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ABSTRACT. We consider abstract non-negative self-adjoint operators on $L^2(X)$ which satisfy the finite speed propagation property for the corresponding wave equation. For such operators we introduce a restriction type condition which in the case of the standard Laplace operator is equivalent to $(p, 2)$ restriction estimate of Stein and Tomas. Next we show that in the considered abstract setting our restriction type condition implies sharp spectral multipliers and endpoint estimates for the Bochner-Riesz summability. We also observe that this restriction estimate holds for operators satisfying dispersive or Strichartz estimates. We obtain new spectral multiplier results for several second order differential operators and recover some known results. Our examples include Schrödinger operators with inverse square potentials on $\mathbb{R}^n$, the harmonic oscillator, elliptic operators on compact manifolds and Schrödinger operators on asymptotically conic manifolds.

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1. Introduction

A celebrated theorem of Hörmander [38] for radial Fourier multipliers on the Euclidean space states that for a given bounded function $F : [0,\infty) \to \mathbb{C}$, the operator $F(-\Delta)$, initially defined by Fourier analysis on $L^2(\mathbb{R}^n)$, extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$ provided the function satisfies

$$\sup_{t>0} \|\eta(\cdot)F(t\cdot)\|_{W^{s,2}} < \infty \tag{1.1}$$

for some $s > n/2$. Here $\eta \in C_c^\infty(0,\infty)$ is a non-trivial auxiliary function. This result is a sharp version of the well known Mikhlin's Fourier multiplier theorem [52]. These results have led to a fruitful research activity on spectral multipliers and new perspectives in harmonic analysis. The Hörmander-Mikhlin theorem has been extended by several authors to other operators than the Laplacian and settings that go beyond the Euclidean case. The bibliography is so broad that it is impossible to provide complete list here. We refer the reader to [2, 12, 14, 15, 16, 17, 21, 24, 25, 26, 27, 28, 29, 32, 35, 37, 50, 53, 54, 56, 59, 60, 62, 64, 65, 67, 69, 70] and the references therein.

Suppose that X is a measure space and that L is a non-negative self-adjoint operator on $L^2(X)$. Such an operator admits a spectral resolution $E_L(\lambda)$ and for any bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$, one can define the operator $F(L)$

$$F(L) = \int_0^\infty F(\lambda)\, \mathrm{d}E_L(\lambda). \tag{1.2}$$

By the spectral theorem, $F(L)$ is well defined and bounded on $L^2(X)$. Spectral multiplier theorems give sufficient conditions on F under which the operator $F(L)$ extends to a bounded operator on $L^p(X)$ for some range of p.

Most of the references mentioned before deal with the case of sub-Laplacians on some Lie groups. The papers [24, 25, 33] deal with a rather general situation where (X,d,μ) is metric measure space of homogeneous type (or even a domain of such space). One of the results there is a spectral multiplier theorem under the sole assumption that the heat kernel of the operator has a Gaussian upper bound. The condition there is however stronger than (1.1) in the sense that the norm in $W^{s,2}$ is replaced by the norm of $W^{s,\infty}$ where s is any constant larger than half of homogeneous dimension. Condition (1.1) with norm $W^{s,2}$ is better than the corresponding one with norm $W^{s,\infty}$. This can be seen from Bochner-Riesz summability which we discuss now.

The theory of spectral multipliers is related to and motivated by the study of convergence of Bochner-Riesz means of self-adjoint operators. For any exponent $\delta \geq 0$ and a parameter $R > 0$ we define the the function $S_R^\delta\colon \mathbb{R}_+ \to \mathbb{R}$ by the formula

$$S_R^\delta(\lambda) = \begin{cases} \left(1 - \frac{\lambda}{R^2}\right)^\delta & \text{for } \lambda \leq R^2 \\ 0 & \text{for } \lambda > R^2. \end{cases} \tag{1.3}$$

Then for any non-negative self-adjoint operator L we define the operator $S_R^\delta(L)$ using (1.2). We call $S_R^\delta(L)$ the Riesz or the Bochner-Riesz means of order δ corresponding to the operator L. The basic question in the theory of Bochner-Riesz means is to establish the critical exponent for the uniform continuity with respect to R and convergence of the Riesz means on L^p spaces for various p with $1 \leq p \leq \infty$.

For $\delta = 0$, this is the spectral projector $E_{\sqrt{L}}[0,R]$, while for $\delta > 0$, $S_R^\delta(L)$ can be seen as a smoothed version of this spectral projector. Bochner-Riesz summability describes the range of δ for which the above operators are bounded on L^p, uniformly in R. If one proves a spectral multiplier result which states that $F(L)$ is bounded on $L^p(X)$ for all $p \in (1,\infty)$ whenever F satisfies

(1.1), then the Bochner-Riesz mean $S_1^\delta(L)$ is bounded on all L^p spaces provided $\delta > \frac{n-1}{2}$. In the case where L is the Euclidean Laplacian and $\delta > \frac{n-1}{2}$, the kernel of $S_R^\delta(L)$ is L^1 and hence Bochner-Riesz means are bounded on L^p for $1 \le p \le \infty$. In this setting more is known. Indeed for $\delta > \max\{n|\frac{1}{2} - \frac{1}{p}| - \frac{1}{2}, 0\}$, it was known for a long time that as a consequence of restriction estimates for the Fourier transform, Bochner-Riesz summability holds on L^p for all $p \le \frac{2n+2}{n+3}$, and by duality for $p \ge \frac{2n+2}{n-1}$. See Stein [67], p. 420. This was extended by Lee [46] to the case $p < \frac{2n+4}{n+4}$ (or $p > \frac{2n+4}{n}$) and recent improvements are proved by Bourgain and Guth [9]. The question whether Bochner-Riesz summability holds on $L^p(\mathbb{R}^n)$ for all p and all $\delta > \max\{n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$ is a long-standing open problem (except for $n = 2$, see Carleson and Sjölin [12] and Hörmander [36]). For all this, see Stein [67], p. 420 and the review paper of Tao [68]. The latter contains many other information and relation of the Bochner-Riesz problem to other open problems in harmonic analysis.

If L is a second order elliptic operator on a compact Riemannian manifold M with dimension n, then the Bochner-Riesz means $\sum_{\lambda_j \le R}(1 - \lambda_j/R)^\delta < \cdot, e_j > e_j$ are uniformly bounded on $L^p(M)$ provided $p \le \frac{2n+2}{n+3}$ or by duality for $p \ge \frac{2n+2}{n-1}$ for $\delta > \max\{n|\frac{1}{2} - \frac{1}{p}| - \frac{1}{2}, 0\}$. Here $\lambda_0 \le \lambda_j \le \lambda_{j+1} \le \dots$ and e_j are the corresponding eigenvalues and normalized L^2 eigenvectors, respectively. See Sogge [62].

The theory of Fourier multipliers and Bochner-Riesz analysis in the setting of the standard Laplace operator on $\mathbb{R}^n$ is related to the so-called *sphere restriction problem* for the Fourier transform: find the pairs (p, q) for which $R_\lambda \in \mathcal{L}(L^p(\mathbb{R}^n), L^q(\mathbb{S}^{n-1}))$ where R_λ is defined by

$$R_\lambda f(\omega) = \hat{f}(\lambda\omega), \ \ \omega \in \mathbb{S}^{n-1}, \lambda > 0.$$

See for example [26, 28, 65, 67, 68]. For $q = 2$ the full description of possible range of p is due to Stein and Tomas. The theorem of Tomas [71], extended by Stein to the endpoint, states that $(p, 2)$ restriction estimates hold if and only if $1 \le p \le 2(n+1)/(n+3)$. The case $q \ne 2$ is not relevant to our discussion so we refer the interested reader to Tao [68] on the subject.

Note that on $\mathbb{R}^n$, the Schwartz kernel of the spectral measure $dE_{\sqrt{-\Delta}}(\lambda)$ of $\sqrt{-\Delta}$ is given by

$$dE_{\sqrt{-\Delta}}(\lambda; z, z') = \frac{\lambda^{n-1}}{(2\pi)^n} \int_{S^{n-1}} e^{i(z-z')\cdot\lambda\omega} d\omega, \quad z, z' \in \mathbb{R}^n,$$

therefore $dE_{\sqrt{-\Delta}}(\lambda) = \frac{\lambda^{n-1}}{(2\pi)^n} R_\lambda^* R_\lambda$ and the restriction theorem for $q = 2$ is equivalent to

$$\|dE_{\sqrt{-\Delta}}(\lambda)\|_{p\to p'} \le C\lambda^{n(1/p-1/p')-1} \tag{1.4}$$

for all $p \in [1, \frac{2n+2}{n+3}]$. In the sequel, we refer to (1.4) as $(p, 2)$ restriction estimate of Stein-Tomas.

In this paper we follow the line of research described above. We deal with the problem of sharp spectral multipliers and Bochner-Riesz summability for other operators than the Euclidean Laplacian and elliptic operators on compact manifolds. Our aim is to build a theory which applies in a general setting of self-adjoint operators on spaces of homogeneous type (i.e., metric measure spaces which satisfy the volume doubling property). Our approach allows us to prove sharp multiplier results and Bochner-Riesz summability in new settings and also unifies several previously known results. In order to do so we introduce a restriction type estimate which in the case of the Laplacian on $\mathbb{R}^n$ turns to be equivalent to the $(p, 2)$ restriction estimate of Stein-Tomas.

Our setting will be the following. We consider a non-negative self-adjoint operator L on $L^2(X)$ where (X, d, μ) is a metric measure space which satisfies the volume doubling condition

$$V(x, \lambda r) \le C\lambda^n V(x, r) \ \ \forall x \in X, \ \lambda \ge 1, \ r > 0,$$

where C and n are positive constants and $V(x,r)$ denotes the volume of the open ball $B(x,r)$ of centre x and radius r. We assume that L satisfies the finite speed propagation property for the corresponding wave equation. We introduce the condition that for any $R>0$ and all Borel functions F supported in $[0,R]$,

$$(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}}) \qquad \big\|F(\sqrt{L})P_{B(x,r)}\big\|_{p\to s} \le CV(x,r)^{\frac{1}{s}-\frac{1}{p}}(Rr)^{n(\frac{1}{p}-\frac{1}{s})}\big\|F(R\cdot)\big\|_q$$

for all $x\in X$ and all $r\ge 1/R$.

We will see that if the volume is polynomial, i.e. $V(x,r)\sim r^n$, then $(\mathrm{ST}^2_{\mathrm{p,2}})$ is equivalent to $(p,2)$ restriction estimate of Stein-Tomas. For this reason, we call $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ a *Stein-Tomas restriction type condition*. One of our main results on sharp spectral multipliers can be stated as follows.

Theorem A. *Assume that X satisfies the volume doubling condition. Suppose that L is a non-negative self-adjoint operator which satisfies the finite speed propagation property and condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ for some p,s,q such that $1\le p<s\le\infty$ and $1\le q\le\infty$.*

(i) **Compactly supported multipliers:** *Let F be an even function such that $\operatorname{supp}F\subseteq[-1,1]$ and $F\in W^{\beta,q}(\mathbb{R})$ for some $\beta>n(1/p-1/s)$. Then $F(\sqrt{L})$ is bounded on $L^p(X)$, and*

$$\sup_{t>0}\|F(t\sqrt{L})\|_{p\to p}\le C\|F\|_{W^{\beta,q}}.$$

(ii) **General multipliers:** *Suppose $s=2$ and F is a bounded Borel function such that estimates $\sup_{t>0}\|\eta(\cdot)F(t\cdot)\|_{W^{\beta,q}}<\infty$ hold for some $\beta>\max\{n(1/p-1/2),1/q\}$ and some non-trivial function $\eta\in C_c^\infty(0,\infty)$. Then $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $p<r<p'$. In addition,*

$$\|F(\sqrt{L})\|_{r\to r}\le C_\beta\Big(\sup_{t>0}\|\eta(\cdot)F(t\cdot)\|_{W^{\beta,q}}+|F(0)|\Big).$$

Assertion (i) of the theorem is inspired by Guillarmou, Hassell and Sikora [29] where a related result is proved under the assumption that the volume is polynomial. Assertion (ii) is in the spirit of Hörmander's multiplier theorem for the Euclidean Laplacian. Here, if $p>1$, the order of differentiability required on F is smaller since we do not search for boundedness of $F(\sqrt{L})$ on L^r for all $r\in(1,\infty)$.

The proof of assertion (i) makes heavy use of the finite speed propagation property. This property together with the classical dyadic decomposition of F allow to reduce the problem of boundedness of $F(\sqrt{L})$ on L^p to boundedness of certain compactly supported operators. The Stein-Tomas restriction type condition will be used to obtain an L^p-L^s estimate of these operators from which we recover the boundedness of $F(\sqrt{L})$ on L^p.

Assertion (ii) appeals as expected to singular integral theory. We shall also make use of the estimate from assertion (i) since $F(\sqrt{L})$ can be written as the sum $\sum F_j(\sqrt{L})$ with compactly supported functions F_j. However the operators $F_j(\sqrt{L})$ do not act independently of each other and hence L^p estimate for $F(\sqrt{L})$ does not hold in a trivial way from the corresponding estimates for $F_j(\sqrt{L})$. As explained by Littman, McCarthy and Rivière [48], we may have $F_j(\sqrt{-\Delta})$ to be uniformly bounded on $L^1(\mathbb{R}^n)$ but $F(\sqrt{-\Delta})$ fails to be a multiplier of any L^p other than L^2. This problem of recovering L^p bounds for $F(\sqrt{-\Delta})$ from those for $F_j(\sqrt{-\Delta})$ is discussed by Carbery, Seeger and Sogge in [11, 58, 59]. We shall follow closely Carbery [11] and adapt some ideas there to our abstract setting.

Our restriction type estimate does not hold when the point spectrum of the operator is not empty. In particular, it does not hold for elliptic operators on compact manifolds or for the harmonic oscillator. In order to treat these situations as well we modify the restriction estimate as follows:

for a fixed natural number κ and for all $N \in \mathbb{N}$ and all even Borel functions F such that $\operatorname{supp} F \subseteq [-N, N]$,

$$\|F(\sqrt{L})P_{B(x,r)}\|_{p\to s} \le CV(x,r)^{\frac{1}{s}-\frac{1}{p}}(Nr)^{n(\frac{1}{p}-\frac{1}{s})}\|F(N\cdot)\|_{N^\kappa,q}, \tag{$\mathrm{SC}^{q,\kappa}_{p,s}$}$$

for all $x \in X$ and all $r \ge 1/N$ where

$$\|F\|_{N,q} = \left(\frac{1}{2N}\sum_{\ell=1-N}^{N}\sup_{\lambda\in[\frac{\ell-1}{N},\frac{\ell}{N})}|F(\lambda)|^q\right)^{1/q}$$

for F supported in $[-1,1]$. For $q=\infty$, we put $\|F\|_{N,\infty} = \|F\|_\infty$. The norm $\|F\|_{N,q}$ was used by Cowling and Sikora [21] and Duong, Ouhabaz and Sikora [24] in the context of spectral multipliers.

In some situations, $(\mathrm{SC}^{2,1}_{p,2})$ is equivalent to the following condition introduced by Sogge (see [62, 63, 64])

$$\|E_{\sqrt{L}}[\lambda,\lambda+1)\|_{p\to p'} \le C(1+\lambda)^{n(\frac{1}{p}-\frac{1}{p'})-1}. \tag{S_p}$$

We call $(\mathrm{SC}^{q,\kappa}_{p,s})$ *Sogge's spectral cluster condition*. In this context we shall prove the following result (see Theorems 3.6 and 4.2 for precise statements).

Theorem B. *Suppose that X has finite measure and satisfies the volume doubling condition. Let L be a non-negative self-adjoint operator which satisfies the finite speed propagation property and Sogge's spectral cluster condition $(\mathrm{SC}^{q,1}_{p,s})$ for some p, s, q such that $1 \le p < s \le \infty$ and $1 \le q \le \infty$. Then both assertions of Theorem A hold provided $\beta > \max\{n(1/p-1/s), 1/q\}$.*

The same conclusion holds in the case where $\mu(X) = \infty$ provided $(\mathrm{SC}^{q,\kappa}_{p,s})$ and an a priori estimate for $\|F(\sqrt{L})\|_{p\to p}$ are satisfied.

As for Theorem A, an appropriate decomposition of $F(\sqrt{L})$ as the sum of operators with compact supports is the backbone of our arguments in proving boundedness of $F(\sqrt{L})$ on L^p for compactly supported F. Passing from compactly supported multipliers to the general case will be done in the same way as for Theorem A. The proof of this part does not make explicit use of $(\mathrm{ST}^q_{p,2})$ or $(\mathrm{SC}^{q,\kappa}_{p,2})$ but the rather weaker condition

$$\|(I+t\sqrt{L})^{-N}P_{B(x,r)}\|_{p\to 2} \le CV(x,r)^{\frac{1}{2}-\frac{1}{p}}\left(\frac{r}{t}\right)^{n(\frac{1}{p}-\frac{1}{2})}, \quad x\in X,\ r\ge t>0.$$

Starting now from Theorem A or Theorem B with $s=2$ and choosing the function $F = S^\delta_R$ yields Bochner-Riesz summability on $L^p(X)$ for $\delta > \delta_q(p)$ where

$$\delta_q(p) = \max\left\{0, n\left|\frac{1}{p}-\frac{1}{2}\right| - \frac{1}{q}\right\}.$$

Now we address the question of endpoint estimates, i.e., estimate for $S^{\delta_q(p)}_R(L)$. It turns out that our Stein-Tomas restriction type condition or Sogge's cluster condition imply that Bochner-Riesz means are weak-type (p,p) operators for $\delta = \delta_q(p)$. More precisely we obtain

Theorem C. *Assume that X satisfies the doubling condition and operator L satisfies the finite speed propagation property.*

(i) *If the restriction condition $(\mathrm{ST}^q_{p,2})$ holds for some p, q satisfying $1 \le p < 2$ and $1 \le q \le \infty$ then $S^{\delta_q(p)}_R(L)$ is of weak-type (p,p) uniformly in R.*

(ii) *The same conclusion as in* (i) *holds if $\mu(X) < \infty$ and $(\mathrm{SC}^{q,1}_{p,2})$ is satisfied for some p, q satisfying $1 \le p < 2$ and $1 \le q \le \infty$.*

In the Euclidean case, it is known that $S_R^{\delta_2(p)}(-\Delta)$ is not bounded on $L^p(\mathbb{R}^n)$ for $p \neq 2$. This was observed by Christ and Sogge [17] who also proved weak-type $(1,1)$ for $S_R^{\delta_2(1)}(-\Delta)$. Weak-type (p,p) estimates of $S_R^{\delta_2(p)}(-\Delta)$ are proved by Christ [14, 15] when $p < \frac{2n+2}{n+3}$. The corresponding result on compact manifolds is proved by Seeger [57]. The endpoint estimates for $p = \frac{2n+2}{n+3}$ are proved by Tao [69] both for $\mathbb{R}^n$ and compact manifolds.

Our approach for endpoint estimates is inspired by Christ and Tao [14, 15, 69]. It is based on L^2 Calderón-Zygmund techniques (as used in Fefferman [26]), a spacial decomposition of the Bochner-Riesz multiplier and the fact that if F has its inverse Fourier transform supported on a set of width R, then by the finite speed propagation property the operator $F(\sqrt{L})$ is supported in a $CR-$ neighbourhood of the diagonal. It is worth to note that our proof of endpoint estimates does not require any cancellation argument. This allows us to consider applications to operators with non-smooth kernels.

The previous theorems are proved in Part 1 of this paper. In Part 2, we investigate the relation of $(\mathrm{ST}^2_{\mathrm{p},2})$ to dispersive or Strichartz estimates for the corresponding Schrödinger equation

$$\partial_t u + iLu = 0, \ \ u(0) = f \in L^2. \tag{1.5}$$

In the setting of Euclidean Laplacian, Strichartz's original proof for $L^p(\mathbb{R}\times\mathbb{R}^n)$ estimates of the solution u of (1.5) uses restriction estimates of the Fourier transform. In some sense we want to do the converse here, we want to take advantage of known dispersive or Strichartz estimates for (1.5) to prove a Stein-Tomas restriction type condition and then obtain sharp spectral multipliers by Theorem A. We are able do this either using directly dispersive estimates for e^{itL} or endpoint Strichartz estimates. We prove the following result.

Theorem D.

(i) *Suppose that L satisfies the Strichartz estimate*

$$\int_{\mathbb{R}} \|e^{itL} f\|^2_{\frac{2n}{n-2}}\, dt \leq C\|f\|_2^2, \ \ f \in L^2$$

for some $n > 2$. Assume also that the smoothing property

$$\|\exp(-tL)\|_{p\to\frac{2n}{n+2}} \leq Kt^{-\frac{n}{2}(\frac{1}{p}-\frac{n+2}{2n})},$$

holds for all $p \in [1, \frac{2n}{n+2}]$. Then for all $\lambda \geq 0$

$$\|dE_{\sqrt{L}}(\lambda)\|_{p\to p'} \leq C\lambda^{n(\frac{1}{p}-\frac{1}{p'})-1}.$$

(ii) *Fix $p \in [1, \frac{2n}{n+2}]$. Suppose that X satisfies the doubling condition and that there exists a positive constants $C > 0$ such that $V(x,r) \leq Cr^n$ for every $x \in X$ and $r > 0$. Assume that L satisfies the finite speed propagation property together with Strichartz and smoothing estimates as in* (i)*. Then for every even compactly supported bounded function F such that $\|F\|_{W^{\beta,2}} < \infty$ for some $\beta > n(\frac{1}{p} - \frac{1}{2})$, the operator $F(\sqrt{L})$ is bounded on L^p and*

$$\sup_{t>0} \|F(t\sqrt{L})\|_{p\to p} \leq C\|F\|_{W^{\beta,2}}.$$

(iii) *Suppose that the conditions of* (ii) *are satisfied. Then for any bounded Borel function F such that $\sup_{t>0}\|\eta(.)F(t\cdot)\|_{W^{\beta,2}} < \infty$ for some $\beta > \max\{n(1/p-1/2), 1/2\}$ and some non-trivial function $\eta \in C_c^\infty(0,\infty)$, the operator $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $r \in (p, p')$.*

The main assertion here is (i). Indeed, once (i) is proved we obtain a Stein-Tomas restriction type estimate and then appeal to Theorem A to prove assertions (ii) and (iii). We can also replace the Strichartz estimates by dispersive estimates

$$\|e^{itL}\|_{1\to\infty} \leq C|t|^{-n/2}, \qquad t \in \mathbb{R},\ t \neq 0.$$

Note that by a result of Keel and Tao [42], endpoint Strichartz estimates follow from these dispersive estimates.

Strichartz estimates have been studied by several authors. For example, Burq, Planchon, Stalker and A. Tahvildar-Zadeh [10] proved such estimates for Schrödinger operators with inverse square potentials, i.e. $L = -\Delta + \frac{c}{|x|^2}$ on $\mathbb{R}^n$. Therefore we obtain sharp multiplier results as well as endpoint Bochner-Riesz estimates for these operators. It is worth to mention that if $-(n-2)^2/4 < c < 0$, the semigroup $\exp(-tL)$ acts on $L^p(\mathbb{R}^n)$ only for $p \in (p_c', p_c)$ with $p_c < \infty$. In particular, the corresponding heat kernel does not enjoy any good upper bounds such as Gaussian upper bounds. Nevertheless we obtain sharp spectral multipliers for L on L^p for $p \in (p_c', 2n/(n+2)]$. We discuss in Part 3 several other examples to which Theorems A, B, C and D apply. This includes radial Schrödinger operators with inverse square potentials, the harmonic oscillator, elliptic operators on compact manifolds, Laplacian on asymptotically conic manifolds.

While this paper was finished we learned that in the recent PhD Thesis of M. Uhl [73] and later in [45] a condition similar to our restriction type condition was introduced and a spectral multiplier result similar to our Theorems 4.1 and 4.2 is proved. The order of differentiability required in Uhl's Thesis is $\beta > n/2$ and hence it is less sharp than our Theorems 4.1 and 4.2. The result from [73] was improved in [45].

Part 1. Restriction estimates imply sharp spectral multipliers

2. Restriction type condition

We start by fixing some notation and assumptions. Throughout this paper, unless we mention the contrary, (X, d, μ) is a metric measure space, that is, μ is a Borel measure with respect to the topology defined by the metric d. We denote by $B(x, r) = \{y \in X,\ d(x, y) < r\}$ the open ball with centre $x \in X$ and radius $r > 0$. We often just use B instead of $B(x, r)$. Given $\lambda > 0$, we write λB for the λ-dilated ball which is the ball with the same centre as B and radius λr. We set $V(x, r) = \mu(B(x, r))$ the volume of $B(x, r)$ and we say that (X, d, μ) satisfies the doubling property (see Chapter 3, [18]) if there exists a constant $C > 0$ such that

$$V(x, 2r) \leq CV(x, r) \quad \forall\, r > 0,\ x \in X. \tag{2.1}$$

If this is the case, there exist C, n such that for all $\lambda \geq 1$ and $x \in X$

$$V(x, \lambda r) \leq C\lambda^n V(x, r). \tag{2.2}$$

In the sequel we want to consider n as small as possible. Note that in general one cannot take infimum over such exponents n in (2.2). In the Euclidean space with Lebesgue measure, n corresponds to the dimension of the space. Observe that if X satisfies (2.1) and has finite measure then it has finite diameter (see, e.g., [3]). Therefore if $\mu(X)$ is finite, then we may assume that $X = B(x_0, 1)$ for some $x_0 \in X$.

For $1 \leq p \leq +\infty$, we denote the norm of a function $f \in L^p(X, \mathrm{d}\mu)$ by $\|f\|_p$, by $\langle ., . \rangle$ the scalar product of $L^2(X, \mathrm{d}\mu)$, and if T is a bounded linear operator from $L^p(X, \mathrm{d}\mu)$ to $L^q(X, \mathrm{d}\mu)$,

$1 \le p,\, q \le +\infty$, we write $\|T\|_{p\to q}$ for the operator norm of T. Given a subset $E \subseteq X$, we denote by χ_E the characteristic function of E and set

$$P_E f(x) = \chi_E(x) f(x).$$

For a given function $F : \mathbb{R} \to \mathbb{C}$ and $R > 0$, we define the function $\delta_R F : \mathbb{R} \to \mathbb{C}$ by putting $\delta_R F(x) = F(Rx)$. For $F \in L^2(\mathbb{R})$, the Fourier transform of F is given by

$$\widehat{F}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\lambda) e^{-it\lambda}\, d\lambda.$$

Finally, C denotes a generic constant, not necessarily the same at each occurrence, which, in the course of a proof, may be taken to depend on any of the quantities assumed to be bounded.

2.1. Finite speed propagation for the wave equation. Set

$$\mathcal{D}_\rho = \{(x,\, y) \in X \times X : d(x,\, y) \le \rho\}.$$

Given an operator T from $L^p(X)$ to $L^q(X)$, we write

$$\operatorname{supp} K_T \subseteq \mathcal{D}_\rho \tag{2.3}$$

if $\langle T f_1, f_2\rangle = 0$ whenever f_k is in $C(X)$ and has support $\operatorname{supp} f_k \subseteq B(x_k, \rho_k)$ when $k = 1, 2$, and $\rho_1 + \rho_2 + \rho < d(x_1, x_2)$. Note that if T is an integral operator with a kernel K_T, then (2.3) coincides with the standard meaning of $\operatorname{supp} K_T \subseteq \mathcal{D}_\rho$, that is $K_T(x,\, y) = 0$ for all $(x,\, y) \notin \mathcal{D}_\rho$.

Given a non-negative self-adjoint operator L on $L^2(X)$. We say that L satisfies the finite speed propagation property if

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subseteq \mathcal{D}_t \quad \forall t > 0\,. \tag{FS}$$

Property (FS) holds for most of second order self-adjoint operators and is equivalent to Davies-Gaffney estimates. See, for example [13], [61] and [19].

We recall the following well-known simple lemma.

Lemma 2.1. *Assume that L satisfies* (FS) *and that F is an even bounded Borel function with Fourier transform $\hat{F} \in L^1(\mathbb{R})$ and that* $\operatorname{supp}\hat{F} \subseteq [-\rho,\, \rho]$. *Then*

$$\operatorname{supp} K_{F(\sqrt{L})} \subseteq \mathcal{D}_\rho.$$

Proof. If F is an even function, then by the Fourier inversion formula,

$$F(\sqrt{L}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{F}(t) \cos(t\sqrt{L})\, dt.$$

But $\operatorname{supp} \hat{F} \subseteq [-\rho, \rho]$, and the lemma follows then from (FS). □

2.2. The Stein-Tomas restriction type condition. Assume that (X, d, μ) satisfies the doubling condition, that is (2.2). Consider a non-negative self-adjoint operator L and numbers p, s and q such that $1 \le p < s \le \infty$ and $1 \le q \le \infty$. We say that L satisfies the *Stein-Tomas restriction type condition* if: for any $R > 0$ and all Borel functions F such that $\operatorname{supp} F \subset [0, R]$,

$$\left\|F(\sqrt{L}) P_{B(x,r)}\right\|_{p\to s} \le C V(x,r)^{\frac{1}{s}-\frac{1}{p}} (Rr)^{n(\frac{1}{p}-\frac{1}{s})} \left\|\delta_R F\right\|_q \tag{$\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}}$}$$

for all $x \in X$ and all $r \ge 1/R$.

Remark 2.2. *Note that if condition* ($\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}}$) *holds for some $q \in [1, \infty)$, then* ($\mathrm{ST}^{\tilde{\mathrm{q}}}_{\mathrm{p,s}}$) *holds for all $\tilde{q} \ge q$ including the case $\tilde{q} = \infty$.*

Proposition 2.3. *Suppose that (X, d, μ) satisfies property* (2.2). *Let $1 \le p < 2$ and $N > n(1/p - 1/2)$. Then* ($\mathrm{ST}^{\infty}_{\mathrm{p,2}}$) *is equivalent to each of the following conditions:*

(a) *For all $x \in X$ and $r \geq t > 0$ we have*

$$\left\|e^{-t^2L}P_{B(x,r)}\right\|_{p\to 2} \leq CV(x,r)^{\frac{1}{2}-\frac{1}{p}}\Big(\frac{r}{t}\Big)^{n(\frac{1}{p}-\frac{1}{2})}. \tag{$G_{p,2}$}$$

(b) *For all $x \in X$ and $r \geq t > 0$ we have*

$$\left\|(I+t\sqrt{L})^{-N}P_{B(x,r)}\right\|_{p\to 2} \leq CV(x,r)^{\frac{1}{2}-\frac{1}{p}}\left(\frac{r}{t}\right)^{n(\frac{1}{p}-\frac{1}{2})}. \tag{$E_{p,2}$}$$

Proof. We shall show that $(E_{p,2}) \Rightarrow (ST^{\infty}_{p,2}) \Rightarrow (G_{p,2}) \Rightarrow (E_{p,2})$.

Suppose that F is a Borel function with $\operatorname{supp} F \subset [0,R]$. Let $1 \leq p < 2$ and $N > n(1/p - 1/2)$. It follows from $(E_{p,2})$ that for every $x \in X$ and $r \geq 1/R$,

$$\begin{aligned}
\left\|F(\sqrt{L})P_{B(x,r)}\right\|_{p\to 2} &= \left\|F(\sqrt{L})(I+\frac{\sqrt{L}}{R})^{N}(I+\frac{\sqrt{L}}{R})^{-N}P_{B(x,r)}\right\|_{p\to 2}\\
&\leq \sup_{\lambda}\left|F(\lambda)(1+\frac{\lambda}{R})^{N}\right| \cdot \left\|(I+\frac{\sqrt{L}}{R})^{-N}P_{B(x,r)}\right\|_{p\to 2}\\
&\leq C2^{N}V(x,r)^{\frac{1}{2}-\frac{1}{p}}(Rr)^{n(\frac{1}{p}-\frac{1}{2})}\|\delta_R F\|_{\infty}.
\end{aligned}$$

This gives condition $(ST^{\infty}_{p,2})$.

Next assume $(ST^{\infty}_{p,2})$. Then

$$\begin{aligned}
\left\|e^{-t^2L}P_{B(x,r)}\right\|_{p\to 2} &= \left\|\int_0^{\infty} e^{-t^2\lambda}dE_L(\lambda)P_{B(x,r)}\right\|_{p\to 2}\\
&= \left\|\int_0^{\infty} t^2e^{-t^2\lambda}E_L[0,\lambda]P_{B(x,r)}d\lambda\right\|_{p\to 2}\\
&\leq \int_0^{\infty} t^2e^{-t^2\lambda}\left\|\chi_{[0,\sqrt{\lambda}]}(\sqrt{L})P_{B(x,r)}\right\|_{p\to 2}\,d\lambda.
\end{aligned}$$

Now if $0 < \lambda \leq 1/r^2$, then $B(x,r) \subseteq B(x,\lambda^{-1/2})$. Therefore, by $(ST^{\infty}_{p,2})$

$$\begin{aligned}
\|\chi_{[0,\sqrt{\lambda}]}(\sqrt{L})P_{B(x,r)}\|_{p\to 2} &\leq \|\chi_{[0,\sqrt{\lambda}]}(\sqrt{L})P_{B(x,\lambda^{-1/2})}\|_{p\to 2}\\
&\leq CV(x,\lambda^{-1/2})^{\frac{1}{2}-\frac{1}{p}} \leq CV(x,r)^{\frac{1}{2}-\frac{1}{p}}.
\end{aligned}$$

If $\lambda > 1/r^2$, then by $(ST^{\infty}_{p,2})$

$$\|\chi_{[0,\sqrt{\lambda}]}(\sqrt{L})P_{B(x,r)}\|_{p\to 2} \leq CV(x,r)^{\frac{1}{2}-\frac{1}{p}}(r\lambda^{1/2})^{n(\frac{1}{p}-\frac{1}{2})}.$$

This proves $(G_{p,2})$.

To finish the proof assume that $(G_{p,2})$ holds. Then

$$\begin{aligned}
\left\|(I+t\sqrt{L})^{-N}P_{B(x,r)}\right\|_{p\to 2} &\leq \left\|(I+t^2L)^{N/2}(I+t\sqrt{L})^{-N}(I+t^2L)^{-N/2}P_{B(x,r)}\right\|_{p\to 2}\\
&\leq \sup_{\lambda}\left|(1+t^2\lambda^2)^{N/2}(1+t\lambda)^{-N}\right| \cdot \left\|(I+t^2L)^{-\frac{N}{2}}P_{B(x,r)}\right\|_{p\to 2}\\
&\leq C\left\|(I+t^2L)^{-N/2}P_{B(x,r)}\right\|_{p\to 2}.
\end{aligned}$$

Next note that for $t > 0$

$$\left\|(I+t^2L)^{-N/2}P_{B(x,r)}\right\|_{p\to 2} = \left\|C_N\int_0^{\infty} e^{-s}s^{N/2-1}e^{-st^2L}P_{B(x,r)}ds\right\|_{p\to 2}$$

$$\leq \quad C_N \int_0^\infty e^{-s} s^{N/2-1} \left\| e^{-st^2L} P_{B(x,r)} \right\|_{p\to 2} ds. \tag{2.4}$$

Hence if $s \leq r^2/t^2$, then by $(\mathrm{G}_{\mathrm{p},2})$

$$\|e^{-st^2L} P_{B(x,r)}\|_{p\to 2} \leq CV(x,r)^{\frac{1}{2}-\frac{1}{p}} \Big(\frac{r}{ts^{1/2}}\Big)^{n(\frac{1}{p}-\frac{1}{2})}.$$

If $s > r^2/t^2$, then $B(x,r) \subseteq B(x, ts^{1/2})$ and $(\mathrm{G}_{\mathrm{p},2})$ implies that

$$\|e^{-st^2L} P_{B(x,r)}\|_{p\to 2} \leq \|e^{-st^2L} P_{B(x,ts^{1/2})}\|_{p\to 2} \leq CV(x, ts^{1/2})^{\frac{1}{2}-\frac{1}{p}} \leq CV(x,r)^{\frac{1}{2}-\frac{1}{p}}.$$

Using these estimates in (2.4) yields $(\mathrm{E}_{\mathrm{p},2})$ for $N > n(1/p - 1/2)$. This ends the proof. □

It is natural to generalise condition (1.4) to abstract self-adjoint operators in the following way (see [29]). One says that L satisfies L^p *to* $L^{p'}$ *restriction estimates* if the spectral measure $dE_{\sqrt{L}}(\lambda)$ maps $L^p(X)$ to $L^{p'}(X)$ for some $p < 2$, with an operator norm estimate

$$\left\| dE_{\sqrt{L}}(\lambda) \right\|_{p\to p'} \leq C\lambda^{n(\frac{1}{p}-\frac{1}{p'})-1} \tag{R_{p}}$$

for all $\lambda \geq 0$, where n is as in condition (2.2) and p' is conjugate of p, i.e., $1/p + 1/p' = 1$.

Proposition 2.4. *Fix $1 \leq p < 2n/(n+1)$. Suppose that there exists a constant $C > 0$ such that $C^{-1}r^n \leq V(x,r) \leq Cr^n$ for all $x \in X$ and $r > 0$. Then conditions $(\mathrm{R}_{\mathrm{p}})$, $(\mathrm{ST}^2_{\mathrm{p},2})$ and $(\mathrm{ST}^1_{\mathrm{p},p'})$ are equivalent.*

Proof. The proof is inspired by estimates (2.12) of [29]. We first show the implication $(\mathrm{R}_{\mathrm{p}}) \Rightarrow (\mathrm{ST}^1_{\mathrm{p},p'})$. Suppose that F is a Borel function such that $\operatorname{supp} F \subset [0,R]$ for some $R > 0$. Then by $(\mathrm{R}_{\mathrm{p}})$

$$\begin{aligned}
\left\| F(\sqrt{L}) P_{B(x,r)} \right\|_{p\to p'} &\leq \int_0^\infty |F(\lambda)| \| dE_{\sqrt{L}}(\lambda) \|_{p\to p'} d\lambda \\
&\leq C \int_0^R |F(\lambda)| \lambda^{n(\frac{1}{p}-\frac{1}{p'})-1} d\lambda \\
&\leq CR^{n(\frac{1}{p}-\frac{1}{p'})-1} \int_0^R |F(\lambda)| d\lambda \\
&\leq CV(x,r)^{\frac{1}{p'}-\frac{1}{p}} (rR)^{n(\frac{1}{p}-\frac{1}{p'})} \|\delta_R F\|_1,
\end{aligned}$$

where in the last inequality we used the assumption that $V(x,r) \leq Cr^n$.

Next we prove that $(\mathrm{ST}^1_{\mathrm{p},p'}) \Rightarrow (\mathrm{ST}^2_{\mathrm{p},2})$. Note that $V(x,r) \sim r^n$ for every $x \in X$ and $r > 0$. Letting $r \to \infty$ we obtain from $(\mathrm{ST}^1_{\mathrm{p},p'})$

$$\left\| F(\sqrt{L}) \right\|_{p\to p'} \leq CR^{n(\frac{1}{p}-\frac{1}{p'})} \|\delta_R F\|_1.$$

By T^*T argument

$$\left\| F(\sqrt{L}) \right\|^2_{p\to 2} = \left\| |F|^2(\sqrt{L}) \right\|_{p\to p'} \leq CR^{2n(\frac{1}{p}-\frac{1}{2})} \|\delta_R F\|_2^2.$$

Hence

$$\left\| F(\sqrt{L}) P_{B(x,r)} \right\|_{p\to 2} \leq \left\| F(\sqrt{L}) \right\|_{p\to 2} \quad \leq \quad CV(x,r)^{\frac{1}{2}-\frac{1}{p}} (Rr)^{n(\frac{1}{p}-\frac{1}{2})} \|\delta_R F\|_2.$$

This gives $(\mathrm{ST}^2_{\mathrm{p},2})$.

Now, we prove the remaining implication $(\mathrm{ST}^2_{\mathrm{p},2}) \Rightarrow (\mathrm{R_p})$. By volume estimate $V(x,r) \geq C^{-1} r^n$ and condition $(\mathrm{ST}^2_{\mathrm{p},2})$

$$\left\| F(\sqrt{L}) P_{B(x,r)} \right\|_{p\to 2} \leq C R^{n(\frac{1}{p}-\frac{1}{2})} \left\| \delta_R F \right\|_2 \tag{2.5}$$

for any $R > 0$, all Borel functions F such that $\operatorname{supp} F \subset [0,R]$, all $x \in X$ and $r \geq 1/R$. Taking the limit $r \to \infty$ gives

$$\left\| F(\sqrt{L}) \right\|_{p\to 2} \leq C R^{n(\frac{1}{p}-\frac{1}{2})} \left\| \delta_R F \right\|_2. \tag{2.6}$$

For $F = \chi_{(\lambda-\varepsilon,\lambda+\varepsilon]}$ and $R = \lambda + \varepsilon$ in (2.6) yields

$$\begin{aligned} \left\| \varepsilon^{-1} E_{\sqrt{L}}(\lambda-\varepsilon,\lambda+\varepsilon] \right\|_{p\to p'} &= \varepsilon^{-1} \left\| E_{\sqrt{L}}(\lambda-\varepsilon,\lambda+\varepsilon] \right\|^2_{p\to 2} \\ &\leq C\varepsilon^{-1} (\lambda+\varepsilon)^{2n(\frac{1}{p}-\frac{1}{2})} \left\| \chi_{(\frac{\lambda-\varepsilon}{\lambda+\varepsilon},1]} \right\|_2^2 \\ &\leq C(\lambda+\varepsilon)^{n(\frac{1}{p}-\frac{1}{p'})-1}. \end{aligned}$$

Taking $\varepsilon \to 0$ yields condition $(\mathrm{R_p})$ (see Proposition 1, Section 5, Chapter XI, [74]). □

3. Sharp spectral multipliers - compactly supported functions

In this section we show that the restriction type condition which we introduce in the previous section can be used to obtain sharp spectral multiplier results in the abstract setting of self-adjoint operators acting on homogeneous spaces. We first consider the case of compactly supported functions. We assume here that (X,d,μ) is a metric measure space satisfying the doubling property and recall that n is the doubling dimension from condition (2.2). We use the standard notation $W^{\beta,q}(\mathbb{R})$ for the Sobolev space $\|F\|_{W^{\beta,q}} = \|(I - d^2/dx^2)^{\beta/2} F\|_q$. The first result and its proof are inspired by Theorem 1.1 of [29].

Theorem 3.1. *Suppose that operator L satisfies property* (FS) *and condition* $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ *for some p, s, q such that $1 \leq p < s \leq \infty$ and $1 \leq q \leq \infty$. Next assume that F is an even function such that $\operatorname{supp} F \subseteq [-1,1]$ and $F \in W^{\beta,q}(\mathbb{R})$ for some $\beta > n(1/p - 1/s)$. Then $F(t\sqrt{L})$ is bounded on $L^p(X)$ for all $t > 0$. In addition,*

$$\sup_{t>0} \|F(t\sqrt{L})\|_{p\to p} \leq C \|F\|_{W^{\beta,q}}. \tag{3.1}$$

We describe the proof of Theorem 3.1 at the end of this section.

A standard application of spectral multiplier theorems is Bochner-Riesz means. Such application is also a good test to check if the considered multiplier result is sharp. Let us recall that Bochner-Riesz means of order δ for a non-negative self-adjoint operator L are defined by the formula

$$S^\delta_R(L) = \Big(I - \frac{L}{R^2}\Big)^\delta_+, \quad R > 0. \tag{3.2}$$

The case $\delta = 0$ corresponds to the spectral projector $E_{\sqrt{L}}[0,R]$. For $\delta > 0$ we think of (3.2) as a smoothed version of this spectral projector; the larger δ, the more smoothing. Bochner-Riesz summability on L^p describes the range of δ for which $S^\delta_R(L)$ are bounded on L^p, uniformly in R.

In Theorem 3.1, if one chooses $F(\lambda) = (1-\lambda^2)^\delta_+$ then $F \in W^{\beta,q}$ if and only if $\delta > \beta - 1/q$. Therefore, we obtain

Corollary 3.2. *Suppose that the operator L satisfies the finite speed propagation property* (FS) *and condition* $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ *with some $1 \le p < s \le \infty$ and $1 \le q \le \infty$. Then for all $\delta > n(1/p-1/s)-1/q$, we have*

$$\left\|\left(I - \frac{L}{R^2}\right)_+^{\delta}\right\|_{p\to p} \le C \tag{3.3}$$

uniformly in $R > 0$.

As a consequence, we obtain the following necessary condition for the restriction condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ (see also Kenig, Stanton and Tomas [43]).

Corollary 3.3. *Suppose that $1/q > n(1/p - 1/s)$ for some $q \ge 1$ and $1 \le p < s \le \infty$. Then condition* $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ *implies that $L = 0$.*

Proof. Note that if $1/q > n(1/p - 1/s)$ for some $q \ge 1$ and $1 \le p < s \le \infty$, then there exist $\delta < 0$ and $\varepsilon > 0$ such that $S^{\delta}_1(\lambda^2) = (1-\lambda^2)^{\delta}_+ \in W^{n(\frac{1}{p}-\frac{1}{s})+\varepsilon,q}$. By Theorem 3.1, the operator $S^{\delta}_R(L)$ is bounded on $L^p(X)$ uniformly in R, i.e., $\|S^{\delta}_R(L)\|_{p\to p} \le C < \infty$ for some constant $C > 0$ independent of R. However, $S^{\delta}_R(L)$ is a self-adjoint operator, so $\|S^{\delta}_R(L)\|_{p'\to p'} < \infty$, and by interpolation, $\|S^{\delta}_R(L)\|_{2\to 2} < \infty$. Set $M = 1 + \|S^{\delta}_R(L)\|_{2\to 2}$. Next note that if $\lambda \in (R^2(1-M^{1/\delta}), R^2]$ then $S^{\delta}_R(\lambda) > M$ so by spectral theorem $E_L(R^2(1-M^{1/\delta}), R^2] = 0$. Because R is arbitrary positive number, this implies that $L = 0$. □

Remark 3.4. *Note that condition* $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ *allows to define the operator $S^{\delta}_R(L)$ even when $\delta < 0$ in which case the function $\lambda \to S^{\delta}_1(\lambda^2)$ is unbounded.*

We return to the discussion of Bochner-Riesz analysis in Section 5 and we now discuss a discrete version of Theorem 3.1.

It is not difficult to see that condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ with some $q < \infty$ implies that the point spectrum of L is empty. Indeed, one has for all $0 \le a < R$ and $x \in X$,

$$\left\|\mathbb{1}_{\{a\}}(\sqrt{L})P_{B(x,r)}\right\|_{p\to s} \le CV(x,r)^{\frac{1}{s}-\frac{1}{p}}(rR)^{n(\frac{1}{p}-\frac{1}{s})}\left\|\mathbb{1}_{\{a\}}(R\cdot)\right\|_q = 0, \quad Rr \ge 1$$

and therefore $\mathbb{1}_{\{a\}}(\sqrt{L}) = 0$. Due to $\sigma(L) \subseteq [0,\infty)$, it follows that the point spectrum of L is empty. In particular, $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ cannot hold for any $q < \infty$ for elliptic operators on compact manifolds or for the harmonic oscillator. To be able to study these operators as well, we introduce a variation of condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$. Following [21, 24], for an even Borel function F with $\operatorname{supp} F \subseteq [-1,1]$ we define the norm $\|F\|_{N,q}$ by

$$\|F\|_{N,q} = \left(\frac{1}{2N}\sum_{\ell=1-N}^{N}\sup_{\lambda\in[\frac{\ell-1}{N},\frac{\ell}{N})}|F(\lambda)|^q\right)^{1/q},$$

where $q \in [1,\infty)$ and $N \in \mathbb{N}$. For $q = \infty$, we put $\|F\|_{N,\infty} = \|F\|_\infty$. It is obvious that $\|F\|_{N,q}$ increases monotonically in q.

Consider a non-negative self-adjoint operator L and numbers p, s and q such that $1 \le p < s \le \infty$ and $1 \le q \le \infty$. We shall say that L satisfies the *Sogge spectral cluster condition* if: for a fixed natural number κ and for all $N \in \mathbb{N}$ and all even Borel functions F such that $\operatorname{supp} F \subseteq [-N,N]$,

$$\left\|F(\sqrt{L})P_{B(x,r)}\right\|_{p\to s} \le CV(x,r)^{\frac{1}{s}-\frac{1}{p}}(Nr)^{n(\frac{1}{p}-\frac{1}{s})}\|\delta_N F\|_{N^\kappa,q} \tag{$\mathrm{SC}^{\mathrm{q},\kappa}_{\mathrm{p,s}}$}$$

for all $x \in X$ and $r \ge 1/N$. For $q = \infty$, $(\mathrm{SC}^{\infty,\kappa}_{\mathrm{p,s}})$ is independent of κ so we write it as $(\mathrm{SC}^{\infty}_{\mathrm{p,s}})$.

Remark 3.5. *It is easy to check that for* $\kappa \geq 1$, $(\mathrm{SC}^{q,\kappa}_{p,s})$ *implies* $(\mathrm{SC}^{q,1}_{p,s})$.

Theorem 3.6. *Suppose the operator* L *satisfies property* (FS) *and condition* $(\mathrm{SC}^{q,\kappa}_{p,s})$ *for a fixed* $\kappa \in \mathbb{N}$ *and some* p, s, q *such that* $1 \leq p < s \leq \infty$ *and* $1 \leq q \leq \infty$. *In addition, we assume that for any* $\varepsilon > 0$ *there exists a constant* C_ε *such that for all* $N \in \mathbb{N}$ *and all even Borel functions* H *such that supp* $H \subset [-N, N]$,

$$(\mathrm{AB}^{q,\kappa}_{p}) \qquad \|H(\sqrt{L})\|_{p\to p} \leq C_\varepsilon N^{\kappa n(\frac{1}{p}-\frac{1}{s})+\varepsilon}\|\delta_N H\|_{N^\kappa,q}.$$

Then for any even function F *such that* $\operatorname{supp} F \subseteq [-1,1]$ *and* $\|F\|_{W^{\beta,q}} < \infty$ *for some* $\beta > \max\{n(1/p-1/s), 1/q\}$, *the operator* $F(t\sqrt{L})$ *is bounded on* $L^p(X)$ *for all* $t > 0$. *In addition*

$$\sup_{t>0}\|F(t\sqrt{L})\|_{p\to p} \leq C\|F\|_{W^{\beta,q}}.$$

Note that condition $(\mathrm{SC}^{q,\kappa}_{p,s})$ is weaker than $(\mathrm{ST}^{q}_{p,s})$ and we need a priori estimate $(\mathrm{AB}^{q,\kappa}_{p})$ in Theorem 3.6. See also [21, Theorem 3.6] and [24, Theorem 3.2] for related results. Once $(\mathrm{SC}^{q,\kappa}_{p,s})$ is proved, a priori estimate $(\mathrm{AB}^{q,\kappa}_{p})$ is not difficult to check in general, see for example the section on the harmonic oscillator.

Following [24], we prove the following result.

Proposition 3.7. *Suppose that* $\mu(X) < \infty$ *and* $(\mathrm{SC}^{q,1}_{p,s})$ *for some* p, s, q *such that* $1 \leq p < s \leq \infty$ *and* $1 \leq q \leq \infty$. *Then*

$$\|F(\sqrt{L})\|_{p\to p} \leq CN^{n(\frac{1}{p}-\frac{1}{s})}\|\delta_N F\|_{N,q}$$

for all $N \in \mathbb{N}$ *and all Borel functions* F *such that supp* $F \subseteq [-N, N]$. *Therefore, for any even function* F *such that* $\operatorname{supp} F \subseteq [-1,1]$ *and* $\|F\|_{W^{\beta,q}} < \infty$ *for some* $\beta > \max\{n(1/p-1/s), 1/q\}$, *the operator* $F(t\sqrt{L})$ *is bounded on* $L^p(X)$ *for all* $t > 0$ *and*

$$\sup_{t>0}\|F(t\sqrt{L})\|_{p\to p} \leq C\|F\|_{W^{\beta,q}}.$$

Proof. Since $\mu(X) < \infty$, we may assume that $X = B(x_0, 1)$ for some $x_0 \in X$ (see [3]). It follows from Hölder's inequality and condition $(\mathrm{SC}^{q,1}_{p,s})$ that

$$\begin{aligned}\|F(\sqrt{L})\|_{p\to p} &\leq \mu(X)^{\frac{1}{p}-\frac{1}{s}}\|F(\sqrt{L})P_{B(x_0,1)}\|_{p\to s}\\ &\leq C\mu(X)^{\frac{1}{p}-\frac{1}{s}}\mu(X)^{\frac{1}{s}-\frac{1}{p}}N^{n(\frac{1}{p}-\frac{1}{s})}\|\delta_N F\|_{N,q}\\ &\leq CN^{n(\frac{1}{p}-\frac{1}{s})}\|\delta_N F\|_{N,q}.\end{aligned}$$

This means that $(\mathrm{AB}^{q,1}_{p})$ is satisfied and thus the last assertion follows from Theorem 3.6. This proves Proposition 3.7. □

The proof of Theorems 3.1 and 3.6 uses the following lemma. In the case where the volume is polynomial this lemma is proved in [29] using a similar argument.

Lemma 3.8. *Suppose that* T *is a linear map such that for all* $x \in X$ *and* $r > 0$ *the operator* $TP_{B(x,r)}$ *is bounded from* $L^p(X)$ *to* $L^s(X)$ *for some* $1 \leq p < s \leq \infty$. *Assume also that*

$$\operatorname{supp} K_T \subseteq \mathcal{D}_\rho$$

for some $\rho > 0$. *Then there exists a constant* $C = C_{p,s}$ *such that*

$$\|T\|_{p\to p} \leq C\sup_{x\in X}\{V(x,\rho)^{\frac{1}{p}-\frac{1}{s}}\|TP_{B(x,\rho)}\|_{p\to s}\}.$$

Proof. We fix $\rho > 0$. Then we choose a sequence $(x_n) \in X$ such that $d(x_i, x_j) > \rho/10$ for $i \neq j$ and $\sup_{x\in X} \inf_i d(x, x_i) \leq \rho/10$. Such sequence exists because X is separable. Second we let $B_i = B(x_i, \rho)$ and define $\widetilde{B}_i$ by the formula

$$\widetilde{B}_i = \bar{B}\Big(x_i, \frac{\rho}{10}\Big) \setminus \bigcup_{j<i} \bar{B}\Big(x_j, \frac{\rho}{10}\Big),$$

where $\bar{B}(x,\rho) = \{y \in X\colon d(x,y) \leq \rho\}$. Third we put $\chi_i = \chi_{\widetilde{B}_i}$, where $\chi_{\widetilde{B}_i}$ is the characteristic function of the set $\widetilde{B}_i$. Note that for $i \neq j$ $B(x_i, \frac{\rho}{20}) \cap B(x_j, \frac{\rho}{20}) = \emptyset$. Hence

$$K = \sup_i \#\{j:\ d(x_i, x_j) \leq 2\rho\} \leq \sup_x \frac{V(x, (2 + \frac{1}{20})\rho)}{V(x, \frac{\rho}{20})} < C41^n < \infty.$$

It is not difficult to see that

$$\mathcal{D}_\rho \subset \bigcup_{\{i,j\colon d(x_i,x_j)<2\rho\}} \widetilde{B}_i \times \widetilde{B}_j \subset \mathcal{D}_{4\rho}$$

so

$$Tf = \sum_{i,j\colon d(x_i,x_j)<2\rho} P_{\widetilde{B}_i} T P_{\widetilde{B}_j} f.$$

Hence by Hölder's inequality

$$\begin{aligned}
\|Tf\|_p^p = \| \sum_{i,j\colon d(x_i,x_j)<2\rho} P_{\widetilde{B}_i} T P_{\widetilde{B}_j} f\|_p^p &= \sum_i \| \sum_{j\colon d(x_i,x_j)<2\rho} P_{\widetilde{B}_i} T P_{\widetilde{B}_j} f\|_p^p \\
&\leq CK^{p-1} \sum_i \sum_{j\colon d(x_i,x_j)<2\rho} \|P_{\widetilde{B}_i} T P_{\widetilde{B}_j} f\|_p^p \\
&\leq CK^{p-1} \sum_i \sum_{j\colon d(x_i,x_j)<2\rho} \mu(\widetilde{B}_i)^{p(\frac{1}{p}-\frac{1}{s})} \|P_{\widetilde{B}_i} T P_{\widetilde{B}_j} f\|_s^p \\
&\leq CK^p \sum_j \mu(B_j)^{p(\frac{1}{p}-\frac{1}{s})} \|T P_{\widetilde{B}_j} f\|_s^p \\
&\leq CK^p \sum_j \mu(B_j)^{p(\frac{1}{p}-\frac{1}{s})} \|T P_{\widetilde{B}_j}\|_{p\to s}^p \|P_{\widetilde{B}_j} f\|_p^p \\
&\leq CK^p \sup_{x\in X} \{V(x,\rho)^{p(\frac{1}{p}-\frac{1}{s})} \|T P_{B(x,\rho)}\|_{p\to s}^p\} \sum_j \|P_{\widetilde{B}_j} f\|_p^p \\
&= CK^p \sup_{x\in X} \{V(x,\rho)^{p(\frac{1}{p}-\frac{1}{s})} \|T P_{B(x,\rho)}\|_{p\to s}^p\} \|f\|_p^p.
\end{aligned}$$

This finishes the proof of Lemma 3.8. □

Proof of Theorem 3.1. Let $\eta \in C_c^\infty(\mathbb{R})$ be even and such that $\operatorname{supp}\eta \subseteq \{\xi : 1/4 \leq |\xi| \leq 1\}$ and

$$\sum_{\ell\in\mathbb{Z}} \eta(2^{-\ell}\lambda) = 1 \qquad \forall \lambda > 0.$$

Then we set $\eta_0(\lambda) = 1 - \sum_{\ell>0} \eta(2^{-\ell}\lambda)$,

$$F^{(0)}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \eta_0(t) \hat{F}(t) \cos(t\lambda)\, dt \tag{3.4}$$

and

$$F^{(\ell)}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \eta(2^{-\ell}t) \hat{F}(t) \cos(t\lambda)\, dt. \tag{3.5}$$

Note that in virtue of the Fourier inversion formula

$$F(\lambda) = \sum_{\ell\geq 0} F^{(\ell)}(\lambda)$$

and by Lemma 2.1

$$\operatorname{supp} K_{F^{(\ell)}(t\sqrt{L})} \subset \mathcal{D}_{2^\ell t}.$$

Now by Lemma 3.8

$$\begin{aligned} \left\|F(t\sqrt{L})\right\|_{p\to p} &\leq \sum_{\ell\geq 0} \left\|F^{(\ell)}(t\sqrt{L})\right\|_{p\to p} \\ &\leq C\sum_{\ell\geq 0} \sup_{x\in X} \{V(x,2^\ell t)^{\frac{1}{p}-\frac{1}{s}} \left\|F^{(\ell)}(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s}\}. \end{aligned} \tag{3.6}$$

Since $F^{(\ell)}$ is not compactly supported we choose a function $\psi \in C_c^\infty(-4,4)$ such that $\psi(\lambda) = 1$ for $\lambda \in (-2,2)$ and note that

$$\begin{aligned} &\left\|F^{(\ell)}(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s} \\ &\leq \left\|(\psi F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s} + \left\|((1-\psi)F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s}. \end{aligned} \tag{3.7}$$

To estimate the norm $\|(\psi F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\|_{p\to s}$, we use condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}})$ and the fact that $\psi \in C_c(-4,4)$ to obtain

$$\left\|(\psi F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s} \leq CV(x,2^\ell t)^{\frac{1}{s}-\frac{1}{p}} 2^{\ell n(\frac{1}{p}-\frac{1}{s})} \left\|\delta_{4t^{-1}}(\psi F^{(\ell)})(t\cdot)\right\|_q$$

for all $t > 0$. Hence

$$\begin{aligned} \sum_{\ell\geq 0} \sup_{x\in X} \{V(x,2^\ell t)^{\frac{1}{p}-\frac{1}{s}} \left\|(\psi F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s}\} &\leq C\sum_{\ell\geq 0} 2^{\ell n(\frac{1}{p}-\frac{1}{s})} \left\|\delta_{4t^{-1}}(\psi F^{(\ell)})(t\cdot)\right\|_q \\ &\leq C\sum_{\ell\geq 0} 2^{\ell n(\frac{1}{p}-\frac{1}{s})} \|F^{(\ell)}\|_q \\ &= C\|F\|_{B^{n(\frac{1}{p}-\frac{1}{s})}_{q,1}}, \end{aligned} \tag{3.8}$$

where the last equality follows from definition of Besov space. See, e.g., [5, Chap. VI]. Recall also that if $\beta > n(1/p - 1/s)$ then $W^{\beta,q} \subseteq B^{n(1/p-1/s)}_{q,1}$ and $\|F\|_{B^{n(1/p-1/s)}_{q,1}} \leq C_\beta \|F\|_{W^{\beta,q}}$, see again [5]. Hence the forgoing estimates give

$$\sum_{\ell\geq 0} \sup_{x\in X} \{V(x,2^\ell t)^{\frac{1}{p}-\frac{1}{s}} \left\|(\psi F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s}\} \leq C\|F\|_{W^{\beta,q}}. \tag{3.9}$$

Next we show bounds for $\left\|((1-\psi)F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s}$. Since the function $1-\psi$ is supported outside the interval $(-2,2)$, we can choose a function $\phi \in C_c^\infty(2,8)$ such that

$$1 = \psi(\lambda) + \sum_{k\geq 0} \phi(2^{-k}\lambda) = \psi(\lambda) + \sum_{k\geq 0} \phi_k(\lambda) \qquad \forall \lambda > 0.$$

Hence

$$((1-\psi)F^{(\ell)})(\lambda) = \sum_{k\geq 0} (\phi_k F^{(\ell)})(\lambda) \qquad \forall \lambda > 0.$$

It follows from the implication $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p,s}}) \Rightarrow (\mathrm{ST}^{\infty}_{\mathrm{p,s}})$ that

$$\left\|((1-\psi)F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s} \leq \sum_{k\geq 0} \left\|(\phi_k F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\right\|_{p\to s}$$

$$\leq \quad C\sum_{k\geq 0} V(x,2^\ell t)^{\frac{1}{s}-\frac{1}{p}} 2^{n(\ell+k)(\frac{1}{p}-\frac{1}{s})} \big\|\delta_{2^{k+3}t^{-1}}(\phi_k F^{(\ell)})(t\cdot)\big\|_\infty.$$

Note that $\operatorname{supp} F \in [-1,1]$, $\operatorname{supp}\phi \subset [2,8]$ and $\check{\eta}$ is in the Schwartz class so

$$\big\|\phi_k F^{(\ell)}\big\|_\infty = 2^\ell \big\|\phi_k (F * \delta_{2^\ell}\check{\eta})\big\|_\infty \leq C2^{-M(\ell+k)}\|F\|_q$$

and similarly, $\big\|\phi_k F^{(0)}\big\|_\infty \leq C2^{-Mk}\|F\|_q$. Therefore

$$\begin{aligned}\big\|((1-\psi)F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\big\|_{p\to s} &\leq C\sum_{k\geq 0} V(x,2^\ell t)^{\frac{1}{s}-\frac{1}{p}} 2^{n(\ell+k)(\frac{1}{p}-\frac{1}{s})} 2^{-M(\ell+k)}\|F\|_q \\ &\leq CV(x,2^\ell t)^{\frac{1}{s}-\frac{1}{p}} 2^{\ell(n(\frac{1}{p}-\frac{1}{s})-M)}\|F\|_q. \qquad (3.10)\end{aligned}$$

Hence

$$\begin{aligned}\sum_{\ell\geq 0}\sup_{x\in X}\{V(x,2^\ell t)^{\frac{1}{p}-\frac{1}{s}}\big\|((1-\psi)F^{(\ell)})(t\sqrt{L})P_{B(x,2^\ell t)}\big\|_{p\to s}\} &\leq C\sum_{\ell\geq 0} 2^{\ell(n(\frac{1}{p}-\frac{1}{s})-M)}\|F\|_q \\ &\leq C\|F\|_q. \qquad (3.11)\end{aligned}$$

Now estimate (3.1) follows from (3.11), (3.6), (3.7) and (3.9). This completes the proof of Theorem 3.1. □

Proof of Theorem 3.6.

Case (1). $t \geq 1/4$.

If $t \geq 1/4$ then $\operatorname{supp}\delta_t F \subset [-4,4]$. By $(\mathrm{AB}_{\mathrm{p}}^{\mathrm{q},\kappa})$,

$$\|F(t\sqrt{L})\|_{p\to p} \quad \leq \quad C4^{\kappa n(\frac{1}{p}-\frac{1}{s})+\varepsilon}\|\delta_4(F(t\cdot))\|_{4^\kappa,q} \leq C\|F\|_\infty.$$

Recall that if $\beta > 1/q$, then $W^{\beta,q}(\mathbb{R}) \subseteq L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ and $\|F\|_\infty \leq C\|F\|_{W^{\beta,q}}$. Hence

$$\sup_{t\geq 1/4}\|F(t\sqrt{L})\|_{p\to p} \leq C\|F\|_\infty \leq C\|F\|_{W^{\beta,q}}.$$

Case (2). $t \leq 1/4$.

Let $\xi \in C_c^\infty$ be an even function such that $\operatorname{supp}\xi \subset [-1,1]$, $\hat{\xi}(0) = 1$ and $\hat{\xi}^{(k)}(0) = 0$ for all $1 \leq k \leq [\beta]+2$. Write $\xi_{N^{\kappa-1}} = N^{\kappa-1}\xi(N^{\kappa-1}\cdot)$ where $N = 8[t^{-1}]+1$ and $[t^{-1}]$ denotes the integer part of t^{-1}. Following [21] we write

$$F(t\sqrt{L}) = (\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)(\sqrt{L}) + (\xi_{N^{\kappa-1}} * \delta_t F)(\sqrt{L}).$$

We first prove that

$$\|(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)(\sqrt{L})\|_{p\to p} \quad \leq \quad C\|F\|_{W^{\beta,q}}. \qquad (3.12)$$

Observe that $\operatorname{supp}(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F) \subseteq [-N,N]$. We apply $(\mathrm{AB}_{\mathrm{p}}^{\mathrm{q},\kappa})$ to obtain

$$\|(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)(\sqrt{L})\|_{p\to p} \leq CN^{\kappa n(\frac{1}{p}-\frac{1}{s})+\varepsilon}\big\|\delta_N(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)\big\|_{N^\kappa,q}. \qquad (3.13)$$

Everything then boils down to estimating $\|\cdot\|_{N^\kappa,q}$ norm of $\delta_N(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F)$. We make the following claim. For its proof, see [21, (3.29)] or [24, Propostion 4.6].

Lemma 3.9. *Suppose that* $\xi \in C_c^\infty$ *is an even function such that* $\operatorname{supp}\xi \subset [-1,1]$, $\hat{\xi}(0) = 1$ *and* $\hat{\xi}^{(k)}(0) = 0$ *for all* $1 \leq k \leq [\beta]+2$. *Next assume that* $\operatorname{supp} H \subset [-1,1]$. *Then*

$$\|H - \xi_N * H\|_{N,q} \leq CN^{-\beta}\|H\|_{W^{\beta,q}} \qquad (3.14)$$

for all $\beta > 1/q$ *and any* $N \in \mathbb{N}$.

Note that $\delta_N(\delta_t F - \xi_{N^{\kappa-1}} * \delta_t F) = \delta_{Nt} F - \xi_{N^\kappa} * \delta_{Nt} F$. Now, if $\beta > \max\{n(1/p - 1/s), 1/q\}$ then (3.12) follows from Lemma 3.9 and estimate (3.13).

It remains to show that

$$\|(\xi_{N^{\kappa-1}} * \delta_t F)(\sqrt{L})\|_{p\to p} \quad \le \quad C\|F\|_{W^{\beta,q}}. \tag{3.15}$$

Let $F^{(\ell)}$ be functions defined in (3.4) and (3.5). Following the proof of Theorem 3.1, we write

$$(\xi_{N^{\kappa-1}} * \delta_t F)(\lambda) = \sum_{\ell\ge 0} (\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(\lambda),$$

and by Lemma 2.1, $\operatorname{supp} K_{(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(\sqrt{L})} \subset \mathcal{D}_{2^\ell t}$. Now by Lemma 3.8

$$\begin{aligned} \big\|(\xi_{N^{\kappa-1}} * \delta_t F)(\sqrt{L})\big\|_{p\to p} \quad &\le \quad \sum_{\ell\ge 0} \big\|(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(\sqrt{L})\big\|_{p\to p} \\ &\le \quad C\sum_{\ell\ge 0} \sup_{x\in X} \{V(x, 2^\ell t)^{\frac{1}{p}-\frac{1}{s}} \big\|(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(\sqrt{L}) P_{B(x,2^\ell t)}\big\|_{p\to s}\}. \end{aligned} \tag{3.16}$$

Take a function $\psi \in C_c^\infty(-4, 4)$ such that $\psi(\lambda) = 1$ for $\lambda \in (-2, 2)$. Then

$$\begin{aligned} \big\|(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(\sqrt{L}) P_{B(x,2^\ell t)}\big\|_{p\to s} \quad &\le \quad \big\|(\delta_t \psi(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)}))(\sqrt{L}) P_{B(x,2^\ell t)}\big\|_{p\to s} \\ &+ \quad \big\|((1 - \delta_t\psi)(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)}))(\sqrt{L}) P_{B(x,2^\ell t)}\big\|_{p\to s} = I_\ell + I\!I_\ell. \end{aligned} \tag{3.17}$$

Note that $(\mathrm{SC}^{q,\kappa}_{p,s}) \Rightarrow (\mathrm{SC}^{\infty}_{p,s})$ and $t \le 1/4$. Using $(\mathrm{SC}^{\infty}_{p,s})$ instead of $(\mathrm{ST}^{\infty}_{p,s})$, we show as in the proof of (3.10) that $I\!I_\ell \le CV(x, 2^\ell t)^{\frac{1}{s}-\frac{1}{p}} 2^{\ell(n(\frac{1}{p}-\frac{1}{s})-M)}\|F\|_q$ for some large $M > n(1/p - 1/s) + 1$.

Next we estimate the term I_ℓ. We assume that $\psi \in C_c(-4, 4)$ so by $(\mathrm{SC}^{q,\kappa}_{p,s})$

$$I_\ell \le CV(x, 2^\ell t)^{\frac{1}{s}-\frac{1}{p}} 2^{\ell n(\frac{1}{p}-\frac{1}{s})} \big\|\delta_N(\delta_t\psi(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)}))\big\|_{N^\kappa,q}.$$

Observe that (see also [21, (3.19)])

$$\big|(\xi * \delta_t F^{(\ell)})(\lambda)\big| \le \|\xi\|_{L^{q'}} \Big(\int_{\lambda-1}^{\lambda+1} |F^{(\ell)}(tu)|^q du\Big)^{1/q}$$

so

$$\begin{aligned} \big\|\delta_N(\delta_t\psi(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)}))\big\|_{N^\kappa,q} \quad &= \quad \Big(\frac{1}{2N^\kappa} \sum_{i=1-N^\kappa}^{N^\kappa} \sup_{\lambda\in[\frac{i-1}{N^\kappa}, \frac{i}{N^\kappa})} |\psi(tN\lambda)(\xi_{N^{\kappa-1}} * \delta_t F^{(\ell)})(N\lambda)|^q\Big)^{1/q} \\ &\le \quad C\Big(\frac{1}{N^\kappa} \sum_{i=1-N^\kappa}^{N^\kappa} \sup_{\lambda\in[i-1,i)} \big|(\xi * \delta_{tN^{1-\kappa}} F^{(\ell)})(\lambda)|^q\Big)^{1/q} \\ &\le \quad C\|\xi\|_{q'}\Big(\frac{1}{N^\kappa} \sum_{i=1-N^\kappa}^{N^\kappa} \sup_{\lambda\in[i-1,i)} \int_{\lambda-1}^{\lambda+1} |F^{(\ell)}(tN^{1-\kappa}u)|^q du\Big)^{1/q} \\ &\le \quad C\Big(\frac{1}{N^\kappa} \sum_{i=1-N^\kappa}^{N^\kappa} \int_{i-2}^{i+1} |F^{(\ell)}(tN^{1-\kappa}u)|^q du\Big)^{1/q} \\ &\le \quad C\Big(\frac{1}{Nt} \int_{-\infty}^{\infty} |F^{(\ell)}(u)|^q du\Big)^{1/q} \le C\|F^{(\ell)}\|_q. \end{aligned}$$

This shows that $I_\ell \le CV(x, 2^\ell t)^{\frac{1}{s}-\frac{1}{p}} 2^{\ell n(\frac{1}{p}-\frac{1}{s})}\|F^{(\ell)}\|_q$. Using the above estimates of I_ℓ and $I\!I_\ell$, together with (3.16) and (3.17), we can argue as in (3.9) and (3.11) to obtain estimate (3.15). This proves Theorem 3.6. □

Now we discuss another condition introduced by C.D. Sogge (see [62, 63, 64]). We say that L satisfies (p, p') *spectral cluster estimate* $(\mathrm{S_p})$ for some $1 \le p < 2$ and its conjugate p' if the spectral projection $E_{\sqrt{L}}[k, k+1)$ maps $L^p(X)$ to $L^{p'}(X)$ and

$$\left\|E_{\sqrt{L}}[k,k+1)\right\|_{p\to p'} \le C(1+k)^{n(\frac{1}{p}-\frac{1}{p'})-1} \tag{$\mathrm{S_p}$}$$

for all $k \ge 0$.

Proposition 3.10. *Suppose that $1 \le p \le 2n/(n+1)$, $\mu(X) < \infty$ and $V(x,r) \le C\min(r^n, 1)$ for every $x \in X$ and $r > 0$. Then conditions $(\mathrm{S_p})$ and $(\mathrm{SC}^{2,1}_{\mathrm{p},2})$ are equivalent.*

Proof. We first prove the implication $(\mathrm{S_p}) \Rightarrow (\mathrm{SC}^{2,1}_{\mathrm{p},2})$. Note that for every even Borel function F such that $\operatorname{supp} F \subset [-N, N]$,

$$\begin{aligned}\left\|F(\sqrt{L})f\right\|_2^2 &= \sum_{k=0}^{N} \langle E_{\sqrt{L}}[k,k+1)F(\sqrt{L})f,\ E_{\sqrt{L}}[k,k+1)F(\sqrt{L})f\rangle \\ &\le \sum_{k=0}^{N} \left\|E_{\sqrt{L}}[k,k+1)F(\sqrt{L})\right\|_{p\to 2}^2 \|f\|_p^2.\end{aligned}$$

Using a T^*T argument and condition $(\mathrm{S_p})$ we obtain

$$\begin{aligned}\left\|F(\sqrt{L})P_{B(x,r)}\right\|_{p\to 2}^2 &\le C\left\|F(\sqrt{L})\right\|_{p\to 2}^2 \\ &\le C\sum_{k=0}^{N} \left\|E_{\sqrt{L}}[k,k+1)F(\sqrt{L})\right\|_{p\to 2}^2 \\ &\le C\sum_{k=0}^{N} \sup_{\lambda\in[k,k+1)} \left|F(\lambda)\right|^2 (1+k)^{n(\frac{1}{p}-\frac{1}{p'})-1} \\ &\le CN^{n(\frac{1}{p}-\frac{1}{p'})}\frac{1}{N}\sum_{k=0}^{N} \sup_{\lambda\in[k,k+1)} \left|F(\lambda)\right|^2 \\ &\le CV(x,r)^{2(\frac{1}{2}-\frac{1}{p})}(Nr)^{2n(\frac{1}{p}-\frac{1}{2})}\left\|\delta_N F\right\|_{N,2}^2,\end{aligned}$$

and hence condition $(\mathrm{SC}^{2,1}_{\mathrm{p},2})$ is satisfied.

Next we prove the implication $(\mathrm{SC}^{2,1}_{\mathrm{p},2}) \Rightarrow (\mathrm{S_p})$. By $(\mathrm{SC}^{2,1}_{\mathrm{p},2})$

$$\left\|\chi_{[k,k+1)}(\sqrt{L})\right\|_{p\to 2} \le C(1+k)^{n(\frac{1}{p}-\frac{1}{2})}\|\delta_{(1+k)}\chi_{[k,k+1)}\|_{1+k,\,2}. \tag{3.18}$$

Hence

$$\begin{aligned}\left\|E_{\sqrt{L}}[k,k+1)\right\|_{p\to p'} &= \left\|E_{\sqrt{L}}[k,k+1)\right\|_{p\to 2}^2 \\ &\le C(1+k)^{2n(\frac{1}{p}-\frac{1}{2})}\left\|\delta_{(1+k)}\chi_{[k,k+1)}\right\|_{1+k,\,2}^2 \\ &\le C(k+1)^{n(\frac{1}{p}-\frac{1}{p'})-1},\end{aligned}$$

which shows $(\mathrm{S_p})$. □

Proposition 3.11. *Assume that $\mu(X) < \infty$. Then conditions $(\mathrm{SC}^{\infty}_{\mathrm{p},2})$ and $(\mathrm{ST}^{\infty}_{\mathrm{p},2})$ are equivalent.*

Proof. Since $\mu(X) < \infty$, we may assume that $X = B(x_0, 1)$ for some $x_0 \in X$. Observing that $\|F\|_{N,\infty} = \|F\|_\infty$, we have the implication $(\mathrm{ST}^\infty_{\mathrm{p},2}) \Rightarrow (\mathrm{SC}^\infty_{\mathrm{p},2})$. Let us show the implication $(\mathrm{SC}^\infty_{\mathrm{p},2}) \Rightarrow (\mathrm{ST}^\infty_{\mathrm{p},2})$. Assume that $\operatorname{supp} F \subseteq [0, R]$. If $R \geq 1$, then we let $N = [R] + 1$ and $(\mathrm{ST}^\infty_{\mathrm{p},2})$ follows readily. Now for $0 < R < 1$, from condition $(\mathrm{SC}^\infty_{\mathrm{p},2})$ we can take $N = 1$ and the ball $B(x_0, 1)$ to obtain

$$\|F(\sqrt{L})\|_{p\to 2} \leq C\|F\|_\infty.$$

Now for any $x \in X$ and $r > 0$, we note that conditions $1/2 - 1/p < 0$ and $\mu(X) < \infty$ give that $V(x, r)^{1/2-1/p} \geq C$. Hence for any $r \geq 1/R$

$$\left\|F(\sqrt{L})P_{B(x,r)}\right\|_{p\to 2} \leq C\|F\|_\infty \leq CV(x, r)^{\frac{1}{2}-\frac{1}{p}}(rR)^{n(\frac{1}{p}-\frac{1}{2})}\|\delta_R F\|_\infty,$$

that is $(\mathrm{ST}^\infty_{\mathrm{p},2})$. This ends the proof of Proposition 3.11. □

4. Sharp spectral multipliers - singular integral case

4.1. Statements. As in Section 3 we discuss two type of results corresponding to estimates $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ or $(\mathrm{SC}^{\mathrm{q},\kappa}_{\mathrm{p},2})$. The aim of this section is to prove singular integral versions of Theorems 3.1 and 3.6. We use the same assumptions and notation as in Section 3. Recall that n is a homogeneous dimension from (2.2). Fix a non-trivial auxiliary function $\eta \in C^\infty_c(0, \infty)$.

Theorem 4.1. *Assume that operator L satisfies property* (FS) *and condition* $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ *for some p, q satisfying $1 \leq p < 2$ and $1 \leq q \leq \infty$. Then for any bounded Borel function F such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,q}} < \infty$ for some $\beta > \max\{n(1/p - 1/2), 1/q\}$ the operator $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $p < r < p'$. In addition,*

$$\|F(\sqrt{L})\|_{r\to r} \leq C_\beta\Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,q}} + |F(0)|\Big).$$

Note that if $q < \infty$ then condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ implies that $E_L(\{0\}) = 0$ (and in fact $E_L(\{\lambda\}) = 0$) so the term $F(0)$ can be omitted in the above statement.

The next theorem is a variation of Theorem 4.1 suitable for the operators satisfying condition $(\mathrm{SC}^{\mathrm{q},\kappa}_{\mathrm{p},2})$. It is a singular integral version of Theorem 3.6 above.

Theorem 4.2. *Suppose the operator L satisfies property* (FS), *conditions* $(\mathrm{E}_{\mathrm{p},2})$ *and* $(\mathrm{SC}^{\mathrm{q},\kappa}_{\mathrm{p},2})$ *for some p, q such that $1 \leq p < 2$ and $1 \leq q \leq \infty$, and a fixed natural number κ. In addition, we assume that for any $\varepsilon > 0$ there exists a constant C_ε such that for all $N \in \mathbb{N}$ and all even Borel functions H such that $\operatorname{supp} H \subset [-N, N]$,*

$$(\mathrm{AB}^{\mathrm{q},\kappa}_{\mathrm{p}}) \qquad \|H(\sqrt{L})\|_{p\to p} \leq C_\varepsilon N^{\kappa n(\frac{1}{p}-\frac{1}{2})+\varepsilon}\|\delta_N H\|_{N^\kappa,q}.$$

Then for any bounded Borel function F such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,q}} < \infty$ for some $\beta > \max\{n(1/p - 1/2), 1/q\}$ the operator $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $p < r < p'$. In addition,

$$\|F(\sqrt{L})\|_{r\to r} \leq C_\beta\Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,q}} + |F(0)|\Big).$$

Remark 4.3. *Suppose that $\mu(X) < \infty$ and $(\mathrm{SC}^{\mathrm{q},\kappa}_{\mathrm{p},2})$ holds for some $\kappa \geq 1$. Then $(\mathrm{SC}^\infty_{\mathrm{p},2})$ and $(\mathrm{E}_{\mathrm{p},2})$ are satisfied by Remark 3.5 and Proposition 3.11. In addition, $(\mathrm{AB}^{\mathrm{q},\kappa}_{\mathrm{p}})$ holds by Proposition 3.7. Therefore, Theorem 4.2 holds in this case without assumptions $(\mathrm{E}_{\mathrm{p},2})$ and $(\mathrm{AB}^{\mathrm{q},\kappa}_{\mathrm{p}})$.*

By a classical dyadic decomposition of F, we can write $F(\sqrt{L})$ as the sum $\sum F_j(\sqrt{L})$. Then we apply Theorems 3.1 and 3.6 to estimate $\|F_j(\sqrt{L})\|_{r\to r}$. However, as mentioned in the introduction, this does not automatically imply that the operator $F(\sqrt{L})$ acts boundedly on L^r. See [11, 58, 59] where this problem is discussed in the Euclidean case. Our proof is almost identical to one in [11]. Nevertheless we give full details because the changes which are required to adapt the arguments to the general setting are not trivial.

Note that condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ implies $(\mathrm{E}_{\mathrm{p},2})$ (see Proposition 2.3). Therefore Theorems 4.1 and 4.2 follow from Theorems 3.1 and 3.6 (with $s = 2$) and the next result.

Theorem 4.4. *Assume that L satisfies the finite speed propagation property* (FS) *and condition* $(\mathrm{E}_{\mathrm{p}_0,2})$ *for some $1 \le p_0 < 2$. Next assume that for all even Borel functions H such that* $\operatorname{supp} H \subseteq [-1,1]$ *and $\|H\|_{W^{\beta,q}} < \infty$ for some β and q satisfying $\beta > \max\{n(1/p_0 - 1/2), 1/q\}$ and $1 \le q \le \infty$,*

$$\sup_{t>0} \|H(t\sqrt{L})\|_{p\to p} \le C\|H\|_{W^{\beta,q}}, \quad p_0 \le p \le p_0'. \tag{4.1}$$

Then for any bounded Borel function F such that

$$\sup_{t>0} \|\eta\delta_t F\|_{W^{\beta,q}} < \infty \tag{4.2}$$

for some $\beta > \max\{n(1/p_0 - 1/2), 1/q\}$, the operator $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $p_0 < r < p_0'$.

4.2. Singular integrals. This subsection is devoted to the proof of Theorem 4.4. We start with the following lemma.

Lemma 4.5. *Suppose that operator L satisfies property* (FS) *and condition* $(\mathrm{E}_{\mathrm{p}_0,2})$ *for some $1 \le p_0 < 2$.*

(a) *Assume in addition that F is a bounded Borel function such that*

$$\sup_{t>0} \|\eta\delta_t F\|_{C^k} < \infty$$

for some integer $k > n/2 + 1$ and some non-trivial function $\eta \in C_c^\infty(0,\infty)$. Then the operator $F(\sqrt{L})$ is bounded on $L^p(X)$ for all $p_0 < p < p_0'$.

(b) *Assume in addition that ψ be an even function in $\mathscr{S}(\mathbb{R})$ such that $\psi(0) = 0$. Define the quadratic functional for $f \in L^2(X)$*

$$\mathcal{G}_L(f)(x) = \Big(\sum_{j\in\mathbb{Z}} |\psi(2^j\sqrt{L})f(x)|^2\Big)^{1/2}.$$

Then $\mathcal{G}_L$ is bounded on $L^p(X)$ for all $p_0 < p < p_0'$.

Proof. The finite speed propagation property implies $L^2 - L^2$ off-diagonal estimate

$$\|P_{X\setminus B(x,2^j r)} e^{-r^2 L} P_{B(x,r)}\|_{2\to 2} \le Ce^{-c2^{2j}},$$

see [19, 61]. It follows from $(\mathrm{E}_{\mathrm{p}_0,2})$ that

$$\|P_{X\setminus B(x,2^j r)} e^{-r^2 L} P_{B(x,r)}\|_{p_0\to 2} \le CV(x,r)^{\frac{1}{2}-\frac{1}{p_0}}.$$

Now the Riesz-Thorin interpolation theorem gives for $p \in (p_0, 2)$ the following $L^p - L^2$ off-diagonal estimate

$$\|P_{X\setminus B(x,2^j r)} e^{-r^2 L} P_{B(x,r)}\|_{p\to 2} \le CV(x,r)^{\frac{1}{2}-\frac{1}{p}} e^{-c'2^{2j}}.$$

Now assertion (a) follows from [6]. The latter off-diagonal estimate implies that L has a bounded holomorphic functional calculus on L^p for $p_0 < p < p_0'$ (see [7]). It is known that the holomorphic functional calculus implies the quadratic estimate of assertion (b) (see [20, 51]). □

Throughout the rest of this section, Φ denotes an even function, $\Phi \in \mathscr{S}(\mathbb{R})$ such that $\Phi(0) = 1$ and whose Fourier transform $\widehat{\Phi}$ is supported in $[-1, 1]$. We take $\eta \in C_c^\infty(\mathbb{R})$ even and such that

$$\operatorname{supp}\eta \subseteq \{\xi : 1/2 \le |\xi| \le 2\} \quad \text{and} \quad \sum_{\ell\in\mathbb{Z}} \eta(2^{-\ell}\lambda) = 1, \ \ \forall \lambda > 0. \tag{4.3}$$

Set $\eta_\ell(\lambda) = \eta(2^\ell\lambda)$ and

$$\widehat{F^{(\ell)}} = \eta_{-\ell}\widehat{F}, \qquad \ell \in \mathbb{Z}. \tag{4.4}$$

Put $Q_\ell(\lambda) = \sum_{k\ge 0} \eta_{k+\ell}(\lambda)$. Then we have the following result.

Proposition 4.6. *Suppose that operator L satisfies property* (FS) *and condition* ($\mathrm{E}_{p_0,2}$) *for $1 \le p_0 < 2$ and let $p_0 < p < 2$. Assume in addition that for a bounded Borel function F the following estimates hold*

$$\sum_{k<0} \sup_j \Big\| \sum_{\ell\ge 0} F^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})\Big\|_{p\to p} < \infty \tag{4.5}$$

and

$$\sup_j \Big\| \sum_{\ell\ge 0} F^{(j+\ell)}(\sqrt{L})(I - \Phi(2^j\sqrt{L}))Q_j(\sqrt{L})\Big\|_{p\to p} < \infty. \tag{4.6}$$

Then $F(\sqrt{L})$ is of weak-type (p, p).

Proof. Let $f \in L^p(X)$ and $\alpha > \mu(X)^{-1/p}\|f\|_p$. A simple variation of the Calderón-Zygmund decomposition of $|f|^p$ at height α shows that there exist constants C and K such that $f = g + b = g + \sum_i b_i$ so that $\|g\|_p \le C\|f\|_p$, $\|g\|_\infty \le C\alpha$, each b_i is supported on ball B_i of radius $2^{j(i)}$, and $\#\{i : x \in 8B_i\} \le K$ for all $x \in X$, $\int_X |b_i|^p d\mu \le C\alpha^p\mu(B_i)$, and $\sum_i \mu(B_i) \le C\alpha^{-p}\|f\|_p^p$. As a consequence, $\alpha^{p-2}\|g\|_2^2 \le C\|f\|_p^p$. As in [11], we choose $2^{j(i)}$ rather than $r(i)$ to be able to sum in j.

We define the "nearly good" and "very bad" functions $\tilde{g}$ and $\tilde{b}$ by

$$\tilde{g} = g + \sum_i 2\Phi(2^{j(i)}\sqrt{L})b_i - \sum_i \Phi^2(2^{j(i)}\sqrt{L})b_i \quad \text{and} \quad \tilde{b} = \sum_i (I - \Phi(2^{j(i)}\sqrt{L}))^2 b_i.$$

By Lemma 2.1, $\operatorname{supp}\Phi(2^{j(i)}\sqrt{L})b_i \subset 4B_i$, and by the Calderón-Zygmund decomposition, every $x \in X$ belongs to no more than K balls $4B_i$. Let $N > n(1/p - 1/2)$. Now by condition ($\mathrm{E}_{p_0,2}$)

$$\begin{aligned}
\Big\| \sum_i \Phi(2^{j(i)}\sqrt{L})b_i\Big\|_2^2 &\le K \sum_i \big\|\Phi(2^{j(i)}\sqrt{L})b_i\big\|_2^2 \\
&\le C \sum_i \sup_\lambda \big|\Phi(2^{j(i)}\lambda)(1 + 2^{j(i)}\lambda)^N\big|^2 \big\|(1 + 2^{j(i)}\sqrt{L})^{-N} b_i\big\|_2^2 \\
&\le C \sum_i \mu(B_i)^{1-\frac{2}{p_0}} \|b_i\|_{p_0}^2 \\
&\le C \sum_i \mu(B_i)^{1-\frac{2}{p_0}} \|b_i\|_p^2 \mu(B_i)^{\frac{2}{p_0}-\frac{2}{p}} \\
&\le C\alpha^2 \sum_i \mu(B_i) \le C\alpha^{2-p}\|f\|_p^p.
\end{aligned}$$

Replacing Φ by Φ^2 yields $\big\| \sum_i \Phi^2(2^{j(i)}\sqrt{L})b_i\big\|_2^2 \le C\alpha^{2-p}\|f\|_p^p$. By the standard L^2 argument

$$\mu(\{x : F(\sqrt{L})(\tilde{g})(x) > \alpha\}) \le C\alpha^{-p}\|f\|_p^p.$$

It remains to treat

$$\begin{aligned} F(\sqrt{L})\Big(\sum_i (I-\Phi(2^{j(i)}\sqrt{L}))^2 b_i\Big) &= \sum_i \sum_{\ell\ge 0} F^{(j(i)+\ell)}(\sqrt{L})(I-\Phi(2^{j(i)}\sqrt{L}))^2 b_i \\ &+ \sum_i \sum_{\ell<0} F^{(j(i)+\ell)}(\sqrt{L})(I-\Phi(2^{j(i)}\sqrt{L}))^2 b_i. \end{aligned} \tag{4.7}$$

By Lemma 2.1

$$\operatorname{supp} F^{(j(i)+\ell)}(\sqrt{L})(I-\Phi(2^{j(i)}\sqrt{L}))^2 b_i \subseteq B(x_i, 2^{j(i)+\ell+1}+3\cdot 2^{j(i)}) \subseteq 8B_i, \quad \forall \ell < 0.$$

Thus the second term of (4.7) is supported in $\cup 8B_i$ and $\sum \mu(8B_i) \le C\sum \mu(B_i) \le C\alpha^{-p}\|f\|_p^p$.

To treat the first term we show that

$$\Big\| \sum_j \sum_{\ell\ge 0} F^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))^2 f_j\Big\|_p \le C\Big\| \sum_j |f_j|\Big\|_p. \tag{4.8}$$

If we apply (4.8) with

$$f_j = \sum_{i:j(i)=j} b_i,$$

we see that

$$\begin{aligned} \Big\| \sum_i \sum_{\ell\ge 0} F^{(j(i)+\ell)}(\sqrt{L})(I-\Phi(2^{j(i)}\sqrt{L}))^2 b_i\Big\|_p &= \Big\| \sum_j \sum_{\ell\ge 0} F^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))^2 f_j\Big\|_p \\ &\le C\Big\| \sum_j \Big| \sum_{i:j(i)=j} b_i\Big|\Big\|_p \\ &\le C\Big\| \sum_i |b_i|\Big\|_p \\ &\le C\Big(\sum_i \|b_i\|_p^p\Big)^{1/p} \le C\|f\|_p, \end{aligned}$$

which completes the proof.

As in [11] we argue that by duality, (4.8) is equivalent to

$$\Big\| \sup_j \Big| \sum_{\ell\ge 0} \bar{F}^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))^2 h\Big|\Big\|_{p'} \le C\|h\|_{p'}. \tag{4.9}$$

Write

$$\begin{aligned} \sum_{\ell\ge 0} \bar{F}^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))^2 &= \sum_{k<0}\sum_{\ell\ge 0} \bar{F}^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))^2 \eta_{j+k}(\sqrt{L}) \\ &+ \sum_{\ell\ge 0} \bar{F}^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))^2 Q_j(\sqrt{L}). \end{aligned} \tag{4.10}$$

Let $\tilde{\eta} \in C_c^\infty(0,\infty)$ be a non-negative function satisfying $\operatorname{supp}\tilde{\eta} \subseteq [1/4,4]$ and $\tilde{\eta} = 1$ on $[1/2,2]$, and let $\tilde{\eta}_j$ denote the function $\tilde{\eta}(2^j\cdot)$. By Lemma 4.5 point (b) with $\psi = \tilde{\eta}$ there exists a constant $C > 0$ independent of $k < 0$ such that for $2 < p' < p_0'$,

$$\Big(\sum_j \big\|\tilde{\eta}_{j+k}(\sqrt{L})h\big\|_{p'}^{p'}\Big)^{1/p'} \le \Big\|\Big(\sum_j \big|\tilde{\eta}_{j+k}(\sqrt{L})h\big|^2\Big)^{1/2}\Big\|_{p'} \le C\|h\|_{p'},$$

and by Lemma 4.5 point (a), it follows that for $2 < p' < p_0'$, $\|(I - \Phi(2^j\sqrt{L}))^2\|_{p'\to p'} \le C$ for some constant $C > 0$ independent of $j \in \mathbb{Z}$. Hence

$$\begin{aligned}
&\Big\| \sup_j \Big| \sum_{k<0}\sum_{\ell\ge 0} \bar F^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))^2\eta_{j+k}(\sqrt{L})h\Big|\Big\|_{p'}\\
&\quad\le \sum_{k<0}\Big(\sum_j \Big\| \sum_{\ell\ge 0} \bar F^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))^2\eta_{j+k}(\sqrt{L})\tilde\eta_{j+k}(\sqrt{L})h\Big\|_{p'}^{p'}\Big)^{1/p'}\\
&\quad\le C\sum_{k<0}\sup_j\Big\{\|(I-\Phi(2^j\sqrt{L}))^2\|_{p'\to p'}\Big\| \sum_{\ell\ge 0} \bar F^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})\Big\|_{p'\to p'}\Big\}\\
&\qquad\qquad\times\Big(\sum_j \big\|\tilde\eta_{j+k}(\sqrt{L})h\big\|_{p'}^{p'}\Big)^{1/p'}\\
&\quad\le C\sum_{k<0}\sup_j\Big\| \sum_{\ell\ge 0} \bar F^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})\Big\|_{p'\to p'}\|h\|_{p'} \le C\|h\|_{p'}. \qquad (4.11)
\end{aligned}$$

The last inequality follows from assumption (4.5). Using assumption (4.6) instead of (4.5) the similar argument as above gives the following estimate

$$\Big\| \sup_j \Big| \sum_{\ell\ge 0} \bar F^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))^2 Q_j(\sqrt{L})h\Big|\Big\|_{p'} \le C\|h\|_{p'}.$$

This shows (4.9) and ends the proof of Proposition 4.6. □

Proposition 4.7. *Suppose that operator L satisfies property* (FS) *and condition* ($\mathrm{E}_{p_0,2}$) *for some $1 \le p_0 < 2$ and fix $p \in (p_0, 2)$. Next assume that for all even Borel functions H such that* $\operatorname{supp} H \subseteq [-1,1]$ *and $\|H\|_{W^{\beta,\infty}} < \infty$ for some $\beta > n/2$,*

$$\sup_{t>0}\|H(t\sqrt{L})\|_{p\to p} \le C\|H\|_{W^{\beta,\infty}}. \qquad (4.12)$$

Then for any bounded Borel function F such that $\|(F\eta_i)^{(j)}(\sqrt{L})\|_{p\to p} \le \alpha(i-j)$ for all $i, j \in \mathbb{Z}$ with

$$\sum_{k\le 0}(|k|+1)\alpha(k) < \infty,$$

the operator $F(\sqrt{L})$ is of weak-type (p,p). Here η_i and $(F\eta_i)^{(j)}$ are defined in (4.3) and (4.4), respectively.

Proof. By Proposition 4.6, it suffices to verify (4.5) and (4.6). Note that we may assume $F(0) = 0$ since $F = F - F(0) + F(0)$.

Firstly we show (4.5). Fix $k \le 0$, $\ell \ge 0$ and $j \in \mathbb{Z}$ and write

$$\begin{aligned}
F^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L}) &= \sum_{i\in\mathbb{Z}}(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})\\
&= \sum_{i<k-2} + \sum_{i=k-2}^{k+2} + \sum_{i>k+2}\cdots = I_{jk\ell} + I\!I_{jk\ell} + I\!I\!I_{jk\ell}.
\end{aligned}$$

The main term is $I\!I_{jk\ell}$; $I_{jk\ell}$ and $I\!I\!I_{jk\ell}$ are error terms. By (4.12), $\|\eta_{j+k}(\sqrt{L})\|_{p\to p} \le C$ so

$$\|I\!I_{jk\ell}\|_{p\to p} \le C\sum_{i=k-2}^{k+2}\|(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\|_{p\to p} \le C\sum_{i=-2}^{2}\alpha(k+i-\ell).$$

Thus

$$\sum_{k\le 0}\sum_{\ell\ge 0}\sup_j \|I\!I_{jk\ell}\|_{p\to p} \le C\sum_{m\le 2}(|m|+1)\alpha(m)<\infty. \tag{4.13}$$

To estimate $\|I_{jk\ell}\|_{p\to p}$ set

$$G(\lambda)=\sum_{i<k-2}(F\eta_{i+j})^{(j+\ell)}(2^{-j-k}\lambda)\eta(\lambda).$$

Observe that

$$\frac{d^\gamma}{d\lambda^\gamma}\Big(\sum_{i<k-2}(F\eta_{i+j})^{(j+\ell)}(2^{-j-k}\lambda)\Big) = \int_{|s|\ge 2^{-k-j+2}}\sum_{i<k-2}F(s)\eta_{i+j}(s)2^{j+\ell}2^{(\ell-k)\gamma}(\check{\eta})^{(\gamma)}(2^{\ell-k}\lambda-2^{j+\ell}s)ds.$$

Now $2^{j+\ell}|s|\ge 2^{\ell-k+2}\ge 2^{\ell-k+1}|\lambda|$ for $\lambda\in[1/2,2]$. We may estimate the integral for each $N\in\mathbb{N}$ by

$$\int_{|s|\ge 2^{-k-j+2}}\|F\|_\infty 2^{j+\ell}2^{(\ell-k)\gamma}\frac{C_N}{(2^{j+\ell}|s|)^N}ds.$$

If N is chosen sufficiently large, this is dominated by $\|F\|_\infty 2^{(\ell-k)(1-N+\gamma)}$. This yields

$$\|G\|_{W^{\gamma,\infty}}\le C\|F\|_\infty 2^{\varepsilon_0(k-\ell)} \tag{4.14}$$

for some $\varepsilon_0>0$ and all $\gamma\in\mathbb{N}$. Then by (4.12) $\|I_{jk\ell}\|_{p\to p}\le C\|F\|_\infty 2^{\varepsilon_0(k-\ell)}$. Hence

$$\sum_{k\le 0}\sum_{\ell\ge 0}\sup_j\|I_{jk\ell}\|_{p\to p}<\infty.$$

We estimate $I\!I\!I_{jk\ell}$ in the similar way as $\|I_{jk\ell}\|_{p\to p}$. This proves (4.5).

Secondly we show (4.6). Fix $\ell\ge 0$ and $j\in\mathbb{Z}$. Write

$$\begin{aligned} F^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))Q_j(\sqrt{L}) &= \sum_{i\in\mathbb{Z}}(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))Q_j(\sqrt{L}) \\ &= \sum_{i\le 0}+\sum_{i>0}\cdots = A_{j\ell}+B_{j\ell}. \end{aligned}$$

We start with the term $A_{j\ell}$. By condition (4.12) there exists a positive constant C independent of j such that $\|(I-\Phi(2^j\sqrt{L}))Q_j(\sqrt{L})\|_{p\to p}\le C$. By assumptions of the proposition, we have $\|(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\|_{p\to p}\le\alpha(i-\ell)$. Hence

$$\sup_j\Big\|\sum_{\ell\ge 0}A_{j\ell}\Big\|_{p\to p}\le C\sum_{\ell\ge 0}\sum_{i\le 0}\alpha(i-\ell)\le C\sum_{m\le 0}(|m|+1)\alpha(m)<\infty.$$

It remains to treat the term $B_{j\ell}$. More precisely, we want to prove that

$$\begin{aligned} &\sup_j\sum_{\ell\ge 0}\|\sum_{i>0}(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))Q_j(\sqrt{L})\|_{p\to p} \\ &=\sup_j\sum_{\ell\ge 0}\|\sum_{i>0}\sum_{k\in\mathbb{Z}}(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})(I-\Phi(2^j\sqrt{L}))Q_j(\sqrt{L})\|_{p\to p}<\infty. \end{aligned}$$

We write

$$\sum_{i>0}\sum_{k\in\mathbb{Z}}$$

as

$$\sum_{k\le 0}\sum_{i=1}^{\infty}+\sum_{k=1}^{\ell}\sum_{i=1}^{k}+\sum_{k=1}^{\ell}\sum_{i=k+1}^{\ell}+\sum_{k=1}^{\ell}\sum_{i=\ell+1}^{\infty}+\sum_{k=\ell+1}^{\infty}\sum_{i=1}^{\ell}+\sum_{k=\ell+1}^{\infty}\sum_{i=\ell+1}^{\infty}\cdots$$

$$= I_{j\ell} + II_{j\ell} + III_{j\ell} + IV_{j\ell} + V_{j\ell} + VI_{j\ell}.$$

Claim 1. For $k \le 0$,

$$\| \sum_{i>0} (F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})\|_{p\to p} \le C2^{-\varepsilon(\ell-k)}$$

for some $\varepsilon > 0$.

Claim 2.

$$\sum_{i=1}^{k} \|(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\|_{p\to p} \le \sum_{i=1}^{k} \alpha(i-\ell).$$

Claim 3. For $k \le i$, $\|(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})\|_{p\to p} \le C2^{-\varepsilon(\ell-k)}$, so that

$$\sum_{i=k+1}^{\ell} \|(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})\|_{p\to p} \le C(\ell-k)2^{-\varepsilon(\ell-k)} \le C2^{-\varepsilon'(\ell-k)}.$$

Claim 4. For $\ell \ge k$,

$$\| \sum_{i=\ell+1}^{\infty} (F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})\|_{p\to p} \le C2^{-\varepsilon'(\ell-k)}.$$

Claim 5.

$$\sum_{i=1}^{\ell} \|(F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\|_{p\to p} \le \sum_{i=1}^{\ell} \alpha(i-\ell).$$

Claim 6. For $k \ge \ell$,

$$\| \sum_{i=\ell+1}^{\infty} (F\eta_{i+j})^{(j+\ell)}(\sqrt{L})\eta_{j+k}(\sqrt{L})\|_{p\to p} \le C.$$

If we agree with the claims 1-6, we can finish the proof of the proposition as follows. Let $\tilde{\eta} \in C_c^\infty(0,\infty)$ be a non-negative function as in (4.11). We note that by condition (4.12),

$$\begin{aligned} \|\tilde{\eta}_{j+k}(\sqrt{L})(I - \Phi(2^j\sqrt{L}))Q_j(\sqrt{L})\|_{p\to p} &\le C\|\tilde{\eta}(1-\delta_{2^{-k}}\Phi)\|_{W^{n,\infty}} \\ &\le C\min\{1, 2^{-k}\}. \end{aligned} \tag{4.15}$$

By *Claim 1*,

$$\sum_{\ell\ge 0} \|I_{j\ell}\|_{p\to p} \le C \sum_{\ell\ge 0}\sum_{k\le 0} 2^{-\varepsilon(\ell-k)} < \infty;$$

by *Claim 2* and (4.15),

$$\|II_{j\ell}\|_{p\to p} \le C \sum_{k=1}^{\ell}\sum_{i=1}^{k} \alpha(i-\ell)2^{-\varepsilon'' k};$$

by *Claim 3* and (4.15),

$$\|III_{j\ell}\|_{p\to p} \le C \sum_{k=1}^{\ell} 2^{-\varepsilon'(\ell-k)}2^{-\varepsilon'' k} \le C2^{-\varepsilon\ell};$$

by *Claim 4* and (4.15),

$$\|IV_{j\ell}\|_{p\to p} \le C \sum_{k=1}^{\ell} 2^{-\varepsilon'(\ell-k)}2^{-\varepsilon'' k} \le C2^{-\varepsilon\ell};$$

by *Claim 5* and (4.15),

$$\|V_{j\ell}\|_{p\to p} \le C \sum_{k=\ell+1}^{\infty} \sum_{i=1}^{\ell} \alpha(i-\ell) 2^{-\varepsilon'' k};$$

by *Claim 6* and (4.15),

$$\|VI_{j\ell}\|_{p\to p} \le C \sum_{k=\ell+1}^{\infty} 2^{-\varepsilon'' k} \le C 2^{-\varepsilon \ell}.$$

Hence

$$\sum_{\ell>0} \|III_{j\ell} + IV_{j\ell} + VI_{j\ell}\|_{p\to p} \le C \sum_{\ell>0} 2^{-\varepsilon\ell} < \infty,$$

and

$$\begin{aligned} \|II_{j\ell} + V_{j\ell}\|_{p\to p} &\le C \sum_{k=1}^{\ell} \sum_{i=1}^{k} \alpha(i-\ell) 2^{-\varepsilon'' k} + C \sum_{k=\ell+1}^{\infty} \sum_{i=1}^{\ell} \alpha(i-\ell) 2^{-\varepsilon'' k} \\ &= C \sum_{i=1}^{\ell} \alpha(i-\ell) \sum_{k=i}^{\infty} 2^{-\varepsilon k} \le C \sum_{i=1}^{\ell} \alpha(i-\ell) 2^{-\varepsilon i}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\ell>0} \|II_{j\ell} + V_{j\ell}\|_{p\to p} &\le C \sum_{\ell>0} \sum_{i=1}^{\ell} 2^{-\varepsilon i} \alpha(i-\ell) \\ &\le C \sum_{m\le 0} \alpha(m) 2^{-\varepsilon m} \sum_{\ell\ge -m} 2^{-\varepsilon\ell} \le C \sum_{m\le 0} \alpha(m) < \infty. \end{aligned}$$

Now *Claims 2* and *5* follow immediately from the definition of $\alpha(j)$. Similarly to the proof of (4.14), we use condition (4.12) to prove *Claims 1, 3, 4* and *6*. So to establish *Claims 1, 3, 4* and *6* we examine

$$\frac{d^\gamma}{d\lambda^\gamma}\Big(\sum_i (F\eta_{i+j})^{(j+\ell)}(2^{-j-k}\lambda)\Big)$$

for $|\lambda| \sim 1$, (where the sum is over a range of i depending on which claim we are proving). In any case, we obtain

$$\int \sum_i F(s)\eta_{i+j}(s) 2^{j+\ell} 2^{(\ell-k)\gamma} \frac{d^\gamma \check{\eta}}{d\lambda^\gamma}(2^{\ell-k}\lambda - 2^{j+\ell}s) ds.$$

For *Claim 6*, we just use $|\frac{d^\gamma\check{\eta}}{d\lambda^\gamma}| \le C$ and the fact that the integrand is supported in a set of measure $\le C2^{-(j+\ell)}$ to estimate the integral by $\|F\|_\infty 2^{(\ell-k)\gamma}$; under the hypotheses of *Claims 1, 3* and *4* we have that (essentially) $2^{\ell-k}|\lambda| \ge 2\cdot 2^{j+\ell}|s|$ if $|\lambda| \sim 1$, and thus we may estimate the integral by $\|F\|_\infty 2^{j+\ell} 2^{(\ell-k)(\gamma-N)}$ multiplied by the measure of the support of the integrand, for each $N \in \mathbb{N}$. These measures are 2^{-j}, 2^{-i-j} and $2^{-j-\ell}$ respectively, and so in each case we can dominate the integral by $C2^{(\ell-k)(\gamma-N+1)}$ for all $N \in \mathbb{N}$. □

Recall that for $0 < \alpha < 1$, Λ_α is the usual Lipschitz space as defined for example in [66]. As a consequence of Proposition 4.7, we have the following result.

Corollary 4.8. *Assume that operator L satisfies property* (FS) *and condition* ($E_{p_0,2}$) *for some $1 \le p_0 < 2$. Next assume that for all even bounded Borel function H such that* $\operatorname{supp} H \subseteq [-1,1]$ *and $\|H\|_{W^{\beta,\infty}} < \infty$ for some $\beta > n/2$,*

$$\sup_{t>0} \|H(t\sqrt{L})\|_{p\to p} \le C\|H\|_{W^{\beta,\infty}}, \quad p_0 < p < p_0'. \tag{4.16}$$

Then for any bounded Borel function F such that for all $i, j \in \mathbb{Z}$,

$$\|(F\eta_i)^{(j)}(\sqrt{L})\|_{r\to r} + \|\eta\delta_{2^{-i}}F\|_{\Lambda_\alpha} \le C \tag{4.17}$$

for some $0 < \alpha < 1$ and all $p_0 < r \le 2$, $F(\sqrt{L})$ is bounded on L^r for $p_0 < r < p_0'$. Here η_i and $(F\eta_i)^{(j)}$ are defined in (4.3) and (4.4), respectively.

Proof. First we note that by definition

$$(F\eta_i)^{(j)}(\lambda) = \int F(2^{-i}t)\eta(t)\widehat{\eta_{i-j}}(2^i\lambda - t)dt = (\eta\delta_{2^{-i}}F)^{(j-i)}(2^i\lambda). \tag{4.18}$$

Since $\|\eta\delta_{2^{-i}}F\|_{\Lambda_\alpha} \le C$, we have that $\|(F\eta_i)^{(j)}\|_\infty \le C2^{\alpha(i-j)}$ and so $\|(F\eta_i)^{(j)}(\sqrt{L})\|_{2\to 2} \le C2^{\alpha(i-j)}$. By interpolation, there exists some $\alpha' > 0$ such that

$$\|(F\eta_i)^{(j)}(\sqrt{L})\|_{r\to r} \le C2^{\alpha'(i-j)}$$

for all $p_0 < r \le 2$. From Proposition 4.7, $F(\sqrt{L})$ is of weak-type (r, r) for all $p_0 < r \le 2$. By duality and interpolation, $F(\sqrt{L})$ is bounded on L^r for $p_0 < r < p_0'$. □

Proof of Theorem 4.4. It is enough to verify conditions (4.17) of Corollary 4.8. First we recall that η_i and $(F\eta_i)^{(j)}$ are functions defined in (4.3) and (4.4), respectively. From (4.18), we have that $\|(F\eta_i)^{(j)}(\sqrt{L})\|_{r\to r} = \|(\delta_{2^{-i}}F\eta)^{(j-i)}(2^i\sqrt{L})\|_{r\to r}$. Let ψ be a C_c^∞ even function which is supported on $[-8, 8]$ and $\psi(\lambda) = 1$ on $[-4, 4]$. Write

$$\|(\delta_{2^{-i}}F\eta)^{(j-i)}(2^i\sqrt{L})\|_{r\to r} \le \|(\psi(\delta_{2^{-i}}F\eta)^{(j-i)})(2^i\sqrt{L})\|_{r\to r} + \|((1-\psi)(\delta_{2^{-i}}F\eta)^{(j-i)})(2^i\sqrt{L})\|_{r\to r}.$$

Observe that the function $\psi(\delta_{2^{-i}}F\eta)^{(j-i)}$ is supported on $[-8, 8]$ and

$$\begin{aligned}
\|\psi(\delta_{2^{-i}}F\eta)^{(j-i)}\|_{W^{\beta,q}} &\le C\|(\delta_{2^{-i}}F\eta)^{(j-i)}\|_{W^{\beta,q}} \\
&= C\|\mathscr{F}^{-1}((1+\xi^2)^{\beta/2}\widehat{\delta_{2^{-i}}F\eta}(\xi)\eta_{i-j}(\xi))\|_q \\
&\le C\|\mathscr{F}^{-1}((1+\xi^2)^{\beta/2}\widehat{\delta_{2^{-i}}F\eta}(\xi))\|_q \\
&= C\|F(2^{-i}\lambda)\eta(\lambda)\|_{W^{\beta,q}}.
\end{aligned}$$

From condition (4.2), we have that $\|\eta(\lambda)F(2^{-i}\lambda)\|_{W^{\beta,q}} < \infty$. Therefore, we use our assumption (4.1) to obtain

$$\|(\psi(\delta_{2^{-i}}F\eta)^{(j-i)})(2^i\sqrt{L})\|_{r\to r} \le C \tag{4.19}$$

with $C > 0$ independent of i and j.

We estimate the term $\|((1-\psi)(\delta_{2^{-i}}F\eta)^{(j-i)})(2^i\sqrt{L})\|_{r\to r}$. For $k \in \mathbb{N}$ and all $\lambda \ne 0$, we have by elementary calculation,

$$\begin{aligned}
&\left|(1-\psi(\lambda))\frac{d^k}{d\lambda^k}((\delta_{2^{-i}}F\eta)^{(j-i)})(\lambda))\right| \\
&\quad\le C2^{(j-i)(k+1)}\left|1-\psi(\lambda)\right|\int\left|F(2^{-i}u)\eta(u)\right|(1+2^{j-i}|\lambda-u|)^{-k-1}du \le C|\lambda|^{-k},
\end{aligned}$$

where we use the fact that $|\lambda| \ge 4$ and $|u| \le 2$. We then apply (a) of Lemma 4.5 to obtain

$$\|((1-\psi)(\delta_{2^{-i}}F\eta)^{(j-i)})(2^i\sqrt{L})\|_{r\to r} \le C.$$

This estimate in combination with (4.19) shows that

$$\|(F\eta_i)^{(j)}(\sqrt{L})\|_{r\to r} = \|(\delta_{2^{-i}}F\eta)^{(j-i)}(2^i\sqrt{L})\|_{r\to r} \le C \tag{4.20}$$

for some constant $C > 0$ independent of i, j.

Now we recall that if $1 \le q \le \infty$ and $\beta - 1/q > 0$, then

$$W^{\beta,q} \subseteq B^{\beta}_{q,\infty} \subseteq B^{\beta-\frac{1}{q}}_{\infty,\infty} \subseteq \Lambda_{\min\{\beta-\frac{1}{q},\,\frac{1}{2}\}}$$

and $\|F\|_{\Lambda_{\min\{\beta-\frac{1}{q},\,\frac{1}{2}\}}} \le C\|F\|_{W^{\beta,q}}$. See, e.g., [5, Chap. VI] for more details. We obtain

$$\|\eta\delta_{2^{-i}}F\|_{\Lambda_{\min\{\beta-1/q,\,1/2\}}} \le C\|\eta\delta_{2^{-i}}F\|_{W^{\beta,q}} \le C'. \tag{4.21}$$

This estimate and (4.20) prove condition (4.17) of Corollary 4.8. The proof of Theorem 4.4 is finished. □

5. Endpoint estimates for Bochner-Riesz means

We have seen in Corollary 3.2 that Bochner-Riesz means are bounded on L^p provided the order δ satisfies $\delta > \max\{n(1/p - 1/s) - 1/q, 0\}$. In this section we prove that our restriction type condition implies endpoint estimates for Bochner-Riesz means. Our approach is inspired by the results of Christ and Tao [14, 15, 69]. As in the rest of the paper we assume that (X, d, μ) is a metric measure space satisfying condition (2.2) with a homogeneous dimension n.

For any given $p \in [1, 2)$ and $q \in [1, \infty]$ we define

$$\delta_q(p) = \max\Big\{0, n\Big|\frac{1}{p} - \frac{1}{2}\Big| - \frac{1}{q}\Big\}.$$

For simplicity we will write $\delta(p)$ instead of $\delta_2(p)$. As in Sections 3 and 4 we discuss two type of results corresponding to estimates $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ or $(\mathrm{SC}^{\mathrm{q},1}_{\mathrm{p},2})$.

Theorem 5.1. *Assume that operator L satisfies property* (FS) *and condition* $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ *for some p, q satisfying $1 \le p < 2$ and $1 \le q \le \infty$. Then the operator $S_R^{\delta_q(p)}(L)$ is of weak-type (p, p) uniformly in R.*

The next theorem is a variation of Theorem 5.1. As in Theorems 3.6 and 4.2, this variation can be used in the case of operators with nonempty pointwise spectrum.

Theorem 5.2. *Assume that $\mu(X) < \infty$. Assume that operator L satisfies property* (FS) *and condition* $(\mathrm{SC}^{\mathrm{q},1}_{\mathrm{p},2})$ *for some p, q satisfying $1 \le p < 2$ and $1 \le q \le \infty$. Then the operator $S_R^{\delta_q(p)}(L)$ is of weak-type (p, p) uniformly in R.*

The proofs of Theorem 5.1 and 5.2 require three technical lemmas which we discuss first. We commence with the following observation.

Lemma 5.3. *For each $k \le 0$ there exists a decomposition*

$$S_R^{\delta_q(p)}(\lambda^2) = \eta_k(\lambda)n_k(\lambda) + S_R^{\delta_q(p)}(\lambda^2)n_k(\lambda) \tag{5.1}$$

such that

(a) *Functions n_k are even and their Fourier transforms are supported in $[-2^k/R,\, 2^k/R]$, i.e., $\operatorname{supp}\widehat{n_k} \subset [-2^k/R,\, 2^k/R]$;*
(b) *Functions η_k are continuous, even and $\sum_{k=-\infty}^{0} |\eta_k(\lambda)|^2 \le C$ with C independent of λ and R;*
(c) *For certain arbitrarily large $N \in \mathbb{N}$ there exists a constant C such that*

$$|n_k(\lambda)| \le C\Big(1 + \frac{2^k|\lambda|}{R}\Big)^{-N}.$$

Proof. Following [69] we consider the function $\Phi(\lambda) = 6\lambda^{-3}(\lambda - \sin\lambda)$. Note that $\widehat{\Phi}(t) = 3\pi(1 - |t|)_+^2$. Set $n_k(\lambda) = \Phi^{N/2}(2^k\lambda(RN)^{-1})$. We use term $(RN)^{-1}$ rather than R^{-1} to ensure that $\operatorname{supp}\widehat{n_k} \subset [-2^k/R,\, 2^k/R]$. Next we write

$$\begin{aligned} S_R^{\delta_q(p)}(\lambda^2) &= \left(\frac{S_R^{\delta_q(p)}(\lambda^2)(1 - n_k(\lambda))}{n_k(\lambda)}\right) n_k(\lambda) + S_R^{\delta_q(p)}(\lambda^2) n_k(\lambda) \\ &= \eta_k(\lambda) n_k(\lambda) + S_R^{\delta_q(p)}(\lambda^2) n_k(\lambda). \end{aligned}$$

Verifying conditions (a) and (c) is straightforward. By definition of $n_k(\lambda)$, there exist a constant c_N, depending only on N, and another constant C such that

$$n_k(\lambda) \geq c_N \quad \text{and} \quad |S_R^{\delta_q(p)}(\lambda^2)(1 - n_k(\lambda))| \leq C2^k$$

for all $|\lambda| \leq R$ and $k \leq 0$. This proves condition (b) because $\operatorname{supp} S_R^{\delta_q(p)}(\lambda^2) \subset [-R, R]$. □

The original statement of the following lemma comes from [69].

Lemma 5.4. *For each $k > 0$, there exists a decomposition $S_R^{\delta_q(p)}(\lambda^2) = m_k(\lambda) + \eta_k(\lambda)n_k(\lambda)$ such that:*

(a) *Functions $\widehat{m_k}$ and $\widehat{n_k}$ are even and supported on $[-2^k/R, 2^k/R]$;*
(b) *η_k are continuous and for all $\lambda > 0$, $\sum_{k=1}^{\infty} |\eta_k(\lambda)|^2 \leq C$ with C independent of λ and R;*
(c) *For certain arbitrarily large $N \in \mathbb{N}$ there exists a constant C such that*

$$|n_k(\lambda)| \leq C2^{-\delta_q(p)k}\Big(1 + 2^k\Big|1 - \frac{|\lambda|}{R}\Big|\Big)^{-N}.$$

Proof. For the proof, we refer the reader to Lemma 2.1 in [69]. □

In the proofs of Theorems 5.1 and 5.2 we will also use the following lemma

Lemma 5.5. *Suppose that L is a self-adjoint operator on $L^2(X)$. Assume that $\{Q_k\}_{k\in\mathbb{N}}$ is a family of continuous real-valued functions such that $\sum_k |Q_k(\lambda)|^2 \leq C$ for some constant C independent of λ. Then for any sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ on X,*

$$\Big\| \sum_k Q_k(\sqrt{L}) f_k \Big\|_2^2 \leq C \sum_k \|f_k\|_2^2. \tag{5.2}$$

Proof. Note that

$$\begin{aligned} \Big\| \sum_k Q_k(\sqrt{L}) f_k \Big\|_2^2 &= \Big\langle \sum_k Q_k(\sqrt{L}) f_k, \sum_k Q_k(\sqrt{L}) f_k \Big\rangle \\ &= \int_{-\infty}^{+\infty} \sum_k \sum_j Q_k(\lambda) Q_j(\lambda) d\langle E_{\sqrt{L}}(\lambda) f_k, f_j\rangle. \end{aligned}$$

Under the assumption of Lemma 5.5 the above integral is a limit of its Riemannian approximations, see page 310 of [74]. Therefore let us consider Riemannian partition of $[\alpha, \beta]$

$$\alpha = \lambda_1 < \lambda_2 < \ldots < \lambda_n = \beta, \quad \lambda_\ell' \in (\lambda_\ell,\, \lambda_{\ell+1}]$$

for some $-\infty < \alpha < \beta < \infty$. Now to finish the proof of Lemma 5.5 it is enough to observe that

$$\left| \sum_\ell \sum_k \sum_j Q_k(\lambda_\ell') Q_j(\lambda_\ell') \langle E_{\sqrt{L}}(\lambda_\ell,\, \lambda_{\ell+1}] f_k,\, f_j\rangle \right|$$

$$\leq \sum_\ell \sum_k \sum_j |Q_k(\lambda_\ell')||Q_j(\lambda_\ell')| \sqrt{\langle E_{\sqrt{L}}(\lambda_\ell,\, \lambda_{\ell+1}]f_k,\, f_k\rangle}\sqrt{\langle E_{\sqrt{L}}(\lambda_\ell,\, \lambda_{\ell+1}]f_j,\, f_j\rangle}$$

$$= \sum_\ell \Big(\sum_k |Q_k(\lambda_\ell')| \sqrt{\langle E_{\sqrt{L}}(\lambda_\ell,\, \lambda_{\ell+1}]f_k,\, f_k\rangle}\Big)^2$$

$$\leq \sum_\ell \Big(\sum_k |Q_k(\lambda_\ell')|^2\Big) \sum_k \langle E_{\sqrt{L}}(\lambda_\ell,\, \lambda_{\ell+1}]f_k,\, f_k\rangle$$

$$\leq C \sum_\ell \sum_k \langle E_{\sqrt{L}}(\lambda_\ell,\, \lambda_{\ell+1}]f_k,\, f_k\rangle$$

$$\leq C \sum_k \|f_k\|_2^2,$$

where we used the fact that $\sum_k |Q_k(\lambda)|^2 \leq C$. □

Proof of Theorem 5.1. Assume that condition $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$ holds for some $1 \leq p < 2$ and $1 \leq q \leq \infty$. Fix a $f \in L^p$ and $\alpha > \mu(X)^{-1/p}\|f\|_p$, and apply the Calderón-Zygmund decomposition at height α to $|f|^p$. There exist constants C and K so that

(i) $f = g + b = g + \sum_j b_j$;
(ii) $\|g\|_p \leq C\|f\|_p$, $\|g\|_\infty \leq C\alpha$;
(iii) b_j is supported in B_j and $\#\{j : x \in 4B_j\} \leq K$ for all $x \in X$;
(iv) $\int_X |b_j|^p d\mu \leq C\alpha^p \mu(B_j)$, and $\sum_j \mu(B_j) \leq C\alpha^{-p}\|f\|_p^p$.

Let r_{B_j} be the radius of B_j and denote by $J_k = \{j :\ 2^k/R \leq r_{B_j} < 2^{k+1}/R\}$. Write

$$f = g + \sum_j b_j = g + \sum_{k\leq 0}\sum_{j\in J_k} b_j + \sum_{k>0}\sum_{j\in J_k} b_j = g + h_1 + h_2.$$

By a standard argument it is enough to show that there exists a constant $C > 0$ independent of R and α such that for every $\alpha > \mu(X)^{-1/p}\|f\|_p$,

$$\mu(\{x : S_R^{\delta_q(p)}(L)(g)(x) > \alpha\}) \leq C\alpha^{-p}\|f\|_p^p \tag{5.3}$$

and that for $i = 1, 2$,

$$\mu(\{x : S_R^{\delta_q(p)}(L)(h_i)(x) > \alpha\}) \leq C\alpha^{-p}\|f\|_p^p. \tag{5.4}$$

Note that $\sup_{\lambda,R>0}\big(1 - \frac{\lambda}{R}\big)_+^{\delta_q(p)} = 1$ and that by (ii) $\alpha^{p-2}\|g\|_2^2 \leq C\|f\|_p^p$. Hence by spectral theory

$$\begin{aligned}\mu(\{x : S_R^{\delta_q(p)}(L)(g)(x) > \alpha\}) &\leq \alpha^{-2}\|S_R^{\delta_q(p)}(L)(g)\|_2^2 \leq \alpha^{-2}\|g\|_2^2\\ &\leq C\alpha^{-p}\|f\|_p^p.\end{aligned} \tag{5.5}$$

Next we prove (5.4) for $i = 1$. By the spectral theorem and equality (5.1)

$$\begin{aligned}\sum_{k\leq 0}\sum_{j\in J_k} S_R^{\delta_q(p)}(L)b_j &= \sum_{k\leq 0} \eta_k(\sqrt{L})\Big(\sum_{j\in J_k} n_k(\sqrt{L})b_j\Big)\\ &+ S_R^{\delta_q(p)}(L)\Big(\sum_{k\leq 0}\sum_{j\in J_k} n_k(\sqrt{L})b_j\Big).\end{aligned}$$

Applying the spectral theorem and Lemma 5.5 with $Q_k(\lambda) = \eta_k(\lambda)$ yields

$$\Big\|\sum_{k\leq 0}\sum_{j\in J_k} S_R^{\delta_q(p)}(L)b_j\Big\|_2^2 \leq C\sum_{k\leq 0}\Big\|\sum_{j\in J_k} n_k(\sqrt{L})b_j\Big\|_2^2 + C\Big\|\sum_{k\leq 0}\sum_{j\in J_k} n_k(\sqrt{L})b_j\Big\|_2^2. \tag{5.6}$$

Next $\operatorname{supp}\widehat{n_k} \subseteq [-2^k/R, 2^k/R]$ so by Lemma 2.1,

$$\operatorname{supp} K_{n_k(\sqrt{L})} \subseteq \{(x,y) \in X \times X : d(x,y) \le 2^k/R\}.$$

Hence if $j \in J_k$, then $\operatorname{supp} n_k(\sqrt{L})b_j \subseteq 4B_j$. Thus by (iii) there exists constant $C > 0$ such that

$$\sum_{k\le 0}\Big\|\sum_{j\in J_k} n_k(\sqrt{L})b_j\Big\|_2^2 + \Big\|\sum_{k\le 0}\sum_{j\in J_k} n_k(\sqrt{L})b_j\Big\|_2^2 \le C\sum_{k\le 0}\sum_{j\in J_k}\big\|n_k(\sqrt{L})b_j\big\|_2^2.$$

Next, by Proposition 2.3 and Remark 2.2,

$$\begin{aligned}
\big\|n_k(\sqrt{L})b_j\big\|_2 &= \big\|n_k(\sqrt{L})P_{B_j}b_j\big\|_2\\
&\le \big\|n_k(\sqrt{L})P_{B_j}\big\|_{p\to 2}\big\|b_j\big\|_p\\
&\le \Big\|n_k(\sqrt{L})\Big(I+2^k\frac{\sqrt{L}}{R}\Big)^N\Big\|_{2\to 2}\Big\|\Big(I+2^k\frac{\sqrt{L}}{R}\Big)^{-N}P_{B_j}\Big\|_{p\to 2}\|b_j\|_p\\
&\le C\Big\|\Big(I+2^k\frac{\sqrt{L}}{R}\Big)^{-N}P_{B_j}\Big\|_{p\to 2}\|b_j\|_p\\
&\le C\mu(B_j)^{\frac12-\frac1p}\alpha\mu(B_j)^{\frac1p}\\
&\le C\alpha\mu(B_j)^{1/2}.
\end{aligned}$$

Hence by (iv)

$$\begin{aligned}
\mu\Big(\Big\{x : \Big|S_R^{\delta_q(p)}(L)\Big(\sum_{k\le 0}\sum_{j\in J_k} b_j\Big)\Big| > \alpha\Big\}\Big) &\le C\alpha^{-2}\Big\|S_R^{\delta_q(p)}(L)\Big(\sum_{k\le 0}\sum_{j\in J_k} b_j\Big)\Big\|_2^2\\
&\le C\alpha^{-p}\|f\|_p^p. \qquad (5.7)
\end{aligned}$$

Now, we prove (5.4) for $i = 2$. Let $\Omega^* = \bigcup_{j\in\mathbb{N}} 4B_j$. By (2.1) that

$$\mu(\Omega^*) \le C\sum_j \mu(B_j) \le C\alpha^{-p}\|f\|_p^p.$$

Hence it is enough to show that

$$\Big\|S_R^{\delta_q(p)}(L)\Big(\sum_{k>0}\sum_{j\in J_k} b_j\Big)\Big\|_{L^2(X\setminus\Omega^*)}^2 \le C\alpha^2\sum_j \mu(B_j) \le C\alpha^{2-p}\|f\|_p^p. \qquad (5.8)$$

Using the decomposition from Lemma 5.4 we write

$$S_R^{\delta_q(p)}(L)\Big(\sum_{k>0}\sum_{j\in J_k} b_j\Big) = \sum_{k>0}\sum_{j\in J_k} m_k(\sqrt{L})b_j + \sum_{k>0}\eta_k(\sqrt{L})n_k(\sqrt{L})\Big(\sum_{j\in J_k} b_j\Big). \qquad (5.9)$$

Recall that $\widehat{m_k}$ is even and supported in $[-2^k/R, 2^k/R]$. By Lemma 2.1

$$\operatorname{supp} K_{m_k(\sqrt{L})} \subset \{(x,y) \in X \times X : d(x,y) \le \frac{2^k}{R}\}.$$

This implies that if $x \in X\setminus\Omega^*$, then $m_k(\sqrt{L})b_j(x) = 0$ for any $j \in J_k$ and $k > 0$ so it makes no contribution to (5.8). By (5.9) and Lemma 5.5

$$\begin{aligned}
\Big\|S_R^{\delta_q(p)}(L)\Big(\sum_{k>0}\sum_{j\in J_k} b_j\Big)\Big\|_{L^2(X\setminus\Omega^*)}^2 &\le \Big\|\sum_{k>0}\eta_k(\sqrt{L})n_k(\sqrt{L})\Big(\sum_{j\in J_k} b_j\Big)\Big\|_2^2\\
&\le C\sum_{k>0}\Big\|n_k(\sqrt{L})\Big(\sum_{j\in J_k} b_j\Big)\Big\|_2^2. \qquad (5.10)
\end{aligned}$$

Next $\widehat{n_k}$ is even and supported on $[-2^k/R, 2^k/R]$ so by Lemma 2.1

$$\operatorname{supp} K_{n_k(\sqrt{L})} \subseteq \{(x,y) \in X \times X : d(x,y) \le 2^k/R\}.$$

Hence $\operatorname{supp} n_k(\sqrt{L}) b_j \subseteq 4B_j$ for $j \in J_k$. By (iii) there exists a constant $C > 0$ such that

$$\sum_{k>0} \Big\| n_k(\sqrt{L}) \Big(\sum_{j \in J_k} b_j \Big) \Big\|_2^2 \;\le\; C \sum_{k>0} \sum_{j \in J_k} \big\| n_k(\sqrt{L}) b_j \big\|_2^2. \tag{5.11}$$

To continue, fix $\psi \in C_c^\infty$, even and supported in $[-2,2]$ such that $\psi(\lambda) = 1$ for $|\lambda| \le 3/2$. Write

$$n_k(\sqrt{L}) = n_k(\sqrt{L}) \psi\Big(\frac{\sqrt{L}}{R}\Big) + n_k(\sqrt{L}) \Big(1 - \psi\Big(\frac{\sqrt{L}}{R}\Big)\Big). \tag{5.12}$$

Then $\operatorname{supp}(1 - \psi(\lambda/R)) \subset (-\infty, -3R/2] \cup [3R/2, \infty)$ so for every $k > 0$,

$$\begin{aligned} \Big| n_k(\lambda) \Big(1 - \psi\Big(\frac{\lambda}{R}\Big)\Big) \Big| &\le C \Big| 1 - \psi\Big(\frac{\lambda}{R}\Big) \Big| 2^{-\delta_q(p)k} \Big(1 + 2^k \Big| 1 - \frac{|\lambda|}{R} \Big|\Big)^{-N} \\ &\le C \Big(1 + \Big| \frac{2^k \lambda}{R} \Big|\Big)^{-N}, \end{aligned}$$

and we use a similar argument as in the proof of (5.4) for $i = 1$ to conclude that

$$\Big\| n_k(\sqrt{L}) \Big(I - \psi\Big(\frac{\sqrt{L}}{R}\Big)\Big) b_j \Big\|_2 \le C \alpha \mu(B_j)^{1/2}. \tag{5.13}$$

On the other hand, $\operatorname{supp} n_k(\lambda)\psi(\lambda/R) \subset [-2R, 2R]$ so by $(\mathrm{ST}^{\mathrm{q}}_{\mathrm{p},2})$

$$\begin{aligned} \Big\| n_k(\sqrt{L}) \psi\Big(\frac{\sqrt{L}}{R}\Big) b_j \Big\|_2 &= \Big\| n_k(\sqrt{L}) \psi\Big(\frac{\sqrt{L}}{R}\Big) P_{B_j} b_j \Big\|_2 \\ &\le \Big\| n_k(\sqrt{L}) \psi\Big(\frac{\sqrt{L}}{R}\Big) P_{B_j} \Big\|_{p \to 2} \|b_j\|_p \\ &\le C \alpha \mu(B_j)^{\frac{1}{2}} 2^{kn(\frac{1}{p} - \frac{1}{2})} \big\| \delta_{2R}(n_k(\lambda)\psi(\lambda/R)) \big\|_q. \end{aligned}$$

Now

$$\begin{aligned} \Big\| \delta_{2R}\Big(n_k(\lambda)\psi\Big(\frac{\lambda}{R}\Big)\Big) \Big\|_q &\le C \Big(\int_0^1 |n_k(2R\lambda)|^q d\lambda \Big)^{1/q} \\ &\le C 2^{-\delta_q(p)k} \Big(\int_0^1 (1 + 2^k |1 - 2\lambda|)^{-Nq} d\lambda \Big)^{1/q} \\ &\le C 2^{-\delta_q(p)k} 2^{-\frac{k}{q}} \le C 2^{-nk(\frac{1}{p} - \frac{1}{2})}. \end{aligned}$$

This yields

$$\Big\| n_k(\sqrt{L}) \psi\Big(\frac{\sqrt{L}}{R}\Big) b_j \Big\|_2 \;\le\; C \alpha \mu(B_j)^{1/2}. \tag{5.14}$$

By (5.13) and (5.14)

$$\big\| n_k(\sqrt{L}) b_j \big\|_2 \le C \alpha \mu(B_j)^{1/2}.$$

The rest of the proof of (5.4) for $i = 2$ is similar to the case $i = 1$. □

Proof of Theorem 5.2. Assume that condition $(\mathrm{SC}^{\mathrm{q},1}_{\mathrm{p},2})$ holds for some $1 \le p < 2$ and $1 \le q \le \infty$. The proof of Theorem 5.2 is almost identical to that of Theorem 5.1 except some minor technical complications so we only give a brief sketch of it.

We apply the Calderón-Zygmund decomposition at height α to $|f|^p$ to get the same decomposition

$$f = g + \sum_{k\le 0}\sum_{j\in J_k} b_j + \sum_{k>0}\sum_{j\in J_k} b_j = g + h_1 + h_2$$

as in Theorem 5.1. The proof of weak type estimates for g and h_1 uses the simple observation that $(\mathrm{SC}^{q,1}_{p,2}) \Rightarrow (\mathrm{SC}^{\infty}_{p,2}) \Leftrightarrow (\mathrm{ST}^{\infty}_{p,2}) \Rightarrow (\mathrm{E}_{p,2})$ (see Propositions 2.3 and 3.11) and is essentially the same as the corresponding argument in the proof of Theorem 5.1.

It remains to show that

$$\mu(\{x : S_R^{\delta_q(p)}(L)(h_2)(x) > \alpha\}) \le C\alpha^{-p}\|f\|_p^p. \tag{5.15}$$

To show (5.15) we note that if $\mu(X)$ is finite, then we may assume $X = B(x_0, 1)$ for some $x_0 \in X$. Recall that $J_k = \{j : 2^k/R \le r_{B_j} < 2^{k+1}/R\}$. Thus the radius of each B_j in the Calderón-Zygmund decomposition satisfies $2^k/R \le 4$. If $R \le 4$, then $k \le 4$. Hence one can use the same argument as in the proof of Theorem 5.1 for $i = 1$.

Next we consider the remaining case $R > 4$. Using the decomposition described in Lemma 5.4 it is not difficult to note that to finish the proof it is enough to show that

$$\Big\|n_k(\sqrt{L})\Big(I - \psi\Big(\frac{\sqrt{L}}{R}\Big)\Big)b_j\Big\|_2 \le C\alpha\mu(B_j)^{1/2}, \tag{5.16}$$

$$\Big\|n_k(\sqrt{L})\psi\Big(\frac{\sqrt{L}}{R}\Big)b_j\Big\|_2 \le C\alpha\mu(B_j)^{1/2}, \tag{5.17}$$

where n_k is defined in Lemma 5.4 and ψ is a function in (5.12). The proof of (5.16) is similar to that of (5.13).

To prove (5.17) set $N = [2R] + 1$. By condition $(\mathrm{SC}^{q,1}_{p,2})$

$$\Big\|n_k(\sqrt{L})\psi\Big(\frac{\sqrt{L}}{R}\Big)b_j\Big\|_2 \le C\mu(B_j)^{(\frac12-\frac1p)}2^{kn(\frac1p-\frac12)}\big\|\delta_N(n_k(\lambda)\psi(\lambda/R))\big\|_{N,q}\|b_j\|_p.$$

Next (assuming that $\sup|\psi| = 1$)

$$\begin{aligned}
2^{1/q}\big\|\delta_N(n_k(\lambda)\psi(\lambda/R))\big\|_{N,q} &\le \Big(\frac1N\sum_{\ell=1-N}^{N}\sup_{\lambda\in[\ell-1,\ell)}|n_k(\lambda)|^q\Big)^{1/q}\\
&\le \Big(\frac1N\sum_{\ell=1-N}^{-[R]-2}\sup_{\lambda\in[\ell-1,\ell)}|n_k(\lambda)|^q\Big)^{1/q} + \Big(\frac1N\sum_{\ell=-[R]-1}^{-[R]+3}\sup_{\lambda\in[\ell-1,\ell)}|n_k(\lambda)|^q\Big)^{1/q}\\
&+ \Big(\frac1N\sum_{\ell=-[R]+4}^{0}\sup_{\lambda\in[\ell-1,\ell)}|n_k(\lambda)|^q\Big)^{1/q} + \Big(\frac1N\sum_{\ell=1}^{[R]-3}\sup_{\lambda\in[\ell-1,\ell)}|n_k(\lambda)|^q\Big)^{1/q}\\
&+ \Big(\frac1N\sum_{\ell=[R]-2}^{[R]+2}\sup_{\lambda\in[\ell-1,\ell)}|n_k(\lambda)|^q\Big)^{1/q} + \Big(\frac1N\sum_{\ell=[R]+3}^{N}\sup_{\lambda\in[\ell-1,\ell)}|n_k(\lambda)|^q\Big)^{1/q}\\
&= I + II + III + IV + V + VI.
\end{aligned}$$

Let M be a sufficiently large natural number. By Lemma 5.4

$$V = \Big(\frac1N\sum_{\ell=[R]-2}^{[R]+2}\sup_{\lambda\in[\ell-1,\ell)}|n_k(\lambda)|^q\Big)^{1/q}$$

$$\begin{aligned}
&\le C2^{-\delta_q(p)k}\Big(\frac{1}{N}\sum_{\ell=[R]-2}^{[R]+2}\sup_{\lambda\in[\ell-1,\ell)}\Big(1+2^k\Big|\frac{\lambda}{R}-1\Big|\Big)^{-Mq}\Big)^{1/q}\\
&\le C2^{-\delta_q(p)k}R^{-1/q}\le C2^{-\delta_q(p)k}2^{-\frac{k}{q}}\Big(\frac{2^k}{R}\Big)^{1/q}\\
&\le C2^{-nk(\frac{1}{p}-\frac{1}{2})}
\end{aligned}$$

and

$$\begin{aligned}
VI &\le C\Big(\frac{1}{N}\sum_{\ell=[R]+3}^{N}\sup_{\lambda\in[\ell-1,\ell)}\Big|2^{-\delta_q(p)k}\Big(1+2^k\Big(\frac{\lambda}{R}-1\Big)\Big)^{-M}\Big|^q\Big)^{1/q}\\
&\le C2^{-\delta_q(p)k}\Big(\int_R^\infty\Big(1+2^k\Big(\frac{\lambda}{R}-1\Big)\Big)^{-Mq}R^{-1}d\lambda\Big)^{1/q}\\
&\le C2^{-nk(\frac{1}{p}-\frac{1}{2})}.
\end{aligned}$$

A similar argument as in V shows that $II \le C2^{-nk(1/p-1/2)}$; the similar argument as in VI shows that each of I, III and IV is less than $C2^{-nk(1/p-1/2)}$. Thus

$$\big\|\delta_N(n_k(\lambda)\psi(\lambda/R))\big\|_{N,q}\le C2^{-nk(\frac{1}{p}-\frac{1}{2})}.$$

This finishes the proof of Theorem 5.2. □

Part 2. Dispersive and restriction estimates

6. Dispersive and Strichartz estimates

Let (X,d,μ) be a metric measure space. Next let L be a non-negative self-adjoint operator acting on $L^2(X)$. In virtue of the spectral theory, we can define the semigroup $\exp(-zL)$ for all $z\in\mathbb{C}$ with $\mathrm{Re}\, z\ge 0$ and such that

$$\|\exp(-zL)\|_{2\to 2}\le 1.$$

We say that the operator L satisfies dispersive type estimates if there exist constants n and C such that

$$\|\exp(isL)\|_{1\to\infty}\le C|s|^{-n/2},\quad \forall\, s\in\mathbb{R}\setminus\{0\}. \tag{6.1}$$

Of course, the standard Laplacian on $\mathbb{R}^n$ satisfies the dispersive estimates. Such estimates are of importance in analysis and PDE. In particular, they imply endpoint Strichartz estimates (see Keel and Tao [42]). We refer to Strichartz endpoint estimates for the corresponding Schrödinger equation as

$$\int_{\mathbb{R}}\|e^{itL}f\|_{\frac{2n}{n-2}}^2\,dt\le C\|f\|_2^2,\ \ f\in L^2. \tag{6.2}$$

This endpoint estimate together with the obvious fact

$$\|\exp(itL)f\|_{L_t^\infty L_x^2}\le\|f\|_{L^2}$$

give $L_t^pL_x^q$ Strichartz estimates. See [42] for more details. Our aim will be to explain how sharp spectral multipliers follow from the dispersive or Strichartz estimates.

It is natural to consider the dispersive estimates (6.1) in conjunction with the smoothing condition

$$\|\exp(-tL)\|_{1\to\infty}\le Kt^{-n/2},\ \forall\, t>0. \tag{6.3}$$

Note that if the self-adjoint contractive semigroup $\exp(-tL)$ on $L^2(X)$ is in addition uniformly bounded on $L^\infty(X)$, which includes the case of a sub-Markovian semigroup, then

$$\|\exp(-(t+is)L)\|_{1\to\infty} \le \|\exp(-tL/2)\|_{\infty\to\infty}\|\exp(-isL)\|_{1\to\infty} \le C\,|s|^{-n/2}.$$

Together with (6.3) this yields

$$\|\exp(-(t+is)L)\|_{1\to\infty} \le C\min\{t^{-n/2}, |s|^{-n/2}\} \le C'|t+is|^{-n/2}$$

for all $t>0$, $s\in\mathbb{R}$. Hence

$$\|\exp(-zL)\|_{1\to\infty} \le C|z|^{-n/2} \tag{6.4}$$

for all $\operatorname{Re} z \ge 0$. Of course this estimate implies (6.1) and (6.3) and the argument above shows that if semigroup $\exp(-tL)$ is uniformly bounded on $L^\infty(X)$ then it is equivalent to conjunction (6.1) and (6.3). It turns out however that this equivalence holds without the boundedness assumption on $L^\infty(X)$. This fact will be used in the next subsection in which we will not assume uniform boundedness of semigroup $\exp(-tL)$ on $L^\infty(X)$.

Lemma 6.1. *Suppose that L is a non-negative self-adjoint operator on $L^2(X)$. Then the dispersive estimates* (6.1) *are equivalent to* (6.4).

Proof. All what we need is to prove that (6.1) is enough to get (6.4) on the positive half-plane. Fix $f, g \in L^1(X)\cap L^2(X)$ and consider the function

$$H(z) = z^{n/2}\langle \exp(-zL)f, g\rangle.$$

The analyticity of the semigroup on L^2 implies analyticity of H on the open right half-plane and continuity on the boundary. Now for $z=is$ with $s\in\mathbb{R}$, the dispersive estimates (6.1) gives

$$|H(is)| \le C\|g\|_1\|f\|_1.$$

For all z with $\operatorname{Re} z \ge 0$, we have

$$|H(z)| \le |z|^{n/2}\|g\|_2\|f\|_2.$$

Therefore, we can apply the Phragmén-Lindelöf theorem and conclude that

$$|H(z)| \le C\|g\|_1\|f\|_1 \tag{6.5}$$

for all z with $\operatorname{Re} z \ge 0$. From this and the density of $L^1(X)\cap L^2(X)$ in $L^1(X)$ we obtain the lemma. □

7. From dispersive and Strichartz estimates to sharp multipliers

We continue with the assumption that (X, d, μ) is a metric measure space. In this section, let us start with the following proposition.

Proposition 7.1. *Let L be a non-negative self-adjoint operator on L^2. Assume that L satisfies estimates* (6.1). *Then for all $1 \le p < \frac{2n}{n+2}$ and all $\lambda \ge 0$*

$$\|dE_{\sqrt{L}}(\lambda)\|_{p\to p'} \le C\lambda^{n(\frac{1}{p}-\frac{1}{p'})-1}, \tag{7.1}$$

where p' is again the conjugate exponent of p.

Proof. We first prove that L satisfies

$$\|F(\sqrt{L})\|_{p\to p'} \le CR^{n(\frac{1}{p}-\frac{1}{p'})}\|\delta_R F\|_1 \tag{7.2}$$

for all bounded $F \in L^1$ with $\operatorname{supp} F \subseteq [0,R]$ and all p with $1 \le p < \frac{2n}{n+2}$. This estimate is very similar to $(\mathrm{ST}^1_{\mathrm{p,p'}})$ studied in Part 1. Note that we do not consider here X to be a doubling space (even a metric d is not needed).

Consider the case where $R = 1$ and fix F with support contained in $[0,1]$. Set $G(\lambda) = F(\sqrt{\lambda})e^{\lambda}$. By the inverse Fourier transform, we have (up to a constant)

$$G(\lambda) = \int_{\mathbb{R}} \hat{G}(\xi)e^{i\xi\lambda}d\xi.$$

This gives

$$F(\sqrt{L}) = \int_{\mathbb{R}} \hat{G}(\xi)\exp(-(1-i\xi)L)d\xi. \tag{7.3}$$

This equality follows immediately from Fubini's theorem if $\hat{G} \in L^1$. One may start by proving (7.3) for smooth functions F_n and then use standard approximation arguments to obtain the equality for all F as above.

Now the dispersive estimates together with Lemma 6.1 imply that for any $p \in [1,2]$

$$\begin{aligned}
\|F(\sqrt{L})\|_{p\to p'} &\le \int_{\mathbb{R}} |\hat{G}(\xi)|\|\exp(-(1-i\xi)L)\|_{p\to p'}d\xi \\
&\le C\int_{\mathbb{R}} |\hat{G}(\xi)|(1+\xi^2)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{p'})}d\xi \\
&\le C\|\hat{G}\|_\infty \int_{\mathbb{R}} (1+\xi^2)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{p'})}d\xi. \qquad (7.4)
\end{aligned}$$

Now we note that

$$\|\hat{G}\|_\infty \le \|G\|_1 \le C\|F\|_1$$

and

$$\int_{\mathbb{R}} (1+\xi^2)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{p'})}d\xi < \infty$$

for $p < \frac{2n}{n+2}$. This shows (7.2) when $R = 1$. Now, for general $R > 0$ and F with support in $[0,R]$ we reproduce the previous arguments with the function $\delta_R F$ and the operator $L' = \frac{L}{R^2}$. This leads to (7.2). Now we argue as in the proof of Proposition 2.4. Fix $\lambda \ge 0$ and $\varepsilon > 0$ small. We use (7.2) to obtain

$$\begin{aligned}
\left\|\varepsilon^{-1}E_{\sqrt{L}}(\lambda-\varepsilon,\lambda+\varepsilon]\right\|_{p\to p'} &= \varepsilon^{-1}\left\|\mathbb{1}_{(\lambda-\varepsilon,\lambda+\varepsilon]}(\sqrt{L})\right\|_{p\to p'} \\
&\le C\varepsilon^{-1}(\lambda+\varepsilon)^{n(\frac{1}{p}-\frac{1}{p'})}\left\|\chi_{(\frac{\lambda-\varepsilon}{\lambda+\varepsilon},1]}\right\|_1 \\
&\le C(\lambda+\varepsilon)^{n(\frac{1}{p}-\frac{1}{p'})-1}.
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$\left\|dE_{\sqrt{L}}(\lambda)\right\|_{p\to p'} \le C\lambda^{n(\frac{1}{p}-\frac{1}{p'})-1},$$

which implies the estimate of the proposition. □

Remark 7.2. *Let $p \in [1, \frac{2n}{n+2}]$ and suppose that G is a distribution such that* $\operatorname{supp} G \subset [0, R]$ *and* $\int_{\mathbb{R}} |\hat{G}(\xi)|(1+\xi^2/R^2)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{p'})}d\xi < \infty$. *It follows from the previous calculations that $G(L)$ is well defined as an operator acting from L^p to $L^{p'}$ and that*

$$\|G(L)\|_{p\to p'} \le C \int_{\mathbb{R}} |\hat{G}(\xi)|(1+\xi^2/R^2)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{p'})}d\xi.$$

Proposition 7.1 does not yield the optimal results for the standard Laplace operator. However in the abstract setting we can include the endpoint $p = \frac{2n}{n+2}$ when $n > 2$ in the following way. We start with the Strichartz estimate (6.2) and repeat the previous proof to get

$$\begin{aligned}
\|F(\sqrt{L})f\|_{\frac{2n}{n-2}} &\le \int_{\mathbb{R}} |\hat{G}(\xi)| \|\exp(-(1-i\xi)L)f\|_{\frac{2n}{n-2}} d\xi \\
&\le \|\hat{G}\|_2 \left(\int_{\mathbb{R}} \|\exp(i\xi L)\exp(-L)f\|^2_{\frac{2n}{n-2}} d\xi \right)^{1/2} \\
&\le C\|F\|_2 \|\exp(-L)f\|_2 \le C\|F\|_2\|f\|_2
\end{aligned}$$

for all F with support in $[0, 1]$. For F supported in $[0, R]$ we apply the previous estimate with $\delta_R F$ and $L' = \frac{L}{R^2}$ to get

$$\|F(\sqrt{L})\|_{2\to\frac{2n}{n-2}} \le CR\|\delta_R F\|_2.$$

Therefore,

$$\|F(\sqrt{L})\|_{\frac{2n}{n+2}\to\frac{2n}{n-2}} \le CR^2\|\delta_R F\|_1^2. \tag{7.5}$$

Similar arguments as above give (7.1) for $p = \frac{2n}{n+2}$.

We can now extend this easily to all $p < \frac{2n}{n+2}$ if the smoothing property (6.3) is satisfied. More precisely, fix $p < \frac{2n}{n+2}$ and assume that

$$\|\exp(-tL)\|_{p\to\frac{2n}{n+2}} \le Kt^{-\frac{n}{2}(\frac{1}{p}-\frac{n+2}{2n})}, \ \forall\, t > 0. \tag{7.6}$$

We introduce as before $G_R(\lambda) = (\delta_R F)(\sqrt{\lambda})e^{\lambda}$ for F supported in $[0, R]$. Then for $q = p'$ we have

$$\begin{aligned}
\|F(\sqrt{L})f\|_q &= \left\| \int_{\mathbb{R}} \widehat{G_R}(\xi) \exp(-(\frac{1}{R^2} - i\frac{\xi}{R^2})L)f\, d\xi \right\|_q \\
&\le \|\exp(-\frac{L}{R^2})\|_{\frac{2n}{n-2}\to q} \int_{\mathbb{R}} |\widehat{G_R}(\xi)| \|\exp(-(i\frac{\xi}{R^2})L)f\|_{\frac{2n}{n-2}} d\xi \\
&\le CR^{n(\frac{1}{2}-\frac{1}{q})-1}\|G_R\|_2 \left(\int_{\mathbb{R}} \|\exp(-(i\frac{\xi}{R^2})L)f\|^2_{\frac{2n}{n-2}} d\xi \right)^{1/2} \\
&\le CR^{n(\frac{1}{2}-\frac{1}{q})}\|\delta_R F\|_{L^2(\mathbb{R})}\|f\|_{L^2(X)},
\end{aligned}$$

where we used the Strichartz estimate (6.2) to obtain the last inequality. As in the last proposition, this gives the Stein-Tomas restriction (7.1) for all $p \in [1, \frac{2n}{n+2}]$. We have proved

Proposition 7.3. *Let L be a non-negative self-adjoint operator on L^2. Assume that L satisfies the Strichartz estimate (6.2) for some $n > 2$. Fix p such that $1 \le p \le \frac{2n}{n+2}$ and assume that the smoothing property (7.6) is satisfied. Then for all $\lambda \ge 0$*

$$\|dE_{\sqrt{L}}(\lambda)\|_{p\to p'} \le C\lambda^{n(\frac{1}{p}-\frac{1}{p'})-1}, \tag{7.7}$$

where p' is again the conjugate exponent of p.

We mentioned above that the dispersive estimates imply the Strichartz estimates ([42]). For this reason we formulate the results below for the case where L satisfies the endpoint Strichartz estimate.

Theorem 7.4. *Suppose that (X, d, μ) satisfies the doubling property (2.2) and there exists a positive constant $C > 0$ such that $V(x, r) \le Cr^n$ for every $x \in X$ and $r > 0$. Assume that L satisfies the finite speed propagation property* (FS) *and the Strichartz estimate* (6.2) *with the same n as in the doubling property. Assume also that $n > 2$. Fix $p \in [1, \frac{2n}{n+2}]$ and assume that* (7.6) *holds. Then for every bounded Borel function F such that* $\operatorname{supp} F \subseteq [-1, 1]$ *and*

$$\|F\|_{W^{\beta,2}} < \infty$$

for some $\beta > n(\frac{1}{p} - \frac{1}{2})$, the operator $F(t\sqrt{L})$ is bounded on L^p for all $t > 0$ and

$$\sup_{t>0} \|F(t\sqrt{L})\|_{p\to p} \le C\|F\|_{W^{\beta,2}}.$$

Proof. If F is supported in $[0, R]$ then by Proposition 7.3, we have

$$\begin{aligned} \|F(\sqrt{L})\|_{p\to p'} &= \|\int_0^R F(\lambda) dE_{\sqrt{L}}(\lambda)\|_{p\to p'} \\ &\le C\int_0^R |F(\lambda)|\lambda^{n(\frac{1}{p}-\frac{1}{p'})-1} d\lambda \\ &\le CR^{n(\frac{1}{p}-\frac{1}{p'})}\|\delta_R F\|_1. \end{aligned}$$

Hence by the T^*T argument

$$\|F(\sqrt{L})\|_{p\to 2} \le CR^{n(\frac{1}{p}-\frac{1}{2})}\|\delta_R F\|_2.$$

Combining this with our assumption on the volume yields $(\mathrm{ST}^2_{\mathrm{p},2})$. We then apply Theorem 3.1 and obtain the result for $p \in [1, \frac{2n}{n+2}]$. □

We have seen that the assumptions of the previous theorem imply $(\mathrm{ST}^2_{\mathrm{p},2})$, we can then apply Theorem 5.1 to obtain endpoint estimate for Bochner-Riesz means. In addition, by applying Theorem 4.1 we obtain under the assumptions of the previous theorem the following result.

Theorem 7.5. *Fix $p \in [1, \frac{2n}{n+2}]$. For any bounded Borel function F such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,2}} < \infty$ for some $\beta > \max\{n(1/p - 1/2), 1/2\}$ and some non-trivial function $\eta \in C_c^\infty(0, \infty)$, the operator $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $r \in (p, p')$. In addition,*

$$\|F(\sqrt{L})\|_{r\to r} \le C_\beta\Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,2}} + |F(0)|\Big).$$

Remark 7.6. *In the general setting of doubling spaces, we can replace the dispersive estimates* (6.1) *by*

$$\|P_{B(x,r)} \exp(isL)\, P_{B(x,r)}\|_{p\to p'} \le CV(x, r)^{\frac{1}{p'}-\frac{1}{p}} \left(\frac{r}{\sqrt{|s|}}\right)^{n(\frac{1}{p}-\frac{1}{p'})}. \tag{7.8}$$

Here the constant n is the same as in the doubling condition. The arguments in the proof of (7.2) show that for $p < \frac{2n}{n+2}$,

$$\|P_{B(x,r)}|F|^2 P_{B(x,r)}\|_{p\to p'} \le CV(x, r)^{\frac{1}{p'}-\frac{1}{p}} (Rr)^{n(\frac{1}{p}-\frac{1}{p'})} \|\delta_R F\|_2^2$$

for all F supported in $[0,R]$. The T^*T argument gives $(\mathrm{ST}^2_{\mathrm{p},2})$ for all $p < \frac{2n}{n+2}$. Therefore we obtain the same conclusion as in Theorems 7.4 and 7.5 for L satisfying property (FS) *and (7.8) on any doubling space.*

Note also that under these two assumptions, we obtain from Theorem 5.1 endpoint estimates for the Bochner-Riesz mean $S_R^{\delta(p)}(L)$ on L^p ($p \in [1, \frac{2n}{n+2})$) for $\delta(p) = \max\{0, n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}\}$.

We finish this section by observing that a local dispersive estimate could also be used to estimates for $\|E_L[k,k+1)\|_{p\to p'}$. More precisely,

Proposition 7.7. *Let L be a non-negative self-adjoint operator on L^2. Assume that L satisfies the local dispersive estimates*

$$\|\exp(itL)\|_{1\to\infty} \le Ct^{-n/2}$$

for all $0 < |t| \le 2$. Assume also that

$$\|\exp(-tL)\|_{\infty\to\infty} \le C \quad \text{and} \quad \|\exp(-tL)\|_{1\to\infty} \le Ct^{-n/2}$$

for all $t > 0$. Then for all $1 \le p < \frac{2n}{n+2}$ and all $k > 0$

$$\|E_L[k,k+1)\|_{p\to p'} \le C(1+k)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{p'})-1}, \tag{7.9}$$

where p' is again the conjugate exponent of p.

Proof. Set $\widehat{G_k}(\xi) = 3\pi(1-|\xi|)_+^2 e^{-ik\xi}$ so that $G_k(\lambda) = 6(\lambda-k)^{-2} - 6\frac{\sin(\lambda-k)}{(\lambda-k)^3}$. Note that there exists a positive constant c such that $G_k(\lambda)e^{-\lambda/(2k)} \ge c\chi_{[k,k+1)}(\lambda)$ for all $\lambda \in [k,k+1)$. So

$$c^2\|E_L[k,k+1)f\|_2^2 \le \|G_k(L)e^{-L/(2k)}f\|_2^2.$$

It follows from the above inequality and the T^*T argument that

$$c^2\|E_L[k,k+1)\|_{p\to p'} \le \||G_k|^2(L)e^{-L/k}\|_{p\to p'}.$$

Next (6.4) holds for $|\mathrm{Im} z| \le 2$. Hence

$$\begin{aligned}
\||G_k|^2(L)e^{-L/k}\|_{p\to p'} &\le \int_{-2}^{2} |\widehat{G_k} * \widehat{G_k}|(\xi)\,\|\exp(-(1/k - i\xi)L)\|_{p\to p'}\,d\xi \\
&\le C\int_{-2}^{2} (k^{-2}+\xi^2)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{p'})}\,d\xi \\
&\le C\int_{\mathbb{R}} (k^{-2}+\xi^2)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{p'})}\,d\xi \\
&\le C(1+k)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{p'})-1}.
\end{aligned}$$

This proves estimate (7.9). □

Remark 7.8. *Using Mehler's formula it is not difficult to verify that the estimates required in Proposition 7.7 hold for the harmonic oscillator. One can use this fact to prove the corresponding spectral cluster estimates* (10.4) *for this case (see Theorem 10.4).*

Part 3. Applications

8. Standard Laplace operator and compact manifolds

As mentioned in the introduction, the restriction estimates (R_p) for standard Laplace operator on $\mathbb{R}^n$ are valid for $1 \le p \le 2(n+1)/(n+3)$. As a consequence of Theorem 5.1, we obtain alternative proof of Theorem 1.1 of [69] by Tao, Theorem 1 [15] and main result of [14] described by M. Christ. These results can be stated in the following way:

Proposition 8.1. *For all $n \ge 2$ and $1 \le p \le 2(n+1)/(n+3)$, the operator $(I-\Delta)_+^{\delta(p)}$ is of weak-type (p,p).*

Proof. This result is a consequence of Proposition 2.4 and Theorem 5.1. □

Similarly, using Theorem 5.2 one can obtain alternative proof of Theorem 1.2 of [69]. Our proof shows that this result holds for all operators on compact manifolds which satisfy property (FS) and condition (S_p) as in the following proposition.

Proposition 8.2. *Suppose the operator L satisfies* (FS) *and condition* (S_p) *for some $1 \le p \le 2(n+1)/(n+3)$. Then the operator $(I-L/R^2)_+^{\delta(p)}$ is of weak-type (p,p) uniformly in R.*

Proof. This result follows from Proposition 3.10 and Theorem 5.2. □

In both cases of compact manifolds with or without boundaries, examples which satisfy condition (S_p) are described in [62, 64] by C.D. Sogge.

We mentioned here endpoint Bochner-Riesz summability results. From Theorems 4.1 and 4.2 we have more general spectral multiplier results for the operators considered in the previous propositions.

9. Asymptotically conic manifolds

Scattering manifolds or asymptotically conic manifolds are defined as the interior of a compact manifold with boundary M, and the metric g is smooth on M° and has the form

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2}$$

in a collar neighbourhood near ∂M, where x is a smooth boundary defining function for M and $h(x)$ a smooth one-parameter family of metrics on ∂M; the function $r := 1/x$ near $x = 0$ can be thought of as a radial coordinate near infinity and the metric there is asymptotic to the exact metric cone $((0,\infty)_r \times \partial M, dr^2 + r^2 h(0))$.

In this subsection we consider the following classical operators:

- Schrödinger operators, i.e. $-\Delta + V$ on $\mathbb{R}^n$, where V smooth and decaying sufficiently at infinity;
- The Laplacian with respect to metric perturbations of the flat metric on $\mathbb{R}^n$, again decaying sufficiently at infinity;
- The Laplacian on asymptotically conic manifolds.

Proposition 9.1. *Let (M,g) be an asymptotically conic manifold of dimension $n \ge 3$, and let x be a smooth boundary defining function of ∂M. Let $L := -\Delta + V$ be a Schrödinger operator with $V \in x^3 C^\infty(M)$ and assume that L has no L^2-eigenvalues and that 0 is not a resonance. Then*

(i) *For any $\lambda_0 > 0$ there exists a constant $C > 0$ such that the spectral measure $dE(\lambda)$ for $\sqrt{L}$ satisfies*

$$\|dE_{\sqrt{L}}(\lambda)\|_{L^p(M)\to L^{p'}(M)} \le C\lambda^{n(\frac{1}{p}-\frac{1}{p'})-1} \tag{9.1}$$

for $1 \le p \le 2(n+1)/(n+3)$ and $0 < \lambda \le \lambda_0$.

(ii) *If (M, g) is nontrapping, then there exists $C > 0$ such that (9.1) holds for all $\lambda > 0$.*

Proposition 9.1 was proved in [29, Theorem 1.2]. This proposition has useful consequence to establish the convergence of the Riesz means up to the critical exponent $\delta(p) = \max\{0, n|1/p - 1/2| - 1/2\}$ for all $1 \le p \le 2(n+1)/(n+3)$.

Corollary 9.2. *Let (M, g) be nontrapping and the operator L satisfies all assumptions of Proposition 9.1. Let $1 \le p \le 2(n+1)/(n+3)$. Then*

(i) *$S_R^{\delta(p)}(L)$ is of weak-type (p, p) uniformly in R.*

(ii) *For any bounded Borel function $F : [0,\infty) \to \mathbb{C}$ such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,2}} < \infty$ for some $\beta > \max\{n(1/p - 1/2), 1/2\}$ and some non-trivial function $\eta \in C_c^\infty(0,\infty)$, the operator $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $p < r < p'$ with*

$$\|F(\sqrt{L})\|_{r\to r} \le C_\beta\Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,2}} + |F(0)|\Big).$$

Proof. Corollary 9.2 follows from Propositions 9.1, 2.4 and Theorems 5.1, 4.1. □

10. Schrödinger operators with rough potentials

This section we discuss new spectral multiplier results for Schrödinger type operators of the form $-\Delta + V$.

10.1. Schrödinger operators with inverse-square potential. We start with inverse square potentials, that is $V(x) = \frac{c}{|x|^2}$. Fix $n > 2$ and assume that $-(n-2)^2/4 < c$. Define by quadratic form method $L = -\Delta + V$ on $L^2(\mathbb{R}^n, dx)$. The classical Hardy inequality

$$-\Delta \ge \frac{(n-2)^2}{4}|x|^{-2}, \tag{10.1}$$

shows that for all $c > -(n-2)^2/4$, the self-adjoint operator L is non-negative. Set $p_c^* = n/\sigma$, $\sigma = \max\{(n-2)/2 - \sqrt{(n-2)^2/4 + c}, 0\}$. If $c \ge 0$ then the semigroup $\exp(-tL)$ is pointwise bounded by the Gaussian semigroup and hence acts on all L^p spaces with $1 \le p \le \infty$. If $c < 0$, then $\exp(-tL)$ acts as a uniformly bounded semigroup on $L^p(\mathbb{R}^n)$ for $p \in ((p_c^*)', p_c^*)$ and the range $((p_c^*)', p_c^*)$ is optimal (see for example [47]).

It is proved in [10] that the solution $u(t) = e^{-itL}f$ of the corresponding Schrödinger equation

$$i\partial_t u + Lu = 0,\ u(0) = f$$

satisfies Strichartz estimates (6.2). The smoothing property (7.6) is proved in [4]. Therefore, we obtain from Proposition 7.3 that L satisfies restriction estimate $(\mathrm{R_p})$ for all $p \in ((p_c^*)', \frac{2n}{n+2}]$. If $c \ge 0$, then $p = (p_c^*)' = 1$ is included. Using the above observation and Theorems 5.1 and 7.4 we obtain

Theorem 10.1. *Suppose that $n > 2$ and $-(n-2)^2/4 < c$ and that $p \in ((p_c^*)', 2n/(n+2)]$ where $p_c^* = n/\sigma$ and $\sigma = \max\{(n-2)/2 - \sqrt{(n-2)^2/4 + c}, 0\}$ and $(p_c^*)'$ its dual exponent. Then*

(i) $S_R^{\delta(p)}(L)$ *is of weak-type* (p,p) *uniformly in* R.

(ii) *For any bounded Borel function* $F:[0,\infty)\to\mathbb{C}$ *such that* $\sup_{t>0}\|\eta\,\delta_t F\|_{W^{\beta,2}}<\infty$ *for some* $\beta>\max\{n(1/p-1/2),1/2\}$ *and some non-trivial function* $\eta\in C_c^\infty(0,\infty)$, *the operator* $F(\sqrt{L})$ *is bounded on* $L^r(X)$ *for all* $p<r<p'$ *with*

$$\|F(\sqrt{L})\|_{r\to r}\le C_\beta\Big(\sup_{t>0}\|\eta\,\delta_t F\|_{W^{\beta,2}}+|F(0)|\Big).$$

10.2. Scattering operators. Assume now that $n=3$ and V is a real-valued measurable function such that

$$\int_{\mathbb{R}^6}\frac{|V(x)|\,|V(y)|}{|x-y|^2}dxdy<(4\pi)^2\qquad\text{and}\qquad\sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}dy<4\pi. \tag{10.2}$$

The following proposition is a consequence of Proposition 7.1 and the main result in Rodnianski and Schlag [55] which gives the dispersive estimates for $\exp(it(-\Delta+V))$ on $\mathbb{R}^3$.

Proposition 10.2. *Suppose that* $L=-\Delta+V$ *on* $\mathbb{R}^3$ *with a real-valued* V *which satisfies* (10.2). *Then* L *satisfies* $(\mathrm{R_p})$ *for all* $1\le p<6/5$.

In the special case $p=1$, Proposition 10.2 was obtained in [24, Theorem 7.15] for compactly supported function $V\ge 0$ which satisfies (10.2). The following result is a consequence of Theorem 5.1, 4.1 and Proposition 10.2.

Corollary 10.3. *Suppose that* $L=-\Delta+V$ *on* $\mathbb{R}^3$ *and that* V *satisfies assumption of Proposition 10.2. Assume also that* $1\le p<6/5$. *Then*

(i) $S_R^{\delta(p)}(L)$ *is of weak-type* (p,p) *uniformly in* R.

(ii) *For any bounded Borel function* $F:[0,\infty)\to\mathbb{C}$ *such that* $\sup_{t>0}\|\eta\,\delta_t F\|_{W^{\beta,2}}<\infty$ *for some* $\beta>\max\{3(1/p-1/2),1/2\}$ *and some non-trivial function* $\eta\in C_c^\infty(0,\infty)$, *the operator* $F(\sqrt{L})$ *is bounded on* $L^r(X)$ *for all* $p<r<p'$.

If $n\ge 3$ and potential $V\in W^{s,2}(\mathbb{R}^n)$ for some $s>\frac{n}{2}-1$ and has fast decay, Bourgain [8] proved the dispersive estimates for $\exp(it(-\Delta+V))$. Our results apply for $L=-\Delta+V$ and allow to obtain sharp spectral multiplier results. We also refer to Rodnianski and Schlag [55] for more references on dispersive estimates for Schrödinger operators.

We also mention that Strichartz estimates are proved for a class of elliptic operators with variable coefficients by J. Marzuola, J. Metcalfe and D. Tataru [49] (see Theorem 1.20). Therefore, the same reasoning as for the Theorem 10.1 allows us to obtain sharp spectral multipliers and endpoint Bochner-Riesz summability for these elliptic operators.

10.3. The harmonic oscillator. In this section we focus on Schrödinger operators such as the harmonic oscillator $-\Delta+|x|^2$ on $L^2(\mathbb{R}^n)$ for $n\ge 2$. As in [44] we can also consider Schrödinger operators $L=-\Delta+V$ with a positive potential V which satisfies the following condition

$$V\sim|x|^2,\quad|\nabla V|\sim|x|,\quad|\partial_x^2 V|\lesssim 1. \tag{10.3}$$

We apply Theorems 3.6 and 4.2 and the results from [44] to prove sharp results on Bochner-Riesz summability and singular spectral multipliers for L. Bochner-Riesz summability results for the harmonic oscillator were obtained before (see for instance, [31, 41, 40, 70]). Here we describe an alternative proof. The corresponding singular integral multiplier is a new result. The following theorem is the main goal of this section.

Theorem 10.4. *Assume that potential V satisfies condition (10.3) and set $L = -\Delta + V$. Let $1 \le p \le 2n/(n+2)$. Then*

(i) *For any even function F such that $\operatorname{supp} F \subseteq [-1,1]$ and $\|F\|_{W^{\beta,2}} < \infty$ for some $\beta > \max\{n(1/p - 1/2), 1/2\}$, the operator $F(t\sqrt{L})$ is bounded on $L^p(X)$ for all $t > 0$ and*

$$\sup_{t>0} \|F(t\sqrt{L})\|_{p\to p} \le C\|F\|_{W^{\beta,2}}.$$

(ii) *For any bounded Borel function $F : [0,\infty) \to \mathbb{C}$ such that $\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,2}} < \infty$ for some $\beta > \max\{n(1/p - 1/2), 1/2\}$ and some non-trivial function $\eta \in C_c^\infty(0,\infty)$, the operator $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $p < r < p'$. In addition,*

$$\|F(\sqrt{L})\|_{r\to r} \le C_\beta\Big(\sup_{t>0} \|\eta\, \delta_t F\|_{W^{\beta,2}} + |F(0)|\Big).$$

Proof. It follows from Theorem 4 in [44] that for all $\lambda \ge 0$ and all $1 \le p \le 2n/(n+2)$

$$\|E_L[\lambda^2, \lambda^2+1)\|_{p\to 2} \le C(1+\lambda)^{n(\frac{1}{p}-\frac{1}{2})-1}. \tag{10.4}$$

Take a function F with support in $[-N, N]$. We have as in the proof of Proposition 3.10

$$\begin{aligned}
\|F(\sqrt{L})\|_{p\to 2}^2 &\le \sum_{\ell=1}^{N^2} \|E_{\sqrt{L}}[\frac{\ell-1}{N}, \frac{\ell}{N})F(\sqrt{L})\|_{p\to 2}^2 \\
&\le \sum_{\ell=1}^{N^2} \sup_{\lambda\in[\frac{\ell-1}{N}, \frac{\ell}{N})} |F(\lambda)|^2 \|E_{\sqrt{L}}[\frac{\ell-1}{N}, \frac{\ell}{N})\|_{p\to 2}^2 \\
&= \sum_{\ell=1}^{N^2} \sup_{\lambda\in[\frac{\ell-1}{N}, \frac{\ell}{N})} |F(\lambda)|^2 \|E_L[(\frac{\ell-1}{N})^2, (\frac{\ell}{N})^2)\|_{p\to 2}^2. \qquad (10.5)
\end{aligned}$$

Now we observe that for all $\ell = 1, 2, \cdots, N^2$

$$\big\|E_L[(\frac{\ell-1}{N})^2, (\frac{\ell}{N})^2)\big\|_{p\to 2}^2 \le \big\|E_L[(\frac{\ell-1}{N})^2, (\frac{\ell-1}{N})^2 + 2)\big\|_{p\to 2}^2.$$

This, in combination with (10.4) and (10.5), shows that for $1 \le p \le 2n/(n+2)$

$$\begin{aligned}
\|F(\sqrt{L})\|_{p\to 2}^2 &\le C\sum_{\ell=1}^{N^2} \sup_{\lambda\in[\frac{\ell-1}{N}, \frac{\ell}{N})} |F(\lambda)|^2 \Big(2 + \frac{\ell-1}{N}\Big)^{2n(\frac{1}{p}-\frac{1}{2})-2} \\
&\le CN^{2n(\frac{1}{p}-\frac{1}{2})} \frac{1}{N^2} \sum_{\ell=1}^{N^2} \sup_{\lambda\in[\frac{\ell-1}{N}, \frac{\ell}{N})} |F(\lambda)|^2 \\
&\le CN^{2n(\frac{1}{p}-\frac{1}{2})} \frac{1}{N^2} \sum_{\ell=1}^{N^2} \sup_{\lambda\in[\frac{\ell-1}{N^2}, \frac{\ell}{N^2})} |F(N\lambda)|^2.
\end{aligned}$$

This proves $(\mathrm{SC}_{\mathrm{p},2}^{2,\kappa})$ for $\kappa = 2$ and p such that $1 \le p \le 2n/(n+2)$. It remains to show $(\mathrm{AB}_{\mathrm{p}}^{2,\kappa})$ and then apply Theorems 3.6 and 4.2. Now condition $(\mathrm{AB}_{\mathrm{p}}^{2,\kappa})$ follows from the following lemma. □

Lemma 10.5. *Let $L = -\Delta + V$, where $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \ge 0$. Suppose that for some $\kappa > 0$ and any $\epsilon > 0$*

$$\int_{\mathbb{R}^n} (1 + V(x))^{n(1-\kappa)/2-\epsilon} dx < \infty. \tag{10.6}$$

Then condition $(\mathrm{SC}^{2,\kappa}_{\mathrm{p},2})$ *implies* $(\mathrm{AB}^{2,\kappa}_{\mathrm{p}})$.

The proof of Lemma 10.5 can be obtained by making a minor modification of the proof of Lemma 7.9 of [24] so we skip it here. □

11. Operators $\Delta_n + \frac{c}{r^2}$ acting on $L^2((0,\infty), r^{n-1}dr)$

In this section we consider a class of Schrödinger operators on $L^2((0,\infty), r^{n-1}dr)$. These operators generate semigroups but do not have the classical Gaussian upper bound for the heat kernel.

Fix $n > 2$ and $c > -(n-2)^2/4$ and consider the space $L^2((0,\infty), r^{n-1}dr)$. For $f, g \in C_c^\infty(0,\infty)$ we define the quadratic form

$$Q^{(0,\infty)}_{n,c}(f,g) = \int_0^\infty f'(r)g'(r)r^{n-1}dr + \int_0^\infty \frac{c}{r^2} f(r)g(r)r^{n-1}dr. \tag{11.1}$$

Using the Friedrichs extension one can define the operator $L_{n,c} = \Delta_n + c/r^2$ as the unique self-adjoint operator corresponding to $Q^{(0,\infty)}_{n,c}$, acting on $L^2((0,\infty), r^{n-1}dr)$. In the sequel we will write L instead of $L_{n,c}$, which is formally given by the following formula

$$Lf = (\Delta_n + \frac{c}{r^2})f = -\frac{d^2}{dr^2}f - \frac{n-1}{r}\frac{d}{dr}f + \frac{c}{r^2}f.$$

The classical Hardy inequality (10.1) shows that for all $c > -(n-2)^2/4$, the self-adjoint operator L is non-negative. Such operators can be seen as radial Schrödinger operators with inverse-square potentials. It follows by Theorem 3.3 of [19] that L satisfies Davies-Gaffney estimate, which in turns implies property (FS).

Now for $-(n-2)^2/4 < c < 0$, we set $p_c^* = n/\sigma$ where $\sigma = (n-2)/2 - \sqrt{(n-2)^2/4 + c}$ and $(p_c^*)'$ its dual exponent. Note that $2 < \frac{2n}{n-2} < p_c^*$. Liskevich, Sobol and Vogt [47] proved that for all $t > 0$ and all $p \in ((p_c^*)', p_c^*)$,

$$\|e^{-tL}\|_{p\to p} \le C.$$

They also proved that range $((p_c^*)', p_c^*)$ is optimal and that for all $p \notin ((p_c^*)', p_c^*)$, the semigroup does not even act on $L^p((0,\infty), r^{n-1}dr)$ (see also [22, 19, 30]).

Proposition 11.1. *Suppose that $n > 2$ and $-(n-2)^2/4 < c$. For $c < 0$, set $p \in ((p_c^*)', \frac{2n}{n+1})$ where $p_c^* = n/\sigma$ and $\sigma = (n-2)/2 - \sqrt{(n-2)^2/4 + c}$ and $(p_c^*)'$ its conjugate exponent. For $c \ge 0$, set $p \in [1, \frac{2n}{n+1})$. Then for any $R > 0$ and all Borel functions F such that* $\operatorname{supp} F \subset [0, R]$,

$$\left\|F(\sqrt{L})P_{B(x_B,r_B)}\right\|_{p\to 2} \le CV(x_B, r_B)^{\frac{1}{2}-\frac{1}{p}}(Rr_B)^{n(\frac{1}{p}-\frac{1}{2})}\left\|\delta_R F\right\|_2 \tag{11.2}$$

for all $x_B \in \mathbb{R}_+$ and $r_B \ge 1/R$.

Proof. In [30] the explicit formula for the resolvent of the operator $L = \Delta_n + \frac{c}{r^2}$ is described. Based on this formula we calculate explicitly the spectral projections $dE_{\sqrt{L}}(\lambda)$ (This calculation was shown to us by Andrew Hassell). Define the number $n' = n'(n,c)$ to be the maximum positive root of the equation $(n'/2-1)^2 = (n/2-1)^2 + c$ so that $p_c^* = n/\sigma = 2n/(n-n')$ where $\sigma = (n-2)/2 - \sqrt{(n-2)^2/4 + c}$. Next let K_α and I_α are modified Bessel function, see [1, §9.6.1 p.

374] or [72, §1.14 p. 16]. Set $m(x) = x^{-n/2+1}I_{n'/2-1}(x)$ and $k(x) = x^{-n/2+1}K_{n'/2-1}(x)$. Then by (4.2) and Section 6.1 of [30] the resolvent kernel for $L = \Delta_n + \frac{c}{r^2}$ is given by

$$R(\lambda)(x,y) = K_{(L+\lambda^2)^{-1}}(x,y) = \begin{cases} \nu\lambda^{n-2}k(\lambda y)m(\lambda x) & \text{if } y \geq x, \\ \nu\lambda^{n-2}m(\lambda y)k(\lambda x) & \text{if } x > y \end{cases} \tag{11.3}$$

for some constant ν. Next recall that $x^{-\alpha}I_\alpha$ is an even analytic function and that

$$K_\alpha(x) = \frac{\pi}{2}\,\frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)},$$

see [1, §9.6.1 p. 374]. Hence by the limiting absorption principle (that is Stone's Theorem, Chapter XIV, [34]) for all $x \leq y$

$$\begin{aligned} K_{dE_{\sqrt{L}}(\lambda)}(x,y) &= \frac{i\lambda}{\pi}(R(i\lambda)(x,y) - R(-i\lambda)(x,y)) \\ &= \nu\frac{i\lambda}{2}x^{1-n/2}y^{1-n/2}I_{n'/2-1}(i\lambda x)\frac{I_{1-n'/2}(i\lambda y) - I_{n'/2-1}(i\lambda y)}{\sin((n'/2-1)\pi)} \\ &\quad -\nu\frac{i\lambda}{2}x^{1-n/2}y^{1-n/2}I_{n'/2-1}(-i\lambda x)\frac{I_{1-n'/2}(-i\lambda y) - I_{n'/2-1}(-i\lambda y)}{\sin((n'/2-1)\pi)}. \end{aligned}$$

Recall next that $x^{-\alpha}I_\alpha$ is an even analytic function so

$$I_{n'/2-1}(i\lambda x)I_{1-n'/2}(i\lambda y) = I_{n'/2-1}(-i\lambda x)I_{1-n'/2}(-i\lambda y)$$

and

$$I_{n'/2-1}(-i\lambda x)I_{n'/2-1}(-i\lambda y) = e^{i\pi(n'-2)}I_{n'/2-1}(i\lambda x)I_{n'/2-1}(i\lambda y).$$

Thus

$$\begin{aligned} K_{dE_{\sqrt{L}}(\lambda)}(x,y) &= \nu\frac{i\lambda}{2}x^{1-n/2}y^{1-n/2}(e^{i\pi(n'-2)} - 1)\frac{I_{n'/2-1}(i\lambda x)I_{n'/2-1}(i\lambda y)}{\sin((n'/2-1)\pi)} \\ &= i\nu e^{i\pi(n'/2-1)}\lambda x^{1-n/2}y^{1-n/2}I_{n'/2-1}(i\lambda x)I_{n'/2-1}(i\lambda y). \end{aligned} \tag{11.4}$$

We prove equality (11.4) under assumption that $x \leq y$ but similar argument shows that (11.4) holds for all x and y. Now if we set $\ell(x) = x^{-n/2+1}I_{n'/2-1}(ix)$ then by (11.4)

$$dE_{\sqrt{L}}(\lambda)f(x) = C\lambda^{n-1}\ell(\lambda x)\int_0^\infty \ell(\lambda y)f(y)y^{n-1}dy \tag{11.5}$$

for some constant C. The function $\ell = \ell_{n,c}(\lambda)$ for $n > 2$ satisfies the following estimates ([1, 72]):

$$|\ell(\lambda)| \leq \begin{cases} \lambda^{\frac{n'-n}{2}} & \text{if} \quad \lambda \leq 1 \\ \lambda^{\frac{1-n}{2}} & \text{if} \quad 1 \leq \lambda. \end{cases}$$

By (11.5)

$$K_{F(\sqrt{L})}(x,y) = C\int_0^\infty F(\lambda)\ell(\lambda x)\ell(\lambda y)\lambda^{n-1}d\lambda.$$

Let us prove our estimate (11.2). We consider only the case $-(n-2)^2/4 < c < 0$. The proof is similar for the case $c \geq 0$. For every $B = B(x_B, r_B)$, one writes

$$F(\sqrt{L})P_Bf(x) = C\int_0^\infty \Big(\int_0^\infty F(\lambda)\ell(\lambda x)\ell(\lambda y)\lambda^{n-1}d\lambda\Big)\chi_B(y)f(y)y^{n-1}dy.$$

Hence

$$\begin{aligned}\|F(\sqrt{L})P_B f\|_2^2 &= C\int_0^\infty \Big|\int_0^\infty \Big(\int_0^\infty F(\lambda)\ell(\lambda x)\ell(\lambda y)\lambda^{n-1}d\lambda\Big)\chi_B(y)f(y)y^{n-1}dy\Big|^2 x^{n-1}dx\\ &= C\int_0^\infty \Big|\int_0^\infty \ell(\lambda x)\lambda^{n-1}\int_0^\infty F(\lambda)\ell(\lambda y)\chi_B(y)f(y)y^{n-1}dyd\lambda\Big|^2 x^{n-1}dx.\end{aligned}$$

Note that the following Plancherel type equality (see, e.g., [16, 24]) is satisfied

$$\int_0^\infty \Big|\int_0^\infty F(\lambda)\ell(\lambda x)\lambda^{n-1}d\lambda\Big|^2 x^{n-1}dx = \int_0^\infty |F(\lambda)|^2\lambda^{n-1}d\lambda,$$

which yields

$$\begin{aligned}\|F(\sqrt{L})P_B f\|_2^2 &= C\int_0^\infty |F(\lambda)|^2\Big|\int_0^\infty \ell(\lambda y)\chi_B(y)f(y)y^{n-1}dy\Big|^2\lambda^{n-1}d\lambda\\ &\le C\int_0^\infty |F(\lambda)|^2\|\ell(\lambda y)\chi_B(y)\|_{p'}^2\|f\|_p^2\lambda^{n-1}d\lambda.\end{aligned}$$

Case I: $B = B(x_B, r_B)$ and $x_B \le 2r_B$. Hence

$$\begin{aligned}\big\|\ell(\lambda y)\chi_B\big\|_{p'}^{p'} &\le C\int_0^{3r_B}\big|\ell(\lambda y)\big|^{p'}y^{n-1}dy\\ &\le C\int_0^\infty\big|\ell(\lambda y)\big|^{p'}y^{n-1}dy\\ &\le C\lambda^{-n}\int_0^\infty\big|\ell(y)\big|^{p'}y^{n-1}dy\\ &\le C\lambda^{-n}\end{aligned}$$

for all $2n/(n-1) < p' < p_c^* = 2n/(n-n')$ (For $c \ge 0$ this condition should be replaced by $2n/(n-1) < p'$). Thus

$$\begin{aligned}\|F(\sqrt{L})P_B f\|_2^2 &\le C\int_0^\infty |F(\lambda)|^2\lambda^{n-1-\frac{2n}{p'}}d\lambda\|f\|_p^2\\ &\le CR^{n-\frac{2n}{p'}}\|\delta_R F\|_2^2\|f\|_p^2.\end{aligned}$$

When $x_B \le 2r_B$, we have that $V(B) \approx r_B^n$ and $V(B)^{1/2-1/p}r_B^{n(1/p-1/2)} \approx 1$. This gives

$$\begin{aligned}\|F(\sqrt{L})\chi_B f\|_2^2 &\le CV(B)^{2(\frac12-\frac1p)}r_B^{2n(\frac1p-\frac12)}R^{n-\frac{2n}{p'}}\|\delta_R F\|_2^2\|f\|_p^2\\ &\le CV(B)^{2(\frac12-\frac1p)}(Rr_B)^{2n(\frac1p-\frac12)}\|\delta_R F\|_2^2\|f\|_p^2.\end{aligned}$$

This proves Case I.

Case II: $B = B(x_B, r_B)$ and $x_B > 2r_B$. Then

$$\begin{aligned}\|\ell(\lambda y)\chi_B\|_{p'}^{p'} &\le \int_{x_B-r_B}^{x_B+r_B}\big|\ell(\lambda y)\big|^{p'}y^{n-1}dy\\ &\le C\int_{x_B-r_B}^{x_B+r_B}(\lambda y)^{\frac{(1-n)p'}{2}}y^{n-1}dy\\ &\le C\lambda^{p'\frac{1-n}{2}}r_B x_B^{n-1+p'\frac{1-n}{2}}.\end{aligned}$$

Hence

$$\begin{aligned}\|F(\sqrt{L})P_B f\|_2^2 &\le C\int_0^\infty |F(\lambda)|^2 r_B^{\frac{2}{p'}} x_B^{2(n-1+\frac{(1-n)p'}{2})/p'} d\lambda \|f\|_p^2\\ &\le C r_B^{\frac{2}{p'}} x_B^{\frac{2(n-1)}{p'}-n+1} R\|\delta_R F\|_2^2\|f\|_p^2.\end{aligned}$$

Note that if $x_B > 2r_B$ then $V(B) \approx x_B^{n-1} r_B$. Thus

$$\begin{aligned}\|F(\sqrt{L})P_B f\|_2^2 &\le C r_B^{\frac{2}{p'}} x_B^{\frac{2(n-1)}{p'}-n+1} R\|\delta_R F\|_2^2\|f\|_p^2\\ &\le CV(B)^{2(\frac12-\frac1p)}(Rr_B)^{2n(\frac1p-\frac12)}(Rr_B)^{1-2n(\frac1p-\frac12)}\|\delta_R F\|_2^2\|f\|_p^2\\ &\le CV(B)^{2(\frac12-\frac1p)}(Rr_B)^{2n(\frac1p-\frac12)}\|\delta_R F\|_2^2\|f\|_p^2\end{aligned}$$

according to the condition $Rr_B \ge 1$ and $p' > 2n/(n-1)$. This proves Case II, and then the proof of Proposition 11.1 is complete. □

Corollary 11.2. *Suppose that $n > 2$ and $-(n-2)^2/4 < c$. For $c < 0$, set $p \in ((p_c^*)', \frac{2n}{n+1})$ where $p_c^* = n/\sigma$ and $\sigma = (n-2)/2 - \sqrt{(n-2)^2/4 + c}$ and $(p_c^*)'$ its conjugate exponent. For $c \ge 0$, set $p \in [1, \frac{2n}{n+1})$. Then*

(i) *$S_R^{\delta(p)}(L)$ is of weak-type (p,p) uniformly in R.*

(ii) *For any bounded Borel function $F : [0,\infty) \to \mathbb{C}$ such that $\sup_{t>0}\|\eta\,\delta_t F\|_{W^{\beta,2}} < \infty$ for some $\beta > \max\{n(1/p - 1/2), 1/2\}$ and some non-trivial function $\eta \in C_c^\infty(0,\infty)$, the operator $F(\sqrt{L})$ is bounded on $L^r(X)$ for all $p < r < p'$ with*

$$\|F(\sqrt{L})\|_{r\to r} \le C_\beta\Big(\sup_{t>0}\|\eta\,\delta_t F\|_{W^{\beta,2}} + |F(0)|\Big).$$

Proof. This result follows from Proposition 11.1 and Theorems 5.1, 4.1. □

Remark 11.3. *Note that for the standard Laplacian Δ on $\mathbb{R}^n$ Stein-Tomas estimate $(\mathrm{R_p})$ holds if and only if $1 \le p \le 2(n+1)/(n+3)$. Surprisingly, if $n > 2$ and $-(n-2)^2/4 < c < 0$, then the restriction estimate $(\mathrm{ST}^2_{\mathrm{p},2})$ for the operator $\Delta_n + \frac{c}{r^2}$ holds only for all $p \in ((p_c^*)', 2n/(n+1))$ where $p_c^* = n/\sigma$ and $\sigma = (n-2)/2 - \sqrt{(n-2)^2/4 + c}$.*

12. Examples of $(1,2)$-restriction type conditions

In [24] the following *Plancherel condition* is introduced: for any $R > 0$ and all even Borel functions F such that $\operatorname{supp} F \subseteq [0,R]$,

$$\int_X |K_{F(\sqrt{L})}(x,y)|^2 d\mu(x) \le CV(y,R^{-1})^{-1}\|\delta_R F\|_q^2 \tag{12.1}$$

for some $q \in [2,\infty]$. Note that for every $x \in X$ and $r \ge 1/R$,

$$\begin{aligned}\big\|F(\sqrt{L})P_{B(x,r)}f\big\|_2 &\le \Big\|\int_X K_{F(\sqrt{L})}(z,w)\chi_{B(x,r)}(w)f(w)d\mu(w)\Big\|_2\\ &\le \int_X \big\|K_{F(\sqrt{L})}(\cdot,w)\big\|_2\chi_{B(x,r)}(w)|f(w)|d\mu(w)\\ &\le C\|\delta_R F\|_q\int_X V(w,R^{-1})^{-1/2}\chi_{B(x,r)}(w)|f(w)|d\mu(w)\\ &\le CV(x,r)^{-1/2}(Rr)^{n/2}\|\delta_R F\|_q\|f\|_1,\end{aligned}$$

where in the last inequality we used the doubling condition (2.2). Therefore, for every $x \in X$ and $r \geq 1/R$

$$\|F(\sqrt{L})P_{B(x,r)}\|_{1\to 2} \leq CV(x,r)^{-1/2}(Rr)^{n/2}\|\delta_R F\|_q, \tag{12.2}$$

and so the condition (12.1) is just a slightly stronger version of condition $(\mathrm{ST}^{\mathrm{q}}_{1,2})$.

It was noted in [16, 23, 39] and used in context of spectral multipliers that condition (12.1) with $q = 2$ holds for homogeneous sub-Laplacian acting on homogeneous Lie groups. Condition (12.1) with $q = 2$ holds also for "quasi-homogeneous" subelliptic and elliptic operators (see [60]). As we note, condition (12.1) is stronger than condition $(\mathrm{ST}^2_{1,2})$ so this implies the following result.

Proposition 12.1. *Let L be a homogeneous sub-Laplacian or "quasi-homogeneous" operator acting on homogeneous Lie group with homogeneous dimension d. Then the Riesz mean $S_R^{(d-1)/2}(L)$ of order $(d-1)/2$ is of weak-type $(1,1)$ uniformly in R.*

Proof. Proposition 12.1 follows directly from Theorem 5.1. □

We believe that in this generality Proposition 12.1 is a new result. However in the case of Heisenberg group it follows from the result obtained by Müller, Stein and Hebisch that the the Riesz means of order $\delta > (d_e - 1)/2$ is bounded on L^1, where $d_e < d$ is the topological dimension of the Heisenberg group (see [53, 32]). Therefore it is likely that Proposition 12.1 is not a genuine endpoint result.

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Peng Chen, Department of Mathematics, Sun Yat-sen (Zhongshan) University, Guangzhou, 510275, P.R. China
E-mail address: achenpeng1981@163.com

El Maati Ouhabaz, Institut de Mathématiques de Bordeaux, Université Bordeaux 1, UMR 5251, 351, Cours de la Libération 33405 Talence, France
E-mail address: Elmaati.Ouhabaz@math.u-bordeaux1.fr

Adam Sikora, Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: sikora@maths.mq.edu.au

Lixin Yan, Department of Mathematics, Sun Yat-sen (Zhongshan) University, Guangzhou, 510275, P.R. China
E-mail address: mcsylx@mail.sysu.edu.cn