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PROVING AND KNOWING IN PUBLIC: THE NATURE OF PROOF IN A CLASSROOM

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Abstract

We consider the question of which classroom performances can count as proof for a teacher. We propose that at least three notions of proof need to be considered. One notion of proof is tied to the habitual uses of words such as *prove* and *proof* in a particular class's discourse as well as to the customary ways in which knowledge is validated and organized in a class. A second notion of proof, which we label *C-proof*, is tied to specific ways in which classes are able to control the fit or the adaptation between problems and their solutions within particular conceptions. A third notion of proof, which we label *K-proof*, addresses the possible role of proof in coming to know new things: It describes how the metaphorical mapping of a conception already known by the class over a new conception can be used to shape and warrant plausible knowledge about the new conception.

PROVING AND KNOWING IN PUBLIC: THE NATURE OF PROOF IN A CLASSROOM

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This essay contributes to the development of a descriptive theory of the mathematical performances of classrooms by proposing three senses in which classroom mathematical practices might include a notion of *proof*. The need for such a theory is based on the hunch that public activity in mathematics classrooms embodies an epistemology, a particular way of knowing, and that understanding this epistemology is key in order to inform the development of curriculum, pedagogy, and assessment. The goal appears ambitious, but is also modest in a key aspect: In considering our problem to device a descriptive theory of classroom activities as public mathematical performances, we bracket the many important questions that one could pose about the individuals who perform. We do not address questions of individual learning or personal meaning for the participants of classroom work; rather, we develop means to describe what their public personae do in the performances open to the public. We do that not as a statement of values, but as part of a strategic plan toward eventually explaining classrooms as collective, complex intelligences where individuals contribute to a distributed cognition that might embody mathematical ideas as public performances.

The focus of this paper is on the notion of proof as a public performance. Specifically, we propose that to understand the various performances that a classroom observer might label after proof, at least three notions of proof need to be considered. One notion of proof is tied to the habitual uses of words such as *prove* and *proof* in a particular class's discourse as well as to the customary ways in which knowledge is validated and organized in a class. A second notion of proof, which we introduce later as *C-proof*, is tied to specific ways in which classes are able to control the fit or the adaptation between problems and their solutions within particular conceptions. A third notion of proof, which we introduce later as *K-proof*, addresses the possible role of proof in coming to know new things: It describes how the metaphorical mapping of a conception already known by the class over a new conception can be used to shape and warrant plausible knowledge about the new conception.

Classroom Mathematical Performances

School systems and parents take classroom activity and teaching as instruments to achieve learning goals of individual students. It is understandable that for other stakeholders, however, the aims of classroom activity can be different. Professional communities, such as those of mathematicians or engineers, are likely to be more concerned with the diffusion of knowledge and the percolation of knowledge experiences on collectives (e.g., a generation). Likewise the classroom teacher is a stakeholder whose first concern is necessarily collective. Unlike a tutor, a teacher rarely teaches individual students—a teacher teaches a class. It is true that a class is not like a soccer team or a ballet company, at least not in terms of their

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institutionally avowed aim to produce individual learning. Yet, that avowed aim is operationally mediated by the public opportunities to learn that are developed in the class. That is, if individual students learn in school, they do so not just from their individual study and not only when they have the individualized attention of their classroom teacher. They learn from being a part of a system of people, constraints, and resources—the class—whose total, systemic activity reframes individual actions as contributions to collective work, one of whose products is opportunity to learn. Students' opportunity to learn depends on classroom performance much in the same way that each of the musicians of an orchestra has the opportunity to claim to have performed a classical piece because of the performance of the complete orchestra, where each musician may have played a (perhaps minute) part of the orchestral arrangement.

Classroom performances represent knowledge as public work. Like the dynamic models of physical phenomena in a museum or the performances of musical compositions by an orchestra, classroom performances can be opportunities to learn because they create representations of knowledge. They embody uses of material and symbolic objects as well as processes, strategies, and heuristics that at least potentially recreate knowledge as part of the lived experience. One could think of those classroom performances as possibly representing ideas, not in any immaterial sense, but rather as ideas embodied or represented in intelligent or informed human action over time. As representations they are important especially for the teacher, because students' actual learning depends on the one hand on the extent to which they take individual advantage of the opportunity to be a part of those representations and on the other hand on the mathematical qualities of what is available for them to be a part of. Classroom performances must be compelling and immersive for students to be taken as opportunities to learn; but they also must be mathematically compelling and important to be culturally valuable. The supervision on that double quality of the opportunity to learn rests on teaching (not just on the human teacher working in the class, but more generally on the system of human and material resources constructed with the intention to teach). In this sense, it is appropriate to think of the activities of classroom instruction as a hotbed of cultural assets being created and managed by people working with cultural tools. The question to be considered in this article is what kinds of representations of *proof* can be constructed in and through the performances of a class and how an observer interested in capturing the perspective of the teacher could describe those representations of proof.

We focus here on developing theoretical means to take the teacher's perspective as observer of the performance of the class as a collective cognitive organization and describe proof as a public performance from that perspective. Specifically, we focus on the teacher as observer of the opportunity to learn created by classroom activity over three key timescales. This perspective is particularly important to inform teacher assessment, teacher education, and curriculum research in which a key activity is to appraise the mathematical qualities of an actual classroom performance against the backdrop of a number of other possible performances that might have happened instead or that might happen in a collection of classrooms. The perspective of the teacher as observer of the opportunity to learn is analogous to that of a soccer coach who might view the tapes of a game already played focusing on the qualities of the game that the team has played as a whole. A description of the mathematical qualities of classroom

performances from this perspective is figuratively speaking the dependent variable whose changes might be caused by changes in the deliberate activity of the teacher.

We have noted (see Balacheff, 2004) that in spite of the large corpus of research on proof in mathematics education, the field is in a sort of deadlock because of the lack of a clear demarcation of our object of study: Is the same notion of proof useful when one wants to know why an individual believes in the truth of a statement, when a reader judges the merit of a piece of mathematical writing, when a curriculum designer expects students to justify their answer in solving a problems associated with a specific conception, when a historian accounts for changes in mathematical ideas, or when a philosopher compares mathematics with natural science? We contend that the lack of agreement as to what is meant by proof in mathematics education research is partially due to lack of clarity in articulating the object of study and the perspective from which each researcher accounts for its object of study. As regards the appraising of classroom mathematical performances, we suggest that the extent to which classroom performances can arguably "count as proof" requires an observer to be apportioned of a special set of meanings for the word *proof*.

We use the word *performance* in alluding to classroom activity in order to stress the point that classroom activity is not merely a set of juxtaposed individual expressions or an emerging social compact, but also a systemic response to reified cultural constructions and an enactment of institutional roles. The word performance attempts to profit from the metaphor of the classroom as an orchestra where the score is the way the mathematical community views the mathematical objects at play but where the meaning is in the experience of the educated listener—which we allude to as we speak from the perspective of the teacher as an observer. Classroom activities are not mathematical performances just because the classroom is a mathematics classroom and not only when their performance is faithful to a mathematically vetted score, yet the observer needs means to support the claim that a classroom activity is a mathematical performance even when they may not have used an accepted definition, a conventional symbol, or a syntactically valid proof.

Four Constraints that Shape Mathematical Activity in School Classrooms

The complexity of classrooms activity and the perspective of the teacher as appraiser of this activity require attention to the singularity of classrooms as knowing organizations. These knowing organizations are singular inasmuch as they exist in response to four conditions, which we will refer to as I^4, and which are also constraints that these organizations have to contend with. Each of them imposes particular demands on the existence of proof in classrooms that need to be attended to in accounting for classroom mathematical performances.[3]

<u>Institutional</u>: The school institution presumes that activity in mathematics classrooms has a subject matter which, rather than being self generated, is issued from a discipline. This discipline exists outside of school both as an academic discipline and as a discipline of live scholarly work and it relies on school to be diffused to the public (for information and recruitment purposes). As members of a culture that sends children to school, teachers and children know that presumption. Thus the discipline of mathematics has a cultural prerogative that school citizens must acknowledge even if to no avail. The discipline of mathematics also has a cultural obligation of making itself available to oversee its representation in instruction—which is often done through the involvement of mathematicians in public opinion and textbook review.

[3] Note that in stating these constraints we are not *prescribing* or endorsing them as obligations that if fulfilled might create any desirable state of affairs. We are merely *describing* them as they appear to play out in school classrooms.

With regard to proof this means that since proof exists in the discipline of mathematics, it might be entitled to exist in classroom activity. And if it were to exist, it would be expected to exist in a form that is accountable to, if not compatible with, how it exists in the discipline. Furthermore, this institutional constraint entails that schools are expected to look into the discipline of mathematics for cues as to what existence and role proof should have in the class. The kind of existence that proof has in the discipline, as a methodological tool used in key functions of knowledge generation and validation, is a stakeholder on the existence that proof might have in classroom activity.

Instructional: The encounter between the student, the teacher, and the subject matter is not a free-formed voluntary get together. What brings them together at a certain time and keeps them together for some time also keeps them in specific places vis-à-vis each other. The student and the teacher are in an asymmetric relationship vis-à-vis the subject of studies: One of them is supposed to know it already, the others are not expected to know it yet, but are expected to come to know it later. There is an unwritten contract or agreement to be in that relationship for some period of time. Since breaches of this relationship are institutionally resented, the relationship has to adapt in order to continue existing for that period. Because they come to the encounter with official roles, as teacher and as student, these individuals know of their relative positions vis-à-vis the subject—what each of them is supposed to do and when it is possible for them to leave each other without penalty. In particular, assuming that proof is part of the subject matter that binds teacher and student together, proof must be "taught," "learned," "known," and "assessed" and the meaning of those verbs that matters is that ascribed to them by the school institution. More generally, the mathematical knowledge of students and of their teacher are supposed to be different and to grow differently: Students' knowledge is supposed to increase, adding new knowledge assiduously, whereas their teacher's knowledge is supposed to be relatively stable, particularly the teacher is expected to know already what the students have yet to know. Those assumptions have particularly important effect on the possibility that proof serves to generate or validate new knowledge: The teacher is expected to know ahead of the students which statements might be relevant to consider, and whether those statements will turn out to be true, even before the act of proving them turns them into theorems that students can then use.

Individual: Particularly after the 20th century, it is accepted that in spite of their shared role as students, individuals are and have the right to be different. It is institutionally accepted (i.e., members of the school institution cannot deny) that individuals differ in how they think, what they like, the rate at which they develop, their talents and their shortcomings. The issues in which they differ can increase in many directions without depriving individuals of their entitlement to be students. With regard to proof this means that it is conceivable that some students may not be convinced of the truth of a statement by (what a mathematician might deem) a mathematically valid proof. Some might be convinced by compelling arguments that, however, nobody would seriously call mathematical; others may not enjoy talking or arguing, others may have difficulties reasoning logically, and yet all of them will have the right to be that way within school. In particular the timing and the timescales by which the many attributes of individuals change[4] are not necessarily in synchrony with, and not expected to follow suit, the timing and timescales of instruction, at which the class is expected to take something "not yet known" as "already known." Hence the mathematically most valued proof of a theorem could quite

[4] Biological growth, identity, learning style, disability status, race, etc.

plausibly not be understood by some or not be convincing to others and these could be deemed acceptable reasons for the undertaking of additional activity to the same end.[5]

Interpersonal: These individuals, students and their teacher, are together under relatively lax external surveillance and over long stretches of time. They don't just enact roles or express selves, they don't just apply decontextualized strategies or else improvise what they do; they actually adapt to each other by developing public identities or personae over time and they manage that interaction tactically (Erickson, 2004). With respect to proof (as much as with any other object of teaching, learning, doing, or knowing) this means that the same (communicative) acts by way of which they relate to proof are also acts by way of which they relate to each other (Watzlawick et al., 1967). Thus a demand for proof, for example, is never just a request to establish truth; it can also be or be taken as an expression of distrust of someone else's word or an admission of weakness of one's understanding or a bid for a rapprochement to someone in power. The giving of a proof can also be or be taken as a gesture of arrogance or pedantry, an assumption of audience incredulousness, or a lack of respect for the audience's time. And, because interpersonal relationships continue over time, those relational bids may be consequential; they may have a cost that affects those relationships.

A setting framed by those I^4 constraints is notably different from other settings where individuals encounter proof such as the setting where a mathematician reviews a manuscript submitted for publication or a person tries to persuade a friend about the truth of a statement. To appraise the role of proof in classroom activity an observer needs to pay attention to several layers of activity. Within each of those layers something that "counts as proof" or a way to decide whether something counts as proof may or may not be found.

It is particularly important for mathematics education researchers to realize that our intellectual tools to describe proof need to go beyond the "institutional" constraint that classrooms are subject to. According to that institutional constraint the discipline of mathematics has the right and the duty to say what a proof is. Institutional researchers (e.g., those who might need to validate a school district's decision to choose a particular textbook on mathematical grounds) may not have other choice than to take the word of the mathematician at face value. But researchers in mathematics education who seek a basic understanding of classroom mathematical work can develop a more complex perspective by taking advantage of what Imre Lakatos taught to philosophers of science. In the preface of *Proofs and Refutations* Lakatos (1976) snickers at the consequence of the formalist presentation of self of 20th century mathematics, according to which there would have been no real mathematics before Boole made explicit the laws of propositional logic. When a mathematics education researcher inquires on the place of proof in classroom mathematical work they confront the same unbearable paradox that a philosopher of science confronts when looking at the role of proof in the historical record of mathematics: The methods by which classroom participants over a year's time (as much as mathematicians over historic time) produce and organize knowledge are as much the product of development over time (either historical or class time) as the knowledge itself. One alternative way of facing that paradox is to exclude any inquiry on the role of proof from any mathematics known by means other than those sanctioned by the discipline at its current state of sophistication. The other alternative is to own up to the informal, heuristic nature of the kinds of proofs that serve to shape mathematical ideas, snickering also at the presumption that all the mathematics students have

[5] Obviously that differs greatly from how a proof is accepted as proof in mathematical publication.

learned before being inducted into those practices that the discipline might recognize as proofs (e.g., before high school geometry or before the first course in linear algebra in college) should be rejected as not being real mathematics. Lakatos suggests that to understand the methodology of mathematics, the second alternative is the best—"even the poverty of historicism is better than the complete absence of it" (Lakatos, 1978, p. 61). But if we will rather do the latter, that is, own up to the informal ways in which mathematical ideas are developed earlier in students' life, how can we gauge the extent to which the performances that they are a part of are mathematical? Do those mathematical performances in school sustain a cultural image of mathematics compatible with Benjamin Peirce's characterization of 'the science that draws necessary conclusions'? (Peterson, 1955). One would like to be able to distinguish between justifications that are mere experiential confirmations of an outcome from justifications that argue for the necessity of a result. One would like to do that without making the distinction rely on the need to require clear definitions and axioms or symbolic formalism. This is what we propose to do here.

Three Timescales

There are three particular timescales (Lemke, 2000) that are important to attend to in describing the nature of proof in the mathematical performances of a class. For each of them, particular descriptive tools may be needed. Yet a comprehensive account for the role of proof in mathematics classroom work in general needs an integration of the analysis across the three timescales. The first timescale is that of the custom of a class—the basic question is what the place of proof is in the mathematical work customarily done in a class, over stretches of time of the order of the year. We will not presume, however, that a characterization of the customary place of proof in a class will provide us with a general notion of proof that individual proofs are particular cases of. After (and with) Lakatos we are highly suspicious of taking the relationship between *proof* and *a proof* as a case of category and element. But we acknowledge that communities, be those classes or working mathematicians, use words like proof and proving *as if* they were general categories. We propose that describing that general, customary usage is one important piece in trying to understand the nature and role of proof in a mathematics class. A class, that is a cognitive organization subject to the constraints I^4, has a custom (Balacheff, 1999), a set of habitual ways of relating to the enterprise of doing mathematics (see also Sumner, 1907/2002). That custom may have a place for proof, which could be reflected in language uses (e.g., the use of words like *prove*) and in habits of relating to knowledge (e.g., the expectation that after stating a claim one will provide support for holding it as true). The need to describe the role of proof at different timescales derives from the critique on the descriptive power of any general notion of proof sketched above.

Lakatos's critique of formalist mathematics entails that the nature of proof in mathematical work is not formal but substantial. Proofs give meaning to (and help refine) concepts and propositions. Consequently there is more to the nature of proof to be found in the examination of the doing of particular proofs than captured by any general idea of proof. Particular proofs of the Euler's conjecture, whether Cauchy's combinatorial proof treating a polyhedron as a net of triangles (Lakatos, 1976, p. 8) or Poincaré's vector algebra proof (Lakatos, 1976, p. 116), are different not just in what they actually prove, but also on what meaning they give to the conjecture, what steps they take as obvious and what ideas they dare to leave unanalyzed. The nature of a proof, for instance, (e.g., whether the proof is closer to a calculation than to an argument) depends on the particular system of representation available to address the ideas involved in the proof. A particular timescale at which to inspect the nature of

proof and design ways of describing this role is that of particular mathematical conceptions; stable practices being done at particular moments in time and where specific means for the control of how problems are solved may be called proofs. We propose the expression C-proofs to designate these.

A third timescale is finally suggested by the classroom counterpoint with which Lakatos narrates the historical development of Euler's conjecture. Specifically, notions of polyhedron, versions of Euler's theorem, their proofs, and their refutations are produced as results along the way that the dynamics of proofs and refutations seeks to prove "the" theorem about polyhedra. That is, in addition to the particular proofs that work to verify results within specific conceptions and the custom of proving that characterizes the mathematical way of relating to knowledge, there is an operational way of coming to know in which every new proof also helps improve the state of knowledge. Proving as a tool for knowing or coming to know, whereby some changes in conceptions or incorporation of new conceptions may be made by appeal to practices that might be called "proving" whereas other changes may not. The purpose of this chapter is to articulate a descriptive protocol for the place and the role of proof at those three timescales, the customary or habitual, the local tied to specific stable practices, and the operational that helps transition to a new state of knowing.

The Possible Place of Proof in the Custom of a Class

A class may or may not trade on the words *proof* and *proving*. That is, the words *proof* and *proving* may or may not be tokens of interaction invested of special meaning. The description of proof as an element of the custom of a class needs to include accounting for the customary uses of those words possibly comparing those uses with the uses that the words would have in mathematical discourse. But this description must also be attuned to other expressions (e.g., justify, explain) that might play similar customary role. Some basic definitions of terms from the observer's perspective may be useful to identify the register in which one could expect a class to craft its customary notions of proof and proving. Some twenty years ago one of us wrote:

> We call *explanation* a certain type of discourse that attempts to make understandable the truth character of a proposition or result acquired by the speaker. The reasons that he or she provides can be discussed, refuted, or accepted. We call *proof* (Fr. preuve) an explanation accepted by a given community at a given moment of time. The decision to accept it can be the object of a debate whose principal objective is to determine a common system of validation for the speakers. Within the mathematical community,[6] only those explanations that adopt a particular form can be accepted as proof. They are sequences of statements organized according to determined rules: A statement is either known to be true or deduced from those that precede it using a rule of deduction from a set of well-defined rules. We call these sorts of proofs *mathematical proofs* (Fr. demonstration[7]). (Our translation of Balacheff, 1987, pp. 147-148)

Thus, we are interested in describing the place of proof (and related words) insofar as they designate discourses applied to statements, (i.e., the contents of sentences that propose an

[6] *The* mathematical community is assumed to be the contemporary community of mathematical researchers that proves theorems and publishes these proofs as scholarly work.

[7] Note that most romance languages afford word couples equivalent to the French *preuve* and *demonstration* that can be used to distinguish these two senses of proof.

assertion as true) and that are aimed at establishing the truth of such statements. To describe the customary place of proof in a class means to describe the relative place of those discourses in relation to statements and compare that place with the relative place they have in customary mathematical work.

The extensive corpus of reflective writings by mathematicians as well as the products of their work could easily be surveyed to complement the notion of mathematical proof given above and its general, customary characteristics within the mathematical community. Much of this work has been surveyed, organized, and elaborated by Gila Hanna (see for example Hanna, 1982, 1987, 1995; Hanna & Jahnke,). Alexsandrov (1956) points to the abstract nature of the objects of inquiry and to the drive to make increasingly abstract claims about those objects that hence resist any method of proof other than calculation and argument. Proofs are given of general statements, often after the statements are made, to establish the truth of a statement, but at times also before the statements are made (as a way to produce the statement). Yet not all true statements are actually given a proof of their own; the proofs of some statements often yield derived corollaries and consequences whose proofs are omitted. Mathematicians' writing is full of "idea" and "sketch" of the proof—thus playing out a custom where showing that a proof exists is as acceptable (and perhaps less pedantic) than providing a complete proof. And yet having an expert check that the idea of the proof can be successfully deployed has been a key gatekeeper for mathematical publication or at least it was so until the era of email (Jaffe, 1997). Thus, contemporary mathematics as much as classic mathematics leaves records that might appropriately be called "descriptions of proof" (Livingston, 1999) since the actual proof is done anew every time a human uses those records to run through the argument, finding and filling the gaps. Mathematician's reflections about their trade and their teaching often address the role and characteristics of proof in their discipline in those general terms (see also Davis, 1972; Haimo, 1995; Thurston, 1994). Some of those reflections are keen to note that the customs of proving in mathematics are subject to change in larger timescales (see for example Jaffe, 1997).

In attempting to find out whether there is a customary notion of proof in a particular classroom, the descriptive focus has to be on whether a class has the habit and resources (semiotic or interactional) for demanding and establishing the truth of statements that participants attribute general characteristics, even if these are not the same kind of general statements that mathematicians would care to prove true. Evidently, it is possible that a class might not recognize its enterprise as one of producing and accumulating general statements which are true and whose truth character the class as community is expected to understand. It is also possible that whereas a class might be involved in acquiring true statements, its means to establish that truth were perceived as completely ad hoc, hence not enabling any general discourse as to the nature of its proofs. And it is also possible that whereas a class might be involved in the acquisition of true statements, that it had a discourse of method that addressed how truth has to be established, and even possibly made use of words like *proof* and *proving* in that context, a comparison with the elements of the customary notion of mathematical proof would reveal it as different. To describe the custom of proof of a class thus one has to track the use of key words such as *prove*, *proof*, and *proving* (as well as others such as justify, justification, explain, explanations, show) and observe whether they are applied to designate discourses that make understandable the truth character of statements.

In the notion of proof from Balacheff (1987) we note that in spite of the important role attributed to community and to time, and in spite of not requiring that a strict form be applied to every proof, this definition does not leave the question of what counts as proof as one of

voluntary agreement within a freeform group. Communities are never freeform groups; hence consensus is never just voluntary agreement. A mathematics classroom is a special community, one that exists tied to the I^4 conditions. The institutional constraint, for example, requires classroom members to act as if the proposition whose truth character is being explained (what is at stake in the doing of a proof) exists as true beyond the communicative acts in which someone is convinced of its truth. The importance of time in that definition cannot be overemphasized: Time is of the essence of what counts as proof because a community that at one moment in time might be interested in knowing that something is true is also one that at a later moment in time might build on their knowledge that such a thing is true. Balacheff's definition thus points to a necessarily circular relationship between "a proof" and "Proof" in any consideration of what is the custom of proof in a class. Historically developed customs for stating general propositions and accepting and recording proofs exist at any one time and are applied in the establishment of a new proposition and their proofs. At the same time those customs are being shaped in and through the activity of proving a new proposition, for example by the development and implementation of new heuristics or by the possibility to leave unsaid some things that can be taken as obvious. Because of that circularity, the habitual notion of proof may be useful for a community to request or expect that a proof be given to an assertion yet not sufficient to judge whether something given as proof counts as proof at a given moment in time.

In American classrooms, the word proof has usually been associated to the high school geometry class and this class has provided a setting for studying customs of proving. Sekiguchi (1991) did a semester-long ethnographic study that accounted for the custom of proving in high school geometry. Herbst (2002) tracked the historical evolution of a custom of proving in high school geometry—documenting what led to the establishment of the two-column form as the standard for proof writing in the early 20th century. At the beginning of the 21st century NCTM's (2000) Principles and Standards for School Mathematics have included Reasoning and Proof as a Standard for all grades and across all subjects. Principles and Standards grants that reasoning and proof might look different across the grades—that many customs of proving might develop depending possibly on factors that might include students' developmental level or available resources (material, symbolic, etc.). Research describing the many customs of proving that could unfold in actual classrooms as curricula, districts, and teachers try to implement this standard needs to be done (and contributions to this volume show that it is being done). But research on proof in classrooms needs to go beyond the description of the customary notion of proof embedded in the way a class approaches the general enterprise of doing mathematics.

Proof as a Regulatory Structure for a Conception

In the preceding section we noted the need to examine the customs of classroom work for indications of whether there are recurrent concerns with the formulation and establishment of general statements and the explanation of their truth character—which could possibly be satisfied with the imperative to prove and the provision of proofs. We raised awareness that a particular class might not have such custom—they may have no recurrent ways of deliberately engaging practices of that kind, let alone a discourse that establishes expectations on method. Yet mathematics classrooms are always involved in some work, and within particular pockets of activity some particular ways of acting deserve being considered proofs. It is a different timescale and a different descriptive register to which we now switch gears.

We do not provide an a priori epistemology of classroom public knowledge, but rather use a set of basic elements and relationships that generate what an observer can take as tools to

describe the mathematical performances of a class in the neighborhood of particular moments in time. We propose a definition of a different notion of proof, ascribable to what a class may be doing at a given moment considering what it has in hand as goal and as resources. In the following we use the expression *conception* following its usage in Balacheff & Gaudin (in press; see also Balacheff, 1995) to refer to stable ways of in which students in a mathematics class relate to particular organizations of their mathematical milieu around a particular moment in time. We realize that the expression "conception" is commonly used in relation to individuals rather than to classes and that to make it ascribable to a class makes sense only when the observer takes on the perspective of the teacher for whom the class can be represented as an abstract student, a *modal* student (Herbst, 1998). Balacheff & Gaudin (in press) argue that the expression *conception* can be used to refer to stable practices (in contrast with, but not incompatible with, its canonical usage to identify mental structures).

Building on Bourdieu's (1990, p. 83) notion of *sphere of practice* we have identified units of mathematical performances and called them *conceptions* (Balacheff, 1995; Herbst, 2005). Balacheff (1995) proposes that what the cognitive-based literature had alternatively called misconceptions, bugs, errors, and alternative conceptions are all comparable cases of adaptation between a cognizing agent (a student, a class) and a milieu, under proscriptive conditions of practical viability. And the relative independence of different conceptions (which an observer could relate to the same concept), which the cognitive literature used to denounce as the coexistence of contradictory conceptions in the same person, is sustained not just by different practical conditions but also by different placements in time (including changes in time across settings such as in and out of school, or within a setting such as 'my class this year' vs. 'my class last year'). This practice-based notion of conception is addressed in what we call the cK¢ model, where a conception is modeled as a quadruplet $C = (P_C, R_C, L_C, \Sigma_C)$, and where

> P_C is a set of problems or tasks undertaken successfully by an agent around a given moment of time,
>
> R_C is a set of operations that the agent could use to solve problems in that set,
>
> L_C is a system of representation or semiotic register within which those problems are posed and their solution expressed, and
>
> Σ_C is a regulatory structure used to control the adequacy of the solution to the problem and warrant its solution

We contend that that model of conceptions can, in particular, be used by an observer to describe molar units of public mathematical performance in the mathematics classroom, on account of the metaphor that the class is, at least for a teacher, like a student (the modal student). We argue that some of the conceptions of a class may be ultimately regulated, or that the adequacy of solution to problem may be ultimately controlled, by what could be called proof.

A canonical example is what could be called the "correspondence" conception of congruence in the high school geometry class. In high school geometry students are often posed problems that require them to decide whether, or to show why, two segments or two angles are congruent. Other problems that mobilize the same conception can conceivably require establishing a ratio between the measures of two segments. Unlike in earlier grades, high school geometry students are not expected to answer those questions by measuring and comparing numbers. Instead, they are expected to visually inspect the diagram to find two triangles about which the measures of some of their elements is known and that can be mapped to each other,

mapping also to each other the elements to be compared. If the corresponding triangles can be claimed congruent, a comparison can be derived about the desired elements from the definition of congruence (which says that corresponding parts of corresponding triangles are congruent). A particular kind of proof, built on the two-column format (Herbst, 2002), is often used to control the congruence of the triangles. This control strategy takes for granted the existence of the triangles to be compared—that is, triangle congruence proofs rarely include any warrant for the drawing of auxiliary lines. The proof format, however, requires the agent to list in separate lines each pair of elements of the two triangles that are known to be congruent, justifying each in reference to "given." Further, the two-column format requires the agent to put those pairs of congruent elements in relative order (e.g., a pair of corresponding congruent sides, then a pair of corresponding angles adjacent to those sides) and to compare the order in which they are listed to a list of sanctioned congruence criteria (e.g., SAS—side, angle, side—but not ASS). More often than not, to perform these congruence comparisons is the bulk of what is done on account of learning proofs in high school geometry. We have described it in that detail to impress upon the reader the point that whereas it is undeniable that the way in which claims to congruence are controlled deserves the name proof, the particular nature of what is expected to be involved and what goes without saying make these "proofs" particular to a conception of congruence—that which reduces congruence to triangulation and uses triangulation to organize an object-to-object correspondence. We use these as a first example of what we would like to call *C-proofs*, to refer to a particular kind of regulatory strategies of conceptions. We want to sensitize the reader to the possibility that particular ways of ascertaining (Harel and Sowder, 1998) that the solution to a problem is correct attest to a kind of reasoning that albeit particular to a specific conception are also singularly mathematical. In the case of triangle congruence, the particularly mathematical feature of that regulatory structure is the preference to reduce the question asked to one of a set of known cases where the information given is sufficient to assert congruence without actually confirming that congruence directly. In other words, that the solution to the problem is necessitated by the conditions of the problem and the affordances and constraints of the representation system, as opposed to contingent on happenstances related to who did it or when it was done. Let's play out that definition for a more elementary case, where the word proof may not even be used.

In early elementary school, students learn to solve multi-digit addition problems by putting numbers in column, taking the columns in order from right to left, adding the digits in each column, putting down the last digit of the sum for each column and carrying the other digits to the top of the next column. Place value, the notion that a number like 176 is 'one hundred plus seven tens plus six units,' is part of the representation system (whose basic elements are the digits 0 to 9 and the powers of ten but also, crucially, the expression "7 tens" and its surface similarity with "7 apples"). If 46 is claimed as the solution to the addition problem 17 + 39 = ?, the adequacy of this solution could be checked by "doing it again" as many students actually do, and getting confirming or disconfirming results. For example a student could get again 46 or else get 416 or even, hopefully, 56. "Doing it again" is a regulatory structure. In some classes "doing stuff again" may be the only customary way to control results. We are not willing to call "doing it again" for the case of multi-digit addition a C-proof: Doing it again hinges on repeating the experience of solving the problem and taking the aggregate of experiences as evidence for the correctness of a decision. It seems as though this is a clear case of empirical induction: If the majority of the times one solves the problem it happens to yield a certain result, then that result has to be the solution to the problem. Note however that in similar circumstances the solution

can be demonstrated as necessitated by the problem and the properties embedded in the system of representation. The calculation could be checked by translating "17 + 39" into "16 + 40," which has to be 56 or at least much bigger than 46. The latter uses the characteristics of the representational system (place value) to avoid repeating the experience of solving the same problem again. Rather, it changes the problem into another, simpler problem whose solution is arguably prior knowledge. The having of that prior knowledge entails the correctness or, as in this case, the incorrectness of the proposed solution. We consider this kind of regulatory structure for multi-digit addition to be analogous to triangle congruence proofs in the correspondence conception of congruence even though there aren't many surface similarities. We are willing to propose both of them as C-proofs in spite of the fact that the "translation to a simpler problem" might rely on reasons which are still implicit (e.g., taking 1 from 17 and putting it with the 39 to make 40 is possible because of the associative property but one would not expect a young child to have to say so).

In terms of describing the nature of proof at the timescale of the conception, the first point to make is that one needs to start from modeling the conception itself. Locating the actual and possible regulatory structures will provide with means to argue that a regulatory strategy that could be considered a C-proof is within reach of the class at a given moment. For the same reason, there are evident ties between those C-proofs and the conceptions they help regulate. Congruence claims can be proved in that way because the correspondence conception of congruence does not require one to specify a global mapping of the plane but just a local mapping object-to-object (proofs would look very different in a conception of congruence regulated by the algebra of transformations of the plane onto itself). In the early arithmetic case, "transforming the problem into a simpler or known problem" can control the result of a calculation because a large amount of information is packaged within the grammar the 10-base numeration system . The observer can ascribe a conception to a class, say that at a given moment in time the class "holds" (i.e., the class works as if it held) a certain conception (e.g., multiplication of polynomials as multiplication of numbers), or that a certain conception is present in the work that a class is doing at a given moment in time. That descriptive claim does not mean that any one student is in possession of that conception, but rather that the modal student is held accountable for performing in accordance with that conception in the sense that cK¢ gives to conception and in the sense given above to the modal student (the normative counterpart of the teacher in classroom interaction).

A first place where an observer may recognize something that deserves to be called proof is thus within a conception. By definition of conception, the control structures of a conception are such that they inject a truth-value on the solution of a problem. If the injection of truth is obtained as a necessity of the formulation of the problem, that is, if the control operates by way of demonstrating that given the objects one had in the definition of the problem, and the means of representing them and handling those representations, the solution found could be reconstructed without reliving the experience of finding it, we say this control structure is a C-proof. Evidently the participants may call what they do to justify the solution to a problem a number of different things, including "proof" but not necessarily so. The process by which a real solution to a problem is rationally reconstructed (using a system of representation and its

[10] Their currency status could be instructional but even before being instructional currency, they could be interpersonal (I have a different idea) or individual currency (I have an idea).

operational grammar) beyond that which one happened to find and into that which one should have found is what we want to call a C-proof.

In particular, for a conception to have a regulatory structure that deserves the name of C-proof, the system of representation must among other things be capable of representing not just the referents in the problem but also the means of operating on those objects (so that the experience of operating cannot add actions on which the result depends that the system of representation cannot handle) and what the collective holds itself accountable for knowing about those objects. These proofs are conception-dependent and not all conceptions would have control structures that satisfy the conditions for being C-proofs in this sense. That is, in some conceptions the validation of solutions to problems is empirical, contingent on experience, in the sense that it requires recurrent engagement in the solution process (and confirmation by reproduction of the same result). Furthermore, in some classes the custom of mathematical work might include a discourse on method that legitimizes determining the truth of general mathematical statements by empirical means.

Thus, the notion of C-proof just defined and the notion of proof proposed by Balacheff (1987) and appropriated here to find and describe the custom of proof in a class are not easily related in terms of object and category. C-proofs are not only conception-specific but also, and for that reason, they may hardly adhere to a common form. Furthermore, classroom participants might not even call them "proofs." To sustain a custom of proof, some form is needed (and by form we are being inclusive of logical form as well as graphical and linguistic form).

Any conception that the class is held accountable for has some means to regulate solutions to problems (at the very least by having the teacher check the student's answer against a key. It seems appropriate however to keep calling the totality of those possibilities "regulatory structures" or "control structures" and to save the word *C-proof* for more special ones. Even a class whose custom of proving avowedly defers to mathematical proof, such as that of the high school geometry class, might potentially regulate some of its conceptions in ways that are contingent on experience. An example of this is in the many conceptions of *figure* that are drawn upon in various kinds of work in high school geometry classes, some of which are controlled by perception and others by measurement (see Chen, forthcoming; González, 2005; Hamlin, 2006). But the presence of visual perception does not automatically make a control strategy empirical, as advocated by Davis and by the tons of examples of "proofs without words." That particular subject only underscores the importance of understanding the conceptions at play in a proof, in particular the semiotics of the diagrammatic objects being perceived.

In general, some conceptions are sufficiently formalized (that is, have a system of representation sufficiently rich and precise) to dedicate control structures that operate on semiotic translations of the experience of solving a problem, translations related to the model or "theory" of the objects and operations involved in the problems, a theory expressed in terms of the system of representation. That is to say, a technical "text" or "discourse" replaces the action of solving a problem by expressing why the result *should* be what it is. Note that this notion of C-proof does not a priori distinguish symbols like ○○ or 17 from symbols like x or pq. They all are signs, even though some may be more amenable to an operational grammar than others. Each of them may or may not be part of a system of representation that supports C-proofs for a certain conception in a certain class, but the extent to which they do depends not on what they look like but on how their grammatical/rhetorical behavior within the system of representation allows them to contain the experience of solving a problem and in such a way obliterate the need to run that experience again to confirm a result.

Proof as a Tool to Know With

The previous consideration of the differences between a custom of proof and the specific notion of C-proof shows that those two notions together do not fully grasp the whole of the work of proving. A custom of proof close to the custom of mathematical proof is epitomized not in the cursory C-proofs of stable conceptions but in the breakthroughs of particular ideas that happen at particular moments. The having of great ideas, such as to draw an auxiliary line in a diagram in order to be able to consider two triangles that were not previously there, is often outside the usual triangle congruence proofs in high school geometry. Yet, those great ideas are what often gets recorded in the "description of proofs" (Livingston, 1999) that mathematicians write. For some researchers those may be aspects relevant to mathematical creativity or insight rather than of proof. Poincaré (2001) referred to them as intuition, which he opposed to rigor; Pölya (1954) dwelled in a similar distinction between discovery and proof. But what Poincaré called intuition was neither necessarily empirical nor necessarily unfathomable. Pölya spoke of the importance of "plausible" reasoning as a mathematically educated way of guessing what could be true. Lakatos (1976) showed that the notion of informal proof is one of the key methodological tools for the mathematician to exercise and control that plausible reasoning. Picking up on Pölya's ideas, Lakatos described mathematical reasoning as neither deductive nor inductive nor intuitive, but heuristic. How can we describe this notion of informal proof in ways that we can use it for classroom observation? If the dialectic of proofs and refutations was to exist in a class as the means by which new knowledge is generated, how could we be attuned to see it?

The previous discussion of conception, which led to defining C-proofs, provides tools for ascribing conceptions to students on the basis of students' being capable of doing some things, not necessarily of being aware of having any knowledge that warranted or entailed such actions as reasonable. Note, however, that the system of practices that an observer models as a conception could also be an object of public reflection for the participants. This is particularly the case in classrooms, at least for some conceptions, as a consequence of the instructional constraint noted above: Students are in classrooms not just to do things but also, fundamentally, to learn. To endure, the contract needs to evolve. The class needs to be able to acknowledge a transition in their responsibilities, from a state in which they were entitled not to know something to a state in which they are accountable for knowing that thing. This key temporal constraint, the chronogenesis of knowledge (Chevallard, 1992), suggests as a necessity of an instructional system that at least some public discourse about public action will be done by participants (usually the teacher) to install some practices as objects of knowing that the class now has. Thus, from an observer's perspective, participants' work does not just attest to the existence of a conception but might also sometimes attest to the class's reflective awareness of *having a conception*. The naming of chunks of action, identifying what students have done with what they had to learn, which includes what Brousseau (1997) calls *institutionalization* (the giving of an official knowledge status to a notion present in students' work), is an important component of classroom work that warrants us being able to speak of classroom mathematics as public knowledge, where the public may be expected to know even if nobody in fact knows. The particular form taken by the knowledge that the class is held accountable for having can of course vary: They might just explicitly acknowledge knowing how to do 'these kind' of problems, or perhaps knowing how to perform a general process instantiated in 'those and other' problems, or even knowing that 'such and such' thing is the case. Of course, frequently classroom discourse makes evident this accountability for knowing and what the class is accountable to know. Our claim is that being accountable for knowing some conceptions (in

addition to acting as if they had many more conceptions) is a necessity entailed by the I^4 conditions that frame a school classroom. On that accountability for knowing one can build the basis for a third sense of proof, anchored in a timeline where transitions are located from one to another state of public knowing. In developing a third way to answer the question of what counts as proof we will be especially attentive to reflective work that involves using a known conception to create or understand a new one.

We use the expression *C-reflection* as a generic term to refer to actions in which a public stock taking of a sphere of practice happens—actions whereby coherent ways of doing become symbolic goods of a public, things held in public and endowed of some currency status.[10] *C-reflection* is a generic term to describe the reflective practices by which a public comes across as making a claim about their state of knowing. C-reflection is the generic for those actions that actualize for a specific conception the reflective capacity of a public or its capacity not just to be a community of doers but also a community of knowers. We intend to locate a third place for proof in those moments of reflection when practices that ordinarily exist separately are juxtaposed, related, and compared. To illustrate this point we refer to the work of the first author on conceptions of equal area (Herbst, 2005).

Within the cK¢ model, the expression "knowing of μ" or Kμ is used to denote a set of conceptions that an observer (provided with a mathematical idea μ) can ascribe to a person or a collective. For example, an observer could note within an interval of time in which a class works on area problems, different conceptions of equal area, the whole set of which constitutes by definition the class's *knowing of equal area*. In Herbst (2005) a corpus of lessons where eight different classes had worked on the "triangle problem" (find a point in a triangle that would split it into three triangles of equal area) was used to demonstrate the existence of four different conceptions of equal area: empirical, congruence, equal dimensions, and quantitative. To remind the reader, the empirical conception (EMP) indicates that two triangles are of equal area if the area formula applied to the measures of their bases and heights yields equal areas, whereas the congruence conception (EAC) indicates that two triangles are of equal area if they are congruent. The equal dimensions conception (EAED) indicates that two triangles are of equal area if their bases and their corresponding heights are congruent (no matter what they measure), and the quantitative conception (QEA) indicates that two triangles are of equal area if they can be shown to be produced by scalar operations (adding or subtracting shapes, and multiplying a shape by a number) on triangles known to be of equal area (see also Herbst, 2003, 2006). An observer could formally establish a set of relationships among the conceptions in that set by defining translation functions among those conceptions. It could be shown, for example, that the quantitative conception subsumes logically the equal dimensions and the congruence conceptions as particular cases. Yet it is interesting to note that chronologically, the quantitative conception emerged after the equal dimensions conception and this one after the congruence conception. Students initially approached the triangle problem with the empirical or the congruence

conceptions—in one case by accepting the problem as one about any triangle and in the other case by choosing a triangle where they could construct a solution that could be proved correct. This work on equal area is relevant to our present purpose because it afforded the following observation: The transition from EAC to EAED and then to QEA was made possible by explanations that, while responding to the demand to prove that was customary of claims in those geometry classes, could not be considered examples of the C-proofs of triangle congruence noted above. We will describe those explanations insofar as their resemblance with and difference from the C-proofs of triangle congruence next, we will then reason abductively from this example to propose a third sense of proof which we'd like to call K-proof.

The EAED-arguments used by students to control why they claimed two triangles were of equal area were rarely written, let alone written in two-columns. They involved little precision of notation (the equal and congruent sign were used without distinction) and incomplete justification (all statements were not justified). Some things were taken for granted, such as the validity of the area formula, which was neither a theorem not a postulate at that time, but just prior knowledge. Rarely those arguments would have been recognized as proofs that meet the standard of a usual two-column, congruence proof. Yet, as we noted in the reports of that study, students responded to the request to prove that non-congruent triangles could be equal in area. They did so by way of using bits and pieces of the control structures normally used to prove triangles congruent. The emergence of the EAED conception was based on resources from both the empirical (EMP) and the congruence (EAC) conceptions. The empirical conception contributed the expression of the area formula (which in EMP is an operator) as an element of the semiotic register of EAED. In EAED the area of a triangle T could be *expressed* as $A_T = \frac{1}{2} b_T \bullet h_{b_T}$ (as opposed to merely *calculated* by executing that formula, as in EMP). The congruence conception contributed a way of tracking corresponding congruent elements across two triangles to be proved of equal area: Students looked for sides, one in each triangle, that were known to be congruent and could be taken as bases, then looked for the corresponding altitudes (which some times had not been drawn) and checked whether they were known or could be argued to be congruent, they marked these congruencies with hash marks, and equated (in speech or using inscriptions) literal expressions of the areas of the two triangles being compared. In view of this, we submit that the C-proofs of triangle congruence were being used as a metaphor to create a viable way of controlling equal area claims for the case when the triangles compared were not congruent. The metaphor mapped congruence into equal area, elements (sides, angles) into measures of dimensions (measure of bases, measure of altitudes), and the practice of marking congruent parts with hash marks mapped into marking or pointing at congruent dimensions with hash marks. Furthermore, the metaphor mapped congruence criteria (ASA, SAS) into the area formula, to produce the sense that it sufficed to show bases and heights are congruent to assert areas are equal without calculating either one of these areas. Thus, in spite of the facts that the explanations of why areas were equal were not in general written in the two column form characteristic of triangle congruence proofs, that the area formula conveniently hid the problems associated to geometric quantity and its measure, and that the area formula for triangles had not yet been proved (and actually no axiomatic definition of area had yet been provided), those explanations were accepted as appropriate responses to the request to prove triangles were equal in area. The productiveness of considering these kinds of mappings between control structures metaphorical becomes more apparent when we consider the control of claims under the quantitative conception.

One problem where we observed what we called the quantitative conception of equal area was that of picking a point *inside* a triangle so that when this point was connected with the three vertices two of the three triangles created would be equal in area. Students picked a point on a median (*O* on *BM*, in Figure 1) and claimed that two of those triangles (△AOB and △BOC in Figure 1) were equal in area. To prove that claim, the proposed argument used the observations that on the one hand △ABM and △CBM and on the other hand △AOM and △COM were equal in area (which students warranted using equal dimensions). They then argued that one could subtract the second pair of equal area triangles from the first pair of equal area triangles to get the targeted pair of triangles and that since they were subtracting equals from equals, the results would be equal. Once again, we submit that the C-proofs of triangle congruence were used as a metaphor for a proof of area equality. In particular, the imperative that statements be warranted by reasons was behind the identification of a property of area--the property that asserts that equals subtracted from equals yield equals. We consider the correspondence to be metaphorical (as opposed to considering this case of proof just an adaptation to area of the kind of C-proofs used in triangle congruence) because on this particular point, the reason given was not a known postulate, theorem, or definition, but rather a new piece of knowledge about equal area. In Herbst (2005, 2006) we provide further evidence to claim this proof was different than the C-proofs of triangle congruence referring to the notation in which equality of areas was inscribed and the mode in which explanations of equal area were expressed.

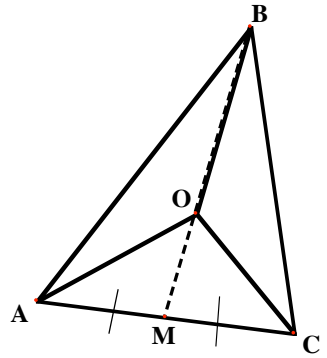

Figure 1. Two triangles of equal area.

The two examples discussing the way equal area claims were explained in the EAED and QEA conceptions of equal area demonstrate that the emergence of awareness of new conceptions of equal area was made possible by controlling claims of equal area in those new conceptions using metaphorically the C-proofs of triangle congruence. We have used the notion of metaphor, of a correspondence that maps systems of meanings (congruence and equal area, in this case; Black, 1962, 1979; Herbst, 1997) to argue for the "likeness" of control structures for different conceptions. We now use that metaphorical mapping between control structures to identify in general a third sense of proof.

Take the particular correspondence between, on the one hand, observing three pairs of congruent parts and deriving that two triangles are congruent, and on the other hand observing two pairs of equal area triangles and deriving that a third pair of triangles (made out from combining those) are of equal area. The presumption that such a correspondence should exist is manifest in the making of one and the other procedure be responses to the same habitual word—*prove*. The taking for granted of that presumption warrants, in this case, the introduction of a

new piece of knowledge—that equals subtracted to equals are equals.[13] It becomes reasonable that such property of areas would be valuable to have insofar as it helps bolster claims about equal areas. We would like to call *K-proofs* those metaphorical mappings between the C-proofs of established, known conceptions and the possible C-proofs of new conceptions. We submit that the existence of those metaphorical mappings, the presumption that one can operate on a new conception as if it were a known conception, and in particular, the presumption that one should be able to control the truth of claims in the new conception as if they were claims in the old conception, witnesses of a practice that one would want to associate with proving. Indeed, the work of such metaphors can both produce new knowledge and establish such knowledge as plausible, reasonable. Unlike C-proofs that control the truths of claims within a conception, K-proofs control the provisional mapping between conceptions and in particular help develop new conceptions. Similarly to C-proofs where the mechanism is one of reducing an unknown problem to a known or simpler problem, the mechanism of K-proofs enlarges a state of knowing into a larger state of knowing by projecting metaphorically the control structures of old, known conceptions into controlling claims about (new) solutions to (new) problems that could not previously be handled. The name K-proofs attempts to identify K-proofs as key contributors to those C-reflections on old known conception when these are brought over to control metaphorically claims that pertain to the new conception.

Thus, we contend that such segments of time when a class engages in non-trivial C-reflections (by nontrivial we mean reflections that actually involve at least two spheres of practice) are places where a third sense of proof can be found. Specifically, we propose that C-reflections are candidate moments to observe mathematical performances that might count as proof and whose purpose is for participants to publicly warrant the bringing in of new ideas, to organize how newly acquired elements of public knowledge integrate with the old, and to argue for the plausibility of work they anticipate doing in the future. Evidently all nontrivial C-reflections will not necessarily include performances that deserve the label "proof": These C-reflections might organize spheres of practice in many different ways, for example as juxtaposition of cases organized in a taxonomy. More important than elaborating on what strategies they might use to structure their old and new public knowledge is to specify what proof might look like when doing this job.

We remind the reader that earlier we introduced the idea of C-proofs to refer to conception-specific control structures characterized by a translation of the experience of solving a problem in terms of the system of representation used to solve the problem and express its solution, that is the obliteration of the need to rerun the experience of solving the problem. To define K-proofs we will use a similar strategy where what we handle are now spheres of practice rather than problems.

We will postulate that for C-reflections that involve different spheres of practice (different systems of meanings) to include K-proofs, the observer must see the class engaged in mapping one system into the other in such a way that the C-proofs of one of those spheres of practice are applied to control the solutions of problems in the other. This kind of mapping we consider to have the same formal structure as what Max Black (1962) called a *metaphor*, whereby parts of two systems of meanings are associated by a mapping that permits to project

[13] This analysis is simplified. To complete it one would have to consider also a metaphorical mapping between semiotic registers, for example between a segment and a triangle, between the measure of a segment and the area of a triangle, and between the properties of the morphism that maps segment union into number addition on the one hand and the properties of the morphism that maps plane figure union into area addition.

the structure of one system onto the other one and, possibly, use the projected structure to "create" elements in the least developed system. In language use, metaphorical projections start from the proposition of a metaphor such as "the poor are the Blacks of Europe" (Black, 1962) which suggests a projection of the known system of social relationships articulated by race in the US onto a system of social relationships articulated by class in Europe. By the use of the known system, race in the US, metaphorical projection warrants the posing of questions such as "which policies sustain the segregation of classes in Europe?" or "do the Europeans have anything like affirmative action?" A similar metaphorical mapping between number and operations on the one hand and polynomial algebra on the other is the one often used in school algebra to warrant the algebra of polynomials: The control structures of number calculations are used to control the good definition of operations among polynomials—at a time in algebra studies when since no formal definition of polynomial has been provided, the operations among them could hardly be defined other than by metaphorical projection. Thus, a metaphorical projection postulates a metaphor between systems of meanings and affords a heuristic to find out things about one of the systems. Participants may engage in mapping different spheres of practice in the context of C-reflections and thus actually develop public knowledge of any one of those spheres of practice.

Public knowledge of new mathematical objects can develop as a result of engagement in proving that hinges on the "making believe" (or "metaphor") that some new unknown objects are like old known objects and can be controlled with similar control structures. For example, in the case of polynomials and numbers this means that the practices associated with polynomial algebra may appear to be rational in spite of the vagueness with which the indeterminate is usually introduced: In spite of the fact that rarely people know what the X means in the polynomial X^2-1, they can accept that $X^2 - 1 = (X+1)\cdot(X-1)$ by virtue of the distributive property which they expect to hold in this domain as a result of the metaphorical projection between numbers and variables. Students can in that way organize a "theory" or proof-organized knowledge of polynomials even when the basic objects and their properties have not been formally constructed. The example discussed at length on the metaphorical mapping between congruence and equal area demonstrates how it is possible for a class to develop proof-organized knowledge of equal area even when the notion of area itself is informally defined, tied to the application of a formula that does not yet rest on an axiomatic definition. Herbst (2005) makes the point that characteristics of two-column proofs of congruent triangles (a C-proof) were projected metaphorically into those new conceptions as a result of the expectation that areas of non-congruent triangles be shown equal. The projection was metaphorical because the two-column proof format was applied on a set of problems where it usually did not apply and where not all conditions for application were available. In making this point we must underscore that the capacity to develop proof-organized knowledge rests on the one hand on the existence of a habit or custom of demanding and providing proofs to decide on truth and falsity and on the other hand on the existence of conception-specific means for control of solutions to problems that replace the experience of solving them with a surveyable record of the solving inscribed in the semiotic register of the conception. K-proofs are therefore informal (metaphorical) instruments of knowing which respond to the presumption that a general sense of proof exists, but operate on the presumption that one particular proof of a particular conception will afford and warrant the new ideas that will flesh that proof out.

What Can an Observer Do with the Theoretical Elements Provided Above?

The previous account provides ways of thinking about the places that proof might occupy in the development, management, and organization of public knowledge in a class. To summarize, one can expect proof to exist at three distinct levels in a class.

First, proof can be a general element of the custom (the habitus or the folkways; Bourdieu, 1990; Sumner, 1907) of a mathematics class, as evidenced by a number of general expectations about official knowledge and uses of language at work: if propositions are made that can be true or false, if speakers can be held accountable for explaining the truth of what they propose, if the request to prove or explain or justify can be met with more than blank stares, we can say that a class has a customary notion of proof and we might be able to describe it in detail. In the most relaxed sense, proof is a word that calls for something, anything; in the most restricted sense, proof is formal proof, where all objects mentioned are primitive terms or already defined, all propositions invoked as reasons are known to be true, all new statements are justified, and so on. Neither extreme characterizes the custom of a significant number of classes—in particular not even the most "traditional" high school geometry class has a customary notion of proof that deserves the label of formal.

Second, proof (as C-proof) can be a specific regulatory structure for some conceptions: the solutions of some problems may be controlled by procedures on the representation of those problems and the objects involved. An example of this from high school geometry--two-column proofs of triangle congruence applied to statements about the congruence of segments and angles of geometric figures that can be decomposed into triangles--has been extensively studied by the first author (Herbst, 2002a, 2002b; Herbst & Brach, 2006). The practice of reducing an addition problem to a known addition fact by way of transparently true operations was proposed as an example of C-proof that could very well exist in a class where no custom of proof exists (and perhaps no use of the word *proof* exists either).

Third, proof (as K-proof) can be a way of relating two or more conceptions, providing ways of operating on one conception "as if it was like" operating in the other. Thus (1) helping impose a(n) (informal) reasonableness onto other new conceptions whose vagueness might compromise their capacity to have native C-proofs as control structures and (2) helping create the impression of rational organization and memorability in the aggregated public knowledge of a class.

The question of what counts as proof really has three possible answers in any given class. The observer can use the foregoing account to locate those levels of activity and describe how the performances of a particular class relate to the performances that the observer judges possible given the conceptions and knowings that the observer can ascribe to the class. When this observer is the teacher, or somebody else, attempting to describe the extent to which the work of teaching in that class has made room for proof, these three notions of proof can help operationalize that description. It can do so without imposing a formal, one size fits all, criterion on what counts as proof.

The observer can tell whether or not the work of teaching in the observed class has fostered a custom of making general statements and requesting proofs of their truth. The observer can bring to the analysis of those practices the set of values that constitute the mathematical habitus: Are memories of past knowledge stored in general statements (statements about general objects) that could be true or false? Are these statements confronted in regard to their truth or falsity? Are expectations of what can be done next expressed in general statements that could be true or false? Is truth or falsity ever addressed as if it could be decided a priori,

before experience? Are participants expected to give or receive explanations of their truth that should convince a modal student (as opposed to actually convince individual students)? Are explanations preferred the more general they are? Are words like "proof" or "explanation" used in reference to the problem of truth? Does the language used attest to the existence of linguistic tokens that might help express generality and necessity (e.g., any, all, if…then, thus, should, has to, would be)?

Independently of whether the word proof has currency in the custom of a class and whether general propositions are made that could be true or false, our observer could also look at the work done to control problem solving within specific conceptions. In each of the **conceptions** held by a class over a certain span of time, proof might or might not exist as the (ultimate) control structure for judging solutions to problems in specific spheres of practice. In examining the performances of a mathematics class at various points in time, the observer can determine whether the students or the teacher could have validated a solutions to a problem by showing that the solution found experientially could have been anticipated. If the solution could have been anticipated from transformations of the problem using the symbols and grammar of the system of representation with which they encode the problem and its solution, our observer could say that such conceptions are controlled by C-proofs. We called C-proofs these uses of the regulatory structures of a conception to emphasize that, insofar as they depend so much on the objects judged relevant by the problems solved and on the signs and grammar of particular representation systems, these control structures, unlike the vague ideas of proof, truth, and generality of the custom, are very specific and conception-dependent.

The two strata noted before, the stratum of the habitus and the stratum of conceptions, are opposed insofar as they possibly allocate a role for proof. The former is general and concerns the values that characterize a mass of knowledge over a long stretch of time, the latter is specific and concerns the validation of specific knowledge claims at given moments in time.

In the production of mathematical knowledge by mathematicians, custom and conceptions support each other: general, shared ideas of what a proof is and what sorts of things are interesting to prove guide the proposition of specific problems and their solutions where original proofs can be found that solve those problems and that at some points inform the customary notion of proof as well. The notion that to incorporate these specific solutions to problems to what the community knows, they need to be represented in ways that make its organization memorable and rational (for example, by defining explicitly new concepts and making explicit which pieces of prior knowledge a proof relies on) could also inherited from that custom. But a class is a very different animal than the mathematical community, particularly as it has to do with the I^4 constraints noted at the beginning that make the developmental paradox cited at the beginning much more of an issue than in the case of history of mathematics. The decision to teach X at a certain point in students' development often entails the need to compromise the discipline's views on X for what can be done in schools. As the experience of the new math demonstrated, there is no way one can teach X without incurring in some serious degree of dishonesty with the discipline as regards to what X means (pace Bruner, 1960). Thus the question—can the public knowledge of a class be mathematical in any meaningful way? Can it be proof-generated knowledge in any sensible use of the word proof?

The previous section, where we develop the notion of K-proofs as metaphorical projections of the C-proofs that control selected conceptions (exemplified in how triangle congruence proofs are metaphorically used to control an emerging conception of equal area or how number and operations are used metaphorically to control an emerging algebra of

polynomials), shows how one could observe whether and how proof plays a role in the development of new knowledge. Specifically, an observer who has used the notion of conceptions to parse the stream of classroom events into spheres of practice that he or she can associate to conceptions, would also, as a result of that parsing reveal moments in classroom discourse where some reflective discourse is engaged. In particular, the observer could identify discourse that elaborates on those spheres of practice by relating some of them to others. Whenever the C-proofs that operate to control problem solving in one of those conceptions are used metaphorically to produce further reflections on another sphere of practice (e.g., to improve a system of representation, to control the solution to a problem, etc.) that work could be called a K-proof. We contend that in these circumstances the work of doing a K-proof can serve the development of new, public, knowledge even when the formalization of such K-proof (for example into a C-proof) might have required that new knowledge to be already acquired.

A customary, general and vague notion of Proof, a local, specific notion of C-proofs, and a reflective, metaphorical notion of K-proofs are the three senses in which proof can have a place in a class. An observer can use each of these notions to inspect the performances of a class over time, at specific times, and over segments of medium duration. The observer can use these notions to express "what counts as proof" in a class and to contrast those descriptive observations with what one might be able to observe in other classroom contexts that work under similar constraints.

If we are in a deadlock as regards to what we mean by proof, we contend this is so partly because we insist on a comprehensive notion of proof that can serve as referent for every use of the word. We suggest that we need to pay particular attention to the arena of mathematical activity where one wants to observe the work of proof. We have indicated four constraints that make classroom work singular insofar as a public mathematical performance. Its critique by an observer requires tools different than a critique of mathematical performances in other arenas, such as the interaction of an expert mathematician and the research paper she or he has to review. We have argued that to make it operational for understanding and appraising the mathematics of classrooms we need at least three meanings for the word. Proof as a general thing is just an expression of the general values that might be the basis of a class's mathematical performances; as those performances are looked individually, proof dissolves into many proofs, there is no such thing as unique and universal characterization of proof; there are specific proofs for specific assertions about specific objects. In some of these practices, objects are sufficiently known so that the particular proofs actualize the general values noted above. In others, the dependence on vague terms or on isolated pieces of empirical or perceptual knowledge might defy any attempt to decide once and for all whether they are mathematical proofs. The notion of K-proofs that establishes a possible lineage for those practices by way of metaphorical projections may help operationalize how the public knowledge in some classes appears to be more rational than that of other classes even if their public knowledge is not more formalized than the public knowledge in others.

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