A Luenberger-type observer for the AM2 model

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Abstract

In this paper, we deal with the problem of designing a new observer for bioreactor models. The main idea is to construct a nonlinear observer with linear errors, which has an adjustable and robust convergence. Simulation results are pesented using a model of chemostat and a model of an anaerobic digestion process for the treatment of wastewater.

Key words: Chemostat model, AM2 model, Nonlinear systems, Nonlinear observers.

1 Introduction

The AM2 model was developed under the European project AMOCO. It is a two-step model (corresponding to two biological cascade reactions, hence its name), which is represented by the following mathematical system:

$$\begin{cases} \dot{s}_1 &= D(t)(s_{1in} - s_1) - k_1\mu_1(s_1)x_1 \\ \dot{x}_1 &= [\mu_1(s_1) - D(t)]x_1 \\ \dot{s}_2 &= D(t)(s_{2in} - s_2) + k_2\mu_1(s_1)x_1 - k_3\mu_2(s_2)x_2 \\ \dot{x}_2 &= [\mu_2(s_2) - D(t)]x_2 \end{cases} \quad (1)$$

In this work we will take as an output

$$y = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

This model was originally proposed in [4] and it is based on two main reactions, where the substrate s_1 is degraded in the substrate s_2 by the biomass x_1 then the substrate s_2 is degraded by the biomass x_2, where $(s_1, s_2) \in R_+^\star \times R_+^\star$ and $(x_1, x_2) \in R_+^\star \times R_+^\star$, $D(t) > 0$ is the dilution rate, k is the growth yield, s_{1in} and s_{2in} are the input substrate concentration of s_1 and s_2 and $\mu_1(s_1)$ and $\mu_2(s_2)$ are the specific growth rates.

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We will assume that a generic specific growth rate function $\mu(s)$ satisfies:

- $\mu(s) = 0 \Leftrightarrow s = 0$,
- $0 \leq \mu(.) \leq \mu_{max} = cst$,
- $\mu(.)$ continue.

In bioprocesses field, the design of nonlinear observers is very challenging, and is an area of intensive research. Several observers have been proposed for anaerobic digestion model, starting with *Extended Kalman Filter* and the *Extended Luenberger observer* which have been proposed and studied by Bastin and Dochain [3]. These approaches are well understood and only require linearization of the model which results in local convergence.
Bastin and Dochain proposed an *asymptotic observer* that allows (under conditions) to observe the state of the system without any knowledge of the kinetic model. This approach is particularly interesting. Nevertheless, the speed of convergence of this observer, contrary to the Luenberger one, cannot be tunned and depends on the dilution rate. Originally, these observers were designed for relatively simple systems, then they were extended by Chen [6], to include more complex ones. In this context, we note that they are characterizied by the simplicity of their design, and preserve some nonlinear aspect of the system, ensuring stability and convergence if the inputs are persistent and bounded. The change of coordinates depends on the stochiometric coefficients making these observers not robust. To overcome these disadvantages, Gouzé *et al.* [9], [11] generalized them and they made the asymptotic observers robust with a speed of convergence partially adjustable.

The *interval observers* should give a good solutions for systems with large uncertainties (see [1] and [10]). They are the combination of two observers, one observes the lower bound and the other observes the upper bound of the state, under a strong property called *cooperativity*, and it is necessary to know the bounds of uncertainties in the model.

Another observer was proposed [5] called *hybrid extended Luenberger-asymptotic observer* which evaluates a level of confidence in the process model, this parameter can vary between two values, 1 and 0, corresponding rigorously to the extended Luenberger observer and the asymptotic observer. The stability of this observer is analyzed but has not been completed so far. In a more general setting, several authors proposed a Luenberger-like observer for nonlinear systems [7] and [8]. The literature in the area of Luenberger Observers for AM2 model is not plentiful.

Thanks to the specific form of the output we were leaded to propose a nonlinear Luenberger observer in its general form with the injection of the outputs in the nonlinear function $\mu(s)$. We get in turn, a linear system for the errors.

This paper is organized as follows: Section 2 is dedicated to get more insight of the behaviour of our observer by applying it on a chemostat model and we will make the hypothesis that μ is completely known to show the convergence. Indeed the chemostat model is a subsystem of the AM2 model and the ideas developed here will be used later. Section 3 is devoted to the application of the proposed observer on the AM2 model as a main result with some simulations to show the effectiveness of the proposed method. In practice, as μ is not completely known, we will show at the end that our observer is robust against the uncertainties.

2 The chemostat model

The chemostat is a laboratory prototype of bioreactors used in waste water treatment, which was introduced by Novick and Szilard [13] and used by Monod [12].
The chemostat is a kind of bioreactor which allows the growth of a population of microorganisms (bacteria, yeast, phytoplankton, zooplankton,...) on some substrates, with suitable environmental conditions (temperature, light, pH and aeration). This device works in continum mode, i.e, the volume of the bioreactor is kept constant.
The nonlinear model of the chemostat obtained by mass balance is given by:

$$\begin{cases} \dot{s} = D(t)(s_{in} - s) - k\mu(s)x \\ \dot{x} = [\mu(s) - D(t)]x \end{cases} \tag{2}$$

We will take as an output $y = s$, where $s \in R_+^\star$ and $x \in R_+^\star$ represent the substrate concentration and the biomass concentration respectively, $D(t) > 0$ is the dilution rate, k is the growth yield, s_{in} is the input substrate concentration and $\mu(s)$ is the specific growth rate per unit of biomass. To avoid the washout we impose the condition $D(t) \leq \mu_{max}$.

We put the following hypothesis:
(H1): $s(t) \geq s_{min} > 0, \forall t \geq 0$

The criterium of observability is

$$rank \frac{\partial}{\partial x}(h(x), L_f h(x)) = rank \begin{pmatrix} 1 & 0 \\ D(t) & -k\mu(s) \end{pmatrix} = 2$$

In order to be observable the chemostat must verify $\mu(s) \neq 0$, $\forall s$ which is equivalent to $s \neq 0$, (ensured by **(H)**). This condition can be fulfilled if the dilution rate is sufficiently exciting, which in turn can be expressed by

$$\int_0^{+\infty} D(\tau)d\tau = +\infty$$

The immediate consequence of **(H1)** is

$$\mu(s) \geq \mu_{min} > 0, \forall s \tag{3}$$

Theorem 1 *For the system (2), the following system:*

$$\begin{cases} \dot{\hat{s}} = D(t)(s_{in} - \hat{s}) - k\mu(s)\hat{x} + a_1(s - \hat{s}) \\ \dot{\hat{x}} = [\mu(s) - D(t)]\hat{x} + a_2(s - \hat{s}) \end{cases} \tag{4}$$

is a globally uniformly asymptotically stable observer, where a_1 and a_2 are two tuning parameters.
Under the condition $\alpha a_1 + 2a_2 = 0$ with $a_1 > 0$ and large enough and $\alpha > \frac{2}{k}$.

PROOF. By putting $e_1 = s - \hat{s}$ and $e_2 = x - \hat{x}$, the error dynamics are given by:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} -D(t) - a_1 & -k\mu(s) \\ -a_2 & \mu(s) - D(t) \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{5}$$

Let us hide $D(t)$ from (5) by putting $p(t) = e^{\int_0^t D(\tau)d\tau}$ and let us multiply $p(t)$ by the two equations of the system (5) then, the later can be written as follows:

$$\begin{cases} p(t)\dot{e}_1 + D(t)p(t)e_1 = -a_1 p(t)e_1 - k\mu(s)p(t)e_2 \\ p(t)\dot{e}_2 + D(t)p(t)e_2 = -a_2 p(t)e_1 + \mu(s)p(t)e_2 \end{cases} \tag{6}$$

i.e.

$$\begin{cases} \frac{d}{dt}(p(t)e_1) = -a_1 p(t)e_1 - k\mu(s)p(t)e_2 \\ \frac{d}{dt}(p(t)e_2) = -a_2 p(t)e_1 + \mu(s)p(t)e_2 \end{cases} \quad (7)$$

Let $u(t) = p(t)e_1(t)$ and $v(t) = p(t)e_2(t)$, convergence of (u, v) to $(0, 0)$ implies the convergence of (e_1, e_2) to $(0, 0)$. Then (7) becomes

$$\begin{cases} \dot{u}(t) = -a_1 u(t) - k\mu(s)v(t) \\ \dot{v}(t) = -a_2 u(t) + \mu(s)v(t) \end{cases} \quad (8)$$

To prove the convergence of the observer we will take the following Lyapunov function candidate:

$$V_\alpha(u, v) = \left\| \begin{pmatrix} \alpha u + v \\ v \end{pmatrix} \right\|^2 = (\alpha u + v)^2 + v^2 \quad (9)$$

where $\alpha \neq 0$.
Note that this function is also a norm and then is radially unbounded. We have

$$\dot{V}_\alpha(u, v) = 2(\alpha u + v)(\alpha \dot{u} + \dot{v}) + 2v\dot{v}$$

By substituting $\dot{u}$ and $\dot{v}$ in $\dot{V}_\alpha(u, v)$, we obtain

$$\dot{V}_\alpha(u, v) = -2\alpha(a_1\alpha + a_2)u^2 + 2\mu(s)(2 - k\alpha)v^2$$

$$+2\left[\alpha(1 - k\alpha)\mu(s) - (a_1\alpha + 2a_2)\right]uv$$

By putting, $a_2 = -\frac{\alpha}{2}a_1$, we have

$$\dot{V}_\alpha(u, v) = -\alpha^2 a_1 u^2 + 2\mu(s)(2 - k\alpha)v^2 + 2\alpha(1 - k\alpha)\mu(s)uv$$

This function can be written as follows

$$\dot{V}_\alpha(u, v) = \begin{pmatrix} u & v \end{pmatrix} \underbrace{\begin{pmatrix} -\alpha^2 a_1 & \alpha(1 - k\alpha)\mu(s) \\ \alpha(1 - k\alpha)\mu(s) & 2\mu(s)(2 - k\alpha) \end{pmatrix}}_{A(t)} \begin{pmatrix} u \\ v \end{pmatrix}$$

Now, let us prove that the matrix $A(t)$ is negative definite i.e. let us prove that $tr(A(t)) < 0$ and $det(A(t)) > 0$.

$$tr(A(t)) = -\alpha^2 a_1 + 2\mu(s)(2 - k\alpha)$$

and

$$det(A(t)) = -2\alpha^2(2 - k\alpha)\mu a_1 - (1 - k\alpha)^2\mu^2(s)\alpha^2$$

Here $tr(A(t))$ will be negative for $a_1 > 0$ and large enough and $det(A(t))$ will be positive with the same condition and $\alpha > \frac{2}{k}$.

Note that $\mu(s) \geq \mu_{min} > 0$, as presented previously, is of great concern for the positivity of $det(A(t))$.

2.1 Simulation results

The simulations were carried out using the parameter values given in [2]. They are recalled in table (1) and the choosen initial conditions are given in table (2).

Parameter	**Value and Unit**
k	6.6 Kg COD/Kg x
μ_{max}	1.2 day^{-1}
K	4.95 Kg COD/m^3
s_{in}	9 Kg/m^3

Table 1
Model parameters for simulation runs

	$s(0)$ (Kg/m^3)	$x(0)$ (Kg/m^3)
Model	3	0.5
Observer	3	5

Table 2
Initial conditions for simulation runs

The time simulation is taken to be 50 days with variable dilution rate as in the figure (1).

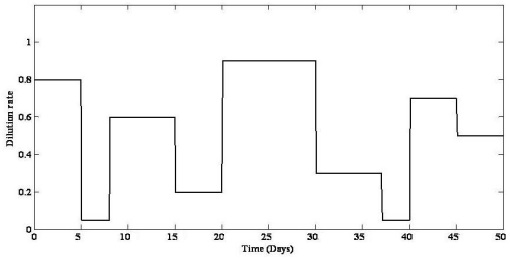

Fig. 1. Dilution rate

We can see from these simulations that on the biomass the convergence is obtained in a half a day with an error equal to 10^{-2}. For comparison, the asymptotic observer converges only after 5 days and cannot be improved [2].

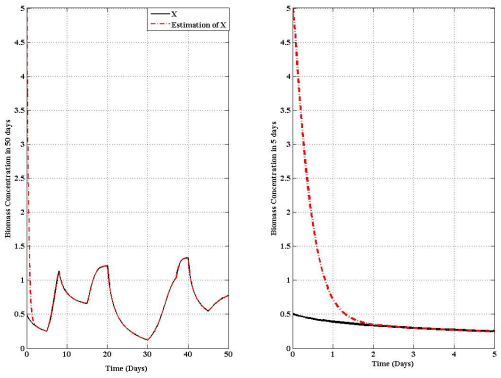

Fig. 2. Biomass Concentration for $a_1 = 30$

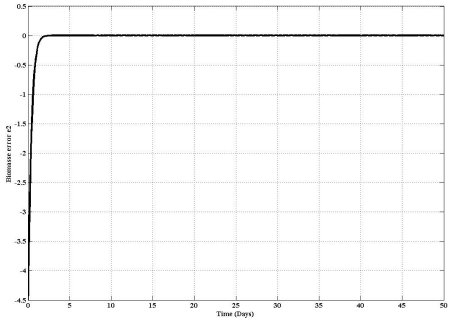

Fig. 3. Biomass error estimation

3 Main result on AM2 model

Theorem 2 *For the system (1):*

$$\begin{cases} \dot{\hat{s}}_1 &= D(t)(s_{1in} - \hat{s}_1) - k_1\mu_1(s_1)\hat{x}_1 + a_1(s_1 - \hat{s}_1) \\ \dot{\hat{x}}_1 &= [\mu_1(s_1) - D(t)]\hat{x}_1 + a_2(s_1 - \hat{s}_1) \\ \dot{\hat{s}}_2 &= D(t)(s_{2in} - \hat{s}_2) + k_2\mu_1(s_1)\hat{x}_1 - k_3\mu_2(s_2)\hat{x}_2 \\ & \quad +a_3(s_2 - \hat{s}_2) \\ \dot{\hat{x}}_2 &= [\mu_2(s_2) - D(t)]\hat{x}_2 + a_4(s_2 - \hat{s}_2) \end{cases} \quad (10)$$

is a globally uniformly asymptotically stable observer, where a_i, $i = \overline{1,4}$ are tuning parameters.
Under the condition $\alpha a_1 + 2a_2 = 0$ with $a_1 > 0$ large enough and $\alpha > \frac{2}{k_1}$ and the condition $\beta a_3 + 2a_4 = 0$ with $a_3 > 0$ large enough and $\beta > \frac{2}{k_3}$

PROOF. Let us define $e_1 = s_1 - \hat{s}_1$, $e_2 = x_1 - \hat{x}_1$, $e_3 = s_2 - \hat{s}_2$ and $e_4 = x_2 - \hat{x}_2$.

The error dynamics are given by:

$$\begin{cases} \dot{e}_1 &= -\left(D(t) + a_1\right) e_1 - k_1\mu_1(s_1)e_2 \\ \dot{e}_2 &= -a_2 e_1 + [\mu_1(s_1) - D(t)]e_2 \\ \dot{e}_3 &= k_2\mu_1(s_1)e_2 - \left(D(t) + a_3\right) e_3 - k_3\mu_2(s_2)e_4 \\ \dot{e}_4 &= -a_4 e_3 + [\mu_2(s_2) - D(t)]e_4 \end{cases} \quad (11)$$

Recall that we have injected the measured variables s_1 and s_2 into $\mu_1(s_1)$ and $\mu_2(s_2)$ to write this observer. Note that the system (11) is linear and non autonomous.

Let us write the system (11) as:

$$\dot{e} = B(t)e$$

where $e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}$ and $B(t) =$

$$\left(\begin{array}{cc:cc} -D(t) - a_1 & -k_1\mu_1(s_1) & 0 & 0 \\ -a_2 & \mu_1(s_1) - D(t) & 0 & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 & k_2\mu_1(s_1) & -D(t) - a_3 & -k_3\mu_2(s_2) \\ 0 & 0 & -a_4 & \mu_2(s_2) - D(t) \end{array}\right)$$

We remark that the diagonal blocks are chemostat-like blocks.

So, let us hide $D(t)$ from the system by putting $p(t) = e^{\int_0^t D(\tau)d\tau}$ and $u_1(t) = p(t)e_1(t)$, $u_2(t) = p(t)e_2(t)$, $u_3(t) = p(t)e_3(t)$ and $u_4(t) = p(t)e_4(t)$.

Then we get

$$\begin{cases} \dot{u}_1 = -a_1 u_1 - k_1\mu_1(s_1)u_2 \\ \dot{u}_2 = -a_2 u_1 + \mu_1(s_1)u_2 \\ \dot{u}_3 = k_2\mu_1(s_1)u_2 - a_3 u_3 - k_3\mu_2(s_2)u_4 \\ \dot{u}_4 = -a_4 u_3 + \mu_2(s_2)u_4 \end{cases} \quad (12)$$

To prove the convergence of this observer we will take

the following Lyapunov function candidate:

$$V_{\alpha,\beta}(u_1,u_2,u_3,u_4) = \left\| \begin{pmatrix} \alpha u_1 + u_2 \\ u_2 \\ \beta u_3 + u_4 \\ u_4 \end{pmatrix} \right\|^2$$

where α and β are positive constants to be chosen later. $V_{\alpha,\beta}$ is also a norm and then is radially unbounded. So

$$\begin{aligned} \dot{V}_{\alpha,\beta}(u_1,u_2,u_3,u_4) = & \ 2(\alpha u_1 + u_2)(\alpha \dot{u}_1 + \dot{u}_2) + 2u_2\dot{u}_2 \\ & +2(\beta u_3 + u_4) + (\beta \dot{u}_3 + \dot{u}_4) + 2u_4\dot{u}_4 \end{aligned}$$

By substituting $\dot{u}_i$, $i = \overline{1,4}$ in $\dot{V}_{\alpha,\beta}(u_i)$, we obtain

$$\begin{aligned} \dot{V}_{\alpha,\beta} = & -2\alpha(a_1\alpha + a_2)u_1^2 - 2[a_1\alpha + 2a_2 + \alpha(\alpha k_1 - 1)\mu_1]u_1u_2 \\ & -2(\alpha k_1 - 2)\mu_1 u_2^2 + 2k_2\beta^2\mu_1 u_2u_3 + 2k_2\beta\mu_1 u_2u_4 \\ & -2\beta(a_3\beta + a_4)u_3^2 - 2[a_3\beta + 2a_4 + \beta(\beta k_3 - 1)\mu_2]u_3u_4 \\ & -2(\beta k_3 - 2)\mu_2 u_4^2 \end{aligned}$$

This function can be written as follows:

$$\dot{V}_{\alpha,\beta} = U^T C(t) U$$

where $U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ and $C(t) = \begin{pmatrix} c_{11} & c_{12} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} \\ 0 & c_{32} & c_{33} & c_{34} \\ 0 & c_{42} & c_{43} & c_{44} \end{pmatrix}$

with

$$\begin{aligned} c_{11} &= -2\alpha(a_1\alpha + a_2), \\ c_{12} = c_{21} &= -[a_1\alpha + 2a_2 + \alpha(\alpha k_1 - 1)\mu_1], \\ c_{22} &= -2(\alpha k_1 - 2)\mu_1, c_{23} = c_{32} = k_2\beta^2\mu_1, \\ c_{24} = c_{42} &= k_2\beta\mu_1, c_{33} = -2\beta(a_3\beta + a_4), \\ c_{34} = c_{43} &= -[a_3\beta + 2a_4 + \beta(\beta k_3 - 1)\mu_2] \end{aligned}$$

and

$$c_{44} = -2(\beta k_3 - 2)\mu_2$$

We have to prove that $C(t)$ is negative definite, i.e. $-C(t)$ positive definite, so we have to prove that all the principal minors of $-C(t)$ are positives.
Let δ_1, δ_2, δ_3 and δ_4 be these minors.

$$\delta_1 = 2\alpha(a_1\alpha + a_2)$$

$$\begin{aligned} \delta_2 = & -\underline{(a_1\alpha + 2a_2)^2} + 2\alpha^2\mu_1(k_1\alpha - 3)a_1 - 4\alpha\mu_1 a_2 \\ & -\alpha^3 k_1(k_1\alpha + 2)\mu_1^2 \end{aligned}$$

Let us put $a_1\alpha + 2a_2 = 0$ then $a_2 = -\frac{\alpha}{2}a_1$ so δ_2 becomes:

$$\delta_2 = 2\alpha^2\mu_1(k_1\alpha - 2)a_1 - \alpha^3 k_1(k_1\alpha + 2)\mu_1^2$$

For $a_1 > 0$ and large enough and for $\alpha > \frac{2}{k_1}$, δ_1 and δ_2 are both positives.

$$\begin{aligned} \delta_3 = & \ 2\alpha^2\beta^2\mu_1[2(\alpha k_1 - 2)a_1 - \mu_1(\alpha k_1 - 1)^2]a_3 \\ & +2\alpha^2\beta\mu_1[2(\alpha k_1 - 2)a_1 - \mu_1(\alpha k_1 - 1)^2]a_4 - \alpha^2\beta^4 k_2^2\mu_1^2 a_1 \end{aligned}$$

$$\begin{aligned} \delta_4 = & \ \alpha^2\mu_1[\mu_1(\alpha k_1 - 1)^2 - 2(\alpha k_1 - 2)a_1]\underline{(\beta a_3 + 2a_4)^2} \\ & +2\mu_1\mu_2\alpha^2\beta^2(\beta k_3 - 3)\left[2(\alpha k_1 - 2)a_1 - \mu_1(\alpha k_1 - 1)^2\right]a_3 \\ & -2\alpha^2\beta\mu_1\left[\left(\beta^2 k_2^2\mu_1 + 4\mu_2(\alpha k_1 - 2)\right)a_1\right. \\ & \left.-2\mu_1\mu_2(\alpha k_1 - 1)^2\right]a_4 - 2\mu_1\mu_2\alpha^2\beta^2\left[\mu_2(\alpha k_1 - 2)\right. \\ & \left.(\beta k_3 - 1)^2 - \beta^2 k_2^2\mu_1\right]a_1 + \mu_1^2\mu_2^2\alpha^2\beta^2(\beta k_3 - 1)^2(\alpha k_1 - 1)^2 \end{aligned}$$

Let us put $a_3\beta + 2a_4 = 0$ then $a_4 = -\frac{\beta}{2}a_3$ so δ_3 and δ_4 become:

$$\begin{aligned} \delta_3 = & \ \alpha^2\beta^2\mu_1[2(\alpha k_1 - 2)a_1 - \mu_1(\alpha k_1 - 1)^2]a_3 \\ & -\alpha^2\beta^4 k_2^2\mu_1^2 a_1 \end{aligned}$$

$$\begin{aligned} \delta_4 = & \ \left[\mu_1\alpha^2\beta^2\left(4(\alpha k_1 - 2)(\beta k_3 - 2)\mu_2 + \beta^2 k_2^2\mu_1\right)a_1\right. \\ & \left.-2\mu_1^2\mu_2\alpha^2\beta^2(\alpha k_1 - 1)^2(\beta k_3 - 2)\right]a_3 - 2\mu_1\mu_2\alpha^2\beta^2\left[\mu_2(\alpha k_1 - 2)\right. \\ & \left.(\beta k_3 - 1)^2 - \beta^2 k_2^2\mu_1\right]a_1 + \mu_1^2\mu_2^2\alpha^2\beta^2(\beta k_3 - 1)^2(\alpha k_1 - 1)^2 \end{aligned}$$

For $a_3 > 0$ and large enough and for $\beta > \frac{2}{k_3}$, δ_3 and δ_4 are both positives. Note that $\mu_1 > \mu_{1min}$ and $\mu_2 > \mu_{2min}$ play a crucial role in the positivity of the δ_i's.

3.1 Simulation results

The simulations were carried out using the parameter values recalled in [2]. They are given in table (3) and the choosen initial conditions are given in table (4). The time simulation is taken to be 50 days with variable dilution rate as in the figure (1).

We can see from these simulations that on the first biomass x_1 the convergence is obtained in a two days with an error equal to 10^{-2} and on the second biomass x_2 the convergence is obtained in a one day with the same error.

4 Robustness

Here we will study the robustness of the proposed observer for the chemostat model only, because it represents the generic part of the other models.

Parameter	Value and Unit
k_1	6.6 Kg COD/Kg x_1
k_2	7.8 mol VFA/Kg x_1
k_3	611.2 mol VFA/Kg x_2
μ_{1max}	1.2 day^{-1}
μ_{2max}	0.69 day^{-1}
K_1	4.95 Kg COD/m^3
K_2	9.28 mol VFA/m^3
K_i	20 (mol VFA/m^3)$^{\frac{1}{2}}$
s_{1in}	15 Kg/m^3
s_{2in}	80 mol/m^3

Table 3
Model parameters for simulation runs

	$s_1(0)$ (Kg/m^3)	$x_1(0)$ (Kg/m^3)	$s_2(0)$ (mol/m^3)	$x_1(0)$ (Kg/m^3)
Model	3	0.5	15	0.12
Observer	3	3	15	1

Table 4
Initial conditions for simulation runs

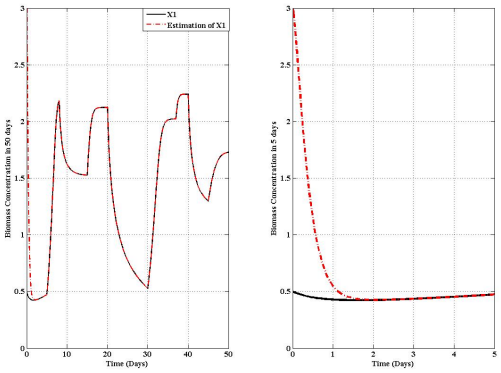

Fig. 4. The first biomass estimation for $a_1 = 20$ and $a_3 = 35$

For the system (2) with $y = s$ as output, the Luenberger-like observer can be written as follows

$$\begin{cases} \dot{\hat{s}} = D(t)(s_{in} - \hat{s}) - k\hat{\mu}(s)\hat{x} + a_1(s - \hat{s}) \\ \dot{\hat{x}} = [\hat{\mu}(s) - D(t)]\,\hat{x} + a_2(s - \hat{s}) \end{cases} \tag{13}$$

where we have injected the output s only in $\hat{\mu}$ which is assumed to be known, contrary to μ which is unknown. The parameters a_1 and a_2 are tuning parameters. We make the following hypothesis

$$(\mathbf{H2}) \qquad \begin{cases} \sup_{s \geq 0} \mid \hat{\mu}(s) - \mu(s) \mid \leq \gamma \\ 0 < \hat{\mu}_{min} \leq \hat{\mu}(s) \leq \hat{\mu}_{max} \end{cases}$$

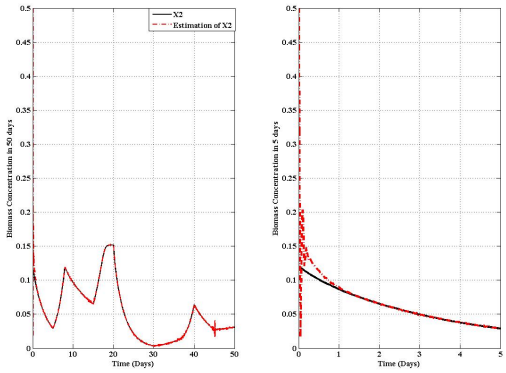

Fig. 5. The second biomass estimation for $a_1 = 20$ and $a_3 = 35$

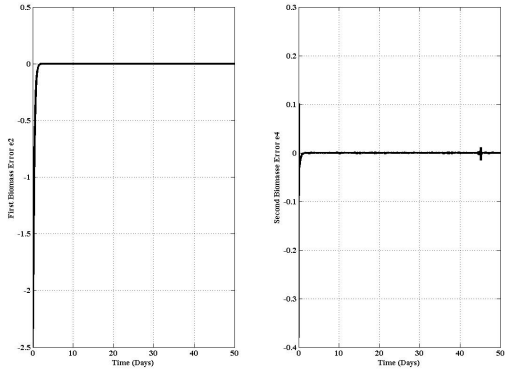

Fig. 6. The biomass errors

By putting $e_1 = s - \hat{s}$ and $e_2 = x - \hat{x}$ we get

$$\begin{cases} \dot{e_1} = -(a_1 + D(t))e_1 - k\hat{\mu}(s)e_2 + k(\hat{\mu}(s) - \mu(s))x \\ \dot{e_2} = -a_2 e_1 + (\hat{\mu}(s) - D(t))e_2 - (\hat{\mu}(s) - \mu(s))x \end{cases}$$

To simplify more the presentation of this last system, we put

$$u_1(t) = e^{\int_0^t D(\tau)d\tau} e_1(t) \qquad u_2(t) = e^{\int_0^t D(\tau)d\tau} e_2(t)$$

and

$$\chi(t) = e^{\int_0^t D(\tau)d\tau} x(t)$$

Then we get

$$\dot{U}(t) = \tilde{A}(t).U(t) + (\hat{\mu}(s) - \mu(s))\chi(t).C$$

where $U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$, $\tilde{A}(t) = \begin{pmatrix} -a_1 & -k\hat{\mu}(s) \\ -a_2 & \hat{\mu}(s) \end{pmatrix}$

and $C = \begin{pmatrix} k \\ -1 \end{pmatrix}$. Let $V_\alpha(\xi_1, \xi_2) = (\alpha\xi_1 + \xi_2)^2 + \xi_2^2$ with

$\alpha \neq 0$, be the Lyapounov candidate fonction used in (9). Note that, according to the equivalence of norms in R^2 we have

$$\theta_-(\xi_1^2+\xi_2^2) \leq V_\alpha(\xi_1,\xi_2) \leq \theta_+(\xi_1^2+\xi_2^2)$$

with

$$\theta_- = \frac{\alpha^2+2-\sqrt{\alpha^4+4}}{2} \;\; and \;\; \theta_+ = \frac{\alpha^2+2+\sqrt{\alpha^4+4}}{2}$$

Note also that $V_\alpha(\xi_1,\xi_2) = \xi^T.M_\alpha.\xi$ where $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and $M_\alpha = \begin{pmatrix} \alpha^2 & \alpha \\ \alpha & 2 \end{pmatrix}$. It is not hard to see that

$$\frac{d}{dt}V_\alpha(U(t)) = U(t)^T \left[\underbrace{M_\alpha \tilde{A}(t) + \tilde{A}(t)^T M_\alpha}_{\tilde{B}(t)} \right] U(t)$$

$$+2(\hat{\mu}(s)-\mu(s))\chi(t)U(t)^T M_\alpha C$$

A straightforward calculation gives

$$\tilde{B}(t) = \begin{pmatrix} -2\alpha(\alpha a_1 + a_2) & -(\alpha a_1 + 2a_2) \\ & +\alpha(1-k\alpha)\hat{\mu}(s) \\ -(\alpha a_1 + 2a_2) & 2(2-k\alpha)\hat{\mu}(s) \\ +\alpha(1-k\alpha)\hat{\mu}(s) & \end{pmatrix}$$

Now we take $\alpha a_1 + 2a_2 = 0$, so $\tilde{B}(t)$ becomes

$$\tilde{B}(t) = \begin{pmatrix} -\alpha^2 a_1 & \alpha(1-k\alpha)\hat{\mu}(s) \\ \alpha(1-k\alpha)\hat{\mu}(s) & 2(2-k\alpha)\hat{\mu}(s) \end{pmatrix}$$

By classical results on quadratic forms we have

$$U(t)^T\tilde{B}(t)U(t) \leq \lambda_{max}(\tilde{B}(t))U(t)^TU(t)$$

$$\leq \frac{\lambda_{max}(\tilde{B}(t))}{\theta_-}V_\alpha(U(t))$$

where $\lambda_{max}(\tilde{B}(t))$ is the largest eigenvalue of the symmetric matrix $\tilde{B}(t)$.
By the Cauchy-Schwarz inequality we have

$$2(\hat{\mu}(s)-\mu(s))\chi(t)U(t)^TM_\alpha C \leq \frac{2\gamma\nu_\alpha}{\sqrt{\theta_-}} \mid \chi(t) \mid \sqrt{V_\alpha(U(t))}$$

where $\nu_\alpha = \parallel M_\alpha C \parallel = \sqrt{\alpha^2(k\alpha-1)^2+(k\alpha-2)^2}$.
We, then get

$\frac{d}{dt}V_\alpha(U(t)) \leq$

$$\frac{\lambda_{max}(\tilde{B}(t))}{\theta_-}V_\alpha(U(t)) + \frac{2\gamma\nu_\alpha}{\sqrt{\theta_-}} \mid \chi(t) \mid \sqrt{V_\alpha(U(t))}$$

By choosing $\alpha > \frac{2}{k}$ and a_1 positive and large enough, one can see that $\lambda_{max}(B(t)) \leq \lambda_* < 0$ where λ_* is a constant, indeed

$$\lambda_* = \frac{2\alpha^2\hat{\mu}_{min}\left[(k\alpha-1)^2\hat{\mu}_{min} - 2a_1(k\alpha-2)\right]}{\alpha^2 a_1 + 2(k\alpha-2)\hat{\mu}_{max} + \sqrt{\Delta}}$$

with $\Delta = (\alpha^2 a_1 + 2(k\alpha-2)\hat{\mu}_{max})^2 + 4\alpha^2(k\alpha-1)^2\hat{\mu}_{max}^2$

One can use a less complicated (approximate) value of λ_* done by

$$\lim_{a_1 \to +\infty} \lambda_* = -2(k\alpha-2)\hat{\mu}_{min}$$

The previous differential inequality becomes

$$\frac{d}{dt}\sqrt{V_\alpha(U(t))} \leq \frac{\lambda_*}{2\theta_-}\sqrt{V_\alpha(U(t))} + \frac{\gamma\nu_\alpha}{\sqrt{\theta_-}} \mid \chi(t) \mid$$

By classical rules we get

$$\sqrt{V_\alpha(U(t))} \leq \sqrt{V_\alpha(U(0))}e^{\frac{\lambda_*}{2\theta_-}t} + \frac{\gamma\nu_\alpha}{\sqrt{\theta_-}}\int_0^t \mid \chi(\tau) \mid e^{\frac{\lambda_*}{2\theta_-}(t-\tau)}d\tau$$

Now we return to the error vector $e(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix}$

$$\sqrt{V_\alpha(e(t))} \leq \sqrt{V_\alpha(e(0))}e^{\frac{\lambda_*}{2\theta_-}t - \int_0^t D(\sigma)d\sigma}$$

$$+\frac{\gamma\nu_\alpha}{\sqrt{\theta_-}}\int_0^t \mid x(\tau) \mid e^{\frac{\lambda_*}{2\theta_-}(t-\tau)}e^{-\int_\tau^t D(\sigma)d\sigma}d\tau$$

As $D(.)$ is a positive function, and $0 < x(t) \leq \frac{2}{k}s_{in} + x(0)$, we have then

$$\sqrt{V_\alpha(e(t))} \leq \sqrt{V_\alpha(e(0))}e^{\frac{\lambda_*}{2\theta_-}t} + \frac{\gamma\nu_\alpha}{\sqrt{\theta_-}}(\frac{2}{k}s_{in}+x(0))\int_0^t e^{\frac{\lambda_*}{2\theta_-}(t-\tau)}d\tau$$

and

$$\sqrt{V_\alpha(e(t))} \leq \sqrt{V_\alpha(e(0))}e^{\frac{\lambda_*}{2\theta_-}t} - \frac{2\gamma\nu_\alpha\sqrt{\theta_-}}{\lambda_*}(\frac{2}{k}s_{in}+x(0))(1-e^{\frac{\lambda_*}{2\theta_-}t})$$

This last inequality provide the boundedness of the error. The upper bound is less than ρ (14).

$$\rho = \sqrt{\frac{\theta_+}{\theta_-}} \left\| (e_1(0), e_2(0)) \right\|_2 - \frac{2\gamma\nu_\alpha}{\lambda_*} \left(\frac{2}{k} s_{in} + x(0) \right) \quad (14)$$

We summarize the previous calculation in the following theorem.

Theorem 3 *Under the hypothesis (**H2**), the system (13) is an observer for the system (2). The observation errors are bounded like: $\|(e_1, e_2)\|_2 \leq \rho$, where ρ is given by (14).*

Remark 4 *In the hypothesis (**H2**), it must be noted that no specific form for $\mu(\cdot)$ and $\hat{\mu}(\cdot)$ are assumed.*

4.1 Simulations results

The simulations were carried out using the same parameter values given in table (1) and the choosen initial conditions given in table (2) and with the same dilution rate as in the figure (1).

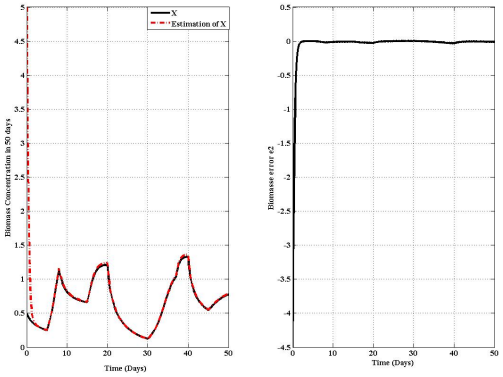

Fig. 7. Biomass Concentration for $a_1 = 30$ for different K

Here we used for $\mu(\cdot)$ and $\hat{\mu}(\cdot)$, a Monod specific function with a differents μ_{max} and K. We added an error of 20% for K (see figure (7)) and the same error for μ_{max} (see figure (8)), then we looked to the behaviour of the system. We note a very satisfactory response of the observer, which means a good robustness against model uncertainties.

5 Conclusion

In this paper, we designed a nonlinear observer for the AM2 model with linear errors, and we proved that this observer is robust with a global convergence and adjustable speed. For large speed of convergence, we noticed that in a small time interval at the beginning, some

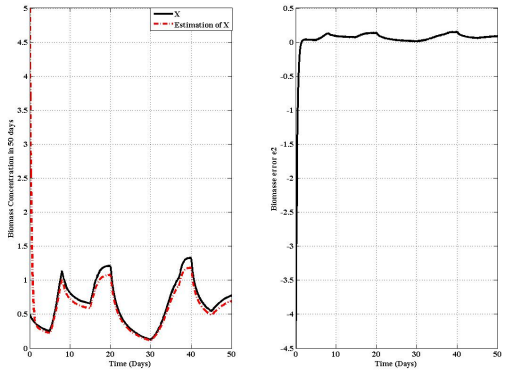

Fig. 8. Biomass Concentration for $a_1 = 30$ for different μ_{max}

variables can take erroneous values (< 0). This is not harmful if the observer is used only as a software sensor, on the other hand, if the observer is used in a control loop, we have to lower the speed of convergence a little bit to ensure that the variables remains positives. It can be noticed, from an application point of view that the implementation of this observer is very easy.

The quantity of methane produced by the system is given by the following expression

$$q_M = k_6 \mu_2(s_2) x_2$$

It can be estimated from the estimated variable x_1 and x_2 which qualifies our observer as a software sensor for the methane.

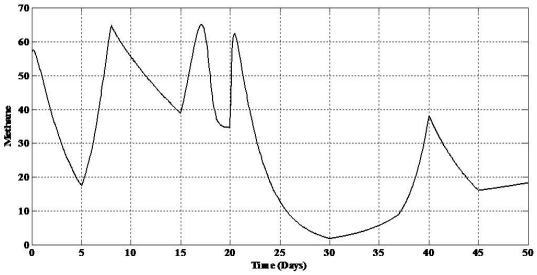

Fig. 9. the output q_M

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