2 f_s\|_\infty \leq r$ , there exists a unique  $C^2$  solution  $f : ]s - T(r), s + T(r)[ \times \mathbb{R}^d \rightarrow \mathbb{R}$ . This solution satisfies

$$\|d^2 f_t\|_\infty \leq Q^{|t-s|}(r).$$

If moreover  $f_s$  is Lipschitz (that is, if  $df_s$  is bounded), we have the estimates

- $Lip(f_t) \leq (Lip(f_s) + 1)e^{A|t-s|} - 1$
- $|\partial_t f(t, x)| \leq A((Lip(f_s) + 1)e^{A|t-s|})^2$
- $|d^2 f(t, x)| \leq D \quad \forall (t, x) \in ]s - T, s + T[ \times \mathbb{R}^d,$

for each  $T < T(r)$ , where  $D$  is some constant depending only on  $A, Lip(f_s), Q^T(B)$ .

This solution is related to the Hamiltonian system as follows: For each  $(t, x) \in ]-T, T[ \times \mathbb{R}^d$ , the Hamiltonian trajectory  $(q(s), p(s))$  which satisfies  $(q(t), p(t)) = (x, \partial_x f(t, x))$  satisfies

$$p(s) = \partial_x f(s, q(s))$$

for each  $s \in ]-T, T[$ , and

$$f(t, x) = f(s, q(s)) + \int_s^t p(s) \cdot \dot{q}(s) - H(s, q(s), p(s)) ds.$$

It is usually not possible to extend  $C^2$  solutions to the whole real line, which led to the introduction of some notions of weak solution.

## 2.2 Variational solutions

See for example [3, 12] for more on variational solutions. This notion of solutions is directly inspired by the method of characteristics.

A function  $g(t, x) : [s, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$  is called a variational solution of the Cauchy problem with Lipschitz initial data  $u(x)$  at time  $s$  if it is locally Lipschitz, satisfies the initial condition  $g_s = u$ , solves the equation almost everywhere, and if, for each  $(t, x) \in ]s, \infty[ \times \mathbb{R}^d$ , there exists a trajectory  $(q(s), p(s))$  of the Hamiltonian system such that  $q(t) = x, p(0) \in Du(q(0))$ , and

$$u(t, x) = u_0(q(0)) + \int_0^t p(s) \cdot \dot{q}(s) - H(s, q(s), p(s)) ds.$$

Here  $Du$  denotes the Clarke differential of  $u$ , see Section 3. In other words,  $g(t, x)$  is a critical value of the functional

$$(q(s), p(s)) \mapsto u(q(0)) + \int_0^t p(s) \cdot \dot{q}(s) - H(s, q(s), p(s)) ds$$

on the space of  $C^1$  curves  $(q(s), p(s)) : [0, t] \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  which satisfy  $q(t) = x$ .

There exists a family of operators  $G_s^t$ ,  $s \leq t$  which map  $C^{0,1}(\mathbb{R}^d)$  (the space of Lipschitz functions) into itself, such that the function  $(t, x) \mapsto G_s^t u(x)$  is a variational solution with initial data  $u$  at time  $s$ , and such that

1.  $Lip(G_s^t u) \leq (Lip(u) + 1)e^{A(t-s)} - 1$ ,  $\|G_s^t u - u\|_\infty \leq A(t-s)((Lip(u) + 1)e^{A(t-s)})^2$
2.  $u \leq v \Rightarrow G_s^t u \leq G_s^t v$
3. If  $f(t, x) : ]T^-, T^+[\times \mathbb{R}^d \rightarrow \mathbb{R}$  is a  $C^2$  solution, then  $G_s^t f_s = f_t$  for each  $s \leq t$  in  $]T^-, T^+[$ .

A family of operators  $G_s^t$  satisfying the properties above is called a variational resolution of (HJ). There is no uniqueness for variational solutions, and not even uniqueness for variational resolutions. It would be tempting to ask in addition that the resolution  $G_s^t$  satisfy the Markov property  $G_s^t \circ G_\tau^s = G_\tau^t$ . However, adding such a condition to the properties (1 – 3) above would lead to the Viscosity resolution, see below, which does not produce variational solutions in general.

We consider that a variational resolution  $G$  is fixed once and for all in the present paper. When we speak of the variational solution emanating from an initial condition  $u_0$ , we mean the function  $g(t, x) = G_0^t u_0(x)$ .

## 2.3 Viscosity solutions

See for example [4] for more on viscosity solutions. There exists a unique family of operators  $V_s^t$  acting on  $C^{0,1}(\mathbb{R}^d)$  and such that

1.  $Lip(V_s^t u) \leq (Lip(u) + 1)e^{A(t-s)} - 1$ ,  $\|V_s^t u - u\|_\infty \leq A(t-s)((Lip(u) + 1)e^{A(t-s)})^2$
2.  $u \leq v \Rightarrow V_s^t u \leq V_s^t v$
3. If  $f(t, x) : ]T^-, T^+[\times \mathbb{R}^d \rightarrow \mathbb{R}$  is a  $C^2$  solution, then  $V_s^t f_s = f_t$  for each  $s \leq t$  in  $]T^-, T^+[$ .

4.  $V_s^t \circ V_\tau^s = V_\tau^t$  for each  $\tau \leq t \leq s$ .

For each Lipschitz initial condition  $u$ , and each initial time  $s$ , the functions

$$]s, \infty[ \times \mathbb{R}^d \ni (t, x) \mapsto V_s^t u(x)$$

is a viscosity solution of (HJ) in the classical sense. It means that each smooth function  $\phi(t, x)$  which has a contact from above (resp. from below) with  $u$  at some point  $(t_0, x_0) \in ]s, \infty[ \times \mathbb{R}^d$  must satisfy

$$\partial_t \phi(t_0, x_0) + H(t_0, x_0, \partial_x \phi(t_0, x_0)) \leq 0, \quad (\text{resp. } \geq 0).$$

Conversely, the function  $(t, x) \mapsto V_s^t u(x)$  is the unique viscosity solution  $v$  such that  $v(s, \cdot) = u$  and such that  $v(t, x) - u(x)$  is bounded and Lipschitz on  $]s, T[ \times \mathbb{R}^d$  for each  $T > s$ .

The theory of viscosity solutions thus provides a "good" resolution of the Cauchy problem (existence and uniqueness). As we mentioned above, apart in some special cases (for example when  $H$  is convex in  $p$ ), the viscosity solution is not in general a variational solution.

It is an interesting problem in general to describe and compare these two notions of solutions. Let us mention in this direction a recent statement recently proved by Qiaoling Wei in [13] (see also [11]). This result had been conjectured by Chaperon and Viterbo. We denote, for each  $k \in \mathbb{N}$ , by  ${}^k G_s^t$  the operator

$${}^k G_s^t := G_{t_{[kt]}}^t \circ G_{t_{[kt]-1}}^{t_{[kt]}} \circ \cdots \circ G_{t_1}^{t_2} \circ G_s^{t_1}$$

where  $t_i = s + i/k$  and  $[kt]$  is the integer part of  $kt$ . We have

$${}^k G_s^t u \longrightarrow V_s^t u$$

locally uniformly (in  $t$  and  $x$ ) for each Lipschitz function  $u$ .

### 3 Nonsmooth Calculus and semi-concave functions as minima of $C^2$ functions

We recall some standard definitions and properties of non-smooth calculus, see [4], Chapter 3 and [5]. We will consider only locally Lipschitz functions  $u : \mathbb{R}^d \rightarrow \mathbb{R}$ .

A super-differential of  $u$  at  $x$  is a vector  $p \in \mathbb{R}^d$  such that there exists a  $C^1$  function  $f$  satisfying  $df(x) = p$  and having a contact from above with  $u$  at  $x$ , which means that

$$f(x) = u(x) \quad , \quad f \geq u.$$

A proximal super-differential of  $u$  at  $x$  is a vector  $p \in \mathbb{R}^d$  such that there exists a  $C^2$  (or, equivalently, smooth) function  $f$  satisfying  $df(x) = p$  and having a contact from above with  $u$  at  $x$ . A proximal super-gradient is obviously a super-gradient, but the converse is not true in general.

For concave functions however, super-differentials in the present sense coincide with super-differentials in the sense of convex analysis (the slopes of affine functions which have a contact from above with  $u$  at  $x$ ), hence with proximal super-differentials.

The vector  $p \in \mathbb{R}^d$  is called a reachable gradient of  $u$  at  $x$  if there exists a sequence  $x_n \rightarrow x$  of points of differentiability of  $u$  such that  $du(x_n) \rightarrow p$ . Recall that the points of differentiability of  $u$  have full measure (hence it is dense). The set  $D^*u(x)$  of reachable gradients is compact and not empty.

The Clarke differential of  $u$ , denoted by  $Du(x)$  is the convex hull, in  $\mathbb{R}^d$ , of  $D^*u(x)$ , see [5], Section 2.5. We denote by  $D^e u(x)$  the set of extremal points of  $Du(x)$ , note that  $D^e u(x) \subset D^*u(x)$ .

For a function  $u(t, x)$  of two variables, we use the notation

$$D_x u(t, x) := \{p, \exists \eta, (\eta, p) \in Du(t, x)\}$$

and similarly for  $D_x^e u(t, x)$ .

The function  $u$  is called semi-concave if  $u(x) - B\|x\|^2/2$  is concave for some  $B$  (we then say that  $u$  is  $B$ -semi-concave). For semi-concave functions, the set of super-differentials, of proximal super-differentials, and of Clarke differentials coincide.

If  $p$  is a super-differential at  $x_0$  of the  $B$ -semi-concave function  $u$ , then the function  $u(x_0) + p \cdot (x - x_0) + B\|x - x_0\|^2/2$  has a contact from above with  $u$  at  $x_0$ .

Let us now consider a set  $F \subset C^2(\mathbb{R}^d, \mathbb{R})$  with uniformly equi-bounded second derivative and assume that the function  $u(x) := \inf_{f \in F} f(x)$  takes finite values.

The function  $u$  is then semi-concave hence locally Lipschitz.

**Definition 3.** We denote by  $dF(x)$  the set of limits of sequences of the form  $df_n(x), f_n \in F, f_n(x) \rightarrow u(x)$ .

Note that  $dF(x)$  is compact. In the case where  $F$  is closed in  $C_{loc}^1$ , this is just  $\{df(x), f \in F, f(x) = u(x)\}$ . The set  $dF(x)$  depends on  $F$ , and not only on the function  $u$ , but it is related to the super-differential  $Du(x)$ :

**Lemma 4.** We have  $dF(x) \subset Du(x)$  and  $dF(x)$  is not empty. If  $u$  is differentiable at  $x$ , then  $dF(x) = \{du(x)\}$ .

PROOF. Let  $p = \lim df_n(x)$ , with  $f_n(x) \rightarrow u(x)$ , be a point of  $dF(x)$ . Since

$$u(x+y) \leq f_n(x+y) \leq f_n(x) + df_n(x) \cdot (y-x) + C\|y-x\|^2,$$

we conclude at the limit that

$$u(x+y) \leq u(x) + p \cdot (y-x) + C\|y-x\|^2$$

hence  $p$  is a proximal super-differential of  $u$  at  $x$ ,  $p \in Du(x)$ . We have proved that  $dF(x) \subset Du(x)$ .

To prove that  $dF(x)$  is not empty, we consider a sequence  $f_n \in F$  such that  $f_n(x) \rightarrow f(x)$ . We have, for each  $y \in \mathbb{R}^d$ ,

$$u(x+y) \leq f_n(x+y) \leq f_n(x) + df_n(x) \cdot y + C\|y\|^2.$$

Applying this inequality with  $y_n = -df_n(x)/\|df_n(x)\|$  yields

$$\|df_n(x)\| \leq f_n(x) - u(x+y_n) + C$$

which implies that the sequence  $df_n(x)$  is bounded. As a consequence, it has converging subsequences, hence  $dF(x)$  is not empty.

If  $u$  is differentiable at  $x$ , then  $Du(x) = \{du(x)\}$ , hence also  $dF(x) = \{du(x)\}$ . □

**Lemma 5.** We have  $D^*u(x) \subset dF(x) \subset Du(x)$ . As a consequence, the super-differential  $Du(x)$  is the convex hull of  $dF(x)$ .

PROOF. let  $p = \lim du(x_n)$  be a reachable gradient, where  $x_n \rightarrow x$  is a sequence of points of differentiability of  $u$ . For each  $n$ , we have  $dF(x_n) = \{du(x_n)\}$  hence there exists a function  $f_n \in F$  such that  $f_n(x_n) \leq u(x_n) + 1/n$  and  $\|df_n(x_n) - du(x_n)\| \leq 1/n$ . Since  $\|df_n(x_n) - df_n(x)\| \leq C\|x_n - x\|$ , we conclude that  $df_n(x) \rightarrow p$ . On the other hand we can estimate

$$\begin{aligned} f_n(x) - u(x) &\leq f_n(x_n) + df_n(x_n) \cdot (x - x_n) + C\|x - x_n\|^2 - u(x) \\ &\leq df_n(x_n) \cdot (x - x_n) + C\|x - x_n\|^2 + f_n(x_n) - u(x_n) + u(x_n) - u(x) \end{aligned}$$

and the right hand side is converging to 0. We conclude that  $p \in dF(x)$ . □

**Lemma 6.** If  $u$  is a  $B$ -semi-concave and  $L$ -Lipschitz function, there exists a set  $F \subset C^2(\mathbb{R}^d, \mathbb{R})$  such that  $|d^2 f(x)| \leq B$  and  $|df(x)| \leq 6L$  for each  $f \in F$  and  $x \in \mathbb{R}^d$ , such that  $u(x) = \min_{f \in F} f(x)$  and such that  $Du(x) = dF(x)$  at each point  $x$ .

PROOF. Let  $\psi(r) : [0, \infty) \rightarrow [0, \infty)$  be a non-negative, non-increasing function which is equal to  $B$  on  $[0, 4L/B]$  and to 0 on  $[5L/B, \infty)$ . Let  $\Psi(r)$  be the primitive of  $\psi$  such that  $\Psi(0) = 0$ . Note that  $\Psi(r) \in [0, 5L]$  for each  $r \in [0, \infty)$ . Let then  $\varphi(r)$  be the primitive of  $\Psi$  such that  $\varphi(0) = 0$ . The function  $\varphi$  is  $5L$ -Lipschitz, convex, and it satisfies  $0 \leq \varphi'' \leq B$ . Note also that

$$\varphi(r) \geq \min(Br^2/2, 2Lr).$$

Let us consider the family  $F$  formed by the functions

$$u(x_0) + p \cdot (x - x_0) + \varphi(|x - x_0|)$$

for  $x_0 \in \mathbb{R}^d$  and  $p \in Du(x_0)$ . Since we have  $|p| \leq L$ , these functions are  $6L$ -Lipschitz. They also satisfy  $|d^2 f| \leq B$ , see Appendix A. It is clear that  $Du(x_0) \subset dF(x_0)$  for each  $x_0$ , we thus have equality provided



that  $u = \min_{f \in F} F$ , which is the last thing we have to prove. It is enough to observe that  $f \geq u$  for each  $f \in F$ . For all  $x_0 \in \mathbb{R}^d$ ,  $p \in Du(x_0)$  and  $x \in \mathbb{R}^d$ , we have

$$u(x) \leq u(x_0) + p \cdot (x - x_0) + B|x - x_0|^2$$

and, since  $|p| \leq L$

$$u(x) \leq u(x_0) + L|x - x_0| \leq u(x_0) + p \cdot (x - x_0) + 2L|x - x_0|,$$

hence

$$u(x) \leq u(x_0) + p \cdot (x - x_0) + \varphi(|x - x_0|).$$

□

## 4 Proof of Theorem 1.

We consider a Lipschitz and  $B$ -semi-concave initial condition  $u_0$ .

We write this initial condition as  $u_0 = \inf_{f_0 \in F_0} f_0$  for an equi-Lipschitz family  $F_0$  such that  $|d^2 f_0(x)| \leq B$  for all  $f_0 \in F_0$  and  $x \in \mathbb{R}^d$ . All the families  $F_0$  considered in this section are assumed to satisfy these conditions. The initial condition  $u_0$  (but not the family  $F_0$ ) and the constant  $B$  are fixed once and for all.

We define the family  $F \subset C^2([0, T(B)] \times \mathbb{R}^d)$  of solutions of (HJ) emanating from elements of  $F_0$ . We recall that  $F$  is equi-Lipschitz and has uniformly equi-bounded second derivative on  $]0, T[ \times \mathbb{R}^d$  for each  $T \in ]0, T(B)[$ .

The inequalities

$$G_0^t u_0 \leq \inf_{f \in F} f_t \quad , \quad V_0^t u_0 \leq \inf_{f \in F} f_t$$

follow from the monotony of the operators  $V_0^t$  and  $G_0^t$  since  $G_0^t f_0 = V_0^t f_0 = f_t$  for each  $f_0 \in F_0$ .

The key step in the proof of Theorem 1 is the following result, which extends to the non-convex setting the main Proposition of [2]:

**Proposition 7.** *If  $dF_0(x) = Du_0(x)$  for each  $x \in \mathbb{R}^d$ , then the inequality  $g \geq \inf f$  holds on  $[0, T[ \times \mathbb{R}^d$  for each variational solution  $g$ . As a consequence, the equality*

$$G_0^t u_0 = \inf_{f \in F} f_t$$

*holds for each  $t \in [0, T[$ .*

PROOF. Let  $g(t, x)$  be a variational solution emanating from  $u_0$ . For each  $(t_0, x_0) \in ]0, T[ \times \mathbb{T}^d$ , there exists a Hamiltonian orbit  $(q(s), p(s))$  such that  $q(t_0) = x_0$ ,  $p(0) \in D_x u_0(q(0))$ , and

$$g(t_0, x_0) = u_0(q(0)) + \int_0^{t_0} p(s) \cdot \dot{q}(s) - H(s, q(s), p(s)) ds.$$

Since  $p(0) \in Du_0(q(0))$ , our hypothesis implies that there exists a sequence  $f_n \in F$  such that  $f_n(0, q(0)) \rightarrow u_0(q(0))$  and  $\partial_x f_n(0, q(0)) \rightarrow p(0)$ . Let  $(q_n(s), p_n(s))$  be the Hamiltonian trajectory such that  $q_n(0) = q(0)$  and  $p_n(0) = \partial_x f_n(0, q(0))$ . The method of characteristics yields

$$f_n(t_0, q_n(t_0)) = f_n(0, q_n(0)) + \int_0^{t_0} p_n(s) \cdot \dot{q}_n(s) - H(s, q_n(s), p_n(s)) ds.$$

At the limit, we obtain,

$$\lim f_n(t_0, x) = u_0(q(0)) + \int_0^{t_0} p(s) \cdot \dot{q}(s) - H(s, q(s), p(s)) ds = g(t_0, x_0).$$

We conclude that  $g \geq \inf f$ .

□



When the (semi-concave and Lipschitz) initial condition  $u_0$  is given, it is possible to chose the family  $F_0$  in such a way that the hypothesis  $dF_0(x) = Du_0(x)$  holds for this family, see Lemma 6. We denote by  $\mathcal{F}_0$  a given family with this property, and by  $\mathcal{F}$  the family of associated solutions of (HJ). For this family  $\mathcal{F}$ , we have the equality  $g = \inf_{f \in \mathcal{F}} f$  on  $[0, T(B)[ \times \mathbb{R}^d$ , this is the conclusion of the Proposition we just proved. Since on the other hand the inequality  $v \leq \inf f$  holds for each family  $F$ , we conclude that

$$v \leq g = \inf_{f \in \mathcal{F}} f \leq \inf_{f \in F} f$$

on  $]0, T(B)[ \times \mathbb{R}^d$ , for each family  $F$ . The equality  $g = \inf_{f \in \mathcal{F}} f$  also implies that  $g_t$  is semi-concave with constant  $Q^t(B)$  (see section 2.1 for the definition of  $Q^t(B)$ ) and that  $g$  is semi-concave on  $]0, T[ \times \mathbb{R}^d$  for each  $T \in ]0, T(B)[$ .

To prove that the viscosity solution  $v$  is semi-concave, we use the result of Qiaoling Wei, see Section 2.3, which states that  ${}^k G_0^t u_0 \rightarrow V_0^t u_0$ . Let us denote by  $L_0$  the Lipschitz constant of  $u_0$ , and as usual by  $B$  its constant of semi-concavity.

For  $t \in [0, 1/k] \cap ]0, T(B)[$ , the function  ${}^k G_0^t u_0 = G_0^t u_0$  is  $Q^t(B)$ -semi-concave and  $L_t$ -Lipschitz, with

$$L_t = (L_0 + 1)e^{At} - 1.$$

Then, for  $t \in [1/k, 2/k] \cap ]0, T(B)[$ , the function  ${}^k G_0^t u_0 = G_{1/k}^t (G_0^{1/k} u_0)$  is semi-concave with constant  $Q^{(t-1/k)}(Q^{1/k}(B)) = Q^t(B)$ , and Lipschitz with constant  $(L_{1/k} + 1)e^{A(t-1/k)} - 1 = L_t$ . We prove similarly by recurrence that the function  ${}^k G_0^t u_0$  is  $Q^t(B)$ -semi-concave and  $L_t$ -Lipschitz for each  $t \in [0, T(B)[$ , independantly of  $k$ . We will use the following Corollary of Proposition 7:

**Corollary 8.** *Let  $s_0 \leq s_1 \leq s_2$  be times in  $]0, T(B)[$ . Let  $u_{s_0}$  be semi-concave with constant  $Q^{s_0}(B)$  and Lipschitz. Then*

$$G_{s_1}^{s_2} \circ G_{s_0}^{s_1} u_{s_0} \leq G_{s_0}^{s_2} u_{s_0}.$$

PROOF. We choose as above, using Lemma 6 and Proposition 7, a family  $\mathcal{F}_{s_0}$  of  $C^2$  functions such that  $G_{s_0}^t u_{s_0} = \inf_{f_t \in \mathcal{F}_t} f_t$  for each  $t \in [s_0, T(B)[$ . We have

$$G_{s_1}^{s_2} \circ G_{s_0}^{s_1} u_{s_0} = G_{s_1}^{s_2} \left( \inf_{f_{s_1} \in \mathcal{F}_{s_1}} f_{s_1} \right) \leq \inf_{f_{s_2} \in \mathcal{F}_{s_2}} f_{s_2} = G_{s_0}^{s_2} u_{s_0}.$$

Let us now fix  $T \in ]0, T(B)[$ . Let  $D$  be a constant such that the variational solution emanating from a  $Q^s(B)$ -semi-concave and  $L_T$ -Lipschitz initial condition  $u_s$  is  $D$ -semi-concave on  $]s, T[ \times \mathbb{R}^d$  for each  $s \in ]0, T[$ . See Section 2.1 for the existence of  $D$ . We claim that the function  $g^k(t, x) := {}^k G_0^t u_0(x)$  is  $D$ -semi-concave for each  $k$ . □

This is true on each interval of the form  $]i/k, (i+1)/k[$  where  ${}^k g(t, x) = G_{i/k}^t ({}^k g_{i/k})(x)$ . In addition, the function  ${}^k g$  is equal to the function  $G_{i/k}^t ({}^k g_{i/k})(x)$ , on  $[i/k, (i+1)/k]$ , and, by Corollary 8, it is not larger than this function on  $[(i+1)/k, T(B)[$ . Since the function  ${}^k g$  can be touched from above by  $D$ -semi-concave functions at each point, it is  $D$ -semi-concave.

We conclude, using the result of Qiaoling Wei, that  $V_0^t u_0$  is  $Q^t(B)$ -semi-concave for each  $t \in [0, T(B)[$ , and that  $v$  is  $D$ -semiconcave on  $[0, T[ \times \mathbb{R}^d$ . □

We now prove the Addendum. Since  $Du_0(x)$  is the convex hull of  $dF_0(x)$ , the hypothesis of Proposition 7 holds if and only if  $dF_0(x)$  is convex. The first statement of the Addendum is thus a restatement of Proposition 7.

If the hypothesis of the second statement is satisfied, then  ${}^k G_0^t u_0 = u_t$  for each  $t$ . Using the result of Qiaoling Wei, we conclude at the limit that  $V_0^t u_0 = u_t$  for each  $t$ . □

## 5 Generalized entropy inequalities

We prove Theorem 2. The Hamiltonian  $H$  is only assumed continuous in the present section. We call semi-concave solution of (HJ) a semi-concave function  $u : ]0, T[ \times \mathbb{R}^d \rightarrow \mathbb{R}$  which solves (HJ) at its points of differentiability (these points form a set of full measure). We recall that  $D^e u(t, x)$  is the set of extreme points of the super-differential  $Du(t, x)$  and that it is contained in the set  $D^* u(t, x)$  of reachable differentials, see Section 3.

**Lemma 9.** *The semi-concave function  $u(t, x)$  is a semi-concave solution if and only if it satisfies the equality  $\eta + H(t, x, p) = 0$  for all  $(t, x)$  and for all  $(\eta, p) \in D^e u(t, x)$ .*

PROOF. If  $u(t, x)$  is a solution, then the equality  $\partial_t u + H(t, x, \partial_x u) = 0$  holds at each point of differentiability of  $u$ . Since  $H$  is continuous we conclude that  $\eta + H(t, x, p) = 0$  for each reachable differential  $(\eta, p)$  hence for each  $(\eta, p) \in D^e u(t, x)$ .

The converse holds because, if  $u$  is differentiable at  $(t, x)$  then  $(\partial_t u, \partial_x u) \in D^e u(t, x)$ .  $\square$

**Lemma 10.** *Let  $u$  be a semi-concave solution of (HJ). Then  $u$  is a viscosity solution if and only each super-differential  $(\eta, p) \in Du(t, x)$  satisfies the inequality  $\eta + H(t, x, p) \leq 0$ .*

PROOF. A semi-concave solution is a viscosity super-solution. Indeed, if  $\phi(t, x)$  has a contact from below with  $u$  at  $(t_0, x_0)$ , then  $u$  is differentiable at  $(t_0, x_0)$ , and

$$\partial_t \phi(t_0, x_0) + H(t_0, x_0, \partial_x \phi(t_0, x_0)) = \partial_t u(t_0, x_0) + H(t_0, x_0, \partial_x u(t_0, x_0)) = 0.$$

It is thus a solution if and only if it is a sub-solution.  $\square$

The function  $\check{H}_{t,x}^u$  is by definition the convex envelop of the function which is equal to  $H_{t,x}$  on  $D_x^e u(t, x)$  and equal to  $+\infty$  at every other points. It follows from [6], Theorem 2.35 that

$$\check{H}_{t,x}^u(p) = \inf_{p = \sum a_i p_i} \left( \sum a_i H(t, x, p_i) \right),$$

where the infimum is taken on all the possibilities to write  $p$  as the convex combination  $\sum_{i=0}^d a_i p_i$  of  $d+1$  points of  $D^e u(t, x)$ . Similarly

$$\hat{H}_{t,x}^u(p) = \sup_{p = \sum a_i p_i} \left( \sum a_i H(t, x, p_i) \right).$$

Let us assume that  $u$  is a viscosity solution, and fix a super-differential  $(\eta, p) \in Du(t, x)$ . Since  $Du(t, x)$  is the convex hull of  $D^e u(t, x)$ , there exist  $d+1$  positive numbers  $a_i, 1 \leq 0 \leq d$  and points  $(\eta_i, p_i) \in D^e u(t, x)$  such that

$$\sum_{i=0}^d a_i = 1 \quad \text{and} \quad \sum_{i=0}^d a_i (\eta_i, p_i) = (\eta, p).$$

Since  $u$  is a semi-concave solution, each of the points  $(\eta_i, p_i)$  solve the equation  $\eta_i + H(t, x, p_i) = 0$ , hence

$$\eta + \sum a_i H(t, x, p_i) = 0.$$

Since  $u$  is a viscosity solution, we have the inequality

$$\eta + H(t, x, p) \leq 0.$$

We conclude that

$$H(t, x, p) \leq \sum a_i H(t, x, p_i) \leq \hat{H}_{t,x}^u(p).$$

Since this holds for each point  $(\eta, p) \in Du(t, x)$ , we have proved (2).

Let us now assume (1) and prove that  $u$  is a viscosity solution. We fix a point  $(t, x) \in ]0, T[ \times \mathbb{R}^d$  and a point  $(\eta, p) \in D^e u(t, x)$  and prove that  $\eta + H(t, x, p) \leq 0$ . Since  $Du(t, x)$  is the convex hull of  $D^e u(t, x)$ , there exists positive coefficients  $a_i, 0 \leq i \leq d$  and points  $(\eta_i, p_i) \in D^e u(t, x)$  such that  $\sum_{i=0}^d a_i = 1$  and  $(\eta, p) = \sum_{i=0}^d a_i (\eta_i, p_i)$ . Using (1), we obtain

$$\eta + H(t, x, p) \leq \eta + \check{H}_{t,x}^u(t, x, p) \leq \sum_{i=0}^d a_i (\eta_i + H(t, x, p_i)) = 0$$

since  $\eta_i + H(t, x, p_i) = 0$ . This ends the proof of Theorem 2.  $\square$

## 6 The Hopf formula

In this section, we consider a continuous Hamiltonian which depends only on the variable  $p$ ,  $H(t, x, p) = H(p)$ , and a (finite valued) concave initial condition  $u_0$ . We write  $u_0$  as an infimum of affine functions with the formula  $u_0(x) = \inf_p (p \cdot x - u_0^*(p))$ , where  $u_0^*(p)$  is the (concave version of the) Legendre transform  $u_0^*(p) := \inf_x (p \cdot x - u_0(x))$ . This dual  $u_0^*$  takes the value  $-\infty$ , outside of its domain, denoted by  $P$ . Since  $u_0$  is finite valued (hence locally bounded), we have

$$\lim_{|p| \rightarrow \infty} \frac{u_0^*(p)}{|p|} = -\infty.$$

Since, in addition, the dual  $u_0^*$  is upper semi-continuous, we have

$$u_0(x) = \min_{p \in P} (p \cdot x - u_0^*(p)).$$

The function  $f(t, x) = px - u_0^*(p) - tH(p)$  is a  $C^2$  solution emanating from the affine initial condition  $f_0(x) = px - u_0^*(p)$ . Let us define the function

$$u(t, x) := \inf_{p \in P} (px - u^*(p) - tH(p))$$

associated to this family. Assume that  $T > 0$  satisfies

$$\lim_{|p| \rightarrow \infty} \frac{u_0^*(p) + tH(p)}{|p|} = -\infty$$

for each  $t \in [0, T[$ . This condition appears for example in [9]. In the case where  $u_0$  is Lipschitz,  $u_0^*$  is equal to  $-\infty$  outside of a compact set and we can take  $T = +\infty$ .

The infimum defining  $u$  is a minimum on  $[0, T[ \times \mathbb{R}^d$ , and we have (see Lemma 5)

$$D^e u(t, x) \subset \{(-H(p), p), \quad p \cdot x - u_0^*(p) - tH(p) = u(t, p)\}$$

for each  $(t, x) \in [0, T[ \times \mathbb{R}^d$ , hence

$$D_x^e u(t, x) \subset \{p \in \mathbb{R}^d, \quad p \cdot x - u_0^*(p) - tH(p) = u(t, p)\}.$$

We conclude that

$$H(p) = \frac{p \cdot x - u_0^*(p) - u(t, x)}{t}$$

on  $D_x^e u(t, x)$  and. Denoting by  $c(p)$  the right hand side we have  $H(p) \leq c(p)$  on  $\mathbb{R}^d$ , hence on  $D_x u(t, x)$ . Since  $c(p)$  is a convex function of  $p$ , this implies the entropy inequality (1). We conclude from Theorem 2 that  $u$  is a viscosity solution on  $[0, T[ \times \mathbb{R}^d$ .

In the case where  $H$  also satisfies Hypothesis 1, and where  $u_0$  is Lipschitz, we conclude that  $v = g = u$  on  $[0, \infty) \times \mathbb{R}^d$ . The equality  $g = u$  could also be deduced from Proposition 7.

## A A hessian computation

We consider the function  $f(x) = \varphi(|x|)$  on  $\mathbb{R}^d$ , where  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  is a  $C^2$  function. Denoting by  $e = x/|x|$  the radial direction, the Hessian of  $f$  at a point  $x \neq 0$  is

$$d^2 f_x[y, z] = \frac{\varphi'(|x|)}{|x|} \langle y, z \rangle + \frac{|x| \varphi''(|x|) - \varphi'(|x|)}{|x|} \langle e, y \rangle \langle e, z \rangle.$$

In an orthonormal basis having the radial direction  $e$  as first vector, this bilinear form is expressed by a diagonal matrix with one diagonal coefficient (corresponding to the radial direction) equal to  $\varphi''(|x|)$  and  $d - 1$  diagonal coefficients equal to  $\varphi'(|x|)/|x|$ . As a consequence, its norm is

$$\|d^2 f_x\| = \max \{|\varphi''(|x|)|, |\varphi'(|x|)/|x||\}.$$

In the case considered in Section 3, we have  $0 \leq \varphi''(r) \leq B$  and  $0 \leq \varphi'(r)/r \leq B$  for each  $r > 0$ , hence  $\|d^2 f_x\| \leq B$  at each point  $x \neq 0$ . This inequality also obviously holds at  $x = 0$  since  $f(x) = B\|x\|^2/2$  near the origin.

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