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WIGNER MEASURES AND OBSERVABILITY FOR THE SCHRÖDINGER EQUATION ON THE DISK

NALINI ANANTHARAMAN, MATTHIEU LÉAUTAUD, AND FABRICIO MACIÀ

Abstract. We analyse the structure of semiclassical and microlocal Wigner measures for solutions to the linear Schrödinger equation on the disk, with Dirichlet boundary conditions.

Our approach links the propagation of singularities beyond geometric optics with the completely integrable nature of the billiard in the disk. We prove a "structure theorem", expressing the restriction of the Wigner measures on each invariant torus in terms of *two-microlocal measures*, propagating according to Schrödinger equations on the circle.

Our construction yields as corollaries (a) that the disintegration of the Wigner measures is absolutely continuous in the angular variable, which is an expression of the dispersive properties of the equation; (b) an observability inequality, saying that the L^2-norm of a solution on any open subset intersecting the boundary (resp. the L^2-norm of the Neumann trace on any nonempty open set of the boundary) controls its full L^2-norm (resp. H^1-norm). These results show in particular that the energy of solutions cannot concentrate on periodic trajectories of the billiard flow different from the boundary.

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1. Introduction

1.1. **Motivations.** We consider the unit disk

$$\mathbb{D} = \{z = (x, y) \in \mathbb{R}^2, |z|^2 = x^2 + y^2 < 1\} \subset \mathbb{R}^2$$

and denote by Δ the euclidean Laplacian. We are interested in understanding dynamical properties related to propagation of singularities by the (time-dependent) linear Schrödinger equation

$$\frac{1}{i}\frac{\partial u}{\partial t}(z,t) = \left(-\frac{1}{2}\Delta + V(t,z)\right) u(z,t) \quad , t \in \mathbb{R}, \quad z = (x,y) \in \mathbb{D}, \tag{1.1}$$

$$u\rceil_{t=0} = u^0 \in L^2(\mathbb{D}) \tag{1.2}$$

with Dirichlet boundary condition[1] (we shall write $\Delta = \Delta_D$ when we want to stress that we are using the Laplacian with that boundary condition). We assume that V is a smooth real-valued potential: $V \in C^\infty\left(\mathbb{R} \times \overline{\mathbb{D}}; \mathbb{R}\right)$ (arguments developed in [AM14] show that all the results of this paper could actually be weakened to $V \in C^0\left(\mathbb{R} \times \overline{\mathbb{D}}; \mathbb{R}\right)$ or even to the case where V is continuous outside a set of zero measure). We shall focus on the propagator starting at time 0, denoted by $U_V(t)$, such that $u(t) = U_V(t)u^0$ is the unique solution of (1.1)-(1.2).

Given a sequence of initial conditions (u_n^0) bounded in $L^2(\mathbb{D})$, we shall investigate the regularity properties of the *Wigner measures* associated with the corresponding solutions of (1.1). They describe how the solutions are distributed over phase space. We will develop both the *microlocal* and *semiclassical* points of view; these are two slightly different, but closely related, approaches to the problem.

Our central results are Theorem 1.8 and 1.14, which provide a detailed structure of the Wigner measures associated to sequences of solutions to the Schrödinger equation, using notions of second-microlocal calculus. As corollaries, we obtain:

[1] The extension of our method to the Neumann boundary condition deserves further investigation.

- Theorem 1.16, and Corollary 1.1, which reflect the dispersive character of the Schrödinger equation (1.1);
- Theorem 1.2 (resp. Theorem 1.3), which states the observability/controllability of the equation from any nonempty open set touching the boundary of the disk (resp. from any nonempty open set of the boundary).

Let us first state these corollaries to motivate the more technical results of this paper.

Corollary 1.1. *Let (u_n^0) be a sequence in $L^2(\mathbb{D})$, such that $\|u_n^0\|_{L^2(\mathbb{D})} = 1$ for all n. Consider the sequence of nonnegative Radon measures ν_n on $\overline{\mathbb{D}} \times \mathbb{R}$, defined by*

$$\nu_n(dz, dt) = |U_V(t)u_n^0(z)|^2 dzdt. \tag{1.3}$$

Let ν be any weak-$*$ limit of the sequence (ν_n): then $\nu(dz, dt) = \nu_t(dz)dt$ where, for almost every t, ν_t is a probability measure on $\overline{\mathbb{D}}$, and $\nu_t\rceil_{\mathbb{D}}$ is absolutely continuous.

This result shows that the weak-$*$ accumulation points of the densities (1.3) possess some regularity in the interior of the disk. This result cannot be extended to $\overline{\mathbb{D}}$, since it is possible to construct sequences of solutions that concentrate singularly on the boundary (the so-called whispering-gallery modes, see Section 1.9). In Theorem 1.16 below, we present a refined version of Corollary 1.1 describing (in phase space) the regularity of microlocal lifts of such limit measures ν. This precise description (as well as all results of this paper) relies on the complete integrability of the billiard flow on the disk. Its statement needs the introduction of action angle coordinates and associated invariant tori, and is postponed to Section 1.7.

The second class of results mentioned above are related to unique continuation-type properties of the Schrödinger equation (1.1). We consider the following condition on an open set $\Omega \subset \overline{\mathbb{D}}$, a time $T > 0$ and a potential V:

$$\left(u^0 \in L^2(\mathbb{D}), \quad U_V(t)u^0\rceil_{(0,T)\times\Omega} = 0\right) \quad \Longrightarrow \quad u^0 = 0. \tag{UCP$_{V,\Omega,T}$}$$

As a consequence of Theorem 1.14, we shall also prove the following quantitative version of (UCP$_{V,\Omega,T}$).

Theorem 1.2. *Let $\Omega \subset \overline{\mathbb{D}}$ be an open set such that $\Omega \cap \partial\mathbb{D} \neq \emptyset$ and $T > 0$. Assume one the the following statements holds:*

- *the potential $V \in C^\infty([0,T]\times\overline{\mathbb{D}};\mathbb{R})$, the time T, and the open set Ω satisfy (UCP$_{V,\Omega,T}$),*
- *the potential $V \in C^\infty(\overline{\mathbb{D}};\mathbb{R})$ does not depend on t.*

Then there exists $C = C(V, \Omega, T) > 0$ such that:

$$\left\|u^0\right\|_{L^2(\mathbb{D})}^2 \leq C \int_0^T \left\|U_V(t)u^0\right\|_{L^2(\Omega)}^2 dt, \tag{1.4}$$

for every initial datum $u^0 \in L^2(\mathbb{D})$.

We are also interested in the boundary analogue of (UCP$_{V,\Omega,T}$) for a given potential V, a time $T > 0$ and an open set $\Gamma \subset \partial\mathbb{D}$:

$$\left(u^0 \in H_0^1(\mathbb{D}), \quad \partial_n(U_V(t)u^0)\rceil_{(0,T)\times\Gamma} = 0\right) \quad \Longrightarrow \quad u^0 = 0, \tag{UCP$_{V,\Gamma,T}$}$$

where $\partial_n = \frac{\partial}{\partial n}$ denotes the exterior normal derivative to $\partial\mathbb{D}$. As a consequence of Theorem 1.14, we shall also prove the following quantitative version of (UCP$_{V,\Gamma,T}$).

Theorem 1.3. *Let Γ be any nonempty subset of $\partial\mathbb{D}$ and $T > 0$. Suppose one of the following holds:*

- *the potential* $V \in C^\infty([0,T]\times\overline{\mathbb{D}})$*, the time* T *and the set* Γ *satisfy* ($\mathrm{UCP}_{V,\Gamma,T}$)*,*
- $V \in C^\infty(\overline{\mathbb{D}})$ *does not depend on* t*.*

Then there exists $C = C(V,\Gamma,T) > 0$ *such that:*

$$\|u^0\|^2_{H^1(\mathbb{D})} \leq C \int_0^T \|\partial_n(U_V(t)u^0)\|^2_{L^2(\Gamma)}\, dt, \tag{1.5}$$

for every initial datum $u^0 \in H^1_0(\mathbb{D})$*.*

Note that the unique continuation properties ($\mathrm{UCP}_{V,\Omega,T}$) and ($\mathrm{UCP}_{V,\Gamma,T}$) are known to hold, for instance, when V is analytic in (t,z), as a consequence of the Holmgren uniqueness theorem as stated by Hörmander (see e.g. [Hör76, Theorem 5.3.1]).

These three results express a delocalization property of the energy of solutions to (1.1). The observation of the localized L^2-norm on *any* open set of the disk touching the boundary is sufficient to recover linearly the norm of the data. In particular, the L^2-mass of solutions cannot concentrate on periodic trajectories of the billiard. Estimates (1.4) and (1.5) are called *observability inequalities* and are especially relevant in control theory (see [Lio88, BLR92, Leb92]). In turn, they are known to imply a controllability result from the set Ω or Γ.

As a consequence of the observability inequality 1.4, we have the following result (where we use the notation of Corollary 1.1).

Corollary 1.4. *For every open set* $\Omega \subset \overline{\mathbb{D}}$ *touching the boundary, for every* $T > 0$*, there exists a constant* $C(T,\Omega) > 0$ *such that for any initial data* (u_n^0) *and any weak-*$*$ *limit* ν *of the sequence* (ν_n) *as in Corollary 1.1, we have*

$$\int_0^T \nu_t(\Omega)dt \geq \frac{1}{C(T,\Omega)}.$$

We would like to stress the fact that the structure theorems, Theorems 1.8 or 1.14, give a unified framework from which to derive simultaneously the absolute continuity of projections of semiclassical measures (a fact that is related to dispersive effects) on one hand, and, on the other hand, the observability estimates (1.4) and (1.5), which are quantitative unique continuation-type properties. This unified approach has already been developed in different geometric settings. We refer the reader to the survey articles [Mac11, AM12].

In the remainder of this introduction, we first define microlocal and semiclassical Wigner measures. We then describe the billiard flow and introduce adapted action-angle coordinates. This allows us to formulate our main results, both in the microlocal and in the semiclassical framework. Next, we define various measures at the boundary of the disk, that will be useful in the proofs, and explain their links with the Wigner measures in the interior. Finally, we compare the results obtained here with the existing literature.

1.2. **Wigner measures: microlocal versus semiclassical point of view.** Let $T^*\mathbb{R}^2 = \mathbb{R}^2\times\mathbb{R}^2$ be the cotangent bundle over $\mathbb{R}^2$, and $T^*\mathbb{R} = \mathbb{R}\times\mathbb{R}$ be the cotangent bundle over $\mathbb{R}$. We shall denote by $z \in \mathbb{R}^2$ (resp. $t \in \mathbb{R}$) the space (resp. time) variable and $\xi \in \mathbb{R}^2$ (resp. $H \in \mathbb{R}$) the associated frequency.

Our main results can be formulated in two different and complementary settings. We first introduce the symbol class needed to formulate their microlocal version, allowing to define *microlocal* Wigner distributions. We then define *semiclassical* Wigner distributions and briefly compare these two objects.

Definition 1.5. *Let us call $\mathcal{S}_0$ the space of functions $a \in C^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R})$, $a(z,\xi,t,H)$ such that*

(a) *a is compactly supported in the variables z, t.*
(b) *a is homogeneous at infinity in (ξ, H) in the following sense: there exists $R_0 > 0$ such that*

$$a(z,\xi,t,H) = a(z,\lambda\xi,t,\lambda^2 H), \quad \text{for } |\xi|^2 + |H| > R_0 \text{ and } \lambda \geq 1. \tag{1.6}$$

Equivalently, there is $a_{\text{hom}} \in C^\infty\left(T^*\mathbb{R}^2 \times T^*\mathbb{R} \setminus \{(\xi,H) = (0,0)\}\right)$ satisfying (1.6) for all $\lambda > 0$, such that

$$a(z,\xi,t,H) = a_{\text{hom}}(z,\xi,t,H), \quad \text{for } |\xi|^2 + |H| > R_0.$$

Such a homogeneous function a_{hom} is entirely determined by its restriction to the set $\{|\xi|^2 + 2|H| = 2\} \subset \mathbb{R}^2 \times \mathbb{R}$, which is homeomorphic to a 2-dimensional sphere $\mathbb{S}^2$. Thus we may (and will, when convenient) identify a_{hom} with a function in the space $C^\infty\left(\mathbb{R}^2_z \times \mathbb{R}_t \times \mathbb{S}^2_{\xi,H}\right)$.

Note that the different homogeneity with respect to the H and ξ variables is adapted to the scaling of the Schrödinger operator.

Let (u_n^0) be a sequence in $L^2(\mathbb{D})$, such that $\|u_n^0\|_{L^2(\mathbb{D})} = 1$ for all n. For $z \in \mathbb{D}$ and $t \in \mathbb{R}$ we denote $u_n(z,t) = U_V(t)u_n^0(z)$. In what follows (e.g. in formula (1.7) below), we shall systematically extend the functions u_n, a priori defined on $\mathbb{D}$, by the value 0 outside $\mathbb{D}$ as done in [GL93], where semiclassical Wigner measures for boundary value problems were first considered. The extended sequence now satisfies the equation

$$\left(-\frac{1}{2}\Delta + V - D_t\right) u_n = \frac{1}{2}\frac{\partial u_n}{\partial n} \otimes \delta_{\partial\mathbb{D}}, \quad (z,t) \in \mathbb{R}^2 \times \mathbb{R},$$

where Δ denotes the Laplacian on $\mathbb{R}^2$.

The *microlocal Wigner distributions* associated to (u_n) act on symbols $a \in \mathcal{S}_0$ by

$$W_{u_n}(a) := \langle u_n, \mathrm{Op}_1(a) u_n \rangle_{L^2(\mathbb{R}^2_z \times \mathbb{R}_t)}, \tag{1.7}$$

where $\mathrm{Op}_1(a) = a(z, D_z, t, D_t)$ (with the standard notation $D = -i\partial$) is a pseudodifferential operator defined by the standard quantization procedure. In what follows, $\mathrm{Op}_\epsilon(a) = a(z, \epsilon D_z, t, \epsilon D_t)$ will stand for the operator acting on $L^2(\mathbb{R}^2 \times \mathbb{R})$ by:

$$\big(\mathrm{Op}_\epsilon(a)u\big)(z,t) = \frac{1}{(2\pi\epsilon)^3} \int_{\mathbb{R}^2\times\mathbb{R}} \int_{\mathbb{R}^2\times\mathbb{R}} e^{\frac{i\xi\cdot(z-z') + iH(t-t')}{\epsilon}} a(z,\xi,t,H)\, u(z',t')\, dz'dt'\, d\xi dH. \tag{1.8}$$

Usual estimates on pseudodifferential operators imply that W_{u_n} is well defined, and forms a bounded sequence in $\mathcal{S}_0'$. The main goal of this article is to understand properties of weak limits of (W_{u_n}) that are valid for any sequence of initial conditions (u_n^0).

The problem also has a semiclassical variant. In this version, one considers $a \in C_c^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R})$, a real parameter $h > 0$, and one defines the *semiclassical Wigner distributions* at scale h by

$$W_{u_n}^h(a) := \langle u_n, \mathrm{Op}_1(a(z, h\xi, t, h^2 H)) u_n \rangle_{L^2(\mathbb{R}^2_z \times \mathbb{R}_t)}, \tag{1.9}$$

where $\mathrm{Op}_1(a(z, h\xi, t, h^2 H)) = a(z, hD_z, t, h^2 D_t) = \mathrm{Op}_h(a(z,\xi,t,hH))$, see (1.8). Note that this scaling relation is the natural one for solutions of (1.1), and its interest will be made clear below. Again $W_{u_n}^h$ is well defined, and forms a bounded sequence in $\mathcal{D}'(T^*\mathbb{R}^2 \times T^*\mathbb{R})$ if h stays

bounded. This formulation is most meaningful if the parameter $h = h_n$ is chosen in relation with the typical scale of oscillation of our sequence of initial conditions (u_n^0).

Definition 1.6. *Given a bounded sequence (w_n) in $L^2(\mathbb{D})$, we shall say that it is (h_n)-oscillating from above (resp. (h_n)-oscillating from below) if the sequence (w_n) extended by zero outside of $\mathbb{D}$ satisfies:*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|\xi|\geq R/h_n} |\widehat{w_n}(\xi)|^2 \, d\xi = 0,$$

(resp.

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \int_{|\xi|\leq \epsilon/h_n} |\widehat{w_n}(\xi)|^2 \, d\xi = 0 \quad)$$

where $\widehat{w_n}$ is the Fourier transform of w_n on $\mathbb{R}^2$.

The property of being (h_n)-oscillating from above is only relevant if $h_n \longrightarrow 0$; if u_n^0 is (h_n)-oscillating for (h_n) bounded away from 0, the (extended) sequence (u_n^0) is compact in L^2 and the structure of the accumulation points of $(W_{u_n}^{h_n})$ is trivial. Therefore, we shall always assume that $h_n \longrightarrow 0$. Note that one can always find (h_n) tending to zero such that (u_n^0) is h_n-oscillating from above (to see that, note that for fixed n one may choose h_n such that $\int_{|\xi|\geq 1/h_n} \left|\widehat{u_n^0}(\xi)\right|^2 d\xi \leq n^{-1}$). However, the choice of the sequence h_n is by no means unique (h_n-oscillating sequences are also h_n'-oscillating as soon as $h_n' \leq h_n$), although in many cases there is a natural scale h_n given by the problem under consideration.

Note that h_n is essentially unique if in addition we require that the sequence $\left(u_n^0\right)$ is (h_n)-oscillating from below. One can find (h_n') such that (u_n^0) is h_n'-oscillating from below if and only if the extended (u_n^0) converges to 0 weakly in $L^2(\mathbb{R}^2)$. It is not always possible to find a common (h_n) such that (u_n^0) is h_n-oscillating both from above and below (see [Gér98] for an example of a sequence with this behavior). However, when it is the case, the semiclassical Wigner distributions contain more information that the microlocal ones (see Section 1.6). On the other hand, if no h_n exists such that (u_n^0) is h_n-oscillating from above and below, the accumulation points of $W_{u_n}^{h_n}$ may fail to describe completely the asymptotic phase-space distribution of the sequence (u_n), either because some mass will escape to $|\xi| = \infty$ or because the fraction of the mass going to infinity at a rate slower that h_n^{-1} will give a contribution concentrated on $\xi = 0$. In those cases, the microlocal formulation is still able to describe the asymptotic distribution of the sequence on the reduced phase-space $\mathbb{R}^2_z \times \mathbb{R}_t \times \mathbb{S}^2_{\xi,H}$.

This is one of the motivations that has lead us to study both points of view, semiclassical and microlocal.

1.3. **The billiard flow.** Microlocal or semiclassical analysis provide a connection between the Schrödinger equation and the billiard on the underlying phase space. In this section we first clarify what we mean by "billiard flow" in the disk. The phase space associated with the billiard flow on the disk can be defined as a quotient of $\overline{\mathbb{D}} \times \mathbb{R}^2$ (position $\times$ frequency). We first define the symmetry with respect to the line tangent to the circle $\partial\mathbb{D}$ at $z \in \partial\mathbb{D}$ by

$$\sigma_z(\xi) = \xi - 2z \cdot \xi, \quad \sigma(z,\xi) = (z, \sigma_z(\xi)), \quad z \in \partial\mathbb{D}.$$

Then, we work on the quotient space

$$\mathbb{W} = \overline{\mathbb{D}} \times \mathbb{R}^2 / \sim \quad \text{where } (z,\xi) \sim \sigma(z,\xi) \text{ for } |z| = 1.$$

We denote by π the canonical projection $\overline{\mathbb{D}} \times \mathbb{R}^2 \to \mathbb{W}$ which maps a point (z, ξ) to its equivalence class modulo $\sim$. Note that π is one-one on $\mathbb{D} \times \mathbb{R}^2$, so that $\mathbb{D} \times \mathbb{R}^2$ may be seen as a subset of $\mathbb{W}$.

A function $a \in C^0(\mathbb{W})$ can be identified with the function $\tilde{a} = a \circ \pi \in C^0(\overline{\mathbb{D}} \times \mathbb{R}^2)$ satisfying $\tilde{a}(z, \xi) = \tilde{a} \circ \sigma(z, \xi)$ for $(z, \xi) \in \partial\mathbb{D} \times \mathbb{R}^2$.

The billiard flow $(\phi^\tau)_{\tau \in \mathbb{R}}$ on $\mathbb{W}$ is the (uniquely defined) action of $\mathbb{R}$ on $\mathbb{W}$ such that the map $(\tau, z, \xi) \mapsto \phi^\tau(z, \xi)$ is continuous on $\mathbb{R} \times \mathbb{W}$, satisfies $\phi^{\tau+\tau'} = \phi^\tau \circ \phi^{\tau'}$, and such that

$$\phi^\tau(z, \xi) = (z + \tau\xi, \xi)$$

whenever $z \in \mathbb{D}$ and $z + \tau\xi \in \mathbb{D}$.

In order to understand how the completely integrable dynamics of the flow ϕ^τ influences the structure of Wigner measures, we need to introduce coordinates adapted to this dynamics. We denote by $\Phi : (s, \theta, E, J) \longrightarrow (x, y, \xi_x, \xi_y)$ the set of "action-angle" coordinates for the billiard flow (see also Section 2), defined by:

$$\begin{cases} x = \frac{J}{E}\cos\theta - s\sin\theta, \\ y = \frac{J}{E}\sin\theta + s\cos\theta, \\ \xi_x = -E\sin\theta, \\ \xi_y = E\cos\theta. \end{cases}$$

These coordinates are illustrated in Figure 1. In other words, we have:

$$\begin{cases} E = |\xi|, \\ J = z \cdot \xi^\perp, \\ s = z \cdot \frac{\xi}{|\xi|}, \end{cases}$$

where $\xi^\perp := (\xi_y, -\xi_x)$, and

$$\begin{cases} \xi = (\xi_x, \xi_y) = E(-\sin(\theta), \cos(\theta)), \\ z = (x, y) = s(-\sin(\theta), \cos(\theta)) + \frac{J}{E}(\cos(\theta), \sin(\theta)) = \left(z \cdot \frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|} + \left(z \cdot \frac{\xi^\perp}{|\xi|}\right)\frac{\xi^\perp}{|\xi|}. \end{cases}$$

The map

$$\Phi : \{(s, \theta, E, J) \ : \ (\theta, E) \in \mathbb{R}/2\pi\mathbb{Z} \times (0, \infty), s \in \mathbb{R}, J \in \mathbb{R}\} \to \{(z, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 \ : \ \xi \neq 0\}$$

is a diffeomorphism. Note that the velocity E and the angular momentum J are preserved along the free transport flow in $\mathbb{R}^2 \times \mathbb{R}^2$, but also along ϕ^τ; the variables s and θ play the role of "angle" coordinates. We call $\alpha = -\arcsin\left(\frac{J}{E}\right)$ the angle that a billiard trajectory makes with the normal to the circle, when it hits the boundary (see Figure 2). The quantity α is preserved by the billiard flow.

Let us denote $T_{(E,J)}$ the level sets of the pair (E, J), namely

$$T_{(E,J)} = \{(z, \xi) \in \overline{\mathbb{D}} \times \mathbb{R}^2 \ : \ (|\xi|, z \cdot \xi^\perp) = (E, J)\}. \tag{1.10}$$

For $E \neq 0$ let us denote $\lambda_{E,J}$ the probability measure on $T_{(E,J)}$ that is both invariant under the billiard flow and invariant under rotations. In the coordinates (s, θ, E, J), we have

$$\lambda_{E,J}(ds, d\theta) = c(E, J)dsd\theta, \quad c(E, J) = \left(\int_{T(E,J)} dsd\theta\right)^{-1} > 0.$$

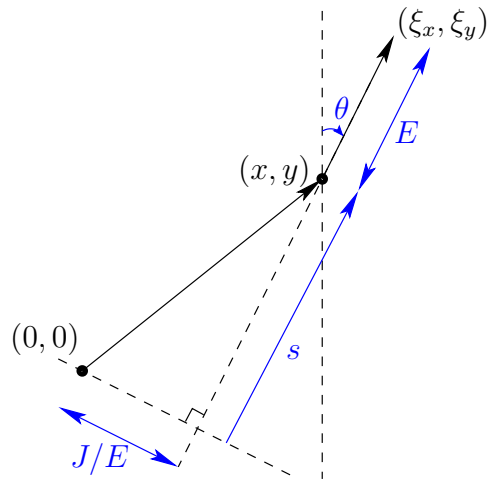

FIGURE 1. Action-angle coordinates for the billiard flow on the disk.

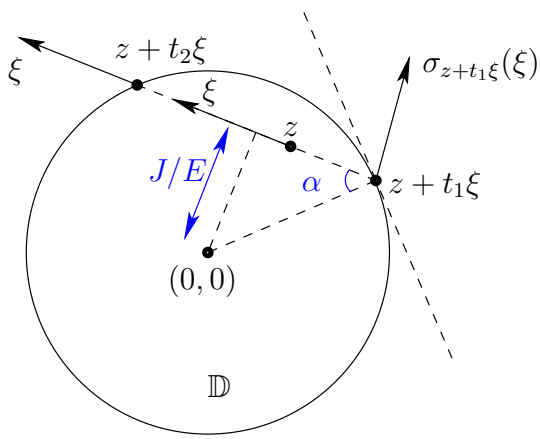

FIGURE 2. Angle α.

Note that for $E \neq 0$ and $\alpha \in \pi\mathbb{Q}$ the billiard flow is periodic on $T_{(E,J)}$ whereas $\alpha \notin \pi\mathbb{Q}$ corresponds to trajectories that hit the boundary on a dense set. More precisely, if $\alpha \notin \pi\mathbb{Q}$ then the billiard flow restricted to $T_{(E,J)}$ has a unique invariant probability measure, namely $\lambda_{E,J}$.

1.4. **Main theorem: semiclassical formulation.** We start formulating the question and results in a semiclassical framework: we have a parameter h_n going to 0, meant to represent the typical scale of oscillation of our sequence of initial conditions (u_n^0).

We simplify the notation by writing $h = h_n$, $u_h^0 = u_n^0$. We will always assume that the functions u_h^0 are normalized in $L^2(\mathbb{D})$. We define $u_h(z,t) = U_V(t)u_h^0(z)$. Since this is a function on $\mathbb{D} \times \mathbb{R}$ it is natural to do a frequency analysis both in z and t. Recall that we keep the notation u_h after the extension by zero outside $\mathbb{D}$. Recall that the *semiclassical Wigner distribution* associated to u_h (at scale h) is a distribution on the cotangent bundle

$T^*\mathbb{R}^2 \times T^*\mathbb{R} = \mathbb{R}^2_z \times \mathbb{R}^2_\xi \times \mathbb{R}_t \times \mathbb{R}_H$, defined by

$$W^h_{u_h} : a \mapsto \left\langle u_h, \mathrm{Op}_1(a(z, h\xi, t, h^2 H))u_h \right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}, \qquad \text{for all } a \in C_c^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R}), \tag{1.11}$$

The scaling $\mathrm{Op}_1(a(z, h\xi, t, h^2 H))$ is performed in order to capture all the information whenever u_h is h-oscillating (if u_h is not h-oscillating, the discussion below remains entirely valid but part of the information about $u_h(z,t)$ is lost when studying $W^h_{u_h}(a)$).

If a is a function on $T^*\mathbb{R}^2 \times T^*\mathbb{R}$ that depends only on (z,t), we note that

$$W^h_{u_h}(a) = \int_{\mathbb{D}} a(z,t)|u_h(z,t)|^2 dz dt. \tag{1.12}$$

It may also be enlightening to rewrite equation (1.1) in the semiclassical time-scale. Putting $v_h(\cdot,t) := h^{1/2}u_h(\cdot, ht)$, equation (1.1) becomes

$$\frac{h}{i}\frac{\partial v_h}{\partial t}(z,t) = h^2\left(-\frac{1}{2}\Delta + V(t,z)\right)v_h(z,t), \tag{1.13}$$

and $W^h_{u_h}(a)$ may also be expressed as

$$\begin{aligned} W^h_{u_h}(a) &= \langle v_h, \mathrm{Op}_1(a(z, h\xi, ht, hH))v_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\ &= \langle v_h, \mathrm{Op}_h(a(z, \xi, ht, H))v_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}. \end{aligned} \tag{1.14}$$

When no confusion arises, we shall denote W_h for $W^h_{u_h}$.

By standard estimates on the norm of $\mathrm{Op}_1(a)$, it follows that W_h belongs to $\mathcal{D}'\left(T^*\mathbb{R}^2 \times T^*\mathbb{R}\right)$, and is uniformly bounded in that space as $h \longrightarrow 0^+$. Thus, one can extract subsequences that converge in the weak-$*$ topology of $\mathcal{D}'\left(T^*\mathbb{R}^2 \times T^*\mathbb{R}\right)$. In other words, after possibly extracting a subsequence, we have

$$W_h(a) \underset{h\longrightarrow 0}{\longrightarrow} \mu_{sc}(a) \tag{1.15}$$

for all $a \in C_c^\infty\left(T^*\mathbb{R}^2 \times T^*\mathbb{R}\right)$.

In this paper such a measure μ_{sc} will be called a *semiclassical Wigner measure*, or in short *semiclassical measure*, associated with the initial conditions (u_h^0) and the scale h.

Remark 1.7. *Remark C.3 tells us that we may without loss of generality assume that $u_h^0 \in H_0^1(\mathbb{D})$ and $\|\nabla u_h^0\|_{L^2(\mathbb{D})} = O(h^{-1})$. In that case Proposition C.1 says that the boundary data $h\partial_n\left(U_V(t)u_h^0\right)$ form a bounded sequence in $L^2_{\mathrm{loc}}(\mathbb{R}\times\partial\mathbb{D})$. We can work under these assumptions when necessary.*

It follows from standard properties of pseudodifferential operators that the limit μ_{sc} in (1.15) has the following properties:

- μ_{sc} is a nonnegative measure, of the form $\mu_{sc}(dz, d\xi, dt, dH) = \mu_{sc}(dz, d\xi, t, dH)dt$ where $t \mapsto \mu_{sc}(t) \in L^\infty(\mathbb{R}_t; \mathcal{M}_+(T^*\mathbb{R}^2 \times \mathbb{R}_H))$. Moreover, for a.e. $t \in \mathbb{R}$, $\mu_{sc}(t)$ is supported in $\{|\xi|^2 = 2H\} \cap \left(\overline{\mathbb{D}} \times \mathbb{R}^2 \times \mathbb{R}_H\right)$. See [GMMP97, LP93] for a proof of nonnegativity; the time regularity and the localization of the support are shown in Proposition D.1.
- From the normalization of u_h^0 in L^2, we have for a.e. t:

$$\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} \mu_{sc}(dz, d\xi, t, dH) \leq 1,$$

the inequality coming from the fact that $\overline{\mathbb{D}} \times \mathbb{R}^2 \times \mathbb{R}$ is not compact, and that there may be an escape of mass to infinity (however, if u_h is h-oscillating from above, escape of mass does not occur and we have $\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} \mu_{sc}(dz, d\xi, t, dH) = 1$).

- The standard quantization enjoys the following property:

$$\left[-\frac{ih}{2}\Delta, \mathrm{Op}_h(a)\right] = \mathrm{Op}_h\left(\xi \cdot \partial_z a - \frac{ih}{2}\Delta_z a\right), \tag{1.16}$$

where Δ is the Laplacian on $\mathbb{R}^2$. From this identity, one can show that

$$\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} \xi \cdot \partial_z a \ \mu_{sc}(dz, d\xi, t, dH) = 0 \tag{1.17}$$

for a.e. t and for every smooth a such that $a(z, \xi, t, H) = a(z, \sigma_z(\xi), t, H)$ for $|z| = 1$. Equivalently,

$$\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} a \circ \phi^\tau \circ \pi(z, \xi, t, H)\mu_{sc}(dz, d\xi, t, dH) = \int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} a \circ \pi(z, \xi, t, H)\mu_{sc}(dz, d\xi, t, dH)$$

for every $a \in C^0(\mathbb{W})$, $\tau \in \mathbb{R}$ – where ϕ^τ is the billiard flow in the disk and $\pi : \overline{\mathbb{D}} \times \mathbb{R}^2 \to \mathbb{W}$ the canonical projection, defined in Section 1.3. In other words, $\pi_*\mu$ is an invariant measure of the billiard flow.

We refer to Section 1.8 for a more general version of (1.16) (as formulated in [GL93]) involving a measure associated to boundary traces.

Main construction.

Now comes our central result, giving the structure of semiclassical measures arising as weak-$*$ limits of sequences (W_h) associated to solutions of (1.1). As a by-product it clarifies the dependence of $\mu_{sc}(t, \cdot)$ on the time parameter t. It was already noted in [Mac09] that the dependence of $\mu_{sc}(t, \cdot)$ on the sequence of initial conditions is a subtle issue.

The statement of Theorem 1.8 is technical and needs introducing some notation. We first restrict our attention to the case where the initial conditions (u_h^0) are h-oscillating from below, or equivalently μ_{sc} does not charge $\{\xi = 0\}$ (otherwise, the restriction of μ_{sc} to $\{\xi = 0\}$ will be better understood at the end of Section 1.6).

The notation (s, θ, E, J), α is as in Section 1.3. Here we restrict our discussion to $E \neq 0$. For each $\alpha_0 \in \pi\mathbb{Q} \cap (-\pi/2, \pi/2)$ we will introduce a flow $(\phi^\tau_{\alpha_0})$ on the billiard phase space $\mathbb{W}$, all of whose orbits are periodic (Lemma 3.3). It coincides with the billiard flow on the set

$$\mathcal{I}_{\alpha_0} = \{(s, \theta, E, J) \in \Phi^{-1}(\overline{\mathbb{D}} \times \mathbb{R}^2), J = -\sin\alpha_0 E\} = \{\alpha = \alpha_0\},$$

which is the union of all the lagrangian manifolds $T_{(E,J)}$ with $J = -\sin\alpha_0 E$. If a is a function on $\mathbb{W}$, we shall denote by $\langle a\rangle_{\alpha_0}$ its average along the orbits of $\phi^\tau_{\alpha_0}$ (actually, $\langle a\rangle_{\alpha_0}$ is well defined even if a is not symmetric with respect to the boundary, since the set of hitting times of the boundary has measure 0). In the coordinates (s, θ, E, J), this is a function whose restriction to $\mathcal{I}_{\alpha_0}$ does not depend on s.

We will denote by

$$m_a^{\alpha_0}(s, E, t, H)$$

the operator on $L^2_{\mathrm{loc}}(\mathbb{R}_\theta)$ acting by multiplication by the function

$$a\left(\Phi(s, \theta, E, -E\sin\alpha_0), t, H\right).$$

If a is a symmetric function (or a function on $\mathbb{W}$), remark that $m^{\alpha_0}_{\langle a\rangle_{\alpha_0}}$ does not depend on the variable s. For our potential V, the function $\langle V\rangle_{\alpha_0}\circ\Phi$ depends only on θ (and t if V is time-dependent).

Given $\omega\in\mathbb{R}/2\pi\mathbb{Z}$, we denote by $U_{\alpha_0,\omega}(t)$ the unitary propagator of the equation

$$-\cos^2\alpha_0 D_t v(t,\theta)+\left(-\frac{1}{2}\partial_\theta^2+\cos^2\alpha_0\langle V\rangle_{\alpha_0}\circ\Phi\right)v(t,\theta)=0$$

acting on the Hilbert space

$$\mathcal{H}_\omega=\{v\in L^2_{\mathrm{loc}}(\mathbb{R}):v(\theta+2\pi)=v(\theta)e^{i\omega},\ \text{for a.e. }\theta\in\mathbb{R}\},$$

i.e. with Floquet-periodic condition. In the statements below, each $\mathcal{H}_\omega$ is identified with $L^2(0,2\pi)$ by taking restriction of functions to $(0,2\pi)$.

Theorem 1.8. *Let (u_h^0) be a family of initial data, assumed to be h-oscillating from below. We can extract a subsequence such that the following hold:*

(i) *The subsequence W_h converges weakly-$*$ to a limit μ_{sc}. The measure μ_{sc} can be decomposed into a countable sum of nonnegative measures, each of them carried by the set $\{H=\frac{E^2}{2}\}$ and invariant under the billiard flow:*

$$\mu_{sc}=\nu_{Leb}+\sum_{\alpha_0\in\pi\mathbb{Q}\cap[-\pi/2,\pi/2]}\nu_{\alpha_0}$$

(ii) *The first term ν_{Leb} is constant in t; ν_{Leb} is of the form $\int_{E>0,|J|\le E}\lambda_{E,J}d\nu'(E,J)dt$ for some nonnegative measure ν' on $\mathbb{R}^2$. In other words ν_{Leb} is a combination of Lebesgue measures on the invariant "tori" $T_{(E,J)}$.*

(iii) *For every $\alpha_0\in\pi\mathbb{Q}\cap(-\pi/2,\pi/2)$, we can build from the sequence of initial conditions (u_h^0) a nonnegative measure $\ell_{\alpha_0}(d\omega,dE,dH,dt)$ (carried by $\{H=E^2/2\}$) on $\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t$, and a function*

$$\sigma_{\alpha_0}:(\mathbb{R}/2\pi\mathbb{Z})_\omega\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t\to\mathcal{L}^1_+\big(L^2(0,2\pi)\big),$$

integrable with respect to ℓ_{α_0}, taking values in the set of nonnegative trace-class operators on $L^2(0,2\pi)$ so that ν_{α_0} is the measure carried by the set $\mathcal{I}_{\alpha_0}\cap\{H=E^2/2\}$ defined by

$$\int a\,d\nu_{\alpha_0}=\int \mathrm{Tr}_{L^2(0,2\pi)}\left(m^{\alpha_0}_{\langle a\rangle_{\alpha_0}}\,\sigma_{\alpha_0}\right)d\ell_{\alpha_0}. \tag{1.18}$$

for all $a\in C_c^\infty(T^*\mathbb{R}^2\times T^*\mathbb{R})$.

Moreover, for ℓ_{α_0}-almost every (ω,E,H), we have

$$\sigma_{\alpha_0}(\omega,E,H,t)=U_{\alpha_0,\omega}(t)\sigma_{\alpha_0}(\omega,E,H,0)U^*_{\alpha_0,\omega}(t). \tag{1.19}$$

This shows in particular that $\sigma_{\alpha_0}\in C^0\left(\mathbb{R}_t;\mathcal{L}^1_+\big(L^2(0,2\pi)\big)\right)$.

(iv) *For $\alpha_0=\pm\frac{\pi}{2}$, ν_{α_0} is a measure that does not depend on t, carried by $\{H=E^2/2\}$ and by $(z,\xi)\in T^*\partial\mathbb{D}$, which is invariant under rotations around the origin.*

Remark 1.9. *The conclusion of the above results (as well as their microlocal counterpart in the next section) also holds for semiclassical measures associated to sequences of approximate solutions of the Schrödinger equation, i.e. satisfying*

$$\left(D_t+\frac{1}{2}\Delta-V(t,z)\right)u_h(z,t)=o_{L^2_{\mathrm{loc}}(\mathbb{D}\times\mathbb{R})}(1).$$

Note that, as in the torus case [AM14, AFKM14]*, Corollary 1.1 also holds for solutions of*

$$\left(D_t + \frac{1}{2}\Delta\right) u_h(z,t) = O_{L^2_{\mathrm{loc}}(\mathbb{D}\times\mathbb{R})}(1),$$

which includes for instance the case of potentials $V \in L^\infty(\mathbb{R}; \mathcal{L}(L^2(\mathbb{D})))$ *(see also* [Bur13] *for related results).*

1.5. **Main theorem: microlocal formulation.** We now give the microlocal version of Theorem 1.8. The main difference is that we now use the class of test functions $\mathcal{S}_0$ defined in Section 1.2.

Let (u_n^0) be a sequence of initial conditions, normalized in $L^2(\mathbb{D})$. Denote $u_n(z,t) = U_V(t)u_n^0(z)$ and recall that u_n also denotes the extended function by zero to whole $\mathbb{R}^2$. All over the paper we let $\chi \in C_c^\infty(\mathbb{R})$ be a nonnegative cut-off function that is identically equal to one near the origin. Let $R > 0$. For $a \in \mathcal{S}_0$, we define

$$\left\langle W_{n,R}^\infty, a\right\rangle := \left\langle u_n, \mathrm{Op}_1\left(\left(1 - \chi\left(\frac{|\xi|^2 + |H|}{R^2}\right)\right) a(z,\xi,t,H)\right) u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})},$$

and

$$\langle W_{c,n,R}, a\rangle := \left\langle u_n, \mathrm{Op}_1\left(\chi\left(\frac{|\xi|^2 + |H|}{R^2}\right) a(z,\xi,t,H)\right) u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \tag{1.20}$$

The Calderón-Vaillancourt theorem [CV71] ensures that both $W_{n,R}^\infty$ and $W_{c,n,R}$ are bounded in $\mathcal{S}_0'$. After possibly extracting subsequences, we have the existence of a limit: for every $a \in \mathcal{S}_0$,

$$\langle \mu^\infty, a\rangle := \lim_{R\to\infty}\lim_{n\to+\infty} \left\langle W_{n,R}^\infty, a\right\rangle,$$

and

$$\langle \mu_c, a\rangle := \lim_{R\to\infty}\lim_{n\to+\infty} \langle W_{c,n,R}, a\rangle. \tag{1.21}$$

The distribution μ_c vanishes if the family (u_n^0) converges weakly to 0 in $L^2(\mathbb{D})$; in other words μ_c reflects the "compact part" of the sequence (u_n^0).

Positivity properties are described in the next proposition.

Proposition 1.10. *μ^∞ is nonnegative, 0-homogeneous and supported at infinity in the variable (ξ, H) (i.e., it vanishes when paired with a compactly supported function). As a consequence, μ^∞ may be identified with a nonnegative Radon measure on $\mathbb{R}^2_z \times \mathbb{R}_t \times \mathbb{S}^2_{\xi,H}$.*

The projection of μ_c on $\mathbb{R}^2_z \times \mathbb{R}_t$ is a nonnegative measure. In other words, $\mu_c(a) \geq 0$ if a is a nonnegative function that does not depend on (ξ, H).

Remark 1.11. *Note that the measure μ^∞ coincides with the microlocal defect measure of* [Gér91b] *(with the appropriate class of symbols $\mathcal{S}_0$).*

Theorem 1.12. *There exists a nonnegative trace class operator ρ_0 on the Hilbert space $L^2(\mathbb{D})$ such that*

$$\langle \mu_c, a\rangle = \int \mathrm{Tr}_{L^2(\mathbb{D})}\left\{U_V(t)^{-1}\mathbb{1}_{\mathbb{D}}\,\mathrm{Op}_1(a(x,\xi,t,H))\mathbb{1}_{\mathbb{D}}U_V(t)\rho_0\right\} dt \tag{1.22}$$

(the meaning of this expression is clarified in Section 7.1).

As a consequence, the projection of μ_c on $\mathbb{D}\times\mathbb{R}_t$ is a nonnegative Radon measure, which is absolutely continuous, and continuous with respect to t.

Note that the ambiguity in the meaning of formula (1.22) arises when a depends on H. If a is independent of H, (1.22) is the well-defined expression

$$\langle \mu_c, a\rangle = \int \mathrm{Tr}_{L^2(\mathbb{D})}\left\{U_V(t)^{-1}\mathbb{1}_{\mathbb{D}}\,\mathrm{Op}_1(a(x,\xi,t))\mathbb{1}_{\mathbb{D}}U_V(t)\rho_0\right\}dt.$$

We now describe μ^∞.

Proposition 1.13. *(i) As a measure on $\mathbb{D}\times\mathbb{R}_t\times\mathbb{S}^2_{\xi,H}$, μ^∞ is supported on the set $\{|\xi|^2=2H\}$.*

(ii) μ^∞ satisfies the invariance property:

$$\left\langle \mu^\infty, \frac{\xi}{\sqrt{2H}}.\partial_z a\right\rangle = 0 \tag{1.23}$$

for a satisfying the symmetry condition $a(z,\xi,t,H)=a(z,\sigma_z(\xi),t,H)$ for $|z|=1$.

Let $\bar\mu^\infty$ be the image of $\mu^\infty(t)$ under the map $(z,\xi,t,H)\mapsto(E,J)$. It is independent of t (see Section 1.7). As previously we call $T_{(E,J)}$ the level sets of (E,J) and $\mathcal{I}_{\alpha_0}=\{J=-\sin\alpha_0 E\}$. The only difference with the semiclassical formalism is that the test functions are homogeneous and thus the measure μ^∞ is naturally defined on $\mathbb{R}^2_z\times\mathbb{R}_t\times\mathbb{S}^2_{\xi,H}$ supported by $\overline{\mathbb{D}}\times\mathbb{R}_t\times\mathbb{S}^2_{\xi,H}$. The microlocal version of Theorem 1.8 reads as follows:

Theorem 1.14. *For any sequence of initial conditions (u_n^0), normalized in $L^2(\mathbb{D})$, we can extract a subsequence such that the following hold:*

(i) *the subsequence W_n converges weakly-$*$ to a limit $\mu_{ml}\in\mathcal{S}_0'$. The measure μ_{ml} can be decomposed into*

$$\mu_{ml}=\mu^\infty+\mu_c$$

where μ_c is as in Theorem 1.12. The measure μ^∞ can be decomposed into a countable sum of nonnegative measures on $\mathbb{R}^2\times\mathbb{R}_t\times\mathbb{S}^2$, each of them carried by $\overline{\mathbb{D}}\times\mathbb{R}_t\times\mathbb{S}^2\cap\{|\xi|^2=2H\}$ and by the cone $\{|J|\le E\}$, and invariant under the billiard flow:

$$\mu^\infty=\mu_{Leb}+\sum_{\alpha_0\in\pi\mathbb{Q}\cap[-\pi/2,\pi/2]}\mu_{\alpha_0}.$$

(ii) *the first term μ_{Leb} does not depend on t; μ_{Leb} is of the form $\int_{E>0,|J|\le E}\lambda_{E,J}d\mu'(E,J)dt$ for some nonnegative measure μ' on $\mathbb{S}^1$ (i.e. the set of pairs (E,J) modulo homothety).*

(iii) *for every $\alpha_0\in\pi\mathbb{Q}\cap(-\pi/2,\pi/2)$, we can build from the sequence of initial conditions (u_n^0) a nonnegative measure $\ell_{\alpha_0}(d\omega,dE,dH,dt)$ (carried by $\{H=E^2/2\}$) on $\mathbb{R}/2\pi\mathbb{Z}\times\{E^2+2|H|=2\}\times\mathbb{R}_t$, and a function*

$$\sigma_{\alpha_0}:(\mathbb{R}/2\pi\mathbb{Z})_\omega\times\{E^2+2|H|=2\}\times\mathbb{R}_t\to\mathcal{L}^1_+\big(L^2(0,2\pi)\big),$$

integrable with respect to ℓ_{α_0}, taking values in the set of nonnegative trace-class operators on $L^2(0,2\pi)$ so that μ_{α_0} is the measure carried by the set $\mathcal{I}_{\alpha_0}\cap\{H=E^2/2\}$ defined by

$$\int a\,d\mu_{\alpha_0}=\int \mathrm{Tr}_{L^2(0,2\pi)}\left(m^{\alpha_0}_{\langle a\rangle_{\alpha_0}}\,\sigma_{\alpha_0}\right)d\ell_{\alpha_0}.$$

for all $a\in\mathcal{S}_0$.

Moreover, for ℓ_{α_0}-almost every (ω,E,H), we have

$$\sigma_{\alpha_0}(\omega,E,H,t)=U_{\alpha_0,\omega}(t)\sigma_{\alpha_0}(\omega,E,H,0)U^*_{\alpha_0,\omega}(t).$$

This shows in particular that $\sigma_{\alpha_0}\in C^0\left(\mathbb{R}_t;\mathcal{L}^1_+\big(L^2(0,2\pi)\big)\right)$.

(iv) *For $\alpha_0 = \pm\frac{\pi}{2}$, μ_{α_0} does not depend on t, it is a measure carried by the set $\mathcal{I}_{\alpha_0} \cap \{H = E^2/2\}$ (which consists of vectors tangent to $\partial\mathbb{D}$) and is invariant under rotations around the origin.*

1.6. **Link between microlocal and semiclassical Wigner measures.** Let us clarify the link between the two approaches in the context of the present article (see also [Gér91a, GL93] for a related discussion).

As was said, if (u_n^0) is h_n-oscillating from above and below, the semiclassical Wigner measures convey more information than the microlocal ones. In fact, if (u_n) is h_n^0-oscillating from above and below (with $h_n \to 0$), we have for $a \in \mathcal{S}_0$

(1.24)
$$\begin{aligned} W_{u_n}(a) &= \langle u_n, \mathrm{Op}_1\left(a(z,\xi,t,H)\left(\chi\left(\frac{h_n^2(|\xi|^2+|H|)}{R}\right) - \chi\left(\frac{h_n^2(|\xi|^2+|H|)}{\epsilon}\right)\right)\right) u_n\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\ &\qquad + o(1)_{\epsilon\longrightarrow 0, R\longrightarrow+\infty} \\ &= \langle u_n, \mathrm{Op}_1\left(a_{\mathrm{hom}}(z,h_n\xi,t,h_n^2H)\left(\chi\left(\frac{h_n^2(|\xi|^2+|H|)}{R}\right) - \chi\left(\frac{h_n^2(|\xi|^2+|H|)}{\epsilon}\right)\right)\right) u_n\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\ &\qquad + o(1)_{\epsilon\longrightarrow 0, R\longrightarrow+\infty} \\ &= W_{u_n}^{h_n}\left(a_{\mathrm{hom}}\left(\chi\left(\frac{(|\xi|^2+|H|)}{R}\right) - \chi\left(\frac{(|\xi|^2+|H|)}{\epsilon}\right)\right)\right) + o(1)_{\epsilon\longrightarrow 0, R\longrightarrow+\infty} \end{aligned}$$

From (1.24), one sees that if W_{u_n} converges weakly to μ_{ml} and $W_{u_n}^{h_n}$ converges weakly to μ_{sc}, and if (u_n^0) is h_n-oscillating from above and below, we have

$$\mu_{ml}(a) = \mu_{sc}(a_{\mathrm{hom}})$$

The right-hand side is well-defined since μ_{sc} is a nonnegative measure which is bounded on sets of the form $\mathbb{D}\times\mathbb{R}^2\times[-T,T]\times\mathbb{R}$ (for any T).

On the other hand, if (u_n^0) is not h_n-oscillating from below, then μ_{sc} does charge the set $\{\xi = 0\}$, and we have for any compactly supported function a :

$$\begin{aligned} \mu_{sc}\rceil_{(\xi,H)=0}(a) &= \lim_{\epsilon\longrightarrow 0}\lim_{n\longrightarrow+\infty} W_{\chi\left(\frac{h_n^2(|D_z|^2+|D_t|)}{\epsilon}\right)u_n}(a(z,0,t,0)) \\ &= \lim_{\epsilon\longrightarrow 0}\lim_{n\longrightarrow+\infty} W_{\chi\left(\frac{3h_n^2|D_t|}{\epsilon}\right)u_n}(a(z,0,t,0)) \\ &= \lim_{\epsilon\longrightarrow 0}\lim_{n\longrightarrow+\infty} W_{U_V(t)v_{n,\epsilon}^0}(a(z,0,t,0)) \end{aligned} \tag{1.25}$$

where $v_{n,\epsilon}^0 = \chi\left(\frac{3h_n^2|D_t|}{\epsilon}\right)u_n\rceil_{t=0}$. Equality of the first two lines comes from the fact that the measures asymptotically concentrate on $\{|\xi|^2 = 2H\}$, and equality of the last two lines is proven in Appendix C. We see that the microlocal Wigner measures associated with $U_V(t)v_{n,\epsilon}^0$ encompass the description of $\mu_{sc}\rceil_{(\xi,H)=0}$.

Finally, if (u_n^0) is not h_n-oscillating from above, we see that

$$\lim_{R\longrightarrow+\infty}\lim_{n\longrightarrow+\infty} W^{h_n}_{(1-\chi)\left(\frac{h_n^2(|D_z|^2+|D_t|)}{R}\right)u_n}(a) = 0$$

for compactly supported a, whereas the limit

$$\lim_{R\longrightarrow+\infty}\lim_{n\longrightarrow+\infty} W_{(1-\chi)\left(\frac{h_n^2(|D_z|^2+|D_t|)}{R}\right)u_n}(a)$$

does not necessarily vanish for homogeneous a. This last limit coincides with

$$\lim_{R\longrightarrow+\infty}\lim_{n\longrightarrow+\infty} W_{(1-\chi)\left(\frac{3h_n^2|D_t|}{R}\right)u_n}(a) = \lim_{R\longrightarrow+\infty}\lim_{n\longrightarrow+\infty} W_{U_V(t)w^0_{n,R}}(a)$$

where $w^0_{n,R} = (1-\chi)\left(\frac{3h_n^2|D_t|}{R}\right)u_n\rceil_{t=0}$, and equality of the limits is proven in Appendix C. Thus, the frequencies of u_n^0 that are of order $\gg h_n^{-1}$ or $\ll h_n^{-1}$ are better captured by the microlocal approach.

Note finally that (1.25) allows to complete the statement of Theorem 1.8 by describing $\mu_{sc}\rceil_{(\xi,H)=0}$ when u_h^0 is not assumed to be h-oscillatory from below: we have

$$\mu_{sc}\rceil_{(\xi,H)=0}(a(z,\xi,t,H)) = \mu_{ml,0}(a(z,0,t,0))$$

where $\mu_{ml,0}$ is a microlocal Wigner measure associated with the sequence of initial data $v^0_{h,\epsilon} = \chi\left(\frac{3h|D_t|}{\epsilon}\right)u_h\rceil_{t=0}$ (in the limits $h\longrightarrow 0$ and $\epsilon\longrightarrow 0$). Thus $\mu_{ml,0}$ possesses the structure described in Theorem 1.14.

1.7. **Application to the regularity of limit measures.** Theorem 1.14, applied to test functions $a\in\mathcal{S}_0$ that do not depend on t and H, implies Corollary 1.1. To state a precise version of this result (say, in the semiclassical setting), we first need the following proposition.

Proposition 1.15. *Suppose that μ_{sc} is a semiclassical measure associated to (u_h) solution of (1.1)-(1.2). Denote by $\bar\mu_{sc}(dE,dJ,t)$ the image of the measure $\mu_{sc}(dz,d\xi,t,dH)$ under the moment map*

$$M:(z=(x,y),\xi,H)\mapsto(E,J)=(|\xi|,x\xi_y-y\xi_x)$$

(velocity and angular momentum). Then $\bar\mu_{sc}$ does not depend on t.

This proposition is proved in Section 6.1.

Arguments developed in [AM14] (and that we do not reproduce here) show that Corollary 1.1 can be refined as follows.

Theorem 1.16. *Define by $\mu_{E,J}(t,\cdot)$ is the disintegration of $\mu_{sc}(t,\cdot)$ with respect to the variables (E,J), carried on the 2-dimensional (lagrangian) manifold $T_{(E,J)}=\{(z,\xi),(|\xi|,x\xi_y-y\xi_x)=(E,J)\}$, i.e.*

$$\int_{\mathbb{R}_H}\int_{\overline{\mathbb{D}}\times\mathbb{R}^2} f(z,\xi,t,H)\mu_{sc}(dz,d\xi,t,dH)$$
$$=\int_{\mathbb{R}^2}\left(\int_{T_{(E,J)}} f\left(z,\xi,t,\frac{E^2}{2}\right)\mu_{E,J}(t,dz,d\xi)\right)\bar\mu_{sc}(dE,dJ),$$

for every bounded measurable function f, for $t\in\mathbb{R}$.

Then for $\bar\mu_{sc}$-almost every (E,J) with $|J|\neq E$, the measure $\mu_{E,J}(t,\cdot)$ is absolutely continuous on $T_{(E,J)}$.

Note that $|J| = E$, with $E \neq 0$, means that $T_{(E,J)} \cap \left(\overline{\mathbb{D}} \times \mathbb{R}^2\right)$ is contained in the set $\{(z,\xi), |z| = 1, z \perp \xi\}$ of tangent rays to the boundary. The restriction of $\mu_{sc}(t)$ to that set may be considered trivial, since (1.17) implies that it is invariant under rotation.

Finally, for $J = E = 0$, we can use the last lines of Section 1.6, combined with Theorem 1.14: the measure μ_{sc} restricted to $\{\xi = 0\} = \overline{\mathbb{D}} \times \{0\}$ is the sum of an absolutely continuous measure carried by the interior $\mathbb{D}$ and a multiple of the Lebesgue measure on $\partial\mathbb{D}$.

Remark 1.17. *The analogues of Proposition 1.15 and Theorem 1.16 hold as well in the microlocal setting. We chose not to state them for the sake of concision.*

1.8. **Measures at the boundary.** In this section, we define and compare different measures on $\partial\mathbb{D}$. Given an invariant measure for the billiard flow, we first define the associated "projected measure" on the boundary. Second, we define semiclassical and microlocal measures associated with the Neumann trace at the boundary of sequences of solutions of (1.1). We finally explain the links between these three objects.

1.8.1. *Projection on the boundary of an invariant measure.* We observe the following standard construction from the theory of Poincaré sections in dynamical systems. Let $S = \{(z,\xi), |z| = 1, \xi \cdot z \neq 0\}$, union of $S^+ = \{(z,\xi), |z| = 1, \xi \cdot z > 0\}$ (vectors pointing outwards) and of $S^- = \{(z,\xi), |z| = 1, \xi \cdot z < 0\}$ (vectors pointing inwards). When $(z,\xi) \in S^+$, we denote as above by $\alpha(z,\xi) = -\arcsin\left(\frac{J(z,\xi)}{|\xi|}\right)$ the angle of the vector ξ with the normal at z to the disk. The map

$$\begin{array}{rcl} P : \{(z,\xi,\tau) \in S^+ \times \mathbb{R}, \tau \in [0, 2\cos\alpha(z,\xi)]\} & \longrightarrow & \overline{\mathbb{D}} \times \mathbb{R}^2 \\ (z,\xi,\tau) & \mapsto & \left(z + \dfrac{\tau}{|\xi|}\sigma_z(\xi), \sigma_z(\xi)\right) \end{array}$$

is a measurable bijection onto its image $S \cup (\mathbb{D} \times \mathbb{R}^2)$, and $\pi \circ P$ is a measurable bijection onto its image (recall that π is the projection from $\overline{\mathbb{D}} \times \mathbb{R}^2$ to $\mathbb{W}$). If μ is a nonnegative measure on $S \cup (\mathbb{D} \times \mathbb{R}^2)$ which does not charge S, and such that $\pi_*\mu$ is invariant under the billiard flow, then $P_*^{-1}\mu$ must be of the form

$$P_*^{-1}\mu = \mu^S \otimes d\tau$$

where μ^S is a measure on S^+ which is invariant under the first return map

$$(z,\xi) \mapsto \left(z + \frac{2\cos\alpha(z,\xi)}{|\xi|}\sigma_z(\xi), \sigma_z(\xi)\right).$$

This implies that

$$\begin{aligned} \int_{\mathbb{D}\times\mathbb{R}^2} \xi.\partial_z a \; d\mu &= \int |\xi| \partial_\tau (a \circ P) d\mu^S \otimes d\tau \\ &= \int_{S^+} |\xi| \left(a\left(z + \frac{2\cos\alpha}{|\xi|}\sigma_z(\xi), \sigma_z(\xi)\right) - a(z, \sigma_z(\xi))\right) \mu^S(dz, d\xi) \\ &= \int_{S^+} |\xi| \left(a(z,\xi) - a(z,\sigma_z(\xi))\right) \mu^S(dz, d\xi). \end{aligned} \tag{1.26}$$

Note that the total mass of μ is $\int d\mu = \int_{S^+} 2\cos\alpha(z,\xi) d\mu^S(z,\xi)$.

1.8.2. *Semiclassical measure associated to Neumann trace.* Let (u_h^0) be a family of initial conditions, normalized in $L^2(\mathbb{D})$. When we look at the semiclassical Wigner distributions (1.14), where we use *compactly supported* symbols, Remarks 1.7 and C.3 show that we may truncate (u_h^0) in frequency and assume, without changing the limit as $h \longrightarrow 0$, that $u_h^0 \in H_0^1(\mathbb{D})$, $\|\nabla u_h^0\|_{L^2(\mathbb{D})} = O(h^{-1})$. Proposition C.1 then entails that the boundary data $h\partial_n \left(U_V(t)u_h^0\right)$ form a bounded sequence in $L^2_{\text{loc}}(\mathbb{R} \times \partial\mathbb{D})$.

Now, let $\mu_{sc}^{\partial} \in \mathcal{M}_+(T^*\partial\mathbb{D} \times T^*\mathbb{R})$ be a semiclassical measure associated with the boundary data $h\partial_n u_h(t)$ defined by quantizing test functions on $T^*\partial\mathbb{D} \times T^*\mathbb{R}$ with the same scaling (hj, h^2H) in the cotangent variables as in (1.11). Then μ_{sc}^{∂} is carried by the set $\{(u, j, t, H) \in T^*\partial\mathbb{D} \times T^*\mathbb{R}, |j| \leq \sqrt{2H}\}$. If μ_{sc} and μ_{sc}^{∂} are obtained through the same sequence of initial data, then we have the relation (see [GL93])

(1.27)
$$\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}} \xi \cdot \partial_z a \; \mu_{sc}(dz, d\xi, t, dH)dt = \int \frac{a(u, \xi^+(j, H)) - a(u, \xi^-(j, H))}{2\sqrt{2H - j^2}} \mu_{sc}^{\partial}(du, dj, dt, dH)$$

valid for any smooth function a. For $(u, j) \in T^*\partial\mathbb{D}$ with $|j| \leq \sqrt{2H}$, the vectors $\xi^{\pm}(j, H)$ are the two vectors (pointing outwards and inwards) in $T_u^*\mathbb{R}^2$ of norm $= \sqrt{2H}$, whose projection to $T_u^*\partial\mathbb{D}$ is j. Note that the expression under the integral on the right hand side of (1.27) has a well-defined finite limit as $|j| \longrightarrow \sqrt{2H}$. Identity (1.27) has three consequences:

- first, that the measure μ_{sc} does not charge the set S defined in Section 1.3 (otherwise the left-hand side of (1.27) would define a distribution of order 1 which is not a measure). Note that (1.27) is stronger than (1.17).
- Second, let $\mu_{sc}^S(t)$ be the measure associated to $\mu_{sc}(t)$ as in Section 1.8.1. Comparing (1.27) with (1.26), we see that for any a defined on S^+,

$$\int_{(u,j)\in T^*\partial\mathbb{D}, |j|<\sqrt{2H}} a(u, \xi^+(j, H), t, H)\mu_{sc}^{\partial}(du, dj, dt, dH)$$
$$= \int_{S^+} 2|\xi|^2 \cos\alpha(z, \xi)a(z, \xi)\mu_{sc}^S(dz, d\xi, t, dH)dt.$$

- Third, (1.27) implies

$$\int_{T^*\partial\mathbb{D}\times\mathbb{R}\times\mathbb{R}} |\xi|^2 a(z, \xi, t, H)\mu_{sc}(dz, d\xi, t, dH)dt = \int_{|j|=\sqrt{2H}} a(u, j, t, H)\mu_{sc}^{\partial}(du, dj, dt, dH).$$

  In particular, note that $\mu_{sc}^{\partial}\rceil_{H=0}$ vanishes, since $H = 0$ corresponds to $\xi = 0$ on the left-hand side.

Identities (1.17) and (1.27) are essentially proven in [GL93] for general domains (for time-independent solutions of (1.1)); we do not reproduce the proofs here.

1.8.3. *Microlocal measure associated to Neumann trace.* The sequences considered here $u_n = U_V(t)u_n^0$ are bounded in $L^\infty(\mathbb{R}; L^2(\mathbb{D}))$. Since normal traces are not convenient to work with at this level of regularity, the definition of associated microlocal measures needs a little care.

For this, let us first define $\psi \in C^\infty(\mathbb{R})$, such that $\psi = 0$ on $(-\infty, 1]$ and $\psi = 1$ on $[2, +\infty)$ and the operator $A(D_t) = \text{Op}_1\left(\frac{\psi(H)}{\sqrt{2H}}\right)$. We have the following regularity result.

Lemma 1.18. *For all $\varphi \in C^\infty(\mathbb{R}_t \times \overline{\mathbb{D}}_z)$ with compact support in the first variable $t \in \mathbb{R}$, there exists a constant $C = C(\varphi, \psi) > 0$ such that for any $u^0 \in L^2(\mathbb{D})$, the associated solution*

$u(t) = U_V(t)u^0$ *satisfies*

$$\|A(D_t)\varphi u\|_{L^2(\mathbb{R};H^1(\mathbb{D}))} \leq C\|u^0\|_{L^2(\mathbb{D})}.$$

This Lemma is proved at the end of Appendix C.

We now define, for any $g \in C_c^\infty(\mathbb{R})$, the sequence $\tilde{u}_n = A(D_t)g(t)u_n$, solution of

$$\left(D_t + \frac{1}{2}\Delta\right)\tilde{u}_n = A(D_t)\left(g(t)V(t,z) + ig'(t)\right)u_n. \tag{1.28}$$

As a consequence of Lemma 1.18, we have $\|\tilde{u}_n\|_{L^2(\mathbb{R};H^1(\mathbb{D}))} \leq C\|u_n^0\|_{L^2(\mathbb{D})}$ together with

$$\|A(D_t)\left(g(t)V(t,z)u_n + ig'(t)\right)u_n\|_{L^2(\mathbb{R};H^1(\mathbb{D}))} \leq C\|u_n^0\|_{L^2(\mathbb{D})}.$$

Equation (1.28) then implies that $\|\tilde{u}_n\|_{L^\infty(\mathbb{R};H^1(\mathbb{D}))} \leq C\|u_n^0\|_{L^2(\mathbb{D})}$ and that $A(D_t)g(t)\partial_n u_n = \partial_n \tilde{u}_n$ is bounded in $L^2(\mathbb{R}\times\partial\mathbb{D})$ by $\|u_n^0\|_{L^2(\mathbb{D})}$, according to the hidden regularity of Proposition C.1. Hence, if we take g to be constant equal to 1 on the support of a, after extraction of subsequences, the following limit exists

$$\langle \mu_{ml}^\partial, a\rangle = \lim_{R\to\infty}\lim_{n\to+\infty}\left\langle \partial_n\tilde{u}_n, \mathrm{Op}_1\left(\left(1-\chi\left(\frac{|H|}{R^2}\right)\right)a(u,j,t,H)\right)\partial_n\tilde{u}_n\right\rangle_{L^2(\partial\mathbb{D}\times\mathbb{R})},$$

for symbols $a \in C^\infty(T^*(\partial\mathbb{D}\times\mathbb{R}))$, compactly supported in the variables z, t, such that

$$a(u,j,t,H) = a(u,\lambda j,t,\lambda^2 H), \quad \text{for } |H| > R_0 \text{ and } \lambda \geq 1.$$

Then μ_{ml}^∂ is carried by the set $\{(u,j,t,H) \in T^*\partial\mathbb{D}\times T^*\mathbb{R}, |j| \leq \sqrt{2H}\}$. If moreover μ_{ml} and μ_{ml}^∂ are obtained through the same sequence of initial data, then we have the relation (see again [GL93])

$$\begin{aligned}\int_{\overline{\mathbb{D}}\times\mathbb{R}_t\times\mathbb{S}^2_{\xi,H}} &\frac{\xi}{\sqrt{2H}}\cdot\partial_z a\ \mu^\infty(dz,d\xi,t,dH)dt \\ &= \int_{|j|\leq\sqrt{2H}}\frac{1}{2}\left(\frac{2H}{2H-j^2}\right)^{\frac{1}{2}}\left(a(u,\xi^+(j,H)) - a(u,\xi^-(j,H))\right)\mu_{ml}^\partial(du,dj,dt,dH)\end{aligned} \tag{1.29}$$

valid for any $a \in \mathcal{S}_0$. The vectors $\xi^\pm(j,H)$ are the two vectors (pointing outwards and inwards) in $T_u^*\mathbb{R}^2$ of norm $= \sqrt{2H}$, whose projection to $T_u^*\partial\mathbb{D}$ is j. As above, this implies that μ^∞ does not charge the set S; we then denote by $\mu_{ml}^S(t)$ the measure associated to $\mu^\infty(t)$ as in Section 1.8.1. Comparing with (1.26), we see that for any $a \in \mathcal{S}_0$, we have

$$\begin{aligned}\int_{(u,j)\in T^*\partial\mathbb{D},|j|<\sqrt{2H}} &a(u,\xi^+(j,H),t,H)\mu_{ml}^\partial(du,dj,dt,dH) \\ &= \int_{S^+} 2\cos\alpha(z,\xi)a(z,\xi)\mu_{ml}^S(dz,d\xi,t,dH)dt.\end{aligned} \tag{1.30}$$

Moreover, (1.29) implies

$$\begin{aligned}\int_{(u,\xi)\in T^*\partial\mathbb{D},|\xi|=\sqrt{2H}} &a(z,\xi,t,H)\mu^\infty(dz,d\xi,t,dH)dt \\ &= \int_{|j|=\sqrt{2H}} a(u,j,t,H)\mu_{ml}^\partial(du,dj,dt,dH).\end{aligned} \tag{1.31}$$

These links between the different measures shall be in particular useful when proving the boundary observability result of Theorem 1.3.

1.9. **Comments and relations to other works.**

1.9.1. *Regularity of semiclassical measures.* This work belongs to the longstanding study of the so-called "quantum-classical correspondence", which aims at understanding the links between high frequency solutions of the Schrödinger equation and the dynamics of the underlying billiard flow (see for instance the survey article [AM12]).

More precisely, it is concerned with a case of completely integrable billiard flow. This particular dynamical situation has already been addressed on compact manifolds in [Mac10] and [AM14] in the case of flat tori (we refer to these papers for a detailed discussion of the existing literature for the Schrödinger equation on flat tori), and in [AFKM14] in the general case. This last paper contains a precise description of Wigner measures associated to solutions of the Schrödinger equation on non-degenerate completely integrable systems.

The proofs of these three papers use in a central way a "second microlocalization" to understand the possible concentration on invariant tori. The main tools are some two microlocal semiclassical measures, introduced in the local Euclidean setting in [Nie96, FK00a, FK00b, FKG02, Mil96, Mil97], and defined in [Mac10, AM14, AFKM14] as global objects according to the dynamics.

The present paper provides the results analogous of [Mac10, AM14, AFKM14] on the disk. The main additional difficulties here come from the fact that we are considering a boundary problem. To our knowledge, there is little work studying two microlocal measures for boundary value problems [Mil96, Mil97]. In these papers though, the second microlocal parameter describes the thickness of the interface, whereas here, it aims at describing the concentration on invariant tori. The difficulties hence arise when comparing action-angle variables (in which we perform the second microlocalization) and polar coordinates (well-adapted to describe the boundary of the disk).

1.9.2. *Observability of the Schrödinger equation.* Since the pioneering work of Lebeau [Leb92], it is known that observability inequalities like (1.4)-(1.5) always hold if all trajectories of the billiard enter the observation region Ω or Γ in finite time. However, since [Har89, Jaf90], we know that this strong geometric control condition is not necessary: (1.4) holds on the torus as soon as $\Omega \neq \emptyset$. These properties seem to deeply depend on the global dynamics of the billiard flow.

To our knowledge, few results are known concerning the observability of the Schrödinger equation in situations where the geometric control condition fails: as already mentioned, the observability inequality (1.4) holds on flat tori for any nonempty open subset (see [Jaf90, Mac11, BZ12] in the 2-D case and [Kom92, AM14] for higher dimensional tori). More generally, it is also true on completely integrable systems under the suitable non-degeneracy assumptions, for open sets that intersect every invariant torus, see [AFKM14]. Note also that boundary observability (1.5) holds in the square if (and only if) the observation region Γ contains both a horizontal and a vertical nonempty segments [RTTT05]. Finally, the observability inequality (1.4) is also valid on manifolds with negative curvature [Ana08, AR12] if the set of uncontrolled trajectories is sufficiently small.

In the present situation, Theorems 1.2 and 1.3 provide necessary and sufficient conditions for the observability of the Schrödinger group on the disk. This is clear in the case of boundary observability since Theorem 1.3 only requires Γ to be non-empty. In the case of internal observability, if $\Omega \subset \mathbb{D}$ is such that $\Omega \cap \partial\mathbb{D} = \emptyset$, the observability inequality fails (1.4). One can indeed construct so-called whispering-gallery modes concentrating on the boundary.

Taking for instance as initial data $u_n^0 := \psi_{n,0}^{\pm}$ (see (1.32) below) with and $n \to \infty$, then it is classical (see for instance [Lag83, Lemma 3.1]) that

$$|u_n^0|^2 dx \rightharpoonup (2\pi)^{-1}\delta_{\partial\mathbb{D}};$$

more precisely, the Wigner measures associated with the initial data $u_n^0 := \psi_{n,0}^{\pm}$ concentrate on the set $\{|J| = E\}$. Combined with Proposition 1.15, this shows that $\bar{\mu}^{ml}$ is entirely carried by the set $\{|J| = E\}$, and thus μ^{ml} itself does not charge the interior of the disk, where $|J| < E$. This shows that (1.4) cannot hold if Ω does not touch the boundary.

Remark 1.19. *Note finally that our results directly yield a polynomial decay rate for the energy of the damped wave equation $(\partial_t^2 - \Delta + b(z)\partial_t)u = 0$ with Dirichlet Boundary conditions on the disk. More precisely,* [AL14, Theorem 2.3] *and Theorem 1.2 imply that if $b \geq 0$ is positive on an open set Ω such that $\Omega \cap \partial\mathbb{D} \neq \emptyset$, then the $H_0^1 \times L^2$ norm of the solution decays at rate $1/\sqrt{t}$ for data in $(H^2 \cap H_0^1) \times H_0^1$. This rate is better than the* a priori *logarithmic decay rate given by the Lebeau theorem* [Leb96].

1.9.3. *Observability of eigenfunctions on the disk in the case $V = 0$.* The localization properties of solutions of the Schrödinger equation are intimately linked to the localization of high energy eigenfunctions of the elliptic operator involved. And in the absence of potential, the eigenfunctions of the Dirichlet laplacian on $\mathbb{D}$ are well identified: they are the functions whose expression in polar coordinates $(x = -r\sin u, y = r\cos u)$ is

$$\psi_{n,k}^{\pm}(re^{iu}) = J_n(\alpha_{n,k}r)e^{\pm inu} \tag{1.32}$$

where n, k are nonnegative integers, J_n is the n-th Bessel function, and the $\alpha_{n,k}$ are its positive zeros ordered increasingly with respect to k. The corresponding eigenvalue is $-\alpha_{n,k}^2$.

It was proved by Siegel [Sie29] that J_n, J_m have no common zeroes for $n \neq m$. Thus, the spectrum of Δ_D has multiplicity ≤ 2, and it follows that for any non empty open set $\Omega_{I_1,I_2} = \{re^{iu}, r \in I_1, u \in I_2\} \subset \mathbb{D}$ (where I_1 is an open subset of $\mathbb{R}$, I_2 an interval of $\mathbb{S}^1$), for any eigenfunction ψ,

$$\|\psi\|_{L^2(\Omega_{I_1,\mathbb{S}^1})} \leq C(|I_2|)\|\psi\|_{L^2(\Omega_{I_1,I_2})}$$

where $C(|I_2|)$ is a positive constant depending only on the size of I_2. In particular, if Ω_{I_1,I_2} touches the boundary ($1 \in I_1$), we have

$$\|\psi\|_{L^2(\mathbb{D})} \leq C(|I_2|)C'(I_1)\|\psi\|_{L^2(\Omega_{I_1,I_2})}$$

since the geometric control condition implies that

$$\|\psi\|_{L^2(\mathbb{D})} \leq C'(I_1)\|\psi\|_{L^2(\Omega_{I_1,\mathbb{S}^1})}.$$

We do not know of a straightforward argument that would yield the observability inequality of Theorem 1.2 or the regularity result of Theorem 1.16 for general initial data, directly from the result for eigenfunctions.

One can imagine that for $V = 0$ one could prove Theorems 1.2 and 1.16 by expanding the kernel of $e^{it\Delta_D}$ into Bessel functions:

$$e^{it\Delta_D} = \sum_{n,k} e^{-it\alpha_{n,k}^2}|\psi_{n,k}\rangle\langle\psi_{n,k}|$$

and by using some of their known properties. Such an approach may be possible, but would require some technical work on the spacings between the $\alpha_{n,k}$. This is not the kind of approach we have in mind here; our goal is to understand how the completely integrable nature of the

dynamics of the billiard flow allows to prove delocalization and dispersion of the solutions to the Schrödinger equation. In particular we understand how to incorporate the presence of a boundary in the construction of [AM14] to obtain a structure theorem such as Theorems 1.8 or 1.14.

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Organization of the paper. Section 2 discusses in more detail the coordinates described in the introduction, in which the dynamics of the billiard can be integrated; it introduces the Fourier Integral Operator corresponding to this change of coordinates.

Section 3 reduces the study of an invariant measure to its restriction to all the sets $\mathcal{I}_\alpha$. Sections 4 and 5 are devoted to the proof of Theorem 1.8 (semiclassical version of the result). In Section 7 we explain how to adapt the proof to obtain the microlocal version, Theorem 1.14.

The observability inequalities are then derived in Section 8.

Appendices A and B are devoted to extremely technical calculations, that seem unavoidable. The problem we face is that the coordinates of Section 2, in which the classical dynamics is very nice, are extremely nasty to treat the boundary condition (for which the use of polar coordinates would be more suitable). It seems that we cannot avoid the problem of going back and forth between the two sets of coordinates. The calculations are made even more technical by the fact that, to treat the second-microlocal calculus, we have to understand the asymptotic expansions of our Fourier Integral Operators not only to leading order, but to next-to-leading order.

Appendix C is a technical elaboration on the "hidden regularities" of solutions of Schrödinger equations.

Finally, Appendix D states and proves the L^∞ regularity in time of Wigner measures associated to solutions of the Schrödinger equation.

2. Action-angle coordinates and their quantization

Recall the symplectic change of coordinates introduced in Section 1.3:

$$\Phi \ : (s, \theta, E, J) \mapsto (x, y, \xi_x, \xi_y),$$

where

$$\begin{cases} x = \frac{J}{E}\cos\theta - s\sin\theta, \\ y = \frac{J}{E}\sin\theta + s\cos\theta, \\ \xi_x = -E\sin\theta, \\ \xi_y = E\cos\theta. \end{cases}$$

The map

$$\Phi : \{(s,\theta,E,J) \ : \ E>0, \theta\in\mathbb{R}/2\pi\mathbb{Z}, s\in\mathbb{R}, J\in\mathbb{R}\} \to \{(z,\xi)\in\mathbb{R}^2\times\mathbb{R}^2 \ : \ \xi\neq 0\}$$

is a diffeomorphism satisfying, in particular,

$$\Phi^{-1}\big(\mathbb{D}\times(\mathbb{R}^2\setminus\{0\})\big) = \left\{(s,\theta,E,J) \ : \ (\theta,E)\in\mathbb{R}/2\pi\mathbb{Z}\times(0,\infty), (J/E)^2+s^2<1\right\}.$$

The inverse map is given by the formulas

$$\begin{cases} E=\sqrt{\xi_x^2+\xi_y^2}, \text{ (velocity)} \\ J = x\xi_y - y\xi_x, \text{ (angular momentum)} \\ \theta = -\arctan\left(\frac{\xi_x}{\xi_y}\right), \text{ (angle of } \xi \text{ with the vertical)} \\ s = -x\sin\theta + y\cos\theta, \text{ (abscissa of } (x,y) \text{ along the line } \left(\frac{J}{E}\cos\theta, \frac{J}{E}\sin\theta\right)+\mathbb{R}\xi \text{)}. \end{cases}$$

Note that the hamiltonian flow of the energy $\frac{E^2}{2}$ (the geodesic flow) reads

$$G^\tau : (s,\theta,E,J)\mapsto(s+\tau E,\theta,J,E), \qquad \tau\in\mathbb{R},$$

and the hamiltonian flow of J (unit speed rotation) reads

$$R^\tau : (s,\theta,E,J)\mapsto(s,\theta+\tau,J,E), \qquad \tau\in\mathbb{R}.$$

Write for $\theta\in\mathbb{R}/2\pi\mathbb{Z}$,

$$\omega(\theta) := (-\sin\theta,\cos\theta);$$

the transformation Φ admits the generating function

$$S(z,\theta,s,E) = E\omega(\theta)\cdot z - Es,$$

meaning that

$$\begin{aligned}\operatorname{Graph}\Phi &= \{(s,\theta,E,J,z,\xi) \ : \ (z,\xi)=\Phi(s,\theta,E,J)\} \\ &= \left\{(s,\theta,E,J,z,\xi) \ : \ \frac{\partial S}{\partial E}=0, \xi=\frac{\partial S}{\partial z}, J=-\frac{\partial S}{\partial\theta}, E=-\frac{\partial S}{\partial s}\right\}.\end{aligned}$$

The existence of such a generating function implies that the diffeomorphism Φ preserves the symplectic form (see for instance [Zwo12, Theorem 2.7]), i.e.

$$d\xi_x\wedge dx + d\xi_y\wedge dy = dE\wedge ds + dJ\wedge d\theta.$$

Using this generating function we define a unitary operator that quantises the canonical transformation Φ. The operator

$$\mathscr{U} f(s,\theta) = (2\pi h)^{-3/2}\int_0^\infty\int_{\mathbb{R}^2} e^{-i\frac{S(z,\theta,s,E)}{h}} f(z)\sqrt{E}\,dz\,dE, \tag{2.1}$$

associating functions on $\mathbb{R}^2_z$ to functions on $\mathbb{R}_s\times\mathbb{R}_\theta$ is in fact a semiclassical Fourier Integral Operator associated with Φ (the choice of the term $\sqrt{E}$ in this expression is explain by Lemma 2.1 below). Note that $\mathscr{U} f$ can be also written independently of h as:

$$\mathscr{U} f(s,\theta) = \int_0^\infty e^{iEs}\widehat{f}(E\omega(\theta))\sqrt{E}\frac{dE}{(2\pi)^{3/2}},$$

where $\widehat{f}$ stands for the Fourier transform of f. Therefore, the Fourier transform with respect to s of $\mathscr{U} f(s,\theta)$ is merely:

$$(2\pi)^{-1/2}\,\widehat{f}(E\omega(\theta))\,\mathbb{1}_{[0,\infty)}(E)\sqrt{E}.$$

From this it is clear that for any symbol $\phi : \mathbb{R} \to \mathbb{R}$ one has:

$$\phi(D_s)\,\mathscr{U} f = \mathscr{U}\,\phi(|D_z|)\,f,$$

and, by Placherel's theorem,

$$\begin{aligned}\int_0^{2\pi}\int_0^{\infty} \mathscr{U} f(s,\theta)\overline{\mathscr{U} g(s,\theta)}dsd\theta &= \int_0^{2\pi}\int_0^{\infty} \widehat{f}(E\omega(\theta))\,\overline{\widehat{g}(E\omega(\theta))}E\frac{dsd\theta}{(2\pi)^2}\\ &= \langle g, f\rangle_{L^2(\mathbb{R}^2)}.\end{aligned}$$

In particular, the following Lemma has been proved:

Lemma 2.1. *(i) The operator $\mathscr{U}$ is unitary from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z})$: $\mathscr{U}^*\mathscr{U} = I$.*
(ii) For $f \in C_c^\infty(\mathbb{R}^2)$, we have $\partial_s^2 \mathscr{U} f = \mathscr{U}\Delta f$.
As a consequence,

$$-h^2\mathscr{U}\Delta\mathscr{U}^* = -h^2\partial_s^2.$$

Notation. We denote by $P_0(z,\xi) = \frac{|\xi|^2}{2}$ the hamiltonian generating the geodesic flow in $\mathbb{R}^2\times\mathbb{R}^2$; and $P_1(z,\xi) = x\xi_y - y\xi_x$ the hamiltonian generating the (unit speed) rotation. We denote by $X_{P_0} = \xi\cdot\partial_z$ and $X_{P_1} = z^\perp\cdot\partial_z + \xi^\perp\cdot\partial_\xi$ the corresponding hamiltonian vector fields on $T^*\mathbb{R}^2$. We denote by $G^\tau(z,\xi) = (z+\tau\xi,\xi)$ the geodesic flow (generated by P_0) and R^τ the flow generated by P_1 (rotation of angle τ of both z and ξ).

In the new coordinates, these hamiltonians and vector fields are slightly simpler since $P_0\circ\Phi = \frac{E^2}{2}$, $P_1\circ\Phi = J$ together with $X_{P_0\circ\Phi} = E\partial_s$ and $X_{P_1\circ\Phi} = \partial_\theta$. Very often, we shall (with a slight abuse of notation) use the letter J to mean the function P_1, and E for the function $\sqrt{2P_0}$.

3. Decomposition of an invariant measure of the billiard

This section aims at describing properties shared by all measures μ invariant by the billiard flow (even if they are not necessarily linked with solutions of a partial differential equation). It essentially collects a few simple facts that will be useful in the next sections when studying measures arising from solutions of the Schrödinger equation (1.1).

Let $(z,\xi)\in\overline{\mathbb{D}}\times\mathbb{R}^2$. There exist $t_1\le 0, t_2\ge 0$ such that $|z+t_1\xi| = |z+t_2\xi| = 1$. Note that if $(z,\xi)\in\mathbb{D}\times\mathbb{R}^2$, then t_1 and t_2 are unique and $t_1>0$, $t_2<0$.

Recall that $\alpha\in[-\pi/2,\pi/2]$ (defined in Section 1.3) is the oriented angle between $-(z+t_1\xi)$ and ξ (that is, the angle between the velocity ξ and the inner normal to the disk, at the point where the oriented straight line $\{z+t\xi, t\in\mathbb{R}\}$ first hits the disk). See Figure 2. One has the expression

$$\alpha = -\arcsin\left(\frac{x\xi_y - y\xi_x}{|\xi|}\right).$$

Our work is based on the following partition of phase space:

$$\overline{\mathbb{D}}\times(\mathbb{R}^2\setminus\{0\}) = \alpha^{-1}(\pi\mathbb{Q}\cap[-\pi/2,\pi/2])\sqcup\alpha^{-1}(\mathbb{R}\setminus\pi\mathbb{Q}),$$

from which the following lemma follows.

Lemma 3.1. *Let μ be any finite, nonnegative Radon measure*[2] *on $\overline{\mathbb{D}}\times\mathbb{R}^2$. Then μ decomposes as a sum of nonnegative measures:*

$$\mu = \mu\rceil_{\alpha\notin\pi\mathbb{Q}} + \sum_{r\in\mathbb{Q}\cap[-1/2,1/2]} \mu\rceil_{\alpha=r\pi} + \mu\rceil_{\xi=0}. \tag{3.1}$$

Note that the functions P_0, P_1, and thus also α, are well-defined on the billiard phase space $\mathbb{W}$. Thus the previous lemma applies as well to measures on $\mathbb{W}$.

In what follows, we shall call *nonnegative invariant measure* a nonnegative Radon measure on $\mathbb{W}$ which is invariant under the billiard flow. We shall extend this terminology to measures μ defined a priori on $\overline{\mathbb{D}}\times\mathbb{R}^2$, to mean that $\pi_*\mu$ (the image of μ under the projection π) is invariant under the billiard flow ϕ^τ on $\mathbb{W}$.

Lemma 3.2. *Let μ be a nonnegative invariant measure on $\mathbb{W}$. Then every term in the decomposition (3.1) is a nonnegative invariant measure, and $\mu\rceil_{\alpha\notin\pi\mathbb{Q}}$ is invariant under the rotation flow (R^τ), as well as $\mu\rceil_{\alpha=\pm\pi/2}$.*

The rotation flow (R^τ) is well defined on $\mathbb{W}$, so the last sentence makes sense. The assertion for $\alpha=\pm\pi/2$ comes from the fact that the rotation flow coincides with the billiard flow (up to time change) on the set $\{\alpha=\pm\pi/2\}$. The assertion for $\alpha\notin\pi\mathbb{Q}$ is a standard fact. It comes from the remark that, for any given value α_0 (such that $\alpha_0\notin\pi\mathbb{Q}$) we can find $T=T(\alpha_0)>0$ such that ϕ^T coincides with an irrational rotation on the set $\{\alpha=\alpha_0\}$.

Thus, for $\alpha\notin\pi\mathbb{Q}$ or $\alpha=\pm\pi/2$, there is nothing to prove to get Theorem 1.16.

Now consider a term $\mu\rceil_{\alpha=r_0\pi}$, where $r_0\in\mathbb{Q}\cap(-1/2,1/2)$ is fixed. Let us denote $\alpha_0=\pi r_0$. Introduce the vector field on $T^*\mathbb{R}^2$:

$$(\alpha_0-\alpha)X_{P_1}+\frac{\cos\alpha}{E}X_{P_0}$$

On the set $\mathcal{I}_{\alpha_0}=\{J=-\sin\alpha_0 E\}$ it coincides with X_{P_0} up to a constant factor. Denote by $\phi^\tau_{\alpha_0}$ the flow on $\mathbb{W}$ generated by $(\alpha_0-\alpha)X_{P_1}+\frac{\cos\alpha}{E}X_{P_0}$ with reflection on the boundary of the disk. More precisely, it is the unique continuous flow defined on $\mathbb{W}$ such that

$$\phi^\tau_{\alpha_0}(z,\xi)=R^{(\alpha_0-\alpha)\tau}\left(z+\tau\frac{\cos\alpha}{|\xi|}\xi,\xi\right)$$

whenever $z\in\mathbb{D}$ and $z+\tau\frac{\cos\alpha}{|\xi|}\xi\in\mathbb{D}$ (with $\alpha=-\arcsin\frac{P_1(z,\xi)}{|\xi|}$ as previously).

In the coordinates (s,θ,E,J), this flow simply reads

$$\phi^\tau_{\alpha_0}\circ\Phi(s,\theta,E,J)=\Phi(s+\tau\cos\alpha,\theta+(\alpha_0-\alpha)\tau,E,J),\quad \alpha=-\arcsin(J/E),$$

with reflection on the boundary of the disk.

All its orbits are periodic: actually, we determined the coefficients $(\alpha_0-\alpha)$ and $\frac{\cos\alpha}{E}$ precisely for that purpose, see Figure 3. Some trajectories of the flow are represented on Figure 4.

The following lemma is now obvious.

Lemma 3.3. *Let μ be a nonnegative invariant measure on $\mathbb{W}$.*

Let $a\in C^0(\mathbb{W})$. Then

$$\int a\circ\phi^\tau_{\alpha_0}d\mu\rceil_{\alpha=\alpha_0}=\int a d\mu\rceil_{\alpha=\alpha_0}$$

for every $t\in\mathbb{R}$.

[2] We denote by $\mathcal{M}_+\left(\overline{\mathbb{D}}\times\mathbb{R}^2\right)$ the set of all such measures.

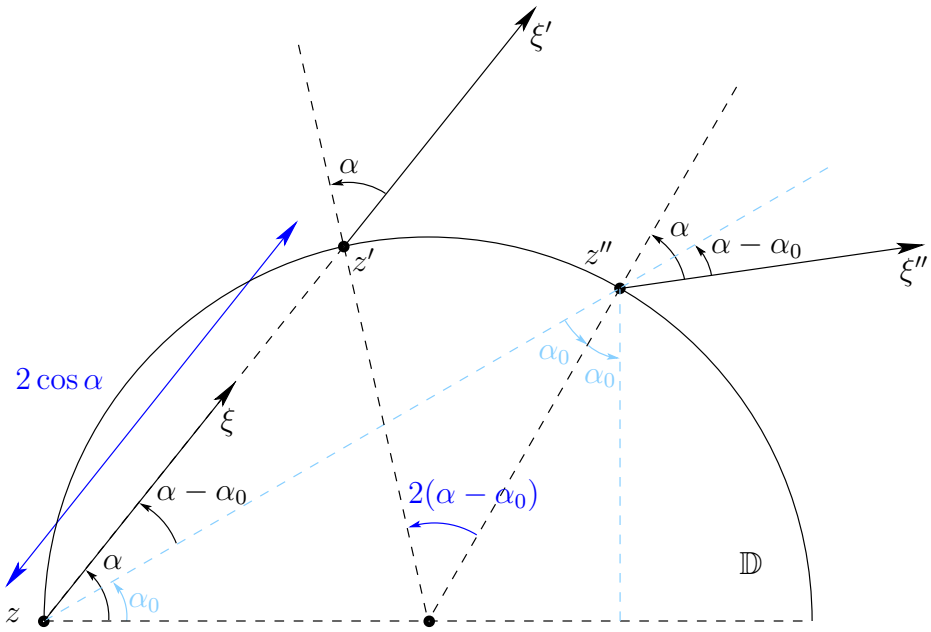

FIGURE 3. Construction of the flow $\phi^\tau_{\alpha_0}$ with $\alpha_0 = \pi/6$.
On the figure, $(z',\xi') = (z + 2\frac{\cos\alpha}{|\xi|}\xi, \xi)$ and $(z'',\xi'') = R^{2(\alpha_0-\alpha)}(z',\xi') = \phi^2_{\alpha_0}(z,\xi)$.

Equivalently, we have

$$\int a d\mu\rceil_{\alpha=\alpha_0} = \int \langle a\rangle_{\alpha_0} d\mu\rceil_{\alpha=\alpha_0}$$

where

$$\langle a\rangle_{\alpha_0} = \lim_{T\longrightarrow+\infty} \frac{1}{T}\int_0^T a\circ\phi^\tau_{\alpha_0} dt.$$

Remark that $\langle a\rangle_{\alpha_0}$ is well defined even if a is a bounded measurable function on $\mathbb{D}$ (the times τ where the trajectories of $\phi^\tau_{\alpha_0}$ hit the boundary form a set of measure 0 in $\mathbb{R}$).

4. Second microlocalization on a rational angle

This section and the next one are devoted to proving the semiclassical version of our result, Theorem 1.8.

Let $\left(u_h^0\right)$ be a bounded family in $L^2(\mathbb{D})$. Denote by $u_h(z,t) = U_V(t)u_h^0(z)$. After extracting a subsequence, we suppose that its Wigner distributions W_h (defined by (1.14)) converge to a semiclassical measure μ_{sc} in the weak-$*$ topology of $\mathcal{D}'\left(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}_t\times\mathbb{R}_H\right)$. The measure $\mu_{sc}\in L^\infty\left(\mathbb{R}_t;\mathcal{M}_+\left(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}_H\right)\right)$ is for a.e. $t\in\mathbb{R}$ supported by $\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}_H\cap\{H=\frac{E^2}{2}\}$.

From now on, we skip the index sc since there is no possible confusion here (only semiclassical measures are considered until Section 7) to lighten the notation.

The aim of this section is to understand the term $\mu\rceil_{\mathcal{I}_{\alpha_0}}$, where $\mathcal{I}_{\alpha_0} = \{\alpha=\alpha_0\}$ and $\alpha_0\in\pi\mathbb{Q}$. In view of Lemma 3.3, it suffices to characterize the action of $\mu\rceil_{\mathcal{I}_{\alpha_0}}$ on test functions that are $(\phi^\tau_{\alpha_0})$-invariant.

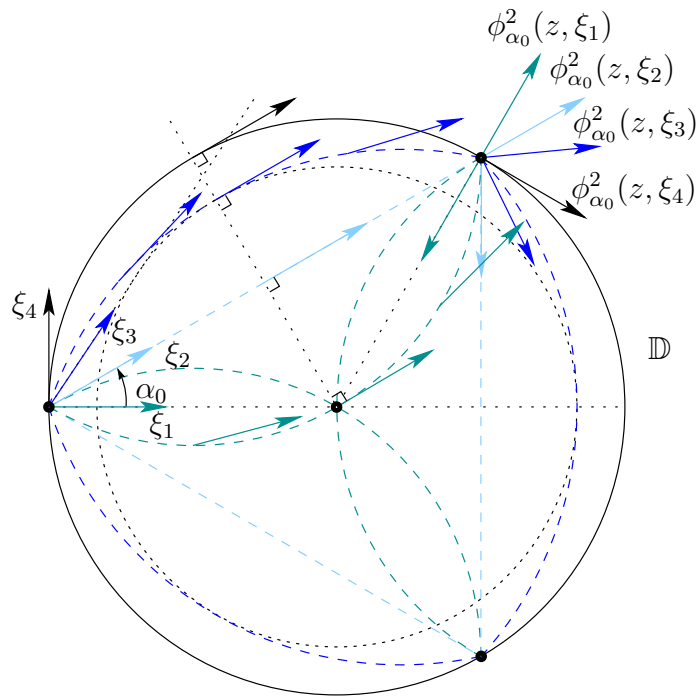

FIGURE 4. Approximate representation of some trajectories of the flow $\phi^\tau_{\alpha_0}$ with $\alpha_0 = \pi/6$ issued from (z, ξ_j) with $z = (-1, 0)$ and $\xi_j, j \in \{1, 2, 3, 4\}$ such that $\alpha(z, \xi_1) = 0$, $\alpha(z, \xi_2) = \alpha_0$, $\alpha(z, \xi_3) \in (\alpha_0, \pi/2)$ and $\alpha(z, \xi_4) = \pi/2$.

4.1. **Classes of test functions.** Here is a list of properties that we may want to impose on our symbols in the course of our proof. We express these properties both in the "old" coordinates (x, y, ξ_x, ξ_y) and in the "new" ones (s, θ, E, J).

Let a be a smooth function of $(x, y, \xi_x, \xi_y, t, H)$, supported away from $\{\xi = 0\}$. Then $a \circ \Phi$ is a smooth function of (s, θ, E, J, t, H) supported away from $\{E = 0\}$. The properties we shall use are the following:

(A) The symbol a is compactly supported w.r.t. ξ, t and H. This is equivalent to $a \circ \Phi$ being compactly supported w.r.t. E, J, t and H. Note also that $a \circ \Phi$ is 2π-periodic w.r.t. θ.

(B) For $|z| = 1$, we have $a(z, \xi) = a \circ \sigma(z, \xi)$ where σ is the orthogonal symmetry with respect to the boundary of the disk at z. In the coordinates of Section 2, this reads (forgetting to write the (t, H)-dependence of a)

$$a \circ \Phi(\cos\alpha, \theta, E, J) = a \circ \Phi(-\cos\alpha, \theta + \pi + 2\alpha, E, J)$$

for all θ, E, J and for $\alpha = -\arcsin\left(\frac{J}{E}\right)$.

Terminology. We shall say that a is a smooth function on $\mathbb{W}$ if a is a smooth functon on $\overline{\mathbb{D}} \times \mathbb{R}^2$ that satisfies (B).

(C) If a satisfies (B), we want in addition that $a \circ \pi \circ \phi^\tau$ defines a smooth function on $\mathbb{W}$ for all τ. This is equivalent to requiring that

$$\partial_s^k(a \circ \Phi)(\cos\alpha, \theta, E, J) = \partial_s^k(a \circ \Phi)(-\cos\alpha, \theta + \pi + 2\alpha, E, J)$$

for all k, for all θ, E, J and for $\alpha = -\arcsin\left(\frac{J}{E}\right)$. In other words, all the derivatives of $a \circ \Phi$ w.r.t. s satisfy the symmetry condition (B).

(D) The function a satisfies (C), and in addition a is $\phi^{\tau}_{\alpha_0}$-invariant, which reads

$$\left((\alpha_0 - \alpha)X_{P_1} + \frac{\cos\alpha}{E}X_{P_0}\right)a = 0,$$

or, in the new coordinates,

$$[(\alpha_0 - \alpha)\partial_\theta + \cos\alpha\partial_s]a \circ \Phi = 0.$$

Furthermore, to fix ideas, let us assume that the support of a with respect to t is contained in $(-1, 1)$. This implies that

$$\begin{aligned} W_h(a) &= \langle g(t)u_h, \mathrm{Op}_1(a(z, h\xi, t, h^2H)g(t)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} + O(h^\infty) \\ &= \langle g(t)u_h, \mathrm{Op}_h(a(z, \xi, t, hH)g(t)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} + O(h^\infty), \end{aligned} \tag{4.1}$$

for any smooth cut-off function g supported in $(-2, 2)$ and taking the value 1 on $(-1, 1)$. In other words, we need only consider the restriction of $u_h(z, t)$ to $t \in (-2, 2)$.

4.2. **Coordinates adapted to the second microlocalization on $\mathcal{I}_{\alpha_0}$.** We wish to study the concentration of W_h around the set $\{J = -E\sin\alpha_0\}$. If the limit measure $(\Phi^{-1})_*\mu$ is supported on the set $\{E = \sqrt{2H}\}$ this is equivalent to studying the concentration of W_h around $\{J = -\sqrt{2H}\sin\alpha_0\}$. Since this assumption is satisfied for sequences u_h satisfying (1.13), we shall study the concentration of W_h around this set.

Thus we make the (symplectic) change of variables

$$\left(s, \theta, E, J' - \sqrt{2H}\sin\alpha_0, t' + \frac{\theta\sin\alpha_0}{\sqrt{2H}}, H\right) = (s, \theta, E, J, t, H)$$

which sends $\{J' = 0\}$ to $\{J = -\sqrt{2H}\sin\alpha_0\}$ and leaves untouched the variables (s, E).

Consider the following corresponding Fourier Integral Operator (which leaves untouched the variables (s, E), omitted here from the notation):

$$\mathscr{V} f(\theta, H) = (2\pi)^{-1/2}e^{i\sqrt{2H}\sin\alpha_0\theta/h}\int f(\theta, ht)e^{-iHt/h}dt.$$

Lemma 4.1. *If $b \in C_c^\infty(\mathbb{R}_\theta \times \mathbb{R}_J \times \mathbb{R}_t \times \mathbb{R}_H)$, we have*

$$\mathscr{V}\,\mathrm{Op}_1(b(\theta, hJ, t, h^2H))\mathscr{V}^* = \mathrm{Op}_h(\tilde{b}(\theta, J', H, ht)) + O(h), \tag{4.2}$$

where $\tilde{b}(\theta, J', H, ht) = b\left(\theta, J' - \sqrt{2H}\sin\alpha_0, -ht, H\right)$, *and*

$$\mathscr{V}\mathscr{V}^* = I. \tag{4.3}$$

Proof. First notice that we have

$$\mathscr{V}^* g(\theta, t) = (2\pi)^{-1/2}h^{-1}\int g(\theta, H)e^{iHt/h^2}e^{-i\sqrt{2H}\sin\alpha_0\theta/h}dH.$$

Second, we may now compute $A := \mathscr{V}\,\mathrm{Op}_1(b(\theta, hJ, t, h^2H))\mathscr{V}^* e^{it_0H/h}e^{iJ_0\theta/h}\rceil_{H=H_0,\theta=\theta_0}$. We have the exact formula

$$A = (2\pi h)^{-1}e^{i\sqrt{2H_0}\sin\alpha_0\theta_0/h}\int b(\theta_0, J_0 - \sqrt{2H}\sin\alpha_0, ht, H)$$
$$e^{iHt/h}e^{iHt_0/h}e^{iJ_0\theta_0/h}e^{-i\sqrt{2H}\sin\alpha_0\theta_0/h}e^{-iH_0t/h}dHdt.$$

Note that this expression is exact and thus does not involve the derivatives of b w.r.t. θ or J. Taking $b=1$ in this expression gives the exact formula $A = e^{it_0H_0/h}e^{iJ_0\theta_0/h}$, which proves (4.3).

We carry on the computations with a general b. After a change of variables, A is now equal to

$$\begin{aligned} A &= (2\pi h)^{-1}e^{i\sqrt{2H_0}\sin\alpha_0\theta_0/h}e^{iJ_0\theta_0/h}\int b(\theta_0, J_0-\sqrt{2H}\sin\alpha_0, ht, H) \\ &\qquad e^{iH(t+t_0)/h}e^{-i\sqrt{2H}\sin\alpha_0\theta_0/h}e^{-iH_0t/h}dHdt \\ &= (2\pi h)^{-1}e^{iH_0t_0/h}e^{i\sqrt{2H_0}\sin\alpha_0\theta_0/h}e^{iJ_0\theta_0/h}\int b(\theta_0, J_0-\sqrt{2(H+H_0)}\sin\alpha_0, h(t-t_0), H_0+H) \\ &\qquad e^{iHt/h}e^{-i\sqrt{2(H+H_0)}\sin\alpha_0\theta_0/h}dHdt. \end{aligned}$$

Standard application of the method of stationary phase shows that this expression is of order $O(h^\infty)$ if H_0 is away from the support of b. Besides, the phase has a single nondegenerate critical point at $(t,H) = (\frac{\sin\alpha_0\theta_0}{\sqrt{2H_0}}, 0)$, so that *uniformly in* $t_0\in\mathbb{R}$ the method of stationary phase yields

$$\begin{aligned} A &= (2\pi h)^{-1}e^{iH_0t_0/h}e^{i\sqrt{2H_0}\sin\alpha_0\theta_0/h}e^{iJ_0\theta_0/h} \\ &\quad \left((2\pi h)e^{-i\sqrt{2H_0}\sin\alpha_0\theta_0/h}b\left(\theta_0, J_0-\sqrt{2H_0}\sin\alpha_0, h(-t_0+\frac{\sin\alpha_0\theta_0}{\sqrt{2H}}), H_0\right)\right)+O(h). \end{aligned}$$

This is

$$A = e^{iH_0t_0/h}e^{iJ_0\theta_0/h}b\left(\theta_0, J_0-\sqrt{2H_0}\sin\alpha_0, -ht_0, H_0\right)+O(h),$$

where $O(h)$ is uniform if θ_0 stays in a fixed compact set. This concludes the proof of (4.2). □

Recalling the definition of $W_h(a)$ in (1.14), we thus have

$$\begin{aligned} W_h(a) &= \langle \mathscr{V}\mathscr{U}u_h, (\mathscr{V}\mathscr{U}\operatorname{Op}_h(a(z,\xi,t,hH)\mathscr{U}^*\mathscr{V}^*)\mathscr{V}\mathscr{U}u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)} \\ &= \langle \mathscr{V}\mathscr{U}u_h, \operatorname{Op}_h(\tilde b(s,\theta,J',E,H,ht))\mathscr{V}\mathscr{U}u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)}+O(h) \end{aligned}$$

where

$$\tilde b(s,\theta,J',E,H,ht)) = a\circ\Phi(s,\theta,E,J'-\sin\alpha_0\sqrt{2H},-ht,H), \tag{4.4}$$

and $\mathbb{T}_\theta = \mathbb{R}/2\pi\mathbb{Z}$ is the circle in which the variable θ takes values. By (4.1), u_h may actually be restricted to $|t|<2$ in this formula, so that it is safe to apply $\mathscr{V}$ to $\mathscr{U}u_h$.

To work in our new coordinates, we now define

$$\langle w_h, b\rangle = \langle \mathscr{V}\mathscr{U}u_h, \operatorname{Op}_h(b(s,\theta,E,J',H,ht))\mathscr{V}\mathscr{U}u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)} \tag{4.5}$$

for symbols b that satisfies

(A) the symbol b is compactly supported w.r.t. E, J', t and H, and 2π-periodic w.r.t. θ.

We then recover $W_h(a) = \langle w_h, \tilde b\rangle$ with $\tilde b$ and a linked by (4.4).

Remark 4.2. *Note that the bracket* (4.5) *can also be written as follows*

$$\langle w_h, b\rangle = \langle \mathscr{V}\mathscr{U}u_h, \operatorname{Op}_h(\chi_0(\theta)b(s,\theta,E,J',H,ht))\mathscr{V}\mathscr{U}u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)} \tag{4.6}$$

for any $\chi_0\in C_c^\infty(\mathbb{R})$ *satisfying* $\sum_{k\in\mathbb{Z}}\chi_0(\theta+2\pi k)\equiv 1$ *on* $\mathbb{R}$. *Indeed, we have*

$$\operatorname{Op}_h(\chi_0(\theta)b) = \chi_0(\theta)\operatorname{Op}_h(b) \tag{4.7}$$

(because Op *denotes the standard quantization) and we write for any* 2π*-periodic function* $f \in L^1_{\mathrm{loc}}(\mathbb{R})$*,* $\int_{\mathbb{R}} \chi_0(\theta) f(\theta) d\theta = \int_{\mathbb{T}} f(\theta) d\theta$*. Because of* (4.7)*, we may also take* $\chi_0 = \mathbb{1}_{(0,2\pi)}$ *when needed.*

4.3. **Second microlocalization.** We now introduce two auxiliary distributions which describe more precisely how w_h concentrates on the set

$$\left\{(s,\theta,E,J,H,t) \in \Phi^{-1}(\overline{\mathbb{D}} \times (\mathbb{R}^2 \setminus \{0\}))) \times \mathbb{R}^2, \text{ such that } -\frac{J}{\sqrt{2H}} = \sin\alpha_0\right\}$$

(which intersection with $\{E = \sqrt{2H}\}$ is equal to $\mathcal{I}_{\alpha_0} \cap \{E = \sqrt{2H}\}$).

For this, we define an appropriate class of symbols depending on an additional variable η, which later in the calculations will be identified with $\frac{J'}{h} = \frac{J+\sqrt{2H}\sin\alpha_0}{h}$.

Definition 4.3.
- *We denote by* $\mathcal{S}$ *the class of smooth functions* $b(s,\theta,E,J',\eta,H,t)$ *on* $\mathbb{R}^7$*, supported away from* $\{E=0\}$ *and that satisfy condition* (A) *in the variables* (s,θ,E,J',H,t)*, and, in addition,*

(E) *b is homogeneous of degree zero at infinity in* $\eta \in \mathbb{R}$*. That is, there exist* $R_0 > 0$ *and* $b_{\mathrm{hom}} \in C^\infty\left(\mathbb{R}^4 \times \{-1,+1\} \times \mathbb{R}^2\right)$ *with*

$$b(s,\theta,E,J',\eta,H,t) = b_{\mathrm{hom}}\left(s,\theta,E,J',\frac{\eta}{|\eta|},H,t\right), \quad \text{for } |\eta| > R_0 \text{ and } (s,\theta,E,J',H,t) \in \mathbb{R}^6.$$

- *We denote by* $\mathcal{S}^\sigma$ *those symbols* $b \in \mathcal{S}$ *that satisfy conditions* (B) *and* (C) *(for all* H,t*):*

(B) $b(\cos\alpha,\theta,E,J') = b(-\cos\alpha,\theta+\pi+2\alpha,E,J')$
for all θ,E,J'*, and for* $\alpha = -\arcsin\left(\frac{J'-\sqrt{2H}\sin\alpha_0}{E}\right)$.

(C) $\partial_s^k b(\cos\alpha,\theta,E,J') = \partial_s^k b(-\cos\alpha,\theta+\pi+2\alpha,E,J')$
for all k*, for all* θ,E,J'*, and for* $\alpha = -\arcsin\left(\frac{J'-\sqrt{2H}\sin\alpha_0}{E}\right)$.

- *We denote by* $\mathcal{S}^\sigma_{\alpha_0}$ *those symbols* $b \in \mathcal{S}^\sigma$ *satisfying the invariance condition* (D)*:*

(D) $[(\alpha_0-\alpha)\partial_\theta + \cos\alpha\partial_s]b(s,\theta,E,J') = 0$
for all s,θ,E,J'*, and for* $\alpha = -\arcsin\left(\frac{J'-\sqrt{2H}\sin\alpha_0}{E}\right)$.

Let $\chi \in C_c^\infty(\mathbb{R})$ be a nonnegative cut-off function that is identically equal to one near the origin and let $R > 0$. For $b \in \mathcal{S}$, we define

$$\left\langle w^{\alpha_0}_{h,R}, b\right\rangle := \left\langle \mathscr{V}\mathscr{U}u_h, \mathrm{Op}_h\left(\left(1-\chi\left(\frac{J'}{Rh}\right)\right)\chi_0(\theta) b(s,\theta,E,J',\frac{J'}{h},H,ht)\right)\mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)},$$

and

(4.8)
$$\langle w_{\alpha_0,h,R}, b\rangle := \left\langle \mathscr{V}\mathscr{U}u_h, \mathrm{Op}_h\left(\chi\left(\frac{J'}{Rh}\right)\chi_0(\theta) b(s,\theta,E,J',\frac{J'}{h},H,ht)\right)\mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)},$$

The Calderón-Vaillancourt theorem [CV71] ensures that both $w^{\alpha_0}_{h,R}$ and $w_{\alpha_0,h,R}$ are bounded in $\mathcal{S}'$. After possibly extracting subsequences, we have the existence of a limit: for every $b \in \mathcal{S}$,

$$\langle \mu^{\alpha_0}, b\rangle := \lim_{R\to\infty}\lim_{h\to 0^+}\left\langle w^{\alpha_0}_{h,R}, b\right\rangle,$$

and

$$\langle \mu_{\alpha_0}, b\rangle := \lim_{R\to\infty}\lim_{h\to 0^+} \langle w_{\alpha_0,h,R}, b\rangle\,. \tag{4.9}$$

Positivity properties are described in the next proposition.

Proposition 4.4. (i) *The distribution μ^{α_0} is a nonnegative Radon measure. In addition, μ^{α_0} is nonnegative, 0-homogeneous and supported at infinity in the variable η (i.e., it vanishes when paired with a compactly supported function). As a consequence, μ^{α_0} may be identified with a nonnegative measure on $\mathbb{R}^4\times\{-1,+1\}\times\mathbb{R}_t\times\mathbb{R}_H$.*

(ii) *The projection of μ_{α_0} on $\mathbb{R}^4_{s,\theta,E,J'}\times\mathbb{R}^2_{H,t}$ (that is, $\int_{\mathbb{R}}\mu_{\alpha_0}(d\eta)$) is a nonnegative measure, carried on $\{J'=0\}$.*

Moreover, both μ_{α_0} and μ^{α_0} are carried by the set $\{E=\sqrt{2H}\}$, as can be seen from (C.12) and (C.13). Note also that the argument of Proposition D.1 proves that μ^{α_0} enjoys L^∞ regularity in the time variable.

Proposition 4.4 (i) is proved at the beginning of Section 4.4, whereas (ii) shall be a consequence of Section 4.5.

Remark 4.5. *If $a=a(z,\xi,t,H)$ is a function on $\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}^2$, let us define*

$$\begin{aligned} m^{\alpha_0}(a) &:= \mu^{\alpha_0}(b\mathbb{1}_{J'=0}))\,,\\ m_{\alpha_0}(a) &:= \mu_{\alpha_0}(b)\,. \end{aligned}$$

where $b(s,\theta,E,J',H,t)=a\circ\Phi(s,\theta,E,J'-\sin\alpha_0\sqrt{2H},-t,H)$ (a function that does not depend on the additional variable η). Then we have

$$\mu\rceil_{\mathcal{I}_{\alpha_0}} = m^{\alpha_0}+m_{\alpha_0}. \tag{4.10}$$

Thus, understanding $\mu\rceil_{\mathcal{I}_{\alpha_0}}$ amounts to understanding both m^{α_0} and m_{α_0}, which we shall do by understanding the structure of μ^{α_0} and μ_{α_0}.

The following proposition states that both distributions μ_{α_0} and μ^{α_0} are invariant under the billiard flow, as μ.

Proposition 4.6. *The distributions μ_{α_0} and μ^{α_0} enjoy the following property:*

$$\langle \mu_{\alpha_0}, E\,\partial_s b\rangle = 0, \quad \langle \mu^{\alpha_0}, E\,\partial_s b\rangle = 0$$

for every $b\in\mathcal{S}^\sigma$.

Proof. We use as a "black-box" the technical calculations developed in Appendix A. The main point of these calculations is to understand how an operator of the form

$$\mathscr{U}^*\,\mathrm{Op}_h\left(P(s,\theta,E,J,t,hH)\right)\mathscr{U}$$

preserves or modifies the Dirichlet boundary condition, according to the properties of P (the technical difficulty is that our new coordinates (s,θ,E,J), well-adapted to the dynamics, are not adapted to express the Dirichlet boundary condition).

In the proof of Proposition 4.6 for μ_{α_0}, we consider the function P

$$P(s,\theta,E,J,t,H) = b\left(s,\theta,E,J+\sin\alpha_0\sqrt{2H},\frac{J+\sin\alpha_0\sqrt{2H}}{h},H,-t\right)\chi\left(\frac{J+\sin\alpha_0\sqrt{2H}}{Rh}\right). \tag{4.11}$$

To prove the result for μ^{α_0}, the argument is the same with the function

$$\tilde{P}(s,\theta,E,J,t,H) = b\left(s,\theta,E,J+\sin\alpha_0\sqrt{2H},\frac{J+\sin\alpha_0\sqrt{2H}}{h},H,-t\right)(1-\chi)\left(\frac{J+\sin\alpha_0\sqrt{2H}}{Rh}\right). \tag{4.12}$$

(1) The operator $\mathscr{U}^*\operatorname{Op}_h(P(s,\theta,E,J,t,hH))\mathscr{U}$ is expressed as a pseudodifferential operator $\mathcal{A}_E(P)$ on $\mathbb{R}^2$ (modulo a small remainder) in polar coordinates

$$z = (-r\sin u, r\cos u).$$

Note that the polar coordinates are the ones adapted to our boundary problem, since the boundary is given by the equation $r=1$.

Thus we have

$$\lim\langle u_h, \mathscr{U}^*\operatorname{Op}_h(P(s,\theta,E,J,t,hH))\mathscr{U}u_h\rangle = \lim\langle u_h, \mathcal{A}_E(P)u_h\rangle.$$

(2) We then introduce a pseudodifferential operator $\mathcal{A}_H(P)$, having the property that the symbols of $\mathcal{A}_E(P)$ and $\mathcal{A}_H(P)$ coincide on $\{|\xi|^2 = 2H\}$. More precisely, we are able to prove (Lemma B.1)

$$\lim\langle u_h, \mathcal{A}_E(P)u_h\rangle = \lim\langle u_h, \mathcal{A}_H(P)u_h\rangle.$$

(3) The explicit expression of $\mathcal{A}_H(P)$ reads

$$A(r,u,\sqrt{2hD_t},hD_u,t) + B(r,u,\sqrt{2hD_t},hD_u,t)\circ hD_r \tag{4.13}$$

modulo terms of order $O(h)$, where $z = (-r\sin u, r\cos u)$ is the decomposition in polar coordinates. The functions A,B,C,D are expressed explicitly in terms of P in Proposition A.3. If P satisfies the symmetry condition (B), then $B\equiv 0$ for $r=1$.

(4) Finally, we show in Proposition B.3 that

$$\lim_{h\longrightarrow 0}\langle u_h, \mathcal{A}_H(E\,\partial_s P)u_h\rangle = \lim_{h\longrightarrow 0}\left\langle u_h, \left[-\frac{ih\Delta}{2}, \mathcal{A}_H(P)\right]u_h\right\rangle$$

where Δ is the laplacian on $\mathbb{R}^2$. On the other hand, if P depends on t, we have

$$[\partial_t, \mathcal{A}_H(P)] = \mathcal{A}_H(\partial_t P).$$

The proof of Proposition 4.6 now goes as follows (with b and P related by (4.11)):

$$\begin{aligned}
\langle \mu_{\alpha_0}, E\,\partial_s b\rangle &= \lim_{R\to\infty}\lim_{h\to 0^+} \langle w_{\alpha_0,h,R}, b\rangle \\
&= \lim_{R\to\infty}\lim_{h\to 0^+} \\
&\quad \left\langle \mathscr{V}\mathscr{U} u_h, \mathrm{Op}_h\left(E\,\partial_s P(s,\theta,E,J'-\sin\alpha_0\sqrt{2H},H,-ht)\right)\mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)} \qquad (4.14)\\
&= \lim_{R\to\infty}\lim_{h\to 0^+} \langle \mathscr{U} u_h, \mathrm{Op}_h\left(E\,\partial_s P(s,\theta,E,J,t,hH)\right)\mathscr{U} u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_t)} \\
&= \lim_{R\to\infty}\lim_{h\to 0^+} \langle u_h, \mathcal{A}_H(E\,\partial_s P) u_h\rangle \\
&= \lim_{R\to\infty}\lim_{h\to 0^+} \left\langle u_h, \left[-\frac{ih}{2}\Delta, \mathcal{A}_H(P)\right] u_h\right\rangle \\
&= \lim_{R\to\infty}\lim_{h\to 0^+} \left\langle u_h, \left[-\frac{ih}{2}\Delta + ihV - h\partial_t, \mathcal{A}_H(P)\right] u_h\right\rangle \\
&= \lim_{R\to\infty}\lim_{h\to 0^+} \left\langle \frac{ih}{2}\frac{\partial u_h}{\partial n}\otimes\delta_{\partial\mathbb{D}}, \mathcal{A}_H(P) u_h\right\rangle - \lim \left\langle u_h, \mathcal{A}_H(P)\frac{ih}{2}\frac{\partial u_h}{\partial n}\otimes\delta_{\partial\mathbb{D}}\right\rangle \qquad (4.15)
\end{aligned}$$

The last line comes from the fact that u_h, extended to $\mathbb{R}^2$ by the value 0 outside $\mathbb{D}$, satisfies

$$\left(-\frac{ih}{2}\Delta + ihV - h\partial_t\right) u_h = \frac{ih}{2}\frac{\partial u_h}{\partial n}\otimes\delta_{\partial\mathbb{D}} \qquad (4.16)$$

where Δ is the laplacian on $\mathbb{R}^2$.

We use now the explicit expression (4.13) of $\mathcal{A}_H(P)$, modulo terms that vanish at the limit. Using the fact that u_h satisfies Dirichlet boundary conditions, and the fact that B vanishes for $r=1$ if P satisfies the symmetry condition (B), we see that the last line (4.15) vanishes.

Note that only the limit $h\longrightarrow 0$ was actually used, so that the result holds even before taking the limit $R\longrightarrow +\infty$. □

Remark 4.7. *More generally, let $b(s,\theta,E,J,\eta,H,t)$ be a smooth function on $\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}^5$ with bounded deratives, and compactly supported w.r.t. s,E,J,H,t. Let $\mathbf{P}(s,\theta,E,J,t,H) = b(s,\theta,E,J+\sin\alpha_0\sqrt{2H},\frac{J+\sin\alpha_0\sqrt{2H}}{h},H,-t)$. Then, the same proof yields,without using the symmetry condition* (B)*, the formula*

$$\begin{aligned}
\lim_h \left\langle \mathscr{V}\mathscr{U} u_h, \mathrm{Op}_h\left(E\,\partial_s b(s,\theta,E,J',\frac{J'}{h},H,ht)\right)\mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)} \qquad (4.17)\\
= \lim_h \langle \mathscr{U} u_h, \mathrm{Op}_h\left(E\,\partial_s \mathbf{P}(s,\theta,E,J,t,hH)\right)\mathscr{U} u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_t)} \\
= -\lim_h \left\langle h\frac{\partial u_h}{\partial n}, \mathbf{B}(1,u,\sqrt{2hD_t},hD_u,t) h\frac{\partial u_h}{\partial n}\right\rangle_{L^2(\partial\mathbb{D}\times\mathbb{R})}
\end{aligned}$$

where $\mathbf{B}$ is the function associated to $\mathbf{P}$ by the formulas of Proposition A.3. Again, if $\mathbf{P}$ satisfies (B)*, the operator $\mathbf{B}(1,u,\sqrt{2hD_t},hD_u,t)$ vanishes.*

This formula, relating the semiclassical measures of boundary data to the semiclassical measures of interior data, is analogous to formula (1.27) *but is expressed in a different set of coordinates.*

Applying (4.17) *to* $\frac{s}{E}\mathbf{P}$ *instead of* $\mathbf{P}$ *(that is,* $\frac{s}{E}b$ *instead of* b*) has the following consequence that will be used later*

$$\lim_h \langle \mathscr{U} u_h, \mathrm{Op}_h\left((\mathbf{P}+s\partial_s\mathbf{P})(s,\theta,E,J,t,hH)\right)\mathscr{U} u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_t)} = -\lim_h \left\langle h\frac{\partial u_h}{\partial n}, (E^{-2}\mathbf{P})^\sigma(1,u,\sqrt{2hD_t},hD_u,t)h\frac{\partial u_h}{\partial n}\right\rangle_{L^2(\partial\mathbb{D}\times\mathbb{R})} \tag{4.18}$$

with the notation (A.4).

The following result states that both μ^{α_0} and μ_{α_0} have some extra regularity (for two different reasons).

Theorem 4.8. (i) *The measure* μ^{α_0} *satisfies the invariance property:*

$$\langle \mu^{\alpha_0}, \partial_\theta b\rangle = 0, \qquad \text{for every } b \text{ in } \mathcal{S}^\sigma_{\alpha_0}. \tag{4.19}$$

(ii) *The distribution* μ_{α_0} *is concentrated on* $\{J'=0\}$ *and its projection onto the variables* (s,θ) *is a nonnegative absolutely continuous measure.*

Section 4.4 is devoted to the study of the properties of μ^{α_0} and gives the proofs of Proposition 4.4 (i) and Theorem 4.8 (i). The study of the structure of μ_{α_0} is performed in Section 4.5 using the notion of two-microlocal measures. This structure will imply (1.18) in Theorem 1.8 (iii). In particular, we prove at the end of Section 4.5 that it yields Theorem 4.8 (ii).

4.4. **Structure and propagation of μ^{α_0}.** In this section, we prove Proposition 4.4 (i) and the invariance property given by Theorem 4.8 (i).

The positivity of μ^{α_0} can be deduced following the lines of [FKG02] Section 2.1, or those of the proof of Theorem 1 in [Gér91b]; or also Corollary 27 in [AM14]. The argument will not be reproduced here. Given $b\in\mathcal{S}$ there exists $R_0>0$ and $b_{\mathrm{hom}}\in C_c^\infty\left(\mathbb{R}^4\times\{-1,+1\}\times\mathbb{R}^2\right)$ such that

$$b\left(s,\theta,E,J',\eta,H,t\right) = b_{\mathrm{hom}}\left(s,\theta,E,J',\frac{\eta}{|\eta|},H,t\right), \quad \text{for } |\eta|\geq R_0.$$

Clearly, for R large enough, the value $\left\langle w^{\alpha_0}_{h,R}, b\right\rangle$ only depends on b_{hom}. Therefore, the limiting distribution μ^{α_0} can be viewed as an element of the dual space of $C_c^\infty\left(\mathbb{R}^4\times\{-1,+1\}\times\mathbb{R}^2\right)$. Its positivity implies that it is a measure, which proves Proposition 4.4 (i).

We now assume that $b\in\mathcal{S}^\sigma_{\alpha_0}$ and prove the invariance property Theorem 4.8 (i).

Let $b\in\mathcal{S}^\sigma_{\alpha_0}$, and define $\tilde{P}$ as in formula (4.12). Because of property (D) in the definition of the class $\mathcal{S}^\sigma_{\alpha_0}$, we have:

$$\partial_\theta\tilde{P}\left(s,\theta,\sqrt{2H},J,t,H\right) = -\frac{\cos\alpha}{\alpha_0-\alpha}\partial_s\tilde{P}\left(s,\theta,\sqrt{2H},J,H,t\right)$$

where $\alpha = -\arcsin\left(\frac{J}{\sqrt{2H}}\right)$. The crucial point in what follows is that $\left|\frac{\cos\alpha}{\alpha_0-\alpha}\right|\leq\frac{C}{hR}$ on the support of $\tilde{P}\left(s,\theta,\sqrt{2H},J,t,H\right)$.

Recall that by definition

$$\langle\mu^{\alpha_0},\partial_\theta b\rangle = \lim_{R\to\infty}\lim_{h\to0^+}\left\langle w^{\alpha_0}_{h,R},\partial_\theta b\right\rangle.$$

Let us first fix R and study the limit $h \longrightarrow 0$. Arguing as in the proof of Proposition 4.6, we have

$$\begin{aligned}
\lim_{h\to 0^+} \left\langle w^{\alpha_0}_{h,R}, \partial_\theta b \right\rangle &= \lim \left\langle \mathscr{U} u_h, \mathrm{Op}_h\left(\partial_\theta \tilde{P}(s,\theta,E,J,t,hH)\right) \mathscr{U} u_h \right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)} \\
&= \lim \left\langle u_h, \mathcal{A}_E\left(\partial_\theta \tilde{P}\right) u_h \right\rangle \\
&= \lim \left\langle u_h, \mathcal{A}_H\left(\partial_\theta \tilde{P}\right) u_h \right\rangle \\
&= \lim \left\langle u_h, \mathcal{A}_H\left(-\frac{\cos\alpha}{(\alpha_0-\alpha)}\partial_s \tilde{P}\right) u_h \right\rangle \\
&= \lim \left\langle u_h, \left[-\frac{ih}{2}\Delta + ihV - h\partial_t, \mathcal{A}_H\left(-\frac{\cos\alpha}{E(\alpha_0-\alpha)}\tilde{P}\right)\right] u_h \right\rangle + O(R^{-1}) \\
&= \lim \left\langle \frac{ih}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}}, \mathcal{A}_H\left(-\frac{\cos\alpha}{E(\alpha_0-\alpha)}\tilde{P}\right) u_h \right\rangle \\
&\quad - \lim \left\langle u_h, \mathcal{A}_H\left(-\frac{\cos\alpha}{E(\alpha_0-\alpha)}\tilde{P}\right) \frac{ih}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}} \right\rangle + O(R^{-1})
\end{aligned} \tag{4.20}$$

where we used again (4.16).

But $\mathcal{A}_H\left(-\frac{\cos\alpha}{E(\alpha_0-\alpha)}\tilde{P}\right)$ equals (modulo terms which only add an error $O(R^{-1})$ to the whole calculation)

$$\begin{aligned}
&- \tilde{A}(r,u,\sqrt{2hD_t},hD_u,t)\frac{\cos\alpha(hD_u,hD_t)}{\sqrt{2hD_t}(\alpha_0-\alpha(hD_u,hD_t))} \\
&\qquad\qquad - \tilde{B}(r,u,\sqrt{2hD_t},hD_u,t)\frac{\cos\alpha(hD_u,hD_t)}{\sqrt{2hD_t}(\alpha_0-\alpha(hD_u,hD_t))} \circ hD_r
\end{aligned}$$

where $z = (-r\sin u, r\cos u)$ is the decomposition in polar coordinates, and $\tilde{A}, \tilde{B}$ are the functions associated to $\tilde{P}$ by the formulas of Proposition A.3.

If $\tilde{P}$ satisfies (B) then $\tilde{B} \equiv 0$ for $r = 1$. Since u_h satisfies Dirichlet boundary conditions, we see that the last terms in (4.20) vanish.

To conclude the proof, we take $R \longrightarrow +\infty$ after taking $h \longrightarrow 0$, so that the terms estimated as $O(R^{-1})$ vanish.

4.5. **Second microlocal structure of μ_{α_0}.** If $\mathcal{H}$ is a Hilbert space, we shall denote by $\mathcal{L}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ and $\mathcal{L}^1(\mathcal{H})$ the spaces of bounded, compact and trace class operators on $\mathcal{H}$. It is well known that $\mathcal{L}^1(\mathcal{H})$ is the dual of $\mathcal{K}(\mathcal{H})$. A measure on a polish space T, taking values in $\mathcal{L}^1(H)$, is defined as a bounded linear functional ρ from $C_c(T)$ to $\mathcal{L}^1(H)$; ρ is said to be nonnegative if, for every nonnegative $b \in C_c(T)$, $\rho(b)$ is a nonnegative hermitian operator. The set of such measures is denoted by $\mathcal{M}_+\left(T;\mathcal{L}^1(H)\right)$; they can be identified in a natural way to nonnegative linear functionals on $C_c(T;\mathcal{K}(H))$. Background and further details on operator-valued measures may be found for instance in [Gér91b].

For each $\omega \in \mathbb{R}/2\pi\mathbb{Z}$, let us define $\mathcal{H}_\omega$, the space of functions f on $\mathbb{R}$ satisfying $f(\theta+2\pi) = f(\theta)e^{i\omega}$ and that are square-integrable on $(0,2\pi)$.

We shall denote by $\mathcal{K}^{2\pi}$ the space of operators on $L^2(\mathbb{R})$ whose kernel K satisfies $K(\theta+2\pi,\theta'+2\pi) = K(\theta,\theta')$ and that define compact operators on each $\mathcal{H}_\omega$. Each Hilbert space $\mathcal{H}_\omega$ is isometric to $L^2(0,2\pi)$ (just by restricting functions to $(0,2\pi)$), and in this identification

the kernel of K acting on $\mathcal{H}_\omega$ is given by

$$K_\omega(\theta,\theta') := \mathbb{1}_{(0,2\pi)}(\theta)\mathbb{1}_{(0,2\pi)}(\theta')\sum_{n\in\mathbb{Z}} K(\theta,\theta'+2\pi n)e^{in\omega}. \tag{4.21}$$

The idea of the Floquet-Bloch theory is that it is completely equivalent to know $K(\theta,\theta')$ and to know $K_\omega(\theta,\theta')$ for almost all ω, by decomposing

$$L^2(\mathbb{R}) = \int_\oplus \mathcal{H}_\omega d\omega.$$

Besides, K is a nonnegative (resp. bounded) operator if and only if K_ω is nonnegative (resp. bounded) for a.e. ω.

An example of an operator in $\mathcal{K}^{2\pi}$ is

$$K_{b,h,R}(s,E,H,t) = b(s,\theta,E,hD_\theta,D_\theta,H,t)\chi(D_\theta/R) \tag{4.22}$$

for $b\in\mathcal{S}$ and fixed $R,h,\ t,H,s,E$. Note that, as $h\longrightarrow 0$, we have $K_{b,h,R}(s,E,H,t) = K_{b,0,R}(s,E,H,t)+O_R(h)$.

If b satisfies the symmetry condition (B), note that the operator $K_{b,0,R}(s,E,H,t)$ has the property

$$K(\cos\alpha,E,H,t) = R^{-1}_{\pi+2\alpha}\circ K(-\cos\alpha,E,H,t)\circ R_{\pi+2\alpha} \tag{4.23}$$

where R is a translation operator on $L^2(\mathbb{R}_\theta)$: $R_\alpha f(\theta) = f(\theta-\alpha)$ and where $\alpha = \arcsin\left(\frac{\sin\alpha_0\sqrt{2H}}{E}\right)$. In particular,

$$K(\cos\alpha_0,\sqrt{2H},H,t) = R^{-1}_{\pi+2\alpha_0}\circ K(-\cos\alpha_0,\sqrt{2H},H,t)\circ R_{\pi+2\alpha_0} \tag{4.24}$$

Remark 4.9. *The fact that the orbits of the billiard flow are periodic on $\mathcal{I}_{\alpha_0}$ ($\alpha_0\in\pi\mathbb{Q}$) is reflected in the fact that the function $s\mapsto K(s,\sqrt{2H},H,t)$ is periodic, if K satisfies (4.24).*

For $K\in C_c^\infty\left(\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t;\mathcal{K}\left(L^2(0,2\pi)\right)\right)$, let us define:

$$\begin{aligned}\left\langle n_h^{\alpha_0},K\right\rangle = (2\pi h)^{-2}\int_{\omega=0}^{2\pi}\sum_{H,\sqrt{2H}\sin\alpha_0/h\equiv\omega(2\pi)}\frac{h}{\sin\alpha_0}\sqrt{\frac{H}{2}}\int_{s,s',E,H',t}\\ \left\langle\chi_0\mathscr{V}\mathscr{U}u_h(s',H'),K\left(\omega,s,E,H',ht\right)\chi_0\mathscr{V}\mathscr{U}u_h(s,H)\right\rangle_{L^2(0,2\pi)}\\ e^{iE(s'-s)/h}e^{it(H'-H)/h}dsdEds'dH'dt\end{aligned} \tag{4.25}$$

where χ_0 is $\mathbb{1}_{(0,2\pi)}$ as in Remark 4.2. This is also

$$\langle\chi_0\mathscr{V}\mathscr{U}u_h,\mathcal{K}\chi_0\mathscr{V}\mathscr{U}u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H,L^2(0,2\pi))}$$

where $\mathcal{K}$ is the pseudodifferential operator with operator-valued symbol:

$$\int_{\omega=0}^{2\pi}\sum_{H,\sqrt{2H}\sin\alpha_0/h\equiv\omega(2\pi)}\frac{h}{\sin\alpha_0}\sqrt{\frac{H}{2}}K\left(\omega,s,E,H',ht\right). \tag{4.26}$$

Remark 4.10. *As noted earlier, it is equivalent (by the relation (4.21)) to consider a family $K(\omega)$ of kernels on $(0,2\pi)^2$ and a kernel K on $\mathbb{R}^2$ satisfying $K(\theta,\theta') = K(\theta+2\pi,\theta'+2\pi)$. With this identification in mind, formula (4.25) amounts to*

$$(2\pi h)^{-2}\int_{s,s',E,H,H',t} \left\langle \chi_0 \mathscr{V}\mathscr{U} u_h(s',H'), K\left(s,E,H',ht\right)\mathscr{V}\mathscr{U} u_h(s,H)\right\rangle_{L^2(\mathbb{R})} e^{iE(s'-s)/h}e^{it(H'-H)/h}dsdEds'dHdH'dt = \left\langle \chi_0\mathscr{V}\mathscr{U} u_h, K\left(s,hD_s,H,h^2D_t\right)\mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H,L^2(\mathbb{R}_\theta))} \tag{4.27}$$

The motivation for rewriting (4.27) in the apparently more complicated form (4.25) is that it will be more convenient to use the compact operators $K(\omega)$ on each $\mathcal{H}_\omega$ than the non-compact operator K on $L^2(\mathbb{R})$.

The relevance of definition (4.25) for us is that we have the relation

$$\begin{aligned}\langle w_{\alpha_0,h,R}, b\rangle &= \left\langle n_h^{\alpha_0}, K_{b,h,R}\right\rangle \\ &= \left\langle n_h^{\alpha_0}, K_{b,0,R}\right\rangle + O_R(h)\end{aligned} \tag{4.28}$$

where $K_{b,h,R}$ was defined in (4.22).

Proposition 4.11. *Suppose $\left(u_h^0\right)$ is bounded in $L^2(\mathbb{D})$. Then, modulo taking subsequences, the following convergence takes place:*

$$\lim_{h\to 0^+}\left\langle n_h^{\alpha_0}, K\right\rangle = \int_0^{2\pi}\int_{\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t} \mathrm{Tr}\left\{K\left(\omega,s,E,H,t\right)\rho_{\alpha_0}\left(d\omega,ds,dE,dH,dt\right)\right\}, \tag{4.29}$$

for every $K\in C_c^\infty\left(\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t;\mathcal{K}\left(L^2(0,2\pi)\right)\right)$. In other words, ρ_{α_0} is the limit of $n_h^{\alpha_0}$ in the weak-$*$ topology of

$$\mathcal{D}'\left(\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t,\mathcal{L}^1\left(L^2(0,2\pi)\right)\right).$$

In fact, ρ_{α_0} is a nonnegative, $\mathcal{L}^1\left(L^2(0,2\pi)\right)$-valued measures on $\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t$. In addition, ρ_{α_0} is supported in $\{s\in[-\cos\alpha_0,\cos\alpha_0], E=\sqrt{2H}\}$.

Proof. Note that $\chi_0\mathscr{V}\mathscr{U} u_h(s,H)$ is bounded in $L^2(\mathbb{R}_s\times\mathbb{R}_H,L^2(0,2\pi))$. The Calderón-Vaillancourt theorem [CV71] gives that the operators $\mathcal{K}$ with symbols of the form (4.26) are uniformly bounded with respect to h. Therefore, the linear map

$$L_h : K\mapsto \int_{\mathbb{R}}\langle n_h^{\alpha_0}, K\rangle$$

is uniformly bounded as $h\longrightarrow 0$. As a consequence, for any K, up to extraction of a subsequence, it has a limit $l(K)$.
Considering a countable dense subset of $C_c^\infty\left(\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t;\mathcal{K}\left(L^2(0,2\pi)\right)\right)$, and using a diagonal extraction process, one finds a sequence (h_n) tending to 0 as n goes to $+\infty$ such that for any $K\in C_c^\infty\left(\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t;\mathcal{K}\left(L^2(0,2\pi)\right)\right)$, the sequence $L_{h_n}(K)$ has a limit as n goes to $+\infty$.
The limit is a linear form on $C_c^\infty\left(\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t;\mathcal{K}\left(L^2(0,2\pi)\right)\right)$, characterized by an element ρ_{α_0} of the dual space $\mathcal{D}'\left(\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t,\mathcal{L}^1\left(L^2(0,2\pi)\right)\right)$.

The positivity of the limit is standard. Note that it is immediately seen in the expression (4.27). □

Comparing with (4.28), we obtain

Corollary 4.12. *For every $b \in \mathcal{S}$,*

$$\int b(s,\theta,E,J,\eta,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt) \\ = \mathrm{Tr}_{L^2(0,2\pi)} \int K_{b,0,\infty}(s,E,H,t)_\omega \ \rho_{\alpha_0}(d\omega,ds,dE,dH,dt)$$

Remember that $K_{b,0,R}(s,E,H,t) = b(s,\theta,0,D_\theta,H,t)\chi(D_\theta/R)$, so that $K_{b,0,\infty}(s,E,H,t) = b(s,\theta,0,D_\theta,H,t)$.

Corollary 4.13. *If b does not depend on η then the above identity can be rewritten as:*

$$\int b(s,\theta,E,J,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt) \\ = \mathrm{Tr}_{L^2(0,2\pi)} \int m_b(s,E,H,t) \ \rho_{\alpha_0}(d\omega,ds,dE,dH,dt)$$

where $m_b(s,E,H,t)$ is the multiplication operator by $b(s,\theta,E,0,H,t)$ acting on $L^2(0,2\pi)$.

Note that $\int b(s,\theta,E,J,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt) \geq 0$ if b does not depend on η and $b \geq 0$. Thus the projection on μ_{α_0} on the variables (s,θ,E,J,H,t) defines a nonnegative measure.

We finish this section by explaining why this implies that the projection of μ_{α_0} on the variables (s,θ) is absolutely continuous. If $b \in \mathcal{S}^\sigma$ does not depend on η, Proposition 4.6 implies that

$$\int b(s,\theta,E,J,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt) \\ = \int \langle b\rangle_{\alpha_0}(\theta,E,J,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt).$$

We know from Section 1.8.2 that $\mu = \mu_{sc}$ does not charge the set S. Since $\mu_{\alpha_0} \leq \mu$ by (4.10), the measure μ_{α_0} does not charge the set $\{s = \pm\cos\alpha_0\}$, and the previous equality actually holds for all $b \in \mathcal{S}$. If b does not depend on (E,J,H,t), we get the formula

$$\int b(s,\theta)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt) = \mathrm{Tr}_{L^2(0,2\pi)} \int m_{\langle b\rangle_{\alpha_0}} \ \rho_{\alpha_0}(d\omega,ds,dE,dH,dt).$$

This formula, defined a priori for continuous b, extends to $b \in L^\infty$. If b vanishes for Lebesgue-almost every (s,θ), the multiplication operator $m_{\langle b\rangle_{\alpha_0}}$ vanishes on $L^2(0,2\pi)$, and

$$\int b(s,\theta)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt) = 0,$$

which proves the absolute continuity.

5. Propagation law for ρ_{α_0}

We now show that the operator-valued measure ρ_{α_0} constructed in the previous section possesses some invariance properties. Below, the notation $\langle V\rangle_{\alpha_0}$ stands short for the function $\langle V\rangle_{\alpha_0} \circ \Phi(s,\theta,E,-E\sin\alpha_0,t)$, a function that actually does not depend on s and is 2π-periodic in θ.

Proposition 5.1. *(i) If K satisfies (4.24), we have*

$$\int \operatorname{Tr} E\, \partial_s K\,(\omega, s, E, H, t)\, \rho_{\alpha_0}\,(d\omega, ds, dE, dH, dt) = 0 \tag{5.1}$$

(ii) If in addition $K(s, \sqrt{2H}, H, t)$ does not depend on s, we have

$$\int \operatorname{Tr}\left(-\cos^2\alpha_0\, \partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0 \langle V\rangle_{\alpha_0}, K\right]_\omega\right)(\omega, s, E, H, t) \rho_{\alpha_0}\,(d\omega, ds, dE, dH, dt) = 0 \tag{5.2}$$

where $\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0\langle V\rangle_{\alpha_0}, K\right]_\omega$ means that we are considering ∂_θ^2 acting on $\mathcal{H}_\omega$ (in other words, $L^2(0, 2\pi)$ with Floquet-periodic boundary condition ($f(\theta + 2\pi) = f(\theta)e^{i\omega}$).

The proof of this key proposition is postponed to the end of this Section. Let us first draw some of its consequences in view of Theorem 1.8.

Remark 5.2. *Proposition 5.1 (ii) implies the following. Take $K = a(\omega, E, H, t) Id_{L^2(0,2\pi)}$ with a a scalar continous function, then*

$$\operatorname{Tr}\left(\int \partial_t a\,(\omega, E, H, t)\, \rho_{\alpha_0}\,(d\omega, ds, dE, dH, dt)\right) = 0.$$

Therefore, the image of ρ_{α_0} by the projection on $\mathbb{R}_s$,

$$\overline{\rho}_{\alpha_0}(d\omega, dE, dH, dt) := \int \rho_{\alpha_0}(d\omega, ds, dE, dH, dt)$$

is such that $\operatorname{Tr}(\overline{\rho}_{\alpha_0})$ does not depend on t.

Remark 5.3. *The Radon-Nikodym theorem* [Gér91b, Appendix] *implies that the operator valued measure $\overline{\rho}_{\alpha_0}$ can also be written as $\overline{\rho}_{\alpha_0} = \sigma_{\alpha_0}\ell_{\alpha_0}$ where $\ell_{\alpha_0} = \operatorname{Tr}(\overline{\rho}_{\alpha_0})$ is a nonnegative scalar measure on $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}_E \times \mathbb{R}_H$, and*

$$\sigma_{\alpha_0} : (\mathbb{R}/2\pi\mathbb{Z})_\omega \times \mathbb{R}_E \times \mathbb{R}_H \times \mathbb{R}_t \to \mathcal{L}^1_+\big(L^2(0, 2\pi)\big),$$

is an integrable function with respect to ℓ_{α_0}, taking values in the set of nonnegative trace-class operators on $L^2(0, 2\pi)$. Note that $\operatorname{Tr}(\sigma_{\alpha_0}) = 1$.

Corollary 5.4. *Let $\overline{\rho}_{\alpha_0}$ as in Remark 5.2 and let ℓ_{α_0} and σ_{α_0} as in Remark 5.3. Then for ℓ_{α_0}-almost every (ω, E, H), we have*

$$-\cos^2\alpha_0 \partial_t \sigma_{\alpha_0} + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0\langle V\rangle_{\alpha_0}, \sigma_{\alpha_0}\right]_\omega = 0$$

in $\mathcal{D}'\left(\mathbb{R}_t; \mathcal{L}^1_+\big(L^2(0, 2\pi)\big)\right)$.

Therefore, for ℓ_{α_0}-almost every (ω, E, H), σ_{α_0} coincides with a continuous function in

$$C^0\left(\mathbb{R}_t; \mathcal{L}^1_+\big(L^2(0, 2\pi)\big)\right)$$

and

$$\sigma_{\alpha_0}(\omega, E, H, t) = U_{\alpha_0,\omega}(t)\sigma_{\alpha_0}(\omega, E, H, 0)U^*_{\alpha_0,\omega}(t).$$

where $U_{\alpha_0,\omega}(t)$ is the unitary propagator of the equation

$$-\cos^2\alpha_0 D_t v(t, \theta) + \left(-\frac{1}{2}\partial_\theta^2 + \cos^2\alpha_0\langle V\rangle_{\alpha_0} \circ \Phi\right) v(t, \theta) = 0$$

Proof. We first rewrite (5.2) for s-independent operators K as

$$\int \mathrm{Tr}\left\{\left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2}+\cos^2\alpha_0\langle V\rangle_{\alpha_0}, K\right]_\omega\right)\sigma_{\alpha_0}\right\}\ell_{\alpha_0}\left(d\omega, dE, dH\right)dt = 0.$$

Therefore, we have

$$\int \mathrm{Tr}\left\{K\left(-\cos^2\alpha_0\,\partial_t\sigma_{\alpha_0} + i\left[-\frac{\partial_\theta^2}{2}+\cos^2\alpha_0\langle V\rangle_{\alpha_0}, \sigma_{\alpha_0}\right]_\omega\right)\right\}\ell_{\alpha_0}\left(d\omega, dE, dH\right)dt = 0,$$

which concludes the proof of Corollary 5.4.

□

To conclude this section, let us now prove its main result.

Proof of Propostion 5.1. As was already mentioned, it is equivalent to consider a family of kernels depending on ω, $K(\omega, s, E, H, t)(\theta, \theta')$ defined for $(\theta, \theta') \in (0, 2\pi)^2$, and a kernel $K(s, E, H, t)(\theta, \theta')$ defined for $(\theta, \theta') \in \mathbb{R}^2$ and satisfying $K(s, E, H, t)(\theta, \theta') = K(s, E, H, t)(\theta + 2\pi, \theta' + 2\pi)$. The link between both representations is the formula

$$K(\omega, s, E, H, t)(\theta, \theta') = \sum_{n\in\mathbb{Z}} K(s, E, H, t)(\theta, \theta' + 2n\pi)e^{in\omega}.$$

By a density argument, it is enough to treat the case where $K(s, E, H, t)$ is smooth in (s, E, H, t) and is a pseudodifferential operator on $L^2(\mathbb{R})$. By this, we mean that there is a $b_0(s, \theta, E, \eta, H, t) \in C_c^\infty(\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}^4)$ such that $K(s, E, H, t) = b_0(s, \theta, E, D_\theta, H, t)$. As ρ_{α_0} is supported by $\{E = \sqrt{2H}\}$, we may further assume that K satisfies (4.23) instead of (4.24).

If K satisfies (4.23), then we have $b_0\left(\cos\alpha, \theta, E, \eta, H, t\right) = b_0\left(-\cos\alpha, \theta + \pi + 2\alpha, E, \eta, H, t\right)$ for $\alpha = \arcsin\left(\frac{\sqrt{2H}\sin\alpha_0}{E}\right)$. We can extend b_0 to a function $b(s, \theta, E, J', \eta, H, t) \in C_c^\infty(\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}^5)$ such that, for $J' = 0$, we have $b(s, \theta, E, 0, \eta, H, t) = b_0(s, \theta, E, \eta, H, t)$, and such that b satisfies the symmetry condition (B) with $\sin\alpha = -\frac{J'-\sqrt{2H}\sin\alpha_0}{E}$. We are now back to our previous notation. The proof Proposition 5.1 (i) goes exactly along the lines of the proof of Proposition 4.6 (see Remark 4.7).

Let us now focus on the proof of (5.2).

If $K(s, \sqrt{2H}, H, t)$ does not depend on s, then $b_0(s, \sqrt{2H}, \theta, \eta, H, t)$ does not depend on s, and we can impose that the function b constructed above satisfy equation (D).

Letting $\eta = \frac{J'}{h}$, we note that, for η in the (compact) support of $b(s, \sqrt{2H}, \theta, J', \eta, H, t)$, we have

$$\alpha - \alpha_0 \sim \frac{-h\eta}{\sqrt{2H}\cos\alpha_0}(1 + O(h))$$

so that

$$\frac{-\eta\cos\alpha}{\sqrt{2H}(\alpha - \alpha_0)} \sim \frac{\cos^2\alpha_0}{h}(1 + O(h)). \tag{5.3}$$

We set

$$\mathcal{Q}_0 := \int \mathrm{Tr}\left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2}+\cos^2\alpha_0\langle V\rangle_{\alpha_0}, K\right]_\omega\right)(\omega, s, E, H, t)$$
$$\rho_{\alpha_0}\left(d\omega, ds, dE, dH, dt\right),$$

so that proving (5.2) amounts to showing that $\mathcal{Q}_0 = 0$.

First note that

$$\begin{aligned}\mathcal{Q}_0 &= \int \mathrm{Tr}\left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0\langle V\rangle_{\alpha_0}, K\right]_\omega\right)\left(\omega, s, \sqrt{2H}, H, t\right) \\ &\qquad\qquad \rho_{\alpha_0}\left(d\omega, ds, dE, dH, dt\right) \\ &= \int \mathrm{Tr}\left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0 V, K\right]_\omega\right)\left(\omega, s, \sqrt{2H}, H, t\right) \\ &\qquad\qquad \rho_{\alpha_0}\left(d\omega, ds, dE, dH, dt\right)\end{aligned} \tag{5.4}$$

since ρ_{α_0} is carried by $E = \sqrt{2H}$. With a slight abuse of notation we denoted by $V = V(s, E, t)$ the operator of multiplication by $V \circ \Phi(s, \theta, E, -\sin\alpha_0 E, t)$ acting on $L^2(0, 2\pi)$. Note that it does not depend on ω. It satisfies the condition (4.24) since the function $V \circ \Phi$ satisfies the symmetry condition (B) (since V is only a function of z in the old coordinates). In (5.4) we used the fact that $K(s, \sqrt{2H}, H, t)$ does not depend on s, and the result of Proposition 5.1 (i), to replace $\langle V\rangle_{\alpha_0}$ by V.

Now, by definition of ρ_{α_0}, we have

$$\begin{aligned}\mathcal{Q}_0 &= \lim \left\langle n_h^{\alpha_0}, -\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0 V, K\right]_\omega\right\rangle \\ &= \lim \Big\langle \chi_0 \mathscr{V}\mathscr{U} u_h, \\ &\qquad \left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0 V, K\right]\right)\left(s, hD_s, H, h^2 D_t\right)\mathscr{V}\mathscr{U} u_h\Big\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H, L^2(\mathbb{R}_\theta))}\end{aligned}$$

Using the fact that $K(s, E, H, t) = b_0(s, \theta, E, D_\theta, H, t)$ and the commutator calculus rule (1.16) for the standard quantization, we obtain

$$\begin{aligned}\mathcal{Q}_0 = &\lim_{h\to 0^+} i\left\langle \chi_0\mathscr{V}\mathscr{U} u_h, \left(\left[\cos^2\alpha_0 V, K\right]\right)\left(s, hD_s, H, h^2 D_t\right)\mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H, L^2(\mathbb{R}_\theta))} \\ &+ \left\langle \chi_0\mathscr{V}\mathscr{U} u_h, \mathrm{Op}_h\left((\eta\partial_\theta - i\frac{\partial_\theta^2}{2} - \cos^2\alpha_0\partial_t) b(s, \theta, E, J', \frac{J'}{h}, H, ht)\right)\mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)}\end{aligned}$$

using the notation $\eta = \frac{J'}{h} = \frac{J + \sin\alpha_0\sqrt{2H}}{h}$.

We now set

$$\begin{aligned}\mathcal{Q}_1 := &\lim_{h\to 0^+}\left\langle \chi_0\mathscr{V}\mathscr{U} u_h, \mathrm{Op}_h\left((\eta\partial_\theta - \cos^2\alpha_0\partial_t) b(s, \theta, E, J', \frac{J'}{h}, H, ht)\right)\mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)} \\ &+ i\left\langle \chi_0\mathscr{V}\mathscr{U} u_h, \left(\left[\cos^2\alpha_0 V, K\right]\right)\left(s, hD_s, H, h^2 D_t\right)\mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H, L^2(\mathbb{R}_\theta))},\end{aligned}$$

so that we have

$$\mathcal{Q}_0 = \mathcal{Q}_1 + \lim_{h\to 0^+}\left\langle \chi_0\mathscr{V}\mathscr{U} u_h, \mathrm{Op}_h\left(-i\frac{\partial_\theta^2}{2} b(s, \theta, E, J', \frac{J'}{h}, H, ht)\right)\mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)} \tag{5.5}$$

Let us for the moment focus on the term $\mathcal{Q}_1$, involving only derivatives of order 1 of b. As in Remark 4.7, we let $\mathbf{P}(s, \theta, E, J, t, H) = b(s, \theta, E, J + \sin\alpha_0\sqrt{2H}, \frac{J + \sin\alpha_0\sqrt{2H}}{h}, H, -t)$. Since

b is compactly supported in the fifth variable, this is also, modulo $O(h)$,

$$\mathbf{P}(s,\theta,E,J,t,H) = b(s,\theta,E,0,\frac{J+\sin\alpha_0\sqrt{2H}}{h},H,-t).$$

Still using the notation $\eta = \frac{J'}{h} = \frac{J+\sin\alpha_0\sqrt{2H}}{h}$, we have

$$\begin{aligned}\mathcal{Q}_1 &= \lim \left\langle \mathscr{U} u_h, \mathrm{Op}_h\left((\eta\partial_\theta - \cos^2\alpha_0\partial_t)\mathbf{P}(s,\theta,E,J,t,hH)\right)\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_t)} \\ &\qquad + i\left\langle \chi_0\mathscr{V}\mathscr{U} u_h, \left(\left[\cos^2\alpha_0 V, K\right]\right)\left(s, hD_s, H, h^2D_t\right)\mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H, L^2(\mathbb{R}_\theta))} \\ &= \lim \left\langle u_h, \mathcal{A}_H\left((\eta\partial_\theta - \cos^2\alpha_0\partial_t)\mathbf{P}\right)u_h\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}_t)} \\ &\qquad + i\left\langle u_h, \left[\cos^2\alpha_0 V, \mathcal{A}_H(\mathbf{P})\right]u_h\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}_t)} \\ &= \lim \left\langle u_h, \mathcal{A}_H\left(-\frac{\eta\cos\alpha}{E(\alpha_0-\alpha)}(E\,\partial_s - h\partial_t)\mathbf{P}\right)u_h\right\rangle + i\left\langle u_h, \left[\cos^2\alpha_0 V, \mathcal{A}_H(\mathbf{P})\right]u_h\right\rangle.\end{aligned}$$

Using that b (and thus also $\mathbf{P}$) satisfies equation (D), together with (5.3), we obtain

$$\mathcal{Q}_1 = \lim \frac{\cos^2\alpha_0}{h}\left\langle u_h, \mathcal{A}_H\left((E\,\partial_s - h\partial_t)\mathbf{P}\right)u_h\right\rangle + i\left\langle u_h, \left[\cos^2\alpha_0 V, \mathcal{A}_H(\mathbf{P})\right]u_h\right\rangle \tag{5.6}$$

Finally, we use again the Schrödinger equation (4.16) satisfied by u_h extended to $\mathbb{R}^2$, and rewrite the last line as

$$\begin{aligned}\mathcal{Q}_1 &= \lim \frac{\cos^2\alpha_0}{h}\left\langle u_h, \left[-\frac{ih}{2}\Delta + ihV - h\partial_t, \mathcal{A}_H(\mathbf{P})\right]u_h\right\rangle \\ &= \lim -\frac{\cos^2\alpha_0}{h}\left\langle \frac{ih}{2}\frac{\partial u_h}{\partial n}\otimes\delta_{\partial\mathbb{D}}, \mathcal{A}_H(\mathbf{P})\,u_h\right\rangle + \lim -\frac{\cos^2\alpha_0}{h}\left\langle u_h, \mathcal{A}_H(\mathbf{P})\,\frac{ih}{2}\frac{\partial u_h}{\partial n}\otimes\delta_{\partial\mathbb{D}}\right\rangle.\end{aligned} \tag{5.7}$$

Here we need the knowledge of $\mathcal{A}_H(\mathbf{P})$ modulo $O(h^2)$ (because of the factor $\frac{\cos^2\alpha_0}{h}$ that appears in the previous expression). Our calculations of Proposition A.3 give us the expression

$$\begin{aligned}\mathcal{A}_H(\mathbf{P}) &= \mathbf{A}(r,u,\sqrt{2hD_t},hD_u,t) + \mathbf{B}(r,u,\sqrt{2hD_t},hD_u,t)\circ hD_r \\ &\qquad + ih\mathbf{C}(r,u,\sqrt{2hD_t},hD_u,t) + ih\mathbf{D}(r,u,\sqrt{2hD_t},hD_u,t)\circ hD_r\end{aligned}$$

if $z = (-r\sin u, r\cos u)$ is the decomposition in polar coordinates and $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ are the functions associated to $\mathbf{P}$ by the formulas of Proposition A.3.

The terms $\mathbf{A},\mathbf{C}$ give a vanishing contribution in formula (5.7) because they are radial operators and u_h satisfies a Dirichlet boundary condition. The term $\mathbf{B}$ gives a vanishing condition if b (and hence $\mathbf{P}$) satisfy the symmetry condition (B): in that case we have $\mathbf{B}(1,u,\sqrt{2hD_t},hD_u,t) = 0$. So there just remains to look at the term $\mathbf{D}(1,u,\sqrt{2hD_t},hD_u,t)$.

Look at formula (A.8) defining the function $\mathbf{D}$. Remember that $\mathbf{P}(s,\theta,\sqrt{2H},J,t,H)$ is supported where $J+\sqrt{2H}\sin\alpha_0 = O(h)$, so that we have $\partial_s\mathbf{P} = O(h)$; also note that, on the set $\{J = -\sin\alpha_0 E\}$, the boundary equation $r=1$ amounts to $s = \pm\cos\alpha_0$, $\cos\theta_1(r,J,E) = \pm\cos\alpha_0$, so that $s\cos\theta_1(r,J,E) = \cos^2\alpha_0$ in formulas (A.8) and the following lines. We see that the function $\mathbf{D}(1,u,\sqrt{2H},J,t)$ coincides, modulo $O(h)$, with $\frac{1}{2H\cos^2\alpha_0}\mathbf{P}^\sigma(1,u,\sqrt{2H},J,t)$,

so that

$$\begin{aligned}\mathcal{Q}_1 &= -\lim \left\langle \frac{h}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}}, \mathbf{D}(1,u,\sqrt{2hD_t}, hD_u, t) \circ hD_r u_h \right\rangle \\ &\quad + h \lim \left\langle u_h, \mathbf{D}(1,u,\sqrt{2hD_t}, hD_u, t) \circ hD_r \frac{h}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}} \right\rangle \\ &= -\frac{1}{2\cos^2\alpha_0} \lim \left\langle h\frac{\partial u_h}{\partial n}, (E^{-2}\partial_2^2 \mathbf{P}^\sigma)(1,u,\sqrt{2hD_t}, -\sqrt{2hD_t}\alpha_0, ht, hD_t) h\frac{\partial u_h}{\partial n} \right\rangle.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\mathcal{Q}_1 &= -\lim \frac{\cos^2\alpha_0}{h} ih \left\langle \frac{h}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}}, \mathbf{D}(1,u,\sqrt{2hD_t}, hD_u, t) \circ hD_r u_h \right\rangle \\ &= -\frac{i}{2} \lim \left\langle h\frac{\partial u_h}{\partial n}, (E^{-2}\partial_2^2 \mathbf{P})(1,u,\sqrt{2hD_t}, -\sqrt{2hD_t}\alpha_0, ht, hD_t) h\frac{\partial u_h}{\partial n} \right\rangle.\end{aligned}$$

Using Remark 4.7, this limit expressed in terms of boundary data can also be expressed in terms of the interior, and we see that it equals

$$\begin{aligned}\mathcal{Q}_1 &= \frac{i}{2} \lim \left\langle \mathscr{U} u_h, \mathrm{Op}_h\left(\partial_2^2 \mathbf{P}(s,\theta,E,J',\frac{J'}{h},H,ht)\right) \mathscr{U} u_h \right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_t)} \\ &= \frac{i}{2} \lim \left\langle \mathscr{V}\mathscr{U} u_h, \mathrm{Op}_h\left(\partial_2^2 b(s,\theta,E,J',\frac{J'}{h},H,ht)\right) \mathscr{V}\mathscr{U} u_h \right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)}.\end{aligned}$$

Finally coming back to (5.5), this yields $\mathcal{Q}_0 = 0$, that is, identity (5.2). This concludes the proof of Proposition 5.1. □

6. End of the semiclassical construction: proof of Theorem 1.8

In this section, we first prove Proposition 1.15, and then conclude the proof of Theorem 1.8.

6.1. **A proof of Proposition 1.15.** For a a smooth compactly supported function on $\mathbb{R}^4$, we show that

$$\lim_{h\longrightarrow 0} \langle u_h, \mathrm{Op}_h(\partial_t a(|\xi|^2, x\xi_y - y\xi_x, t, hH) u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} = 0.$$

This limit is the same as

$$\lim_{h\longrightarrow 0} \langle u_h, \mathrm{Op}_h(\partial_t a(2H, x\xi_y - y\xi_x, t, hH) u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}$$

which is

$$\lim_{h\to 0} \langle u_h, [\partial_t, \mathrm{Op}_h(a(2H, x\xi_y - y\xi_x, t, hH)] u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} = \lim_{h\to 0} \langle u_h, [\partial_t, a(2hD_t, hD_u, t, h^2 D_t)] u_h\rangle$$

where $z = (-r\sin u, r\cos u)$ is the decomposition of $z = (x,y)$ into polar coordinates. Because of the equation satisfied by u_h, this is also (with Δ_D the Dirichlet laplacian)

$$\lim_{h\longrightarrow 0} \left\langle u_h, \left[-i\frac{\Delta_D}{2} + iV, a(2hD_t, hD_u, t, h^2 D_t)\right] u_h \right\rangle$$

Note that $a(2hD_t, hD_u, t, h^2 D_t)$ actually defines an operator on $L^2(\mathbb{D})$ as it is tangential to $\partial\mathbb{D}$. This limit vanishes, because Δ_D commutes with $a(2hD_t, hD_u, t, h^2 D_t)$ and because

$$\left[V, a(2hD_t, hD_u, t, h^2 D_t)\right] = O(h).$$

This concludes the proof of Proposition 1.15.

6.2. **End of the proof of Theorem 1.8.** We are now in a position to prove Theorem 1.8.

The measure ℓ_{α_0} and the function σ_{α_0} of Theorem 1.8 (iii) are the ones appearing in Remark 5.3. The object called ν_{Leb} in Theorem 1.8 (ii) is defined as

$$\nu_{Leb} = \mu_{sc}\rceil_{\alpha\notin\pi\mathbb{Q}} + \sum_{\alpha_0\in\pi\mathbb{Q}} m^{\alpha_0}$$

where m^{α_0} was defined in Remark 4.5. For $\alpha \notin \pi\mathbb{Q}$, we must have

$$\mu_{sc}\rceil_{\mathcal{I}_\alpha}(t) = \int c_1(t,E,J)\lambda_{E,J}\, d\nu_1(E,J)$$

for some nonnegative measure ν_1 (carried by $\{J = -\sin\alpha E\}$) and some measurable function $c_1(t,E,J)$. But, because the image of μ_{sc} under the map $M : (z,\xi) \mapsto (E,J)$ does not depend on t (see below), the function $c_1(t,E,J)$ actually does not depend on t.

The two invariance properties Proposition 4.6 (invariance w.r.t. s) and Theorem 4.8 (i) (invariance w.r.t. θ) also imply that m^{α_0} is of the form

$$m^{\alpha_0}(t) = \int c_2(t,E,J)\lambda_{E,J}\, d\nu_2(E,J)$$

for some nonnegative measure ν_2 (carried by $\{J = -\sin\alpha_0 E\}$).

We now prove that the function $c_2(t,E,J)$ actually does not depend on t. For this, we remark that the same proof as that of Proposition 1.15 above applies if we replace

$$\mathrm{Op}_h\left(\partial_t a(|\xi|^2, x\xi_y - y\xi_x, t, hH)\right)$$

in the first line by

$$\mathscr{U}^*\,\mathrm{Op}_h\left(\partial_t a(E^2,J,t,hH)(1-\chi)\left(\frac{J+\sin\alpha_0\sqrt{2H}}{hR}\right)\right)\mathscr{U}$$

in the limits $h \longrightarrow 0$ followed by $R \longrightarrow +\infty$.

Using the notation of Remark 4.5, this shows that the image of m^{α_0} under the map M is independent of t. Since we already know that $m^{\alpha_0}(t)$ is of the form $\int c_2(t,E,J)\lambda_{E,J}d\nu_2(E,J)$ for some nonnegative measure ν_2, we conclude that $c_2(t,E,J)$ actually does not depend on t.

The proof of Theorem 1.8 is now complete.

7. The microlocal construction: sketch of the proof of Theorem 1.14

Herein we use the definitions and notation introduced in Sections 1.2 and 1.5.

Let (u_n^0) be a sequence of initial data, normalized in L^2, and as above denote $u_n(z,t) = U_V(t)u_n^0(z)$ or in short $u_n = U_V u_n^0$.

7.1. **Structure of μ_c.** Let $a \in \mathcal{S}_0$. Recall that $\langle \mu_c, a\rangle$ is defined as the limit (after extraction of subsequences) as $n \longrightarrow +\infty$ followed by $R \longrightarrow +\infty$ of

$$\langle W_{c,n,R}, a\rangle := \left\langle u_n, \mathrm{Op}_1\left(\chi\left(\frac{|\xi|^2+|H|}{R^2}\right)a(z,\xi,t,H)\right)u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}. \tag{7.1}$$

On the support of $\chi\left(\frac{|\xi|^2+|H|}{R^2}\right)a(z,\xi,t,H)$ note that $|H| \le KR^2$ if χ is supported in $[-K,K]$.

Let g be a smooth compactly supported function on $\mathbb{R}$, taking the value 1 on $[-K,K]$. For $R>0$ define the operator $P_R : L^2(\mathbb{D}) \mapsto L^2(\mathbb{D})$ by

$$P_R u = g\left(\frac{D_t}{R^2}\right) U_V(t)u]_{t=0}.$$

By the results of Appendix C (in particular Remark C.3), we know that P_R has the following properties:

- For fixed R, P_R is compact, and we have $\|\nabla P_R u\|_{L^2(\mathbb{D})} \leq CR\|u\|_{L^2(\mathbb{D})}$ for some constant C (that depends on the function g, but not on R).
- For any u, we have $P_R u \longrightarrow u$ in $L^2(\mathbb{D})$ as $R \longrightarrow +\infty$.
- $\langle \mu_c, a\rangle$ is the limit as $n \longrightarrow +\infty$ followed by $R \longrightarrow +\infty$ of

$$\left\langle u_n, \mathrm{Op}_1\left(\chi\left(\frac{|\xi|^2+|H|}{R^2}\right) a(z,\xi,t,H)\right) U_V P_R u_n^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}. \tag{7.2}$$

Let $K_t(z,z')$ be the kernel of $U_V(t)$. Let B_R be the bounded operator on $L^2(\mathbb{D})$ with kernel

$$B_R(z_1,z_2) = \frac{1}{(2\pi)^3}\int \overline{K_t(z,z_1)}\mathbb{1}_{\mathbb{D}}(z)a(z,\xi,t,H)\chi\left(\frac{|\xi|^2+|H|}{R^2}\right) e^{i\xi(z-z')}e^{iH(t-t')}\mathbb{1}_{\mathbb{D}}(z')K_{t'}(z',z_2)dzdz'dtdt'dHd\xi \tag{7.3}$$

so that (7.1) is $\langle u_n^0, B_R u_n^0\rangle_{L^2(\mathbb{D})}$ and (7.2) is $\langle u_n^0, B_R P_R u_n^0\rangle_{L^2(\mathbb{D})}$.

Call ρ_0 a weak-$*$ limit of the sequence of trace class operators $|u_n^0\rangle\langle u_n^0|$ on $L^2(\mathbb{D})$. Then for fixed R $\langle W_{c,n,R}, a\rangle$ converges to $\mathrm{Tr}_{L^2(\mathbb{D})}(B_R P_R \rho_0)$. Letting now $R \longrightarrow +\infty$ we find the expression

$$\langle \mu_c, a\rangle = \mathrm{Tr}_{L^2(\mathbb{D})}(B_\infty \rho_0) \tag{7.4}$$

where B_∞ is defined as in (7.3) with $\chi\left(\frac{|\xi|^2+|H|}{\infty}\right)=1$. The expression of B_∞ is simpler when a does not depend on H, in this case we have

$$B_\infty = \int U_V(t)^* \mathbb{1}_{\mathbb{D}} a(z,D_z,t)\mathbb{1}_{\mathbb{D}} U_V(t)dt.$$

In particular if a does not depend on H we have

$$\langle \mu_c, a\rangle = \mathrm{Tr}_{L^2(\mathbb{D})}\left(\int U_V(t)^*\mathbb{1}_{\mathbb{D}} a(z,D_z,t)\mathbb{1}_{\mathbb{D}} U_V(t)\rho_0 dt\right)$$

and more generally we write (in a somewhat fuzzy notation)

$$\langle \mu_c, a\rangle = \mathrm{Tr}_{L^2(\mathbb{D})}\left(\int U_V(t)^*\mathbb{1}_{\mathbb{D}} a(x,D_z,t,D_t)\mathbb{1}_{\mathbb{D}} U_V(t)\rho_0 dt\right)$$

to mean the well-defined expression (7.4).

7.2. **Structure of μ^∞ and proof of Theorem 1.14.** Proposition 1.13 is a well-known fact and won't be proven in detail here (the fact that it is carried on $\{H=\frac{|\xi|^2}{2}\}$ follows from Appendix C and the proof of invariance is essentially contained in [GL93]). We focus on the structure theorem 1.14.

Let $\eta = \frac{\xi}{\sqrt{2H}}$. On the support of μ^∞, η has norm 1. To any pair $(z,\eta)\in\mathbb{D}\times\mathbb{S}^1$ we now associate $j = x\eta_y - y\eta_x$ and $\alpha = -\arcsin j$ which is the angle that the billiard ray issued from

(z,η) makes with the inner normal when it bounces on the boundary of the disk. Exactly as in Lemma 3.1 we decompose μ^∞ as a sum of nonnegative measures:

$$\mu^\infty = \mu^\infty\rceil_{\alpha\notin\pi\mathbb{Q}} + \sum_{r\in\mathbb{Q}\cap[-1/2,1/2]} \mu^\infty\rceil_{\alpha=r\pi}. \tag{7.5}$$

The invariance (1.23) implies that $\mu^\infty\rceil_{\alpha\notin\pi\mathbb{Q}}$ is of the form $\int_{E>0,|J|\leq E,\alpha\notin\pi\mathbb{Q}} \lambda_{E,J} d\bar{\mu}^\infty(E,J)$ The fact that $\bar{\mu}^\infty$ does not depend on t is the microlocal version of Proposition 1.15.

We now fix $r_0\in\mathbb{Q}\cap(-1/2,1/2)$, write $\alpha_0 = r_0\pi$ and wish to study $\mu^\infty\rceil_{\alpha=\alpha_0}$. We define

$$\begin{aligned}
&\langle \mu^\infty_{\alpha_0}, a\rangle \\
&:= \lim_{R'}\lim_{R\to\infty}\lim_{n\to+\infty} \\
&\left\langle u_n, \mathrm{Op}_1\left(\left(1-\chi\left(\frac{|\xi|^2+H}{R^2}\right)\right) a(z,\xi,t,H)\chi\left(\frac{J+\sin\alpha_0\sqrt{2H}}{R'}\right)\right) u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\
&= \lim_{R'}\lim_{R}\lim_{n} \\
&\left\langle \mathscr{U} u_n, \mathrm{Op}_1\left(\left(1-\chi\left(\frac{H}{R^2}\right)\right) a\circ\Phi(s,\theta,\sqrt{2H},J,t,H)\chi\left(\frac{J+\sin\alpha_0\sqrt{2H}}{R'}\right)\right) \mathscr{U} u_n\right\rangle_{L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{R})}
\end{aligned}$$

and

$$\begin{aligned}
&\langle \mu^{\infty,\alpha_0}, a\rangle \\
&:= \lim_{R'}\lim_{R\to\infty}\lim_{n\to+\infty} \\
&\left\langle u_n, \mathrm{Op}_1\left(\left(1-\chi\left(\frac{|\xi|^2+H}{R^2}\right)\right) a(z,\xi,t,H)\left(1-\chi\left(\frac{J+\sin\alpha_0\sqrt{2H}}{R'}\right)\right)\right) u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\
&= \lim_{R'}\lim_{R}\lim_{n} \\
&\left\langle \mathscr{U} u_n, \mathrm{Op}_1\left(\left(1-\chi\left(\frac{H}{R^2}\right)\right) a\circ\Phi(s,\theta,\sqrt{2H},J,t,H)\left(1-\chi\left(\frac{J+\sin\alpha_0\sqrt{2H}}{R'}\right)\right)\right) \mathscr{U} u_n\right\rangle.
\end{aligned}$$

The following theorem is proven in essentially the same way as in the semiclassical case:

Theorem 7.1. *(i)μ^{∞,α_0} is invariant under rotations (that is, under the flow of P_1). It is a multiple of the Lebesgue measure $\lambda_{(1,-\sin\alpha_0)}$. That is, $\mu^{\infty,\alpha_0} = c(\alpha_0)\lambda_{(1,-\sin\alpha_0)}$ with $c(\alpha_0)\geq 0$ independent of t.*

(ii) for every $\alpha_0\in\pi\mathbb{Q}\cap(-\pi/2,\pi/2)$, we can build from the sequence of initial conditions (u_n) a nonnegative measure $\sigma_{\alpha_0}(d\omega,ds,dE,dH,dt)$ (carried by $\{H=E^2/2\}$) on $\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{S}^2_{E,H}\times\mathbb{R}_t$, taking values in the trace-class operators on $L^2(0,2\pi)$, so that $\mu^\infty_{\alpha_0}$ is the measure carried by the set $\{j=-\sin\alpha_0\}\cap\{H=E^2/2\}$ such that

$$\begin{aligned}
&\int a_{\mathrm{hom}}(z,\xi,t)\mu^\infty_{\alpha_0}(dz,d\xi,dt,dH) \\
&\qquad = \mathrm{Tr}_{L^2(0,2\pi)}\left(\int m_{a_{\mathrm{hom}}\circ\Phi}(s,\cdot,1,-\sin\alpha_0,t)\,\sigma_{\alpha_0}(d\omega,ds,dE,dH,dt)\right).
\end{aligned}$$

If in addition a is symmetric w.r.t. the boundary, we have

$$\int a_{\mathrm{hom}}(z,\xi)\mu^{\infty}_{\alpha_0}(dz,d\xi,t,dH)$$
$$= \mathrm{Tr}_{L^2(0,2\pi)}\left(\int U_{\alpha_0,\omega}(t)^* m_{\langle a_{\mathrm{hom}}\rangle_{\alpha_0}\circ\Phi}(\cdot,1,-\sin\alpha_0)\, U_{\alpha_0,\omega}(t)\,\sigma_{\alpha_0}(d\omega,ds,dE,dH,0)\right).$$

The decomposition forlmula of Theorem 1.14 (i) now holds with

- the distribution $\mu_c \in \mathcal{S}_0'$ described in Section 7.1;
- the measure μ_{Leb} given by
$$\mu_{Leb} = \mu^{\infty}\rceil_{\alpha\notin\pi\mathbb{Q}} + \sum_{\alpha_0\in\pi\mathbb{Q}\cap(-\pi/2,\pi/2)} \mu^{\infty,\alpha_0};$$
- for $\alpha_0 \in \pi\mathbb{Q}\cap(-\pi/2,\pi/2)$ the measure μ_{α_0} given by
$$\mu_{\alpha_0}(t) = \mu^{\infty}_{\alpha_0}(t);$$
- for $\alpha_0 = \pm\pi/2$ the measure μ_{α_0} given by
$$\mu_{\alpha_0} = \mu^{\infty}\rceil_{\alpha=\alpha_0}.$$

Theorem 7.1 then implies Theorem 1.14.

8. Proof of Theorems 1.2 and 1.3: Observability inequalities

In this section, we prove Theorems 1.2 and 1.3 using the microlocal version of our results. We could have chosen to do it with semiclassical measures as well. However, since there is no natural frequency-scale, it would have required to perform a dyadic decomposition in frequency (see for instance [Leb92, AM14]). Note that the idea of proving observability inequalities using microlocal defect measures is due to Lebeau [Leb96].

8.1. **Unique continuation for microlocal measures.** The goal of this section is to prove a unique continuation result for microlocal measures μ_{ml} associated to solutions of the Schrödinger equation (1.1). According to Theorem 1.14, such a measure decomposes as

$$\mu_{ml} = \mu^{\infty} + \mu_c,$$

that we shall study independently.

In order to state the result for μ^{∞}, we introduce the following notation. For $z \in \partial\mathbb{D}$, we define

$$S^+_z = \{\xi\in\mathbb{R}^2, \xi\cdot z > 0\}, \qquad \overline{S}^+_z = \{\xi\in\mathbb{R}^2, \xi\cdot z \geq 0\}.$$

The set S^+ defined in Section 1.3 is $S^+ = \bigcup_{z\in\partial\mathbb{D}} S^+_z$ and

$$\bigcup_{z\in\partial\mathbb{D}} \overline{S}^+_z = \left\{\Phi\left(\left(1-(J/E)^2\right)^{\frac{1}{2}}, \theta, E, J\right), E>0, |J|\leq E, \theta\in\mathbb{R}/2\pi\mathbb{Z}\right\}.$$

The following two lemmas are respectively useful for the proof of internal and boundary observability.

Lemma 8.1. *Fix $T>0$. Take $b\in\mathcal{S}^0$ independent of (t,H) and assume that*

$$\text{there exists } z_0\in\partial\mathbb{D} \text{ such that } b>0 \text{ in a neighbourhood of } \overline{S}^+_{z_0}. \tag{8.1}$$

Then $\int_0^T\int_{\mathbb{R}^2\times\mathbb{S}^2_{H,\xi}} b^2_{\mathrm{hom}}(z,\xi)\mu^{\infty}(dz,dH,d\xi,t)dt = 0$ implies $\mu^{\infty}=0$ on $\mathbb{R}_t\times\mathbb{R}^2_z\times\mathbb{S}^2_{H,\xi}$.

Lemma 8.2. *Take any nonempty set $\Gamma \subset \partial\mathbb{D}$ and $T > 0$. Then $\mu^{\partial}_{ml} = 0$ on $T^*((0,T)\times\Gamma)$ implies $\mu^\infty = 0$ on $\mathbb{R}_t \times \mathbb{R}^2_z \times \mathbb{S}^2_{H,\xi}$.*

The proof of these lemmas relies on the properties of μ^∞ together with a unique continuation result for the one dimensional Schrödinger flows $U_{\alpha_0,\omega}(t)$ on $L^2(0,2\pi)$ from any nonempty open set $(0,T)\times\Omega$, where $\Omega \subset (0,2\pi)$. Such unique continuation property holds as soon as $\langle V\rangle_{\alpha_0} \in L^\infty((0,T)\times(0,2\pi))$, for instance as a consequence of [Lau10, Appendix B] (see also the references therein).

Concerning μ_c we have the following result.

Lemma 8.3. *Let $\Omega \subset \mathbb{D}$ be a nonempty open set. Assume that the unique continuation property ($\mathrm{UCP}_{V,\Omega,T}$) holds. Then we have*

$$\langle \mu_c, \mathbb{1}_{(0,T)\times\Omega}\rangle = 0 \Longrightarrow \mu_c = 0.$$

The unique continuation property ($\mathrm{UCP}_{V,\Omega,T}$) is foxnce known to hold (in any time $T > 0$ and for any nonempty nomempty open set Ω) if V is analytic in (t,z) as a consequence of the Holmgren theorem (as stated by Hörmander [Hör76, Theorem 5.3.1]). If $V = V(z)$ is smooth and does not depend on t, it is proved in the next section. Note that this last result can be extended to the case where V is continuous outside a set of zero measure zero, see [AM14].

Remark 8.4. *Note that the analogues of the unique continuation results of Lemmas 8.1, 8.2 and 8.3 also hold for semiclassical measures. We chose not to state them here for the sake of brevity.*

Proof of Lemma 8.1. We decompose μ^∞ as in Theorem 1.14

$$\mu^\infty(t,\cdot) = \mu_{Leb} + \sum_{\alpha_0\in\pi\mathbb{Q}\cap[-\pi/2,\pi/2]} \mu_{\alpha_0}(t,\cdot).$$

As every term in this sum is a non-negative measure, the assumption on μ^∞ implies

$$\int_0^T \int_{\mathbb{R}^2\times\mathbb{S}^2} b^2_{\mathrm{hom}}(z,\xi)\mu_{Leb}(dz,dH,d\xi,t)dt = 0, \tag{8.2}$$

$$\int_0^T \int_{\mathbb{R}^2\times\mathbb{S}^2} b^2_{\mathrm{hom}}(z,\xi)\mu_{\alpha_0}(dz,dH,d\xi,t)dt = 0, \tag{8.3}$$

for all $\alpha_0 \in \pi\mathbb{Q}\cap[-\pi/2,\pi/2]$.

Still according to Theorem 1.14, μ_{Leb} is of the form $\int_{E>0,|J|\le E} \lambda_{E,J} d\mu'(E,J)$ for some nonnegative measure μ' on $\mathbb{R}P^1$. Together with (8.2), this reads

$$0 = \int_{E>0,|J|\le E} \int_{T(E,J)} b^2_{\mathrm{hom}} \circ \Phi(s,\theta,E,J)\lambda_{E,J}(ds,d\theta)\mu'(dE,dJ).$$

Recall (see Section 1.3) that $\lambda_{E,J}(ds,d\theta) = c(E,J)dsd\theta$ where $c(E,J) = \left(\int_{T(E,J)} dsd\theta\right)^{-1} > 0$, so that we have

$$0 = \int_{E>0,|J|\le E} \left(\int_{T(E,J)} b^2_{\mathrm{hom}} \circ \Phi(s,\theta,E,J)dsd\theta\right) c(E,J)\mu'(dE,dJ).$$

Now, for any (E,J) such that $E > 0, |J| \le E$, there exists $\theta \in \mathbb{S}^1$ (depending only on J/E), such that

$$\Phi\left(\left(1-(J/E)^2\right)^{\frac{1}{2}}, \theta, E, J\right) \in \overline{S}^+_{z_0}.$$

Assumption (8.1) then implies that $\int_{T(E,J)} b^2_{\text{hom}} \circ \Phi(s,\theta,E,J)dsd\theta > 0$ for any (E,J). As a consequence, μ_{Leb} vanishes identically.

Let us now consider $\alpha_0 = \pm\pi/2$. The rotation invariance given by Theorem 1.14 together with Assumption (8.1) imply that $\mu_{\pm\pi/2}$ vanish.

Let us now consider $\alpha_0 \in \pi\mathbb{Q} \cap (-\pi/2, \pi/2)$. The measure μ_{α_0} is supported by $\mathcal{I}_{\alpha_0}$ and invariant by the billiard flow, so that

$$\int_0^T \int_{\mathbb{R}^2\times\mathbb{S}^2} b^2_{\text{hom}}\, \mu_{\alpha_0}(dz, dH, d\xi, t)dt = \int_0^T \int_{\mathbb{R}^2\times\mathbb{S}^2} \langle b^2_{\text{hom}}\rangle_{\alpha_0}\, \mu_{\alpha_0}(dz, dH, d\xi, t)dt$$

Using Theorem 1.14 with (8.3), we obtain

$$0 = \int \operatorname{Tr}_{L^2(0,2\pi)} \left(B_{\alpha_0}\, \sigma_{\alpha_0}\right) d\ell_{\alpha_0} dt, \quad \text{with } B_{\alpha_0} := m^{\alpha_0}_{\langle b^2_{\text{hom}}\rangle_{\alpha_0}}.$$

According to Corollary 5.4, this yields

$$0 = \int \operatorname{Tr}_{L^2(0,2\pi)} \left(B_{\alpha_0} U_{\alpha_0,\omega}(t)\sigma_{\alpha_0}(\omega,E,H,0)U^*_{\alpha_0,\omega}(t)\right) d\ell_{\alpha_0} dt.$$

Since the integrand is non-negative, we have for ℓ_{α_0}-almost every (ω, E, H),

$$0 = \int_0^T \operatorname{Tr}_{L^2(0,2\pi)} \left(B_{\alpha_0} U_{\alpha_0,\omega}(t)\sigma_{\alpha_0}(\omega,E,H,0)U^*_{\alpha_0,\omega}(t)\right) dt. \tag{8.4}$$

For ℓ_{α_0}-almost every (ω, E, H), $\sigma_{\alpha_0}(\omega, E, H, 0)$ is a non-negative trace-class operator. We can decompose it as a sum of of orthogonal projectors on its eigenfunctions:

$$\sigma_{\alpha_0}(\cdot, 0) = \sum_{k\in\mathbb{N}} \lambda_k |\varphi_k\rangle\langle\varphi_k|, \quad \text{with} \quad \lambda_k \geq 0, \quad \sum_{k\in\mathbb{N}} \lambda_k = 1, \quad \langle\varphi_k|\varphi_j\rangle_{L^2(0,2\pi)} = \delta_{kj}.$$

Note that λ_k, φ_k depend on (ω, E, H). Now Equation (8.4) is equivalent to having, for all $k \in \mathbb{N}$, such that $\lambda_k > 0$,

$$0 = \int_0^T \int_0^{2\pi} \langle b^2_{\text{hom}}\rangle_{\alpha_0} \circ \Phi(s,\theta,E,-E\sin\alpha_0)\, |U_{\alpha_0,\omega}(t)\varphi_k|^2\, (\theta)d\theta dt. \tag{8.5}$$

As above, there exists $\theta \in \mathbb{S}^1$ depending only on α_0, such that

$$\Phi\left(\cos\alpha_0, \theta, E, -E\sin\alpha_0\right) \in \overline{S}^+_{z_0}.$$

Hence, $\langle b^2_{\text{hom}}\rangle_{\alpha_0} > 0$ in a neighborhood of this θ. Then (8.5) implies that $U_{\alpha_0,\omega}(t)\varphi_k$ vanishes in a nonempty open subset of $(0,T) \times (0, 2\pi)$. One dimensional unique continuation (see e.g. [Lau10, Appendix B] and the references therein) then implies that $\varphi_k = 0$. Therefore, $\sigma_{\alpha_0}(\omega, E, H, 0)$ vanishes ℓ_{α_0}-almost everywhere, which yields $\mu_{\alpha_0} = 0$.

This finally proves that $\mu^\infty = 0$ and concludes the proof of the lemma.

□

Proof of Lemma 8.2. Let us fix $z_0 \in \Gamma$ and prove that μ^∞ vanishes in a neighborhood of $\overline{S}^+_{z_0}$. The result shall then follow from Lemma 8.1. First, according to (1.31), the assumption implies that $\mu^\infty\rceil_{T^*((0,T)\times\Gamma)}$ vanishes. Second, as a consequence of the assumption together with (1.30), we have $\mu^S_{ml}\rceil_{(z,\xi)\in S^+, z\in\Gamma} = 0$. Coming back to the definition of the measure μ^S_{ml} in Section 1.8.1, this implies that μ^∞ vanishes on all trajectories of the billiard flow touching the boundary on $\{(z,\xi) \in S^+, z \in \Gamma\}$. In particular, this yields $\mu^\infty\rceil_{\overline{S^+_{z_0}}}$ and the result follows from Lemma 8.1. □

Proof of Lemma 8.3. Theorem 1.14 (and Theorem 1.12) together with $\langle \mu_c, \mathbb{1}_{(0,T)\times\Omega}\rangle = 0$ imply that

$$0 = \mathrm{Tr}_{L^2(\mathbb{D})}\left(\int_0^T U_V(t)^* m_{\mathbb{1}_\Omega} U_V(t)\rho_0 dt\right),$$

where $m_{\mathbb{1}_\Omega}$ is the multiplication operator in $L^2(\mathbb{D})$ by the function $\mathbb{1}_\Omega$. As in the proof of Lemma 8.1, this implies that for any eigenfunction φ of ρ_0, we have

$$0 = \int_0^T \int_\Omega |U_V(t)\varphi|^2(z) dz dt.$$

The unique continuation property ($\mathrm{UCP}_{V,\Omega,T}$) then implies $\varphi = 0$. This proves that $\rho_0 = 0$ which concludes the proof of the lemma. □

8.2. **Interior observability inequality: proof of Theorem 1.2.**

8.2.1. *Unique continuation implies observability.* In this section, we prove the observability inequality (1.4) assuming that ($\mathrm{UCP}_{V,\Omega,T}$) holds. Instead of proving (1.4) for any open set $\Omega \subset \mathbb{D}$ containing a neighbourhood in $\mathbb{D}$ of a point of $\partial\mathbb{D}$, we prove the equivalent statement: for any function $b \in C^0(\mathbb{R}^2)$ (also considered as a function in $C^0(\overline{\mathbb{D}})$) which is positive on a nonempty open subset of $\partial\mathbb{D}$, for any $T > 0$, there exists $C > 0$ such that the following inequality holds:

$$\left\|u^0\right\|^2_{L^2(\mathbb{D})} \leq C \int_0^T \left\|b(z)U_V(t)u^0\right\|^2_{L^2(\mathbb{D})} dt. \tag{8.6}$$

Note that under these conditions on Ω and b, inequalities (8.9) and (8.6) are equivalent.

We proceed by contradiction and suppose that the observability inequality (8.6) is not satisfied. Thus, there exists a sequence $(u_n^0)_{n\in\mathbb{N}}$ in $L^2(\mathbb{D})$ such that

$$\|u_n^0\|_{L^2(\mathbb{D})} = 1, \tag{8.7}$$

$$\int_0^T \left\|b(z)U_V(t)u_n^0\right\|^2_{L^2(\mathbb{D})} dt \to 0, \quad n \to \infty, \tag{8.8}$$

We write $u_n(t) = U_V(t)u_n^0$ the associated solution of (1.1)-(1.2). As in Section 1, we extend u_n to $\mathbb{R}^2$ by zero outside $\mathbb{D}$ (and still use the notation u_n for its extension).

After having extracted a subsequence, we associate to (u_n) a microlocal measure

$$\mu_{ml} = \mu^\infty + \mu_c$$

as in Theorem 1.14. Equation (8.8) implies that

$$\int_0^T \int_{\mathbb{R}^2\times\mathbb{S}^2} b^2(z)\mu^\infty(dz, dH, d\xi, t)dt = 0, \qquad \langle \mu_c, \mathbb{1}_{(0,T)} \otimes b^2\rangle = 0.$$

Lemmas 8.1 and 8.3 imply that $\mu^\infty = 0$ and $\mu_c = 0$ respectively. However, equation (8.7) implies that

$$\langle \mu_{ml}, \mathbb{1}_{(0,T)} \otimes 1\rangle = T.$$

This yields a contradiction and concludes the proof. Note that ($\mathrm{UCP}_{V,\Omega,T}$) has only be used to apply Lemma 8.3 in order to get rid of the term μ_c.

8.2.2. *Observability for time independent potentials.* The structure of the proof in this setting is classical [BLR92, Leb92]. In a first step, we prove the following weakened observability inequality:

$$\left\|u^0\right\|^2_{L^2(\mathbb{D})} \leq C \int_0^T \left\|U_V(t)u^0\right\|^2_{L^2(\Omega)} dt + C \left\|u^0\right\|^2_{H^{-1}(\mathbb{D})}, \tag{8.9}$$

In a second step, we conclude the proof of Theorem 1.2 using a unique continuation property for eigenfunctions of the elliptic operator $-\Delta_D + V$.

The first step is similar to Section 8.2.1. We consider a sequence of initial data (u_n^0) contradicting (8.9). It satisfies (8.7), (8.8), together with

$$\|u_n^0\|_{H^{-1}(\mathbb{D})} \to 0, \quad n \to \infty. \tag{8.10}$$

As before, we consider the associated microlocal measure $\mu_{ml} = \mu^\infty + \mu_c$. Note now that (8.10) implies that $\mu_c = 0$. The rest of the proof is completely similar.

We now prove that (8.9) implies the observability inequality (1.4): this step is by now classical [BLR92, Leb92] but we include it for the sake of completeness. We proceed again by contradiction and suppose that the inequality

$$\|u^0\|_{H^{-1}(\mathbb{D})} \leq C \int_0^T \left\|b(z)U_V(t)u^0\right\|^2_{L^2(\Omega)} dt \tag{8.11}$$

is not satisfied. Then, there exists a sequence $(u_n^0)_{n\in\mathbb{N}}$ in $L^2(\mathbb{D})$ such that

$$\|u_n^0\|_{H^{-1}(\mathbb{D})} = 1, \qquad \int_0^T \left\|b(z)U_V(t)u_n^0\right\|^2_{L^2(\mathbb{D})} dt \to 0, \quad n \to \infty. \tag{8.12}$$

Inequality (8.6) implies that u_n^0 is bounded in $L^2(\mathbb{D})$, so that, after having extracted a subsequence, we have $u_0^n \rightharpoonup u^0$ in $L^2(\mathbb{D})$ and $u_n^0 \to u^0$ in $H^{-1}(\mathbb{D})$. We deduce from (8.12) that

$$\|u^0\|_{H^{-1}(\mathbb{D})} = 1, \qquad U_V(t)u^0 = 0 \text{ on } \{b^2 > 0\} \text{ for all } t \in (0,T).$$

The weak limit u^0 belongs to the set

$$\mathcal{N} = \{f \in L^2(\mathbb{D}), U_V(t)f = 0 \text{ on } \{b^2 > 0\} \text{ for all } t \in (0,T)\}.$$

Then, by linearity, $\mathcal{N}$ is a closed vector subspace of $L^2(\mathbb{D})$. Inequality (8.9) proves that $\mathcal{N}$ is finite dimensional and the time independence of V implies that it is a subspace of $H^2(\mathbb{D}) \cap H_0^1(\mathbb{D})$, stable by the action of the operator $-\Delta_D + V$. If not reduced to $\{0\}$, the space $\mathcal{N}$ hence contains an eigenfunction of $-\Delta_D + V$, vanishing on $\{b^2 > 0\}$. A classical uniqueness result for elliptic operators then implies that this does not occur. This yields $\mathcal{N} = \{0\}$ and thus $u^0 = 0$, which contradicts $\|u^0\|_{H^{-1}(\mathbb{D})} = 1$.

8.3. **Boundary observability inequality: proof of Theorem 1.3.** We proceed as in the previous section: in a first step, we prove the following weakened observability inequality:

Lemma 8.5. *For all $T > 0$, there exists $C > 0$ such that for all $u^0 \in H_0^1(\mathbb{D})$, we have*

$$\left\|u^0\right\|^2_{H^1(\mathbb{D})} \leq C \int_0^T \left\|\partial_n(U_V(t)u^0)\right\|^2_{L^2(\Gamma)} dt + C \left\|u^0\right\|^2_{L^2(\mathbb{D})}, \tag{8.13}$$

With this lemma, we now conclude the proof of the observability inequality (1.5). We proceed by contradiction and suppose that the inequality

$$\|u^0\|_{L^2(\mathbb{D})} \leq \int_0^T \left\|\partial_n(U_V(t)u^0)\right\|^2_{L^2(\Gamma)} dt \tag{8.14}$$

is not satisfied. Then, there exists a sequence $(u_n^0)_{n\in\mathbb{N}}$ in $L^2(\mathbb{D})$ such that

$$\|u_n^0\|_{L^2(\mathbb{D})} = 1, \qquad \int_0^T \left\|\partial_n(U_V(t)u_n^0)\right\|^2_{L^2(\Gamma)} dt \to 0, \quad n\to\infty. \tag{8.15}$$

Then, (8.13) implies that u_n^0 is bounded in $H^1(\mathbb{D})$, so that, after having extracted a subsequence, we have $u_n^0 \rightharpoonup u^0$ in $H_0^1(\mathbb{D})$ and $u_n^0 \to u^0$ in $L^2(\mathbb{D})$. We deduce from (8.15) that

$$\|u^0\|_{L^2(\mathbb{D})} = 1, \qquad \partial_n(U_V(t)u^0) = 0 \text{ on } \Gamma \text{ for all } t\in(0,T).$$

From here, we discuss the two cases with different uniqueness arguments. In the case $V(t,z) = V(z)$, the proof of $u=0$ follows exactly Section 8.2.2 (using unique continuation from the boundary for elliptic operators). The same conclusion holds if we assume ($\mathrm{UCP}_{V,\Gamma,T}$). This contradicts $\|u^0\|_{L^2(\mathbb{D})} = 1$, and proves (8.14). Then, (8.14) and (8.13) imply the sought observability inequality (1.5).

Proof of Lemma 8.5. Assume that (8.13) is not satisfied. Then, there exists a sequence u_n^0 such that

$$\|u_n^0\|^2_{H^1(\mathbb{D})} = 1, \tag{8.16}$$

$$\|u_n^0\|^2_{L^2(\mathbb{D})} \to 0, \tag{8.17}$$

$$\int_0^T \|\partial_n(u_n(t))\|^2_{L^2(\Gamma)} dt \to 0, \tag{8.18}$$

where, as usual, $u_n(t) = U_V(t)u_n^0$. Let us now fix $\chi_T \in C^\infty(\mathbb{R})$ such that $\chi_T = 1$ in a neighbourhood of $[0,T]$, $\psi\in C^\infty(\mathbb{R})$, such that $\psi = 0$ on $(-\infty,1]$ and $\psi = 1$ on $[2,+\infty)$ and set

$$w_n = B(D_t)\chi_T(t)u_n, \qquad B(D_t) = \mathrm{Op}_1(\psi(H)\sqrt{2H}).$$

Lemma 8.6. *We set $A(D_t) = \mathrm{Op}_1\left(\frac{\psi(H)}{\sqrt{2H}}\right)$. For any $R>0$ and $\varepsilon>0$, we have*

$$\left\|\left(D_t + \frac{1}{2}\Delta - V\right) w_n\right\|_{L^2((0,T)\times\mathbb{D})} \to 0 \tag{8.19}$$

$$\|A(D_t)w_n\|_{L^2((-R,R)\times\mathbb{D})} \to 0, \tag{8.20}$$

$$T/2 + o_\varepsilon(1) \leq \|w_n\|^2_{L^2((-\varepsilon,T+\varepsilon)\times\mathbb{D})} \tag{8.21}$$

$$\|w_n\|^2_{L^2((-R,R)\times\mathbb{D})} \leq R + \varepsilon + o_{R,\varepsilon}(1), \tag{8.22}$$

$$\|\partial_n(A(D_t)w_n)\|_{L^2((-R,R)\times\partial\mathbb{D})} \leq C, \tag{8.23}$$

$$\|\partial_n(A(D_t)w_n)\|_{L^2((\varepsilon,T-\varepsilon)\times\Gamma)} \to 0. \tag{8.24}$$

Now, as (w_n) forms a bounded sequence of $L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{D})$, we associate to a subsequence a microlocal measure $\mu_{ml} = \mu^\infty + \mu_c$ as in Section 1.5. According to (8.19) and Remark 1.9 μ_{ml} satisfies the conclusions of Theorem 1.14. The measure μ_c vanishes as a consequence of (8.20). According to (8.23), the sequence $\partial_n(A(D_t)w_n)$ is bounded in $L^2((0,T)\times\mathbb{D})$, so we may again extract another subsequence and associate a microlocal measure μ_{ml}^∂ as in

Section 1.8.3. According to (8.24), $\mu^{\partial}_{ml} = 0$ on $T^*((\varepsilon, T-\varepsilon)\times\Gamma)$. As a consequence of Lemma (8.2), μ^∞ vanishes identically on $\mathbb{R}_t\times\mathbb{R}^2_z\times\mathbb{S}^2_{H,\xi}$. Thus, w_n converges to zero in $L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{D})$, which is contradiction with (8.21). This concludes the proof of Lemma 8.5. □

Proof of Lemma 8.6. Take $\tilde\chi\in C^\infty_c(\mathbb{R})$ such that $\tilde\chi = 1$ on $(0,T)$ and $\chi_T = 1$ on $\mathrm{supp}(\tilde\chi)$. Using that $\left(D_t+\frac12\Delta - V\right)u_n = 0$, we have

$$\begin{aligned}\left\|\left(D_t+\frac12\Delta-V\right)w_n\right\|_{L^2(0,T\times\mathbb{D})} &\le \left\|\tilde\chi\left(D_t+\frac12\Delta-V\right)w_n\right\|_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\le \left\|\tilde\chi B(D_t)\chi_T' u_n\right\|_{L^2(\mathbb{R}\times\mathbb{D})} + \left\|\tilde\chi[V,B(D_t)]\chi_T u_n\right\|_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\le C\left\|\chi_T' u_n\right\|_{L^2(\mathbb{R}\times\mathbb{D})} + C\left\|\chi_T u_n\right\|_{L^2(\mathbb{R}\times\mathbb{D})} \le C\left\|u_n^0\right\|_{L^2(\mathbb{D})}\to 0,\end{aligned}$$

as $\tilde\chi = 0$ on $\mathrm{supp}(\chi_T')$ and $[V,B(D_t)]$ is bounded on $L^2(\mathbb{R}\times\mathbb{D})$. This proves (8.19).

Let us now take $\tilde\chi\in C^\infty_c(\mathbb{R})$ such that $\tilde\chi = 1$ on $(-R,R)$. We have

$$\begin{aligned}\|A(D_t)w_n\|_{L^2((-R,R)\times\mathbb{D})} &\le \left\|\tilde\chi\,\mathrm{Op}_1(\psi^2(H))\chi_T u_n\right\|_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\le \|\chi_T u_n\|_{L^2(\mathbb{R}\times\mathbb{D})}\to 0,\end{aligned}$$

which proves (8.20).

Let us fix now $\check\chi\in C^\infty_c(-\varepsilon,T+\varepsilon)$ such that $\check\chi = 1$ in a neighbourhood of $[0,T]$, and compute

$$\begin{aligned}\|w_n\|^2_{L^2((-\varepsilon,T+\varepsilon)\times\mathbb{D})} \ge \|\check\chi w_n\|^2_{L^2(\mathbb{R}\times\mathbb{D})} &= \langle B(D_t)\check\chi^2 B(D_t)\chi_T u_n,\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\ge \langle\check\chi^2\,\mathrm{Op}_1(\psi^2(H))D_t(\chi_T u_n),\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\quad+\langle[B(D_t),\check\chi^2]B(D_t)\chi_T u_n,\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})},\end{aligned}$$

where $\left|\langle[B(D_t),\check\chi^2]B(D_t)\chi_T u_n,\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| \le \|\chi_T u_n\|_{L^2(\mathbb{R}\times\mathbb{D})}\to 0$. As a consequence, we have

$$\|w_n\|^2_{L^2((-\varepsilon,T+\varepsilon)\times\mathbb{D})} \ge \langle\check\chi^2\,\mathrm{Op}_1(\psi^2(H))D_t(\chi_T u_n),\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1). \tag{8.25}$$

On the other hand, we have

$$\begin{aligned}\langle\check\chi^2 D_t(\chi_T u_n),\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} &= \langle\check\chi^2\,\mathrm{Op}_1(\psi^2(H))D_t(\chi_T u_n),\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\quad+\langle\check\chi^2\,\mathrm{Op}_1(\psi^2(-H))D_t(\chi_T u_n),\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1),\end{aligned}\tag{8.26}$$

where

$$\langle\check\chi^2\,\mathrm{Op}_1(\psi^2(-H))D_t(\chi_T u_n),\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} = \langle\check\chi^2\,\mathrm{Op}_1(\psi^2(-H)H)(\chi_T u_n),\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})}$$

Since $\check\chi^2\psi^2(-H)H\le 0$ the sharp Gårding inequality then provides

$$\langle\check\chi^2\,\mathrm{Op}_1(\psi^2(-H))D_t(\chi_T u_n),\chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} \le \|\chi_T u_n\|^2_{L^2(\mathbb{R}\times\mathbb{D})} = o(1)$$

This, combined with (8.25) and (8.26) now yields

$$\begin{aligned}\|w_n\|^2_{L^2((-\varepsilon,T+\varepsilon)\times\mathbb{D})} &\geq \langle \check{\chi}^2 D_t(\chi_T u_n), \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1)\\ &\geq \langle \check{\chi}^2 \chi_T D_t u_n, \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1)\\ &\geq \langle \check{\chi}^2 \chi_T (-\frac{1}{2}\Delta + V) u_n, \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1)\\ &\geq \frac{1}{2}\langle \check{\chi}^2 \chi_T \nabla u_n, \chi_T \nabla u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1)\\ &\geq \frac{T}{2}\|\nabla u_n^0\|^2_{L^2(\mathbb{D})} + o(1) = \frac{T}{2} + o(1)\end{aligned}$$

This concludes the proof of (8.21). The proof of (8.22) follows the same arguments.

Finally, according to (8.16) and the hidden regularity result of Proposition C.1 the sequence $\partial_n(u_n)$ is bounded in $L^2((-R,R)\times\mathbb{D})$. Moreover, we have

$$\begin{aligned}\|\partial_n(A(D_t)w_n)\|_{L^2((-R,R)\times\partial\mathbb{D})} &\leq \left\|\tilde{\chi}\operatorname{Op}_1(\psi^2(H))\chi_T\partial_n(u_n)\right\|_{L^2(\mathbb{R}\times\partial\mathbb{D})}\\ &\leq \|\chi_T\partial_n(u_n)\|_{L^2(\mathbb{R}\times\partial\mathbb{D})} \leq C.\end{aligned}$$

This proves (8.23). The proof of (8.24) comes from a similar computation combined with (8.18). □

Appendix A. From action-angle coordinates to polar coordinates

Here we develop the (painful) calculations leading to the definitions of the operators $\mathcal{A}_E(P)$ and $\mathcal{A}_H(P)$ used as a black-box in the paper. The point is that our "action-angle" coordinates (s,θ,E,J), well adapted to integrate the dynamics of the billiard flow, are not so convenient to express the Dirichlet boundary condition ($v(z)=0$ for $|z|=1$). Actually the best coordinates in which to write the boundary condition are the polar coordinates (which below will be written as $(x=-r\sin u, y=r\cos u)$) since the boundary is simply expressed as the set $\{r=1\}$.

Let $P(s,\theta,E,J)$ be a function expressed in the new coordinates and let $\mathscr{U}$ be the Fourier integral operator defined in (2.1). The technical calculations done below are aimed at understanding how $\mathscr{U}^*\operatorname{Op}_h(P)\mathscr{U}$ acts in polar coordinates; in particular, under which conditions on the symbol P the boundary condition is preserved by $\mathscr{U}^*\operatorname{Op}_h(P)\mathscr{U}$.

For our purposes we need to understand the operator $\mathscr{U}^*\operatorname{Op}_h(P)\mathscr{U}$ modulo $O(h^2)$. Ideally we would like to separate it into a "tangential part" (involving only angular derivation $\frac{\partial}{\partial u}$) and a "radial part" involving the radial derivative $\frac{\partial}{\partial r}$ in a simple way. Below we calculate the action of the operator $\mathscr{U}^*\operatorname{Op}_h(P)\mathscr{U}$ on a plane wave $e_\xi(z) := e^{i\frac{(\xi_x x+\xi_y y)}{h}}$ (where we use $z=(x,y)$, $\xi=(\xi_x,\xi_y)$ and $|\xi|^2=\xi_x^2+\xi_y^2$) and apply the method of stationary phase. The length of the calculation comes from the fact that we explicitly need the term of order h in the expansion.

Let $P(s,\theta,E,J)$ be a smooth function (possibly depending on h), with support away from $\{E=0\}$ and inside $\{|J|<E\}$. We assume that it satisfies $\|\partial_s^\alpha\partial_\theta^\beta\partial_E^\gamma\partial_J^\delta P\|_\infty \leq C_{\alpha,\beta,\gamma,\delta}h^{-\gamma-\delta}$ for all integers $\alpha,\beta,\gamma,\delta$. The function P may also depend on the time variable t and its dual H, but here we omit them from the notation since they are transparent in the calculation.

Using the definition (2.1) and unfolding all the integrals, we write

$$\begin{aligned}
&\mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U} e_\xi(x,y)\\
&= (2\pi h)^{-5}\int P\left(s,\theta,E',j\right) e^{\frac{ij(\theta-\theta')}{h}} e^{\frac{iE'(s-s')}{h}} e^{-i\frac{S(x',y',\theta',s',E)}{h}} e^{i\frac{(\xi_x x'+\xi_y y')}{h}} e^{i\frac{S(x,y,\theta,s,E'')}{h}}\\
&a(E)a(E'')d\theta dsdE''dx'dy'dEd\theta' ds'dE'dj\\
&= (2\pi h)^{-3}\int P\left(s,\theta,E',j\right) e^{\frac{ij(\theta-\theta_0)}{h}} e^{\frac{iE'(s-s')}{h}} e^{i\frac{s'|\xi|}{h}} e^{i\frac{S(x,y,\theta,s,E'')}{h}} \frac{a(|\xi|)}{|\xi|} a(E'')d\theta dsdE''ds'dE'dj\\
&= (2\pi h)^{-2}\int P\left(s,\theta,|\xi|,j\right) e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{s|\xi|}{h}} e^{i\frac{S(x,y,\theta,s,E'')}{h}} \frac{a(|\xi|)}{|\xi|} a(E'')d\theta dE''dsdj\\
&= (2\pi h)^{-1}\int \left(P\left(s(x,y,\theta),\theta,|\xi|,j\right)a(|\xi|) - ih\partial_s P\left(s(x,y,\theta),\theta,|\xi|,j\right)a'(|\xi|)\right)\\
&e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{-|\xi|\sin\theta x+|\xi|\cos\theta y}{h}} \frac{a(|\xi|)}{|\xi|} d\theta dj \qquad + \frac{O(h^2)}{\inf_{P(s,\theta,E,J)\neq 0}|E|^2},
\end{aligned}$$

where $a(E)=\sqrt{E}$, $(s_0,\theta_0,|\xi|,j_0)=\Phi^{-1}(x,y,\xi_x,\xi_y)$ and $s(x,y,\theta)=-x\sin\theta+y\cos\theta$. By standard estimates on pseudodifferential operators, the remainder term will correspond to an estimate in the $L^2_{\text{comp}}\longrightarrow L^2_{\text{loc}}$ norm of operators.

Let $(x,y)=(-r\sin u, r\cos u)$, so that $r=\sqrt{x^2+y^2}$, $u=\arccos y/r$, and $s(x,y,\theta)=r\cos(\theta-u)$. We are left with

$$\begin{aligned}
(2\pi h)^{-1}\frac{a(|\xi|)}{|\xi|}\int &\left(P\left(r\cos(\theta-u),\theta,|\xi|,j\right)a(|\xi|)\right.\\
&\left.- ih\partial_s P\left(r\cos(\theta-u),\theta,|\xi|,j\right)a'(|\xi|)\right)e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{|\xi|r\cos(\theta-u)}{h}} d\theta dj.
\end{aligned}$$

Now we apply stationary phase w.r.t. θ (while j is kept fixed, since our symbols can be rapidly oscillating in j). We start with the P term. The $ih\partial_s P$-term can be treated exactly the same way. Fixing j and looking at the θ-integral, we let

$$\begin{aligned}
I &= (2\pi h)^{-1/2}\int_0^{2\pi} P\left(r\cos(\theta-u),\theta,|\xi|,j\right) e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{|\xi|r\cos(\theta-u)}{h}} d\theta\\
&= (2\pi h)^{-1/2}\int_{u-\pi/2}^{u+\pi/2}\cdots + (2\pi h)^{-1/2}\int_{u+\pi/2}^{u+3\pi/2}\cdots .
\end{aligned}$$

The phase in I has 2 critical points $\theta=u+\theta_1, u+\theta_2$, where θ_i are the solutions of $j-|\xi|r\sin\theta=0$. Since we are assuming that $P(s,\theta,E,j)$ is supported in $\{|j|<E\}$, these two solutions are distinct for r close to 1, and correspond to non-degenerate stationary points (in all that follows we consider that r is close to 1 since this calculation only serves to understand $\mathscr{U}^*\operatorname{Op}_h(P)\mathscr{U}$ near the boundary of the disk). We will denote by $\theta_1(r,E,j),\theta_2(r,E,j)$ the solutions of $j-Er\sin\theta=0$. To fix ideas, θ_1 will be the one with $\cos\theta_1>0$ and θ_2 the one with $\cos\theta_2<0$ (that is, $\theta_1\in(-\pi/2,\pi/2),\theta_2\in(\pi/2,3\pi/2)$). Below, E will always take the value $E=|\xi|$.

According to the method of stationary phase, the asymptotic expansion of the integral I may be obtained, modulo $O(h^2)$, by replacing P by its Taylor expansion at order 2 at each

critical point:

$$P(r\cos(\theta-u),\theta) \sim P(r\cos\theta_i, u+\theta_i) + (\theta-u-\theta_i)\frac{d}{d\theta}P(r\cos\theta_i, u+\theta_i) + (\theta-u-\theta_i)^2\frac{1}{2}\frac{d^2}{d\theta^2}P(r\cos\theta_i, u+\theta_i). \tag{A.1}$$

(we momentarily drop the j and E variables from the notation since they are fixed in the upcoming calculation).

Remark A.1. *Denoting by $\partial_1 = \partial_s, \partial_2 = \partial_\theta$ (to avoid possible confusion), we have*

$$\frac{d}{d\theta}P(r\cos\theta, u+\theta) = \partial_2 P(r\cos\theta, u+\theta) - r\sin\theta\,\partial_1 P(r\cos\theta, u+\theta),$$

and

$$\frac{d^2}{d\theta^2}P(r\cos\theta, u+\theta) = \partial_2^2 P(r\cos\theta, u+\theta) - r\cos\theta\,\partial_1 P(r\cos\theta, u+\theta) - r\sin\theta\,\partial_2\partial_1 P(r\cos\theta, u+\theta) + r^2\sin^2\theta\,\partial_1^2 P(r\cos\theta, u+\theta).$$

To use integration by parts, it is convenient to rewrite (A.1) using $j - Er\sin(\theta-u)$ (the derivative of the phase) instead of $(\theta-u-\theta_i)$ (where θ_i stands short for $\theta_i(r,E,j)$). Starting with

$$j - Er\sin(\theta-u) \sim Er\left[-\cos\theta_i(\theta-u-\theta_i) + \frac{\sin\theta_i}{2}(\theta-u-\theta_i)^2\right], \tag{A.2}$$

equation (A.1) can be rewritten as

$$P(r\cos(\theta-u),\theta) \sim P(r\cos\theta_i, u+\theta_i) - \frac{(j-Er\sin(\theta-u))}{Er\cos\theta_i}\frac{d}{d\theta}P(r\cos\theta_i, u+\theta_i) + \frac{(j-Er\sin(\theta-u))^2}{(Er\cos\theta_i)^2}\left[\frac{\sin\theta_i}{2\cos\theta_i}\frac{d}{d\theta}P(r\cos\theta_i, u+\theta_i) + \frac{1}{2}\frac{d^2}{d\theta^2}P(r\cos\theta_i, u+\theta_i)\right]. \tag{A.3}$$

Remark A.2. *Important remark about symmetry. We keep denoting θ_i for $\theta_i(r,E,j)$. We first note that $\theta_2 = \pi - \theta_1$, $\cos\theta_1 = -\cos\theta_2, \sin\theta_1 = \sin\theta_2$.*

Moreover, if P satisfies the symmetry condition (B), we have for $r = 1$ (restoring in our notation the dependence of P on the full set of variables)

$$P(\cos\theta_1, u+\theta_1, E, j) = P(\cos\theta_2, u+\theta_2, E, j).$$

And similarly for all partial derivatives of P if we assume the stronger symmetry condition (C).

Here we don't necessarily want to assume that P is symmetric; but, motivated by the previous remark, we introduce the following notation:

$$\begin{aligned} P^\sigma(r,\theta,E,j) &:= \frac{P(r\cos\theta_1, \theta+\theta_1, E, j) + P(-r\cos\theta_1, \theta+\pi-\theta_1, E, j)}{2} \\ &= \frac{P(r\cos\theta_1, \theta+\theta_1, E, j) + P(r\cos\theta_2, \theta+\theta_2, E, j)}{2} \end{aligned} \tag{A.4}$$

$$\begin{aligned} P^\alpha(r,\theta,E,j) &:= \frac{P(r\cos\theta_1,\theta+\theta_1,E,j)-P(-r\cos\theta_1,\theta+\pi-\theta_1,E,j)}{2} \\ &= \frac{P(r\cos\theta_1,\theta+\theta_1,E,j)-P(r\cos\theta_2,\theta+\theta_2,E,j)}{2} \end{aligned} \tag{A.5}$$

for $\theta_1=\theta_1(r,E,j),\theta_2=\theta_2(r,E,j)$ defined previously, so that

$$P(r\cos\theta_1,\theta+\theta_1,E,j)=P^\sigma(r,\theta,E,j)+P^\alpha(r,\theta,E,j),$$

$$P(r\cos\theta_2,\theta+\theta_2,E,j)=P^\sigma(r,\theta,E,j)-P^\alpha(r,\theta,E,j).$$

Modulo $O(h)$ the asymptotic expansion of the integral I looks as follows:

$$\begin{aligned} &(2\pi h)^{-1/2}\int_{u-\pi/2}^{u+\pi/2} P(r\cos\theta_1,u+\theta_1,E,j)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta \\ &\quad+(2\pi h)^{-1/2}\int_{u+\pi/2}^{u+3\pi/2} P(r\cos\theta_2,u+\theta_2,E,j)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta \\ &\quad=(2\pi h)^{-1/2}\int_0^{2\pi} P^\sigma(r,u,E,j)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta \\ &+(2\pi h)^{-1/2}\int_0^{2\pi}\frac{P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)}\left(E\cos(\theta-u)-ih\frac{1}{2r\cos^2\theta_1(r,E,j)}\right)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta \\ &\quad=(2\pi h)^{-1/2}\int_0^{2\pi} P^\sigma(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta \\ &+(2\pi h)^{-1/2}\int_0^{2\pi}\frac{P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)}\left(E\cos(\theta-\theta_0)-ih\frac{1}{2r\cos^2\theta_1(r,E,j)}\right)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta \end{aligned} \tag{A.6}$$

We note that $e^{i\frac{Er\cos(\theta-\theta_0)}{h}}=e^{i\frac{(\xi_x x'+\xi_y y')}{h}}$ if $(x',y')=(-r\sin\theta,r\cos\theta)$, and $E\cos(\theta-\theta_0)e^{i\frac{Er\cos(\theta-\theta_0)}{h}}=hD_re^{i(\xi_x x'+\xi_y y')/h}$ where $D_r=\frac{1}{i}\partial_r$.

Other terms of order h. Apart from the term of order h arising in the last line of (A.6), other terms of order h come from evaluation of the integrals

$$\begin{aligned} &(2\pi h)^{-1/2}\int_{u-\pi/2}^{u+\pi/2}\Bigg[\frac{-(j-Er\sin(\theta-u))}{Er\cos\theta_i}\frac{d}{d\theta}P(u+\theta_1,r\cos\theta_1) \\ &\quad+\frac{(j-Er\sin(\theta-u))^2}{(Er\cos\theta_1)^2}\left(\frac{\sin\theta_1}{2\cos\theta_1}\frac{d}{d\theta}P(u+\theta_1,r\cos\theta_1)+\frac{1}{2}\frac{d^2}{d\theta^2}P(u+\theta_1,r\cos\theta_1)\right)\Bigg] \\ &\qquad e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta \\ &\quad+(2\pi h)^{-1/2}\int_{u+\pi/2}^{u+3\pi/2}\Bigg[\frac{-(j-Er\sin(\theta-u))}{Er\cos\theta_2}\frac{d}{d\theta}P(u+\theta_2,r\cos\theta_2) \\ &\quad+\frac{(j-Er\sin(\theta-u))^2}{(Er\cos\theta_2)^2}\left(\frac{\sin\theta_2}{2\cos\theta_2}\frac{d}{d\theta}P(u+\theta_2,r\cos\theta_2)+\frac{1}{2}\frac{d^2}{d\theta^2}P(u+\theta_2,r\cos\theta_2)\right)\Bigg] \\ &\qquad e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta \end{aligned}$$

After integrations by parts w.r.t. θ (using the fact that $j - Er\sin(\theta - u)$ is the derivative of the phase $j(\theta-\theta_0) + Er\cos(\theta-u)$), this becomes

$$
\begin{aligned}
(\mathrm{A.7}) \quad & (2\pi h)^{-1/2} ih \int_{u-\pi/2}^{u+\pi/2} \frac{1}{(Er\cos\theta_1)^2} Er\cos(\theta-u) \\
& \left[\frac{\sin\theta_1}{2\cos\theta_1}\frac{d}{d\theta}P(u+\theta_1, r\cos\theta_1) + \frac{1}{2}\frac{d^2}{d\theta^2}P(u+\theta_1, r\cos\theta_1)\right] e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{Er\cos(\theta-u)}{h}} d\theta \\
& \qquad + (2\pi h)^{-1/2} ih \int_{u+\pi/2}^{u+3\pi/2} \frac{1}{(Er\cos\theta_2)^2} Er\cos(\theta-u) \\
& \left[\frac{\sin\theta_2}{2\cos\theta_2}\frac{d}{d\theta}P(u+\theta_2, r\cos\theta_2) + \frac{1}{2}\frac{d^2}{d\theta^2}P(u+\theta_2, r\cos\theta_2)\right] e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{Er\cos(\theta-u)}{h}} d\theta \\
& \qquad = (2\pi h)^{-1/2} ih \int_{u-\pi/2}^{u+\pi/2} \frac{1}{(Er\cos\theta_1)} \\
& \left[\frac{\sin\theta_1}{2\cos\theta_1}\frac{d}{d\theta}P(u+\theta_1, r\cos\theta_1) + \frac{1}{2}\frac{d^2}{d\theta^2}P(u+\theta_1, r\cos\theta_1)\right] e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{Er\cos(\theta-u)}{h}} d\theta \\
& \qquad + (2\pi h)^{-1/2} ih \int_{u+\pi/2}^{u+3\pi/2} \frac{1}{(Er\cos\theta_2)} \\
& \left[\frac{\sin\theta_2}{2\cos\theta_2}\frac{d}{d\theta}P(u+\theta_2, r\cos\theta_2) + \frac{1}{2}\frac{d^2}{d\theta^2}P(u+\theta_2, r\cos\theta_2)\right] e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{Er\cos(\theta-u)}{h}} d\theta + O(h^2)
\end{aligned}
$$

Here $\frac{d}{d\theta}P(u+\theta_i, r\cos\theta_i)$ and $\frac{d^2}{d\theta^2}P(u+\theta_i, r\cos\theta_i)$ may be replaced by their expressions in terms of partial derivatives of P, as in Remark A.1.

We summarize our calculations in the following proposition.

Proposition A.3. *Modulo a term of order* $\frac{O(h^2)}{\inf_{P(s,\theta,E,J)\neq 0}|E|^2}$ *in the* $L^2_{\mathrm{comp}} \longrightarrow L^2_{\mathrm{loc}}$*-norm of operators,* $\mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}$ *acts as follows. For* $\xi = (\xi_x, \xi_y)$, $E = |\xi|$ *and* $(x,y) = (-r\sin u, r\cos u)$, *we have*

$$
\begin{aligned}
\mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U} e_\xi(x,y) = & \frac{1}{2\pi h}\int A(r,u,E,j) e^{\frac{ij(u-\theta)}{h}} e_\xi(-r\sin\theta, r\cos\theta) d\theta dj \\
& + \frac{1}{2\pi h}\int B(r,u,E,j) e^{\frac{ij(u-\theta)}{h}} hD_r e_\xi(-r\sin\theta, r\cos\theta) d\theta dj \\
& + \frac{ih}{2\pi h}\int C(r,u,E,j) e^{\frac{ij(u-\theta)}{h}} e_\xi(-r\sin\theta, r\cos\theta) d\theta dj \\
& + \frac{ih}{2\pi h}\int D(r,u,E,j) e^{\frac{ij(u-\theta)}{h}} hD_r e_\xi(-r\sin\theta, r\cos\theta) d\theta dj
\end{aligned}
$$

where

$$
\begin{aligned}
A(r,u,E,j) &= P^\sigma(r,u,E,j) \\
B(r,u,E,j) &= \frac{P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)} \\
C(r,u,E,j) &= -\frac{1}{2E}\partial_s P^\sigma(r,u,E,j) + c^\sigma(r,u,E,j) - \frac{1}{2r\cos^2\theta_1(r,E,j)}\frac{P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)}
\end{aligned}
$$

$$D(r,u,E,j) = -\frac{1}{2E}\frac{\partial_s P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)} + \frac{c^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)} \tag{A.8}$$

with the notation P^σ, P^α of (A.4), (A.5), and where, in addition

$$\begin{aligned} c(s,\theta,E,j) = \frac{1}{(Es)}\Bigg[&\frac{j}{2Es}\left(\partial_2 P(s,\theta,E,j) - \frac{j}{E}\partial_1 P(s,\theta,E,j)\right) \\ &+ \frac{1}{2}\left(\partial_2^2 P(s,\theta,E,j) - s\partial_1 P(s,\theta,E,j) - \frac{j}{E}\partial_2\partial_1 P(s,\theta,E,j) + \frac{j^2}{E^2}\partial_1^2 P(s,\theta,E,j)\right)\Bigg] \end{aligned} \tag{A.9}$$

is calculated so that $c(u+\theta_i, r\cos\theta_i, E, j)$ equals

$$\frac{1}{(Er\cos\theta_i)}\left[\frac{\sin\theta_i}{2\cos\theta_i}\frac{d}{d\theta}P(u+\theta_i, r\cos\theta_i) + \frac{1}{2}\frac{d^2}{d\theta^2}P(u+\theta_i, r\cos\theta_i)\right]$$

(the expression appearing in the last line of (A.7)).

Note that A, B, C, D are real-valued functions if P is.

Appendix B. Commutators

In the following formal calculations, it will be convenient to introduce the following notation. $\mathcal{A}_E(P)$ is the operator whose action on e_ξ at $(-r\sin u, r\cos u)$ is defined by

$$\begin{aligned} &\frac{1}{2\pi h}\int A(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e_\xi(-r\sin\theta, r\cos\theta)d\theta dj \\ &\qquad + \frac{1}{2\pi h}\int B(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}hD_r e_\xi(-r\sin\theta, r\cos\theta)d\theta dj \\ &\qquad + \frac{ih}{2\pi h}\int C(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e_\xi(-r\sin\theta, r\cos\theta)d\theta dj \\ &\qquad + \frac{ih}{2\pi h}\int D(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}hD_r e_\xi(-r\sin\theta, r\cos\theta)d\theta dj \\ &\qquad\qquad =: I_E(P) + II_E(P) + ihIII_E(P) + ihIV_E(P) \end{aligned}$$

where $E = |\xi|$. We have shown that $\mathcal{A}_E(P)$ coincides with $\mathscr{U}^*\operatorname{Op}(P)\mathscr{U}$ modulo $\frac{O(h^2)}{\inf_{P(s,\theta,E,J)\neq 0}|E|^2}$ in the $L^2_{\text{comp}} \longrightarrow L^2_{\text{loc}}$-norm of operators.

We now define $\mathcal{A}_H(P)$ as the operator whose action on e_ξ at $(-r\sin u, r\cos u)$ is defined by

$$\begin{aligned} &\frac{1}{2\pi h}\int A(r,u,\sqrt{2H},j)e^{\frac{ij(u-\theta)}{h}}e_\xi(-r\sin\theta, r\cos\theta)d\theta dj \\ &\qquad + \frac{1}{2\pi h}\int B(r,u,\sqrt{2H},j)e^{\frac{ij(u-\theta)}{h}}hD_r e_\xi(-r\sin\theta, r\cos\theta)d\theta dj \\ &\qquad + \frac{ih}{2\pi h}\int C(r,u,\sqrt{2H},j)e^{\frac{ij(u-\theta)}{h}}e_\xi(-r\sin\theta, r\cos\theta)d\theta dj \\ &\qquad + \frac{ih}{2\pi h}\int D(r,u,\sqrt{2H},j)e^{\frac{ij(u-\theta)}{h}}hD_r e_\xi(-r\sin\theta, r\cos\theta)d\theta dj \\ &\qquad\qquad =: I_H(P) + II_H(P) + ihIII_H(P) + ihIV_H(P) \end{aligned}$$

In other words, in the definition of $\mathcal{A}_E(P)$ we have replaced E with $\sqrt{2H}$ in the symbols. For us, $\mathcal{A}_H(P)$ is a very convenient operator since we have

$$I_H(P) = A(r, u, \sqrt{2hD_t}, hD_u), \qquad III_H(P) = C(r, u, \sqrt{2hD_t}, hD_u)$$

(so that they do not involve any derivative w.r.t. r) and

$$II_H(P) = B(r, u, \sqrt{2hD_t}, hD_u) \circ hD_r, \quad IV_H(P) = D(r, u, \sqrt{2hD_t}, hD_u) \circ hD_r$$

which is is only of degree 1 w.r.t. r.

We would like to use everywhere $\mathcal{A}_H(P)$ instead of $\mathcal{A}_E(P)$. This is possible thanks to the following lemma:

Lemma B.1. *If u_h is a solution to the Schrödinger equation (1.1) satisfying in addition the assumptions of Remark 1.7, then we have*

$$\langle u_h, \mathcal{A}_E(P)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R})} - \langle u_h, \mathcal{A}_H(P)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R})} = O\Big(h\|\partial_E P\|_\infty (h \inf_{P(s,\theta,E,J)\neq 0} |E|)^{-1/2-\epsilon}\Big).$$

For instance if P has bounded derivatives and $\inf_{P(s,\theta,E,J)\neq 0}|E|$ is bounded away from 0 independently of h, the error above is $O(h^{1/2-\epsilon})$.

The goal of this section is to calculate explicitly (in terms of P) the expression of the commutator $[\Delta, \mathcal{A}_H(P)]$, where Δ is the laplacian on $\mathbb{R}^2$. This could, in principle, be done by brutal calculation, using the expression of the laplacian in polar coordinates ($\Delta_{r,u} = \frac{\partial^2}{\partial_r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial_u^2}$). But this is too cumbersome and we try a less frontal approach. We want to use the fact that $[\Delta, \mathcal{A}_E(P)]$ is known (from the exact Egorov theorem, equation (B.1) below) and to see how the calculus is modified when we replace $\mathcal{A}_E(P)$ by $\mathcal{A}_H(P)$.

Recall from Lemma 2.1 and formula (1.16) that we have the exact formula (without remainder term)

$$\left[-\frac{ih\Delta}{2}, \mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}\right] = \mathscr{U}^* \operatorname{Op}_h\left(E\partial_1 b - \frac{ih}{2}\partial_1^2 b\right)\mathscr{U} \tag{B.1}$$

B.1. **Formal calculation of $[\Delta, \mathcal{A}_E(P)]$.** We use the expression of ∇ in polar coordinates: $\nabla = (\partial_r, r^{-1}\partial_u)$ in the orthonormal frame (e_r, e_u). We also use the formula $\Delta(fg) = f\Delta g + 2\nabla f\cdot\nabla g + g\Delta f$. We obtain the following expression of $[\Delta, I_E(P)]$ applied to e_ξ at $(x, y) = (-r\sin u, r\cos u)$:

$$\begin{aligned}(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2} &\Delta_{r,u}A(r,u,E,j)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj \\ &+ \frac{2i}{h}(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2}\partial_r A(r,u,E,j)E\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj \\ &+ \frac{2i}{h}(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2} r^{-2}\partial_u A(r,u,E,j)je^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\end{aligned} \tag{B.2}$$

Note that the details of the calculations are actually not important, we only need to know "what the calculations look like" at a formal level (in particular, small errors of calculation are harmless).

Similarly, $[\Delta, II_E(P)]$ has the expression

$$
\begin{aligned}
(\mathrm{B.3})\quad & (2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2}\Delta_{r,u}B(r,u,E,j)E\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&+\frac{2i}{h}(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2}\partial_r B(r,u,E,j)(E\cos(\theta-u))^2e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&+\frac{2i}{h}(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2}r^{-2}\partial_u B(r,u,E,j)jE\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&=(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2}\Delta_{r,u}B(r,u,E,j)E\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&+\frac{2i}{h}(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2}\partial_r B(r,u,E,j)[(E\cos(\theta_1))^2+ih\frac{\cos\theta_1}{(Er)^2}]e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&+\frac{2i}{h}(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2}r^{-2}\partial_u B(r,u,E,j)jE\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj+O(h^2)
\end{aligned}
$$

Similar calculations can be done for $[\Delta, III_E(P)]$ and $[\Delta, IV_E(P)]$. We do not need the explicit expressions, but need only to note that it gives a final expression of $[-ih\Delta/2, \mathcal{A}_E(P)]$ applied to e_ξ at $(-r\sin u, r\cos u)$ in the form:

$$
\begin{aligned}
&\frac{1}{2\pi h}\int K(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj\\
&\qquad+\frac{1}{2\pi h}\int L(r,u,E,j)E\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj\\
&\qquad\qquad+\frac{ih}{2\pi h}\int M(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj\\
&\qquad+\frac{ih}{2\pi h}\int N(r,u,E,j)E\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj\\
&+(2\pi h)^{-1}\int\partial_r B(r,u,E,j)\left[(E\cos(\theta_1))^2+ih\frac{\cos\theta_1}{(Er)^2}\right]e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&\qquad\qquad+(2\pi h)^{-1}\int ih\partial_r D(r,u,E,j)(E\cos(\theta_1))^2e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj
\end{aligned}
$$

Note that the two last lines may obviously be incorporated into the previous terms; but we shall see later why it is convenient to keep them separate.

The functions K, L, M, N are partial differential operators applied to A, B, C, D, and could in principle be expressed explicitly in terms of P, but we actually do not need these expressions.

B.2. **Identification.** We know from (B.1) that $[-\frac{ih\Delta}{2}, \mathscr{U}^*\operatorname{Op}_h(P)\mathscr{U}] = \mathscr{U}^*\operatorname{Op}_h(E\partial_1 P - \frac{ih}{2}\partial_1^2 P)\mathscr{U} = \mathcal{A}_E(E\partial_1 P - \frac{ih}{2}\partial_1^2 P) + O(h^2)$.

Using the identification lemma B.2 below, this leads directly to the identifications:

$$
\begin{aligned}
K(r,u,E,j)+\partial_r B(r,u,E,j)(E\cos(\theta_1))^2 &= A_{E\partial_1 P}\\
L(r,u,E,j) &= B_{E\partial_1 P}\\
M(r,u,E,j)+\partial_r B(r,u,E,j)\frac{\cos\theta_1}{(Er)^2}+\partial_r D(r,u,E,j)(E\cos(\theta_1))^2 &= C_{E\partial_1 P}-\frac{1}{2}A_{\partial_1^2 P}
\end{aligned}
$$

$$N(r,u,E,j) = D_{E\partial_1 P} - \frac{1}{2} B_{\partial_1^2 P}$$

where $\theta_1 = \theta_1(r,E,j)$ denotes as before the solution in $[-\pi/2, \pi/2)$ of $\sin\theta_1 = j/Er$. On the right-hand sides, notation such as $A_{E\partial_1 P}$, $B_{E\partial_1 P}$ etc. means "the functions A, B etc. associated to $E\partial_1 P$ by the formulas of Proposition A.3".

To justify these identifications we are using the following:

Lemma B.2. *Let A and B be two* smooth *real-valued functions. Then the values of*

$$\frac{1}{2\pi h}\int A(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj + \frac{1}{2\pi h}\int B(r,u,E,j)\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj \tag{B.4}$$

for all r, u, θ_0, E determine A and B uniquely.

Proof. Integrating (B.4) along $e^{in\theta_0}d\theta_0$ ($\theta_0 \in [0, 2\pi]$, n an arbitrary integer) yields the value

$$\int A(r,u,E,nh)e^{in(u-\theta)}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta + \int B(r,u,E,nh)\cos(\theta-\theta_0)e^{in(u-\theta)}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta \tag{B.5}$$

If we take $n = n(h)$ a family of even integers growing like $1/h$, application of the method of stationary phase yields that this is (up to $O(h)$)

$$2e^{inu}(2\pi h)^{1/2}[\sin^{1/2}\theta_1\, A(r,u,E,hn(h)\cos(-n\theta_1 + Erh^{-1}\cos\theta_1 + \pi/4) + iB(r,u,E,hn(h))\sin(-n\theta_1 + Erh^{-1}\cos\theta_1 + \pi/4)] \tag{B.6}$$

where θ_1 is the solution in $[-\pi/2, \pi/2)$ of $\sin\theta_1 = \frac{hn(h)}{Er}$. If A and B are continuous and real-valued then (B.6) suffices to determine A and B. □

B.3. **Formal calculation of $[\Delta, \mathcal{A}_H(P)]$.** We want to use the previous identities to find the formal expression of $[\Delta, \mathcal{A}_H(P)]$. Remember that $\mathcal{A}_H(P)$ is the operator we want to use in all our proofs, because it comes naturally into a "tangential" part and a "radial" part of degree 1.

If we compare the formal calculations leading to the expressions of $[\Delta, \mathcal{A}_E(P)]$ and $[\Delta, \mathcal{A}_H(P)]$, we see that they are identical and thus $[-ih\Delta/2, \mathcal{A}_H(P)]$ applied to e_ξ at $(-r\sin u, r\cos u)$

has the form

$$\frac{1}{2\pi h}\int K(r,u,\sqrt{2H},j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj$$
$$+\frac{1}{2\pi h}\int L(r,u,\sqrt{2H},j)E\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj$$
$$+\frac{ih}{2\pi h}\int M(r,u,\sqrt{2H},j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj$$
$$+\frac{ih}{2\pi h}\int N(r,u,\sqrt{2H},j)E\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj$$
$$+(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2}\partial_r B(r,u,\sqrt{2H},j)\left[(E\cos(\theta_1))^2+ih\frac{\cos\theta_1}{(Er)^2}\right]e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj$$
$$+(2\pi h)^{-1}\int_{u-\pi/2}^{u+\pi/2}ih\partial_r D(r,u,\sqrt{2H},j)(E\cos(\theta_1))^2e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj$$

Note that $\theta_1=\theta_1(r,E,j)$ and that the symbol in the last two lines still depends on E (this is why we treat it separately). Everywhere else in the symbol, E has been replaced by $\sqrt{2H}$. Note also that $(E\cos(\theta_1))^2=E^2-\frac{j^2}{r^2}$.

From this and from the identifications of Section B.2, we deduce the final formula

Proposition B.3. *There exists a function $R(r,u,E,\sqrt{2H},j)$ such that*

$$[-ih\Delta/2,\mathcal{A}_H(P)]=\mathcal{A}_H(E\partial_1 P)-\frac{ih}{2}\mathcal{A}_H(\partial_1^2 P)+O(h^2) \tag{B.7}$$
$$+\partial_r B(r,u,\sqrt{2hD_t},hD_u)\circ(-h^2\Delta-2hD_t)+ihR(r,u,\sqrt{-h^2\Delta},\sqrt{2hD_t},hD_u)\circ(-h^2\Delta-2hD_t)$$

Proof. Indeed, the identifications of Section B.2 yield

$$[-ih\Delta/2,\mathcal{A}_H(P)]=I_{\sqrt{2H}}(E\partial_1 P)+II_{\sqrt{2H}}(E\partial_1 P) \tag{B.8}$$
$$+ih\left(III_{\sqrt{2H}}(E\partial_1 P)-1/2I_{\sqrt{2H}}(\partial_1^2 P)\right)$$
$$+ih\left(IV_{\sqrt{2H}}(E\partial_1 P)-1/2II_{\sqrt{2H}}(\partial_1^2 P)\right)$$
$$+\partial_r B(r,u,\sqrt{2hD_t},hD_u)\circ(-h^2\Delta-2hD_t)+ihR(r,u,\sqrt{-h^2\Delta},\sqrt{2hD_t},hD_u)\circ(-h^2\Delta-2hD_t)$$

where the function R is defined by the identity $R(r,u,E,\sqrt{2H},j)(E^2-2H)$ is

$$R(r,u,E,\sqrt{2H},j)(E^2-2H)=\partial_r B(r,u,\sqrt{2H},j)\left[\frac{\cos\theta_1(r,E,j)}{(Er)^2}-\frac{\cos\theta_1(r,\sqrt{2H},j)}{2Hr^2}\right]$$

We can apply a simple division lemma (actually the Taylor integral formula) to write

$$\frac{\cos\theta_1(r,E,j)}{(Er)^2}-\frac{\cos\theta_1(r,\sqrt{2H},j)}{2Hr^2}=S(r,u,E,j,\sqrt{2H})(E^2-2H),$$

and thus

$$R(r,u,E,\sqrt{2H},j)=\partial_r B(r,u,\sqrt{2H},j)S(r,u,E,j,\sqrt{2H}).$$

□

Appendix C. Regularity of boundary data and consequences

We recall the following classical "hidden regularity" of the boundary data of solutions of (1.1):

Proposition C.1. *For every $T > 0$ there exists a constant $C > 0$ such that, for every $u^0 \in H_0^1(\mathbb{D})$ and every $f \in L^1(0,T;H_0^1(\mathbb{D}))$, the solution $u \in C^0([0,T];H_0^1(\mathbb{D}))$ of*

$$\begin{aligned} \frac{1}{i}\frac{\partial u}{\partial t} &= \left(-\frac{1}{2}\Delta + V\right)u + f \quad , t \in \mathbb{R}, \quad z \in \mathbb{D}, \\ u\rceil_{\partial\mathbb{D}} &= 0 \\ u\rceil_{t=0} &= u^0 \end{aligned}$$

satisfies

$$\|\partial_n u\|_{L^2((0,T)\times\partial\mathbb{D})} \leq C\left(\|\nabla u^0\|_{L^2(\mathbb{D})} + \|f\|_{L^1(0,T;H^1(\mathbb{D}))}\right). \tag{C.1}$$

We refer to [Leb92, p. 284] or [GL93, Lemma 2.1] for a proof.

Remark C.2. *Because we have*

$$\frac{d}{dt}\left\langle\left(-\frac{\Delta}{2}+V\right)U_V(t)u^0, U_V(t)u^0\right\rangle = \left\langle \partial_t V\ U_V(t)u^0, U_V(t)u^0\right\rangle,$$

we see that there exists C (depending on T, $\|V\|_\infty$ and $\|\partial_t V\|_\infty$ such that

$$T\left\|\nabla u^0\right\|^2_{L^2(\mathbb{D})} \leq C\int_0^T \|U_V(t)u^0\|^2_{L^2(\mathbb{D})}dt + \int_0^T \left\|\nabla U_V(t)u^0\right\|^2_{L^2(\mathbb{D})}dt \tag{C.2}$$

$$\int_0^T \left\|\nabla U_V(t)u^0\right\|^2_{L^2(\mathbb{D})}dt \leq T\left\|\nabla u^0\right\|^2_{L^2(\mathbb{D})}dt + C\int_0^T \|U_V(t)u^0\|^2_{L^2(\mathbb{D})}dt \tag{C.3}$$

hence

$$T\left\|\nabla u^0\right\|^2_{L^2(\mathbb{D})} - CT\|u^0\|^2_{L^2(\mathbb{D})} \leq \int_0^T \left\|\nabla U_V(t)u^0\right\|^2_{L^2(\mathbb{D})}dt \leq T\left\|\nabla u^0\right\|^2_{L^2(\mathbb{D})}dt + CT\|u^0\|^2_{L^2(\mathbb{D})}$$

Proposition C.1 has the following consequences, which are used everywhere in the paper.

Let A be an operator. For $T > 0$ let $\chi_T(t)$ be a smooth compactly supported function on $\mathbb{R}$ such that $|\chi_T(t)| \leq 1$, taking the value 0 outside $(0,T)$. Obviously, for every $u \in H_0^1(\mathbb{D})$, for every $s > 0$, we can write

$$\begin{aligned} &\left\langle \chi_T U_V(t)u^0, A\,\chi_T\left(\partial_n\left(U_V(t)u^0\right)\otimes\delta_{\partial\mathbb{D}}\right)\right\rangle_{L^2(\mathbb{D}\times\mathbb{R})} \\ &\leq \|A\|_{L^2((0,T),H^{-s}(\mathbb{R}^2))\longrightarrow L^2((0,T)\times\mathbb{R}^2)}\,\|U_V(t)u^0\|_{L^2((0,T)\times\mathbb{D})}\,\|\partial_n U_V(t)u^0\otimes\delta_{\partial\mathbb{D}}\|_{L^2((0,T),H^{-s}(\mathbb{R}^2))}. \end{aligned} \tag{C.4}$$

For any $s > 1/2$, the standard trace estimates (see for instance [CP82, Chapter 2, Section 4]) imply that

$$\|\partial_n U_V(t)u^0\otimes\delta_{\partial\mathbb{D}}\|_{L^2((0,T),H^{-s}(\mathbb{R}^2))} \leq \left\|\partial_n\left(U_V(t)u^0\right)\right\|_{L^2((0,T)\times\partial\mathbb{D})}$$

and by Proposition C.1 this is bounded by $C\left\|\nabla u^0\right\|_{L^2(\mathbb{D})}$.

Let now be $\epsilon > 0$ (in the context of this paper, think of ϵ as being h or R^{-1}). In the previous inequality we take $A = \mathrm{Op}_1(a(z,\epsilon\xi,t,\epsilon^2 H)) = a(z,\epsilon D_z,t,\epsilon^2 D_t)$ where a is smooth,

compactly supported in all variables, and is such that $\chi_T(t) \equiv 1$ on the support of a. We note then that $\chi_T A = A$ and $A\chi_T = A + O(\epsilon^{-\infty})$. Using the fact that

$$\|A\|_{L^2((0,T),H^{-s}(\mathbb{R}^2)) \longrightarrow L^2((0,T)\times\mathbb{R}^2)} \leq C(T,a)\epsilon^{-s},$$

we obtain

(C.5)
$$\frac{\epsilon^2}{2}\left\langle \chi_T U_V(t)u^0, A\left(\partial_n\left(U_V(t)u^0\right)\otimes\delta_{\partial\mathbb{D}}\right)\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \leq C(T,a)\epsilon^{1-s}\|u^0\|_{L^2(\mathbb{D})}\left\|\epsilon\nabla u^0\right\|_{L^2(\mathbb{D})}.$$

If we extend $U_V(t)u^0$ to take the value 0 outside $\mathbb{D}$, we have already noted that

$$\frac{\epsilon^2}{2}\partial_n\left(U_V(t)u^0\right)\otimes\delta_{\partial\mathbb{D}} = \epsilon^2\left(-\frac{\Delta}{2}+V+i\partial_t\right)U_V(t)u^0$$

and thus equation (C.5) (always for $A = \mathrm{Op}_1(a(z,\epsilon\xi,t,\epsilon^2 H))$ may be rewritten as

$$\left\langle \chi_T U_V(t)u^0, A\left(-\frac{\epsilon^2\Delta}{2}+\epsilon^2 V+i\epsilon^2\partial_t\right)U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \leq C(T,a)\epsilon^{1-s}\|u^0\|_{L^2(\mathbb{D})}\left\|\epsilon\nabla u^0\right\|_{L^2(\mathbb{D})}. \tag{C.6}$$

As a consequence

$$\begin{aligned}\left\langle \chi_T U_V(t)u^0, \mathrm{Op}_1\left(a(z,\epsilon\xi,t,\epsilon^2 H)\left(\frac{\epsilon^2|\xi|^2}{2}-\epsilon^2 H\right)\right)U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\ \leq C(T,a)\epsilon^{1-s}\|u^0\|_{L^2(\mathbb{D})}\left\|\epsilon\nabla u^0\right\|_{L^2(\mathbb{D})} + \epsilon^2\left|\int \chi_T(t)\left\langle V U_V(t)u^0, U_V(t)u^0\right\rangle_{L^2(\mathbb{D})}dt\right|.\end{aligned} \tag{C.7}$$

Finally, if a is supported away from $\{H=0\}$ this implies

$$\begin{aligned}\left\langle \chi_T U_V(t)u^0, \mathrm{Op}_1\left(a(z,\epsilon\xi,t,\epsilon^2 H)\left(\frac{|\xi|^2}{2H}-1\right)\right)U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\ \leq C(T,a)\epsilon^{1-s}\|u^0\|_{L^2(\mathbb{D})}\left(\left\|\epsilon\nabla u^0\right\|_{L^2(\mathbb{D})}+\|u^0\|_{L^2(\mathbb{D})}\right).\end{aligned} \tag{C.8}$$

Since a is compactly supported with respect to H, let us now show that we can actually replace $\left\|\epsilon\nabla u^0\right\|_{L^2(\mathbb{D})}$ by $\|u^0\|_{L^2(\mathbb{D})}$ in the last estimate (up to a constant depending on the support of a). The argument is easy if V does not depend on time, but requires some care for time-dependent V. To see that, introduce a compactly supported function g on $\mathbb{R}$ such that $a(z,\xi,t,H)g(H) = a(z,\xi,t,H)$ (i.e. $g=1$ in a neighborhood of $\mathrm{supp}(a)$).

Let

$$w(t) = g(\epsilon^2 D_t)U_V(t)u^0, \quad \text{and} \quad w^0 = w\rceil_{t=0}. \tag{C.9}$$

If V does not depend on time, we have

$$w = U_V(t)g\left(-\frac{\epsilon^2\Delta_D}{2}+\epsilon^2 V\right)u^0, \quad \text{and} \quad w^0 = g\left(-\frac{\epsilon^2\Delta_D}{2}+\epsilon^2 V\right)u^0$$

so we can replace u^0 by $\tilde{u}^0 = g\left(-\frac{\epsilon^2\Delta}{2}+\epsilon^2 V\right)u^0$ in the previous argument, and it is obvious that $\left\|\epsilon\nabla\tilde{u}^0\right\|_{L^2(\mathbb{D})} \leq C(g)\|u^0\|_{L^2(\mathbb{D})}$.

For general V, we write $[g(\epsilon^2 D_t), V] = O(\epsilon^2)\|\partial_t V\|_\infty$, where the estimate of the remainder holds in the operator norm from $L^\infty_{\text{comp}}(\mathbb{R}_t, L^2(\mathbb{D}))$ to $L^\infty_{\text{loc}}(\mathbb{R}_t, L^2(\mathbb{D}))$. We have

$$\frac{1}{i}\frac{\partial}{\partial t} w = \left(-\frac{\Delta}{2} + V\right) w + [g(\epsilon^2 D_t), V] w.$$

Thus

$$\begin{aligned}
\text{(C.10)} \qquad & \left\langle \chi_T U_V(t) u^0, \mathrm{Op}_1\left(a(z, \epsilon\xi, t, \epsilon^2 H)\left(\frac{\epsilon^2|\xi|^2}{2} - \epsilon^2 H\right)\right) U_V(t) u^0 \right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\
&= \left\langle \chi_T U_V(t) u^0, \mathrm{Op}_1\left(a(z, \epsilon\xi, t, \epsilon^2 H)\left(\frac{\epsilon^2|\xi|^2}{2} - \epsilon^2 H\right)\right) w \right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\
&= \left\langle \chi_T U_V(t) u^0, \mathrm{Op}_1\left(a(z, \epsilon\xi, t, \epsilon^2 H)\left(\frac{\epsilon^2|\xi|^2}{2} - \epsilon^2 H\right)\right) U_V(t) w^0 \right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\
&\quad + O_T(\epsilon^2)\|\partial_t V\|_\infty \|u^0\|^2_{L^2(\mathbb{D})},
\end{aligned}$$

where w^0 is defined in C.9.

We know from the bilinear version of inequality (C.7) that

$$\begin{aligned}
\text{(C.11)} \quad & \left\langle \chi_T U_V(t) u^0, \mathrm{Op}_1\left(a(x, \epsilon\xi, t, \epsilon^2 H)\left(\frac{\epsilon^2|\xi|^2}{2} - \epsilon^2 H\right)\right) U_V(t) w^0 \right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\
&\le C(T,a)\epsilon^{1-s}\|u^0\|_{L^2(\mathbb{D})} \left\|\epsilon\nabla w^0\right\|_{L^2(\mathbb{D})} + \epsilon^2 \left|\int \chi_T(t) \left\langle V U_V(t) u^0, U_V(t) w^0\right\rangle_{L^2(\mathbb{D})} dt\right| \\
&\le C(T,a)\epsilon^{1-s}\|u^0\|_{L^2(\mathbb{D})} \left\|\epsilon\nabla w^0\right\|_{L^2(\mathbb{D})} + \epsilon^2 C T \|u^0\|^2_{L^2(\mathbb{D})}.
\end{aligned}$$

Now we can use (C.2)-(C.3) in the form

$$\begin{aligned}
T\left\|\nabla w^0\right\|^2_{L^2(\mathbb{D})} &\le CT(1+\epsilon^2)\|u^0\|^2_{L^2(\mathbb{D})} + \int_0^T \|\nabla w(t)\|^2_{L^2(\mathbb{D})} \\
&\le CT(1+\epsilon^2)\|u^0\|^2_{L^2(\mathbb{D})} + 2\int_0^T \left\langle \left(-\frac{\Delta}{2} + V\right) w(t), w(t)\right\rangle_{L^2(\mathbb{D})} dt \\
&\le CT(1+\epsilon^2)\|u^0\|^2_{L^2(\mathbb{D})} + 2\left|\int_0^T \langle D_t w(t), w(t)\rangle_{L^2(\mathbb{D})} dt\right| \\
&\le CT(1+\epsilon^2)\|u^0\|^2_{L^2(\mathbb{D})} + 2\left|\int_0^T \left\langle D_t g(\epsilon^2 D_t) u(t), g(\epsilon^2 D_t) u(t)\right\rangle_{L^2(\mathbb{D})} dt\right| \\
&\le CT(1+\epsilon^2)\|u^0\|^2_{L^2(\mathbb{D})} + CT\epsilon^{-2}\|u^0\|^2_{L^2(\mathbb{D})}
\end{aligned}$$

This finally shows that the term $\left\|\epsilon\nabla w^0\right\|_{L^2(\mathbb{D})}$ on the right-hand side of (C.11) is bounded by $CT\|u^0\|_{L^2(\mathbb{D})}$. Finally, what we have shown is that, for $a(x,\xi,t,H)$ compactly supported, and supported away from $H=0$, we have

$$\begin{aligned}
\text{(C.12)} \quad & \left\langle \chi_T U_V(t) u^0, \mathrm{Op}_1\left(a(z, \epsilon\xi, t, \epsilon^2 H)\left(\frac{|\xi|^2}{2H} - 1\right)\right) U_V(t) u^0 \right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\
&\le C(T,a)\epsilon^{1-s}\|u^0\|^2_{L^2(\mathbb{D})}
\end{aligned}$$

for all $u^0 \in L^2(\mathbb{D})$ (that is to say, we do not need to assume a priori that u^0 is ϵ-oscillating).

Similarly, if a is supported away from $\xi = 0$ and compactly supported, we have

$$\left\langle \chi_T U_V(t)u^0, \mathrm{Op}_1\left(a(z,\epsilon\xi,t,\epsilon^2 H)\left(\frac{2H}{|\xi|^2}-1\right)\right)U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \leq C(T,a)\epsilon^{1-s}\|u^0\|^2_{L^2(\mathbb{D})} \tag{C.13}$$

Remark C.3. *Let us summarize the properties of the function $w^0 = g(\epsilon^2 D_t)U_V(t)u^0\rceil_{t=0}$.*

- *For fixed $\epsilon > 0$, the operator $u \mapsto w^0$ is compact on $L^2(\mathbb{D})$ since we have proved $\|\epsilon\nabla w^0\|_{L^2(\mathbb{D})} \leq C\|u\|_{L^2(\mathbb{D})}$.*
- *By continuity of $t \mapsto U_V(t)u$, we have $w^0 \underset{L^2(\mathbb{D})}{\longrightarrow} u$ as $\epsilon \longrightarrow 0$.*
- *For compactly supported a, we have*

$$\begin{aligned}\left\langle U_V(t)w^0, \mathrm{Op}_1\left(a(x,\epsilon\xi,t,\epsilon^2 H)\right)U_V(t)w^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \\ = \left\langle U_V(t)u^0, \mathrm{Op}_1\left(a(x,\epsilon\xi,t,\epsilon^2 H)\right)U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} + o(1)_\epsilon\|u^0\|^2_{L^2(\mathbb{D})}\end{aligned}$$

Finally, we prove by dyadic decomposition a statement similar to (C.12) and (C.13) for homogeneous functions:

Proposition C.4. *Let $a(z,\xi,t,H)$ be a smooth function, compactly supported in (z,t), and with the following homogeneity property : there exist $R_0 > 0$ such that*

$$a(z,\xi,t,H) = a(z,\lambda\xi,t,\lambda^2 H), \quad \text{for } |\xi|^2 + |H| > R_0 \text{ and } \lambda \geq 1.$$

Fix $s \in (1/2, 1)$.

If a vanishes in a neighbourhood of the set $\{|\xi|^2 = 2H\}$ then for R large enough

$$\left\langle U_V(t)u^0, \mathrm{Op}_1\left(a(z,\xi,t,H)\left(1-\chi\left(\frac{|\xi|^2+|H|}{R^2}\right)\right)\right)U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \leq C(T,a)R^{s-1}\|u^0\|^2_{L^2(\mathbb{D})} \tag{C.14}$$

Proof. To see that, decompose

$$a(z,\xi,t,H)\left(1-\chi\left(\frac{|\xi|^2+|H|}{R^2}\right)\right) = \sum_{k=0}^{\infty} a(z,2^{-k}R^{-1}\xi,t,2^{-2k}R^{-2}H)\left(\chi\left(\frac{|\xi|^2+|H|}{2^{2(k+1)}R^2}\right)-\chi\left(\frac{|\xi|^2+|H|}{2^{2k}R^2}\right)\right).$$

For each k in the sum above, decompose further

$$\begin{aligned}\chi\left(\frac{|\xi|^2+|H|}{2^{2(k+1)}R^2}\right)-\chi\left(\frac{|\xi|^2+|H|}{2^{2k}R^2}\right) \\ = \left(\chi\left(\frac{|\xi|^2+|H|}{2^{2(k+1)}R^2}\right)-\chi\left(\frac{|\xi|^2+|H|}{2^{2k}R^2}\right)\right)\left(\chi\left(\frac{H}{2^{2k-1}R^2}\right)+(1-\chi)\left(\frac{H}{2^{2k-1}R^2}\right)\right)\end{aligned}$$

and note that we must have $|\xi|^2 \geq 2^{2k-1}R^2$ or $H \geq 2^{2k-1}R^2$ on the support of this function.

If a vanishes in a neighbourhood of the set $\{|\xi|^2 = 2H\}$, we can write

$$a(z,\xi,t,H) = b(z,\xi,t,H)\left(\frac{2H}{|\xi|^2}-1\right)$$

where $|\xi|^2 \geq 2^{2k-1}R^2$ and

$$a(z,\xi,t,H) = b(z,\xi,t,H)\left(\frac{|\xi|^2}{2H} - 1\right)$$

where $H \geq 2^{2k-1}R^2$. Applying (C.12) and (C.13) for each k (with $\epsilon = 2^{-k}R^{-1}$), we finally obtain

$$\left\langle U_V(t)u^0, \mathrm{Op}_1\left(a(z,\xi,t,H)\left(1-\chi\left(\frac{|\xi|^2+|H|}{R^2}\right)\right)\right)U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \leq C(T,a)\sum_{k=0}^{+\infty} R^{s-1}2^{k(s-1)}\|u^0\|^2_{L^2(\mathbb{D})} \tag{C.15}$$

which proves the proposition. □

To conclude this section, we give a proof of Lemma 1.18.

Proof of Lemma 1.18. The operator $A(D_t)\varphi$ is bounded on $L^2(\mathbb{R}\times\mathbb{D})$, so

$$\begin{aligned}\|\nabla A(D_t)\varphi u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} &= \langle -\Delta A(D_t)\varphi u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\\ &= \langle A(D_t)\varphi(-\Delta)u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\quad + \langle[-\Delta, A(D_t)\varphi]u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}.\end{aligned} \tag{C.16}$$

One the one hand, we have

$$\begin{aligned}\left|\langle[-\Delta, A(D_t)\varphi]u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| &= \left|-\langle 2\nabla\varphi\cdot\nabla u + u\Delta\varphi, A(D_t)^2\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right|\\ &\leq 2\left|\langle u, div\left\{\nabla\varphi\left(A(D_t)^2\varphi u\right)\right\}\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| + C\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\leq 2\left|\langle u, \nabla\varphi\cdot\nabla\left(A(D_t)^2\varphi u\right)\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| + C\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\leq \varepsilon\|\nabla A(D_t)\varphi u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} + C(1+\varepsilon^{-1})\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})}\end{aligned}$$

for some $\tilde{\varphi}$ equal to one on the support of φ, for all $\varepsilon > 0$.

On the other hand, since u solves (1.1), we have

$$\begin{aligned}\left|\langle A(D_t)\varphi(-\Delta)u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| &= \left|\langle A(D_t)\varphi(2D_t - V)u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right|\\ &\leq \left|\langle A(D_t)\varphi(2D_t - V)u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right|\\ &\leq 2\left|\langle A(D_t)^2\varphi D_t u, \varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| + C\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\leq 2\left|\langle A(D_t)^2 D_t\varphi u, \varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| + C\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\leq C\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})},\end{aligned}$$

since $A(D_t)^2 D_t = 1/2\,\mathrm{Op}_1(\psi^2(H))$ is bounded. Collecting these estimates in (C.16), recalling that $\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} \leq C\|u^0\|^2_{L^2(\mathbb{D})}$, and taking ε sufficiently small concludes the proof of Lemma 1.18. □

Appendix D. Time regularity of Wigner measures

In this section we present a proof of the following (general) result on time regularity of semiclassical measures associated to solutions of the Schrödinger equation (1.1). Even if not stated here, its microlocal counterpart also holds.

Proposition D.1. *Let μ_{sc} be obtained as a limit (1.14). Then there exists $\mu \in L^\infty\left(\mathbb{R}_t; \mathcal{M}_+(T^*\mathbb{R}^2)\right)$ such that, for every $a \in C_c^\infty\left(T^*\mathbb{R}^2 \times T^*\mathbb{R}\right)$ we have:*

$$\int_{T^*\mathbb{R}^2\times T^*\mathbb{R}} a\left(z,\xi,t,H\right)\mu_{sc}\left(dz,d\xi,dt,dH\right) = \int_{\mathbb{R}}\int_{T^*\mathbb{R}^2} a\left(z,\xi,t,\frac{|\xi|^2}{2}\right)\mu\left(t,dz,d\xi\right)dt.$$

Proof. Let $u_h\left(\cdot,t\right) := U_V\left(t\right)u_h^0$ and note that the Wigner distributions:

$$\tilde{W}_{u_h}^h\left(t\right) : C_c^\infty\left(T^*\mathbb{R}^2\right) \ni l \longmapsto \left\langle U_V\left(t\right)u_h^0, \mathrm{Op}_h\left(l\right)U_V\left(t\right)u_h^0\right\rangle_{L^2(\mathbb{R}^2)} \in \mathbb{C}$$

are uniformly bounded in $L^\infty\left(\mathbb{R}_t;\mathcal{D}'(T^*\mathbb{R}^2)\right)$. Hence, possibly after extracting a subsequence, we can assume that, for every $b \in C_c^\infty\left(T^*\mathbb{R}^2\times\mathbb{R}\right)$:

$$\lim_{h\to 0^+}\int_{\mathbb{R}}\left\langle U_V\left(t\right)u_h^0, \mathrm{Op}_h\left(b\left(\cdot,t\right)\right)U_V\left(t\right)u_h^0\right\rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}}\int_{T^*\mathbb{R}^2} b\left(z,\xi,t\right)\tilde{\mu}_{sc}\left(t,dz,d\xi\right)dt$$

and (using the sharp Gårding inequality) the limiting Wigner distribution is a nonnegative measure $\tilde{\mu}_{sc} \in L^\infty\left(\mathbb{R}_t;\mathcal{M}_+(T^*\mathbb{R}^2)\right)$. We next show that for any $b \in C_c^\infty\left(T^*\mathbb{R}^2\times\mathbb{R}\right)$ with $b \geq 0$ one has:

$$\int_{T^*\mathbb{R}^2\times T^*\mathbb{R}} b\left(z,\xi,t\right)\mu_{sc}\left(dz,d\xi,dt,dH\right) \leq \int_{\mathbb{R}}\int_{T^*\mathbb{R}^2} b\left(z,\xi,t\right)\tilde{\mu}_{sc}\left(t,dz,d\xi\right)dt. \tag{D.1}$$

To see this, let $\chi \in C_c^\infty\left(\mathbb{R}\right)$ be a cut-off function satisfying $0 \leq \chi \leq 1$, strictly positive in $(-3/2,3/2)$, vanishing outside that interval, and such that $\chi\rceil_{(-1,1)} \equiv 1$. Write, for $R > 0$, $\chi_R := \chi\left(\cdot/R\right)$ and $\sigma_R := \sqrt{1-\chi_R}$. Then we have:

$$\left\langle u_h, \mathrm{Op}_h\left(b\right)\chi_R\left(h^2D_t\right)u_h\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} = \left\langle u_h, \mathrm{Op}_h\left(b\right)u_h\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} + k_{h,R}\left(b\right) + O\left(h\right), \tag{D.2}$$

where:

$$k_{h,R}\left(b\right) := \left\langle \sigma_R\left(h^2D_t\right)u_h, \mathrm{Op}_h\left(b\right)\sigma_R\left(h^2D_t\right)u_h\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}.$$

Taking limits in (D.2) as $h \to 0^+$ we find that:

$$\begin{aligned}\int_{T^*\mathbb{R}^2\times T^*\mathbb{R}} & b\left(z,\xi,t\right)\chi_R\left(H\right)\mu_{sc}\left(dz,d\xi,dt,dH\right) \\ &= \int_{\mathbb{R}}\int_{T^*\mathbb{R}^2} b\left(z,\xi,t\right)\tilde{\mu}_{sc}\left(t,dz,d\xi\right)dt + \lim_{h\to 0^+} k_{h,R}\left(b\right).\end{aligned} \tag{D.3}$$

But clearly, as $b \geq 0$, we always have

$$\lim_{h\to 0^+} k_{h,R}(b) = \lim_{h\to 0^+}\int_{\mathbb{R}} \tilde{W}_{\sigma_R(h^2D_t)u_h}^h\left(b\left(t,\cdot\right)\right)dt \geq 0,$$

for every $R > 0$. Taking this into account and letting $R \to \infty$ in (D.3) proves (D.1).

Now, as a consequence of (D.1) we have that the image of μ_{sc} under the projection onto the H-component is of the form $\mu\left(t,\cdot\right)dt$ for some $\mu \in L^\infty\left(\mathbb{R}_t;\mathcal{M}_+(T^*\mathbb{R}^2)\right)$. The disintegration theorem then ensures that μ_{sc} can be written as:

$$\mu_{sc}\left(dz,d\xi,dt,dH\right) = \mu_{x,\xi,t}\left(dH\right)\mu\left(t,dz,d\xi\right)dt.$$

Since μ_{sc} is supported on the characteristic set $|\xi|^2 = 2H$ we conclude that $\mu_{x,\xi,t}(dH) = \delta_{|\xi|^2/2}(dH)$ and the result follows. □

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