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SUPERDIFFUSION OF ENERGY IN A CHAIN OF HARMONIC OSCILLATORS WITH NOISE

MILTON JARA, TOMASZ KOMOROWSKI, AND STEFANO OLLA

ABSTRACT. *We consider a one dimensional infinite chain of harmonic oscillators whose dynamics is perturbed by a stochastic term conserving energy and momentum. We prove that in the unpinned case the macroscopic evolution of the energy converges to a fractional diffusion governed by $-|\Delta|^{3/4}$. For a pinned system we prove that energy evolves diffusively, generalizing some of the results of [4].*

1. INTRODUCTION

Superdiffusion of energy and corresponding anomalous thermal conductivity have been observed numerically in the dynamics of unpinned FPU chains [17, 16]. This is generally attributed to the small scattering rate for low modes, due to momentum conservation. When the interaction has a pinning potential, it is expected that the system undergoes a normal diffusion. More recently the problem has been studied in models where the Hamiltonian dynamics is perturbed by stochastic terms that conserve energy and momentum, like random exchange of velocity between nearest neighbors particles [1, 2]. In these models, if the interaction is purely harmonic, the Green-Kubo formula for thermal conductivity κ can be studied explicitly. It diverges for one and two dimensional lattices in case of no-pinning, while thermal conductivity stays finite for pinned systems or in dimension $d \geq 3$. In the cases when the conductivity is finite it is proven in [4] that energy fluctuations evolve diffusively following the heat equation.

The main result of the present article concerns the nature of the superdiffusion in dimension 1 when the chain is unpinned. It has already been proven that in the weak noise limit (where the average number of *stochastic collisions* is kept finite as in the Grad limit) the Wigner distribution converges to an inhomogeneous phonon linear Boltzmann equation [5]. Since the corresponding scattering kernel $R(k, k')$ is positive, the resulting Boltzmann equation can be interpreted probabilistically as the evolution of the density for some Markov process: in this limit a *phonon* of mode k moves with the velocity given by the gradient of the dispersion relation $\nabla\omega(k)$ and change mode with rate $R(k, k')$. Under a proper space-time rescaling, this process converges to a Lévy

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superdiffusion generated by the fractional laplacian $-|\Delta|^{3/4}$. This is proven in [11, 3], using probabilistic techniques such as coupling and martingale convergence theorems. A completely analytic proof of the convergence, from a kinetic to a fractional diffusion equation, without the use of the probabilistic representation, has been proposed in [19]. All these results provide a *two-step* solution: first take a kinetic limit, then use a *hydrodynamic* rescaling of the kinetic equation. A kind of diagonal procedure is treated in [13]: one can push the time scale a little longer (matching suitably the size of a still small scattering rate) than in the kinetic limit case. As a result it is possible to obtain the diffusive limit under the pinning potential and superdiffusive in the unpinned case.

In the present paper we prove a direct limit to the fractional superdiffusion, just by rescaling space and time, without the weak noise assumption. We also recover the diffusive limit results of [4] in the case of finite diffusivity and study the cases of intermediate weaker noise limits. The rigorous formulations of our main results are listed in Section 3.

In a recent article [23], Herbert Spohn predicts the same fractional superdiffusive behavior for the heat mode in the β -FPU at zero pressure. This follows from an application of mode coupling approximation procedure to fluctuating hydrodynamic equations. Our present results concern a model, that has also three conserved quantities. They are in agreement with predictions of [23], confirming that the harmonic stochastic model is a good approximation of some non-linear models, at least for symmetric interactions.

The strategy of the proof is as follows: first, we formulate the result for the limit evolution of the Wigner distribution $W_\epsilon(t)$ of the energy, when the initial data are in L^2 , see Theorems 5.1 and 5.2, proven in Sections 9 and 10. These results concern the system with a non-equilibrium initial data but finite total energy. The extension to homogeneous initial data, in particular the equilibrium dynamics with Gibbs distributed initial data, is possible by a simple duality argument, see Section 11.

Our results can be formulated in terms of a local energy functional, see Theorems 3.2 and 3.3 in the case of the L^2 integrable initial data, and Theorem 3.4 for the initial data in equilibrium respectively. This is possible thanks to the asymptotic equivalence of the relevant energy functionals proven in Propositions 5.3 and 6.3 below.

Concerning the proof when the initial data are in L^2 , which is the crucial part of our argument, we study first the time evolution of the

Wigner distribution of the energy $W_\epsilon(t)$. As it has been already remarked in [5], it is not autonomous but involves another distribution $Y_\epsilon(t)$ that represents the Wigner distribution of the difference between kinetic and potential energy. Due to the high number of random collisions in the time scales considered, both $W_\epsilon(t)$ and $Y_\epsilon(t)$ homogenize (unlike the case of the kinetic limit considered in [5]), and their limits will not depend on the fast k -mode variable. This homogenization is proven in Section 7. In addition, because of fast fluctuations, the time integral of $Y_\epsilon(t)$ will disappear from the final equation, as in the case of the kinetic limit in [5]. The above implies that the phonon-Boltzmann equation gives a good approximation of the evolution of $W_\epsilon(t)$, but the presence of the error term, that is of order $o(1)$, as ϵ tends to 0, does not allow for a direct application of the probabilistic approach of [11, 3]. Instead, we use a version of the analytical approach of [19], based on projections on the product components of the scattering kernel $R(k, k')$.

In our choice of the dynamics a diffusive random exchange takes place between three nearest neighbor particles in such a way that total kinetic energy and momentum are conserved. However, our method can be applied to other linear models with quite general stochastic scattering mechanisms, generating different scattering rates and the result does not depend on the particular type of stochastic perturbations, as long as it conserves the appropriate quantities. E.g. we could consider a model with a simple Poissonian exchange of two nearest neighbor velocities, see Section 2.1.3 below. Then, the scattering kernel is of the form $R(k, k') = R(k)R(k')$ for some function $R(k)$, see (5.17) below, and our argument (in fact a bit less complex, due to the simpler form of the scattering kernel) leads to an analogous result for that model. The above shows that our method applies to relatively general scattering rates and the result does not depend on the particular stochastic perturbations, as long as it conserves energy and momentum.

In the companion article [7], a similar result is proven, independently and with an entirely different method, for a dynamics with two conserved quantities in the case when the initial data is given by a Gibbs equilibrium measure.

2. THE DYNAMICS

2.1. Infinite chain of interacting harmonic oscillators.

2.1.1. *Hamiltonian system.* The dynamics of the system of oscillators can be written formally as a Hamiltonian system of differential equations

$$\dot{\mathbf{q}}_y(t) = \partial_{\mathbf{p}_y} \mathcal{H}(\mathbf{p}(t), \mathbf{q}(t)) \quad (2.1)$$

$$\dot{\mathbf{p}}_y(t) = -\partial_{\mathbf{q}_y} \mathcal{H}(\mathbf{p}(t), \mathbf{q}(t)), \quad y \in \mathbb{Z}.$$

The formal Hamiltonian is given by

$$\mathcal{H}(\mathbf{p}, \mathbf{q}) := \frac{1}{2} \sum_{y \in \mathbb{Z}} \mathbf{p}_y^2 + \frac{1}{2} \sum_{y, y' \in \mathbb{Z}} \alpha_{y-y'} \mathbf{q}_y \mathbf{q}_{y'} \quad (2.2)$$

and we assume also (cf [5]) that

- a1) $\{\alpha_y, y \in \mathbb{Z}\}$ is real valued and there exists $C > 0$ such that $|\alpha_y| \leq C e^{-|y|/C}$ for all $y \in \mathbb{Z}$,
- a2) $\hat{\alpha}(k)$ is also real valued and $\hat{\alpha}(k) > 0$ for $k \neq 0$ and in case $\hat{\alpha}(0) = 0$ we have $\hat{\alpha}''(0) > 0$.
- a3) To guarantee that the local energy functional, see (3.21) below, is non-negative we assume that

$$\alpha_x \leq 0, \quad x \neq 0. \quad (2.3)$$

Here $\hat{\alpha}(k)$, $k \in \mathbb{T}$ is a Fourier transform of sequence (α_x) , defined as

$$\hat{\alpha}(k) = \sum_x \alpha_x e_x^*(k),$$

where z^* denotes the complex conjugate of a given $z \in \mathbb{C}$ and

$$e_x(k) := \exp\{2\pi i x k\}, \quad x \in \mathbb{Z}.$$

The above conditions imply that both functions $y \mapsto \alpha_y$ and $k \mapsto \hat{\alpha}(k)$ are even. In addition, $\hat{\alpha} \in C^\infty(\mathbb{T})$. Define the *dispersion relation* as $\omega(k) := \hat{\alpha}^{1/2}(k)$. In case when $\hat{\alpha}(0) > 0$ the dispersion relation belongs to $C^\infty(\mathbb{T})$. If $\hat{\alpha}(0) = 0$ we can write

$$\omega(k) = |\sin(\pi k)| \sqrt{\frac{\hat{\alpha}''(0)}{2\pi^2} \varphi(\sin^2(\pi k))}, \quad (2.4)$$

where $\varphi : [0, +\infty) \rightarrow (0, +\infty)$ is of C^2 class and such that $\varphi(0) = 1$.

The \mathbf{p}_y is the velocity (or momentum, as mass is taken equal to 1) of the particle y , while \mathbf{q}_y should be interpreted as the displacement of the position of the particle y from the point ay , where $a \geq 0$ is an arbitrary

equilibrium interdistance. Since the dynamics is linear, it does not depend on a . In the pinned case, $\hat{\alpha}(0) > 0$, the particle y feels a pinning harmonic potential centered at ay . In the unpinned case, $\hat{\alpha}(0) = 0$, the system is translation invariant, and only the interparticle distances are relevant for the dynamics. So in the unpinned case the variables \mathbf{q}_y are defined up to a common additive constant. In the following we will only consider functionals defined on the configuration space that depend only on the relative distances between the particles. In particular we will introduce the concept of a wave function, see Section 5.1 below. Its definition and corresponding dynamics are unambiguous, see (5.4), both in the pinned and unpinned cases.

Furthermore, in the unpinned case, the total momentum and the energy of the chain are formally conserved, (besides the volume of course) and, since we insist on preserving these properties, we choose a stochastic perturbation with the same conservation laws. This can be done either locally, via a time continuous stochastic exchange of momentum (see Section 2.1.2 below), or through a time discontinuous random exchange of momentum mechanism (see Section 2.1.3).

2.1.2. *Continuous time noise.* Now we add to the right hand side of (2.1) a local stochastic term that conserves $\mathbf{p}_{y-1}^2 + \mathbf{p}_y^2 + \mathbf{p}_{y+1}^2$ and $\mathbf{p}_{y-1} + \mathbf{p}_y + \mathbf{p}_{y+1}$. Namely, we write the stochastic differential equations:

$$d\mathbf{q}_y(t) = \mathbf{p}_y(t) dt, \quad (2.5)$$

$$\begin{aligned} d\mathbf{p}_y(t) &= \left[-(\alpha * \mathbf{q}(t))_y - \frac{\gamma}{2}(\beta * \mathbf{p}(t))_y \right] dt \\ &\quad + \gamma^{1/2} \sum_{z=-1,0,1} (Y_{y+z} \mathbf{p}_y(t)) dw_{y+z}(t), \quad y \in \mathbb{Z}. \end{aligned}$$

with the parameter $\gamma > 0$ that determines the strength of the noise in the system, and (Y_x) are vector fields given by

$$Y_x := (\mathbf{p}_x - \mathbf{p}_{x+1})\partial_{\mathbf{p}_{x-1}} + (\mathbf{p}_{x+1} - \mathbf{p}_{x-1})\partial_{\mathbf{p}_x} + (\mathbf{p}_{x-1} - \mathbf{p}_x)\partial_{\mathbf{p}_{x+1}}. \quad (2.6)$$

Here $\{w_y(t), t \geq 0\}$, $y \in \mathbb{Z}$ are i.i.d. one dimensional, real valued, standard Brownian motions, that are non-anticipative over some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Furthermore, we assume that $\beta_y = \Delta\beta_y^{(0)}$, where

$$\beta_y^{(0)} = \begin{cases} -4, & y = 0 \\ -1, & y = \pm 1 \\ 0, & \text{if otherwise.} \end{cases}$$

Here, the lattice Laplacian of a given $g : \mathbb{Z} \rightarrow \mathbb{C}$ is defined as $\Delta g_y := g_{y+1} + g_{y-1} - 2g_y$. We shall also denote $\nabla g_y := g_{y+1} - g_y$ and $\nabla^* g_y := g_{y-1} - g_y$.

For a future reference we introduce $\beta_{1,y} := \nabla^* \beta_y^{(0)}$. A simple calculation shows that

$$\hat{\beta}(k) = 8\mathfrak{s}^2(k) [1 + 2\mathfrak{c}^2(k)] = 8\mathfrak{s}^2(k) + 4\mathfrak{s}^2(2k), \quad (2.7)$$

and

$$\hat{\beta}_1(k) = (1 - e^{-2i\pi k}) (4 + e^{2\pi i k} + e^{-2\pi i k}), \quad (2.8)$$

where, for the abbreviation sake, we have writtten

$$\mathfrak{s}(k) := \sin(\pi k) \quad \text{and} \quad \mathfrak{c}(k) := \cos(\pi k), \quad k \in \mathbb{T}.$$

2.1.3. *Random momentum exchange.* Let $(N_{x,x+1}(t))$, $x \in \mathbb{Z}$ are i.i.d. Poisson processes with intensity $3\gamma/2$. The dynamics of the position component \mathbf{q}_y is the same as in (2.5), while the momentum $(\mathbf{p}_y(t))$, $y \in \mathbb{Z}$ is a cadlag process given by

$$\begin{aligned} d\mathbf{p}_y(t) = & -(\alpha * \mathbf{q}(t))_y dt \\ & + [\nabla \mathbf{p}_y(t-) dN_{y,y+1}(t) + \nabla^* \mathbf{p}_y(t-) dN_{y-1,y}(t)], \quad y \in \mathbb{Z}. \end{aligned} \quad (2.9)$$

3. MAIN RESULTS: MACROSCOPIC EVOLUTION

3.1. **Hyperbolic scaling: Euler equations.** Consider now the unpinned case $\hat{\alpha}(0) = 0$. For a configuration $(\mathbf{p}(t), \mathbf{r}(t))$ we define the energy per atom:

$$\mathbf{e}_x(t) = \frac{\mathbf{p}_x^2(t)}{2} - \frac{1}{4} \sum_y \alpha_{x-y} (\mathbf{q}_x(t) - \mathbf{q}_y(t))^2. \quad (3.1)$$

Thanks to condition a3) on the potential (α_x) we have $\mathbf{e}_x(t) \geq 0$. Notice also that, since $\sum_x \alpha_x = 0$, formally we have $\sum_x \mathbf{e}_x(t) = \mathcal{H}(\mathbf{p}(t), \mathbf{q}(t))$.

Define $\mathbf{r}_x = \mathbf{q}_x - \mathbf{q}_{x-1}$. Then $\sum_x \mathbf{r}_x$, when finite, represent the total *length* of the system when the equilibrium interparticle distance $a = 0$.

In the unpinned case the chain has three formally conserved (also called balanced) quantities

$$\begin{aligned} \sum_x \mathbf{r}_x(t) & \quad - \text{volume (length)}, \\ \sum_x \mathbf{p}_x(t) & \quad - \text{momentum}, \\ \sum_x \mathbf{e}_x(t) & \quad - \text{energy.} \end{aligned} \tag{3.2}$$

Observe that under the finite energy condition $\mathcal{H}(\mathbf{p}, \mathbf{q}) < +\infty$, these quantities are finite.

Because the noise is added to the system, these are the 'only' conserved quantities, in the sense that the only stationary probability measures that are also translation invariant, are given by the Gibbs measures

$$d\mu_{T, \bar{p}, \tau} = \frac{1}{Z_{T, \bar{p}, \tau}} \exp \left\{ -T^{-1} \left(\mathcal{H} - \bar{p} \sum_x \mathbf{p}_x - \tau \sum_x \mathbf{r}_x \right) \right\} \prod_x d\mathbf{r}_x d\mathbf{p}_x$$

parametrized by the temperature T , momentum \bar{p} and tension τ , properly defined locally by the appropriate DLR equations on their conditional distributions (see Section 4 of [8]).

It can be proven, that at the hyperbolic space-time scaling, these conserved quantities evolve deterministically following the system of Euler equations:

$$\begin{aligned} \partial_t \bar{r}(t, y) &= \partial_y \bar{p}(t, y), \\ \partial_t \bar{p}(t, y) &= \tau_1 \partial_y \bar{r}(t, y), \\ \partial_t \bar{e}(t, y) &= \tau_1 \partial_y (\bar{r}(t, y) \bar{p}(t, y)), \end{aligned} \tag{3.3}$$

with the initial data

$$\bar{r}(0, y) = \bar{r}_0(y), \quad \bar{p}(0, y) = \bar{p}_0(y), \quad \bar{e}(0, y) = \bar{e}_0(y).$$

The sound speed τ_1 is defined by

$$\tau_1 := \frac{\hat{\alpha}''(0)}{8\pi^2}. \tag{3.4}$$

Note that in the case under consideration the system (3.3) decouples. Quantities $(\bar{r}(t, y), \bar{p}(t, y))$ satisfy the linear wave equation and the energy $\bar{e}(t, y)$ is given by

$$\bar{e}(t, y) = \frac{\tau_1 \bar{r}^2(t, y)}{2} + \frac{\bar{p}^2(t, y)}{2} + T(y)$$

and that the profile of temperature $T(y)$ does not change in time. In this case the tension is linear and equals $\tau(\bar{r}) = \tau_1 \bar{r}$.

More precisely, consider the empirical distributions associated to the conserved quantities:

$$\epsilon \sum_x J(\epsilon x) \mathbf{u}_x(\epsilon^{-1}t),$$

where

$$\mathbf{u}_x(t) = (\mathbf{r}_x(t), \mathbf{p}_x(t), \mathbf{e}_x(t))$$

and $J \in C_0^\infty(\mathbb{R})$ is a smooth test function with compact support. Then,

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \sum_x J(\epsilon x) \mathbf{u}_x(\epsilon^{-1}t) = \int_{\mathbb{R}} J(y) \bar{\mathbf{u}}(t, y) dy,$$

where the convergence holds in probability, and the vector function $\bar{\mathbf{u}}(t, y) := (\bar{r}(t, y), \bar{p}(t, y), \bar{e}(t, y))$ satisfies (3.3), provided that the same convergence hold for the initial distribution at $t = 0$.

In the case of a finite number of particles $N = [\epsilon^{-1}]$, with periodic or other boundary conditions, this can be proven by using relative entropy methods (cf. [20] and [10]), also in the non-linear case in the smooth regime of the Euler equations. In the infinite volume, starting with the initial distribution μ_ϵ such that

$$\limsup_{\epsilon \rightarrow 0^+} \epsilon \langle \mathcal{H}(\mathbf{p}, \mathbf{q}) \rangle_{\mu_\epsilon} < +\infty, \quad (3.5)$$

with $\langle \cdot \rangle_{\mu_\epsilon}$ denoting the expectation with respect to μ_ϵ , relative entropy method cannot be applied. In the present article we show, see Theorem 3.1 below, that

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \sum_x J(\epsilon x) \mathbb{E}_\epsilon \mathbf{u}_x(\epsilon^{-1}t) = \int_{\mathbb{R}} J(y) \bar{\mathbf{u}}(t, y) dy, \quad (3.6)$$

where \mathbb{E}_ϵ denotes the expectation with respect to the product measure corresponding to the initial data and the noise in (2.5).

3.2. Superdiffusive behavior of the unpinned chain. In fact our purpose is to go beyond the hyperbolic time scale and understand the behavior of the quantities listed in (3.2), and in particular of the energy, on the diffusive or (if necessary) superdiffusive space-time scale. To understand the evolution at a larger time scale, let us consider the Riemann invariants of the system (3.3), given by

$$\bar{f}^{(\pm)}(t, y) = \bar{p}(t, y) \pm \sqrt{\tau_1} \bar{r}(t, y).$$

They are constant along the lines $y \mp \sqrt{\tau_1}t = \text{const}$, respectively, therefore

$$\bar{f}^{(\pm)}(t, y) = \bar{f}_0^{(\pm)}(y \pm \sqrt{\tau_1}t). \quad (3.7)$$

with $\bar{f}_0^{(\pm)}(y)$ determined from the initial datum. This motivates an introduction of the microscopic *phonon modes* given by

$$\mathfrak{f}_y^{(+)}(t) := \mathfrak{p}_y(t) + \sqrt{\tau_1} \left[1 + \frac{1}{2} \left(\frac{3\gamma}{\sqrt{\tau_1}} - 1 \right) \nabla^* \right] \mathfrak{r}_y(t), \quad (3.8)$$

$$\mathfrak{f}_y^{(-)}(t) := \mathfrak{p}_y(t) - \sqrt{\tau_1} \left[1 - \frac{1}{2} \left(\frac{3\gamma}{\sqrt{\tau_1}} + 1 \right) \nabla^* \right] \mathfrak{r}_y(t),$$

that are the second order approximations (up to a diffusive scale) of $\bar{f}^{(\pm)}(t, y)$. More precisely, assume that $\mu_\epsilon, \epsilon \in (0, 1]$ - the initial laws of the wave function - are such that for any $J \in C_0(\mathbb{R})$ we have

$$\lim_{\epsilon \rightarrow 0+} \epsilon \sum_y J(\epsilon y) \langle \mathfrak{r}_y \rangle_{\mu_\epsilon} = \int_{\mathbb{R}} J(y) \bar{r}_0(y) dy \quad (3.9)$$

and

$$\lim_{\epsilon \rightarrow 0+} \epsilon \sum_y J(\epsilon y) \langle \mathfrak{p}_y \rangle_{\mu_\epsilon} = \int_{\mathbb{R}} J(y) \bar{p}_0(y) dy. \quad (3.10)$$

Denote also

$$P_t(y) := \frac{1}{\sqrt{4\pi\bar{D}t}} \exp \left\{ -\frac{y^2}{4\bar{D}t} \right\}, \quad (3.11)$$

where $\bar{D} := 3\gamma$ and $\bar{f}_\iota^{(d)}(t, y) := P_t * \bar{f}_0^{(\iota)}(y)$ and

$$\bar{f}_0^{(\pm)}(y) := \bar{p}_0(y) \pm \sqrt{\tau_1} \bar{r}_0(y). \quad (3.12)$$

With the above notation we can formulate the following result.

Theorem 3.1. *Suppose that $\mu_\epsilon, \epsilon \in (0, 1]$ - the initial laws of the wave function - fulfill (3.9) and (3.10). Then, the phonon modes $\bar{f}_y^{(\pm)}(t)$ satisfy*

$$\lim_{\epsilon \rightarrow 0+} \epsilon \sum_y J(\epsilon y) \mathbb{E}_\epsilon \bar{f}_y^{(\pm)} \left(\frac{t}{\epsilon} \right) = \int_{\mathbb{R}} J(y) \bar{f}^{(\pm)}(t, y) dy, \quad (3.13)$$

for any $J \in C_0^\infty(\mathbb{R})$. In addition, for any $\iota \in \{-, +\}$

$$\lim_{\epsilon \rightarrow 0+} \epsilon \sum_y J \left(\epsilon y - \iota \sqrt{\tau_1} \frac{t}{\epsilon} \right) \mathbb{E}_\epsilon \bar{f}_y^{(\iota)} \left(\frac{t}{\epsilon^2} \right) = \int_{\mathbb{R}} \bar{f}_\iota^{(d)}(t, y) J(y) dy, \quad (3.14)$$

The proof of this theorem is contained in Section 12.

Our main result concerns the macroscopic behavior of the energy functional. Suppose that the initial distribution of energy satisfies the following assumptions:

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \sum_y J(\epsilon y) \langle \mathbf{e}_y \rangle_{\mu_\epsilon} = \int_{\mathbb{R}} J(y) W_0(y) dy, \quad (3.15)$$

where $W_0 \in L^1(\mathbb{R})$ is non-negative. Define

$$\mathbf{w}_\epsilon(k) := \langle |\hat{\mathbf{p}}(k)|^2 + \hat{\alpha}(k) |\hat{\mathbf{q}}(k)|^2 \rangle_{\mu_\epsilon},$$

where $\hat{\mathbf{p}}(k)$, $\hat{\mathbf{q}}(k)$ and $\hat{\alpha}(k)$ are Fourier transforms of (\mathbf{p}_x) , (\mathbf{q}_x) and (α_x) , respectively, see Section 4 below. Assumption (3.5) is equivalent to

$$K_1 := \sup_{\epsilon \in (0,1]} \frac{\epsilon}{2} \int_{\mathbb{T}} \mathbf{w}_\epsilon(k) dk < +\infty. \quad (3.16)$$

We shall further suppose that the energy is uniformly integrable, i.e.

$$\lim_{L \rightarrow +\infty} \sup_{\epsilon \in (0,1]} \frac{\epsilon}{2} \int_{|\mathbf{w}_\epsilon(k)| \geq L} \mathbf{w}_\epsilon(k) dk = 0. \quad (3.17)$$

Theorem 3.2. *Let $\delta = 3/2$, then, under the conditions on the initial distribution stated in the foregoing, for any test function $J \in C_0^\infty([0, +\infty) \times \mathbb{R})$ we have:*

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \sum_y \int_0^{+\infty} J(t, \epsilon y) \mathbb{E}_\epsilon \mathbf{e}_y \left(\frac{t}{\epsilon^\delta} \right) dt = \int_0^{+\infty} \int_{\mathbb{R}} W(t, y) J(t, y) dt dy, \quad (3.18)$$

where $W(t, y)$ satisfies the fractional heat equation:

$$\partial_t W(t, y) = -\hat{c} |\partial_y^2|^{3/2} W(t, y) \quad (3.19)$$

with the initial condition $W(0, y) = W_0(y)$ and

$$\hat{c} := \frac{[\alpha''(0)]^{3/4}}{2^{9/4} (3\gamma)^{1/2}}. \quad (3.20)$$

Remark. The proof of this result is a direct consequence of Theorem 5.2 and Proposition 5.3 formulated below. In fact, (as can be seen from the aforementioned results) it can be formulated in a more general way to cover also the case of a *weaker* noise, i.e. parameter γ can be replaced by $\epsilon^s \gamma_0$, for some $s \in [0, 1)$ and $\gamma_0 > 0$. Then, the result is still valid at the time scale $\delta = (3 - s)/2$. The limiting function $W(t, y)$ is the same as in the case $s = 0$, covered by Theorem 3.2.

3.3. Diffusive behavior of the pinned chain. If $\hat{\alpha}(0) > 0$ there is a pinning potential and the Hamiltonian dynamics does not conserve the momentum. Energy is the only relevant conserved quantity but there is no macroscopic evolution in the hyperbolic space-time scale. In this case, for a given configuration $(\mathbf{p}(t), \mathbf{q}(t))$, the energy per site is defined as

$$\mathbf{e}_x(t) := \frac{\mathbf{p}_x^2(t)}{2} - \frac{1}{4} \sum_y \alpha_{x-y} (\mathbf{q}_x(t) - \mathbf{q}_y(t))^2 + \frac{\hat{\alpha}(0)}{2} \mathbf{q}_x^2(t). \quad (3.21)$$

We assume that both (2.3) and (3.16) are in force. Define

$$\hat{\sigma}^2 := \int_{\mathbb{T}} \frac{\omega'(k)^2}{R(k)} dk \quad (3.22)$$

where

$$R(k) := \frac{\hat{\beta}(k)}{4}. \quad (3.23)$$

Since $\omega'(k) \approx k$ and $R(k) \approx k^2$, as $k \ll 1$ (see (2.4) and (2.7)), we have $\hat{\sigma}^2 < +\infty$ (while it is infinite in the unpinned case, due to $\omega'(k) \approx \text{sign } k$). As a result, the evolution is always diffusive and we have the following:

Theorem 3.3. *Let $s \in [0, 1)$, $\gamma = \epsilon^s \gamma_0$, and $\hat{\alpha}(0) > 0$ then for any $J(t, y)$ as in Theorem 3.2 we have*

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \sum_x \int_0^{+\infty} J(t, \epsilon x) \mathbb{E}_\epsilon \mathbf{e}_x \left(\frac{t}{\epsilon^\delta} \right) = \int_0^{+\infty} \int_{\mathbb{R}} W(t, y) J(t, y) dt dy,$$

with $\delta = 2 - s$, where $W(t, y)$ satisfy the heat equation:

$$\partial_t W(t, y) = D \partial_y^2 W(t, y) \quad (3.24)$$

with

$$D = \frac{\hat{\sigma}^2}{\gamma_0} + 8\gamma_0 \pi^2 \quad \text{if } s = 0,$$

and

$$D = \frac{\hat{\sigma}^2}{\gamma_0} \quad \text{if } 0 < s < 1.$$

3.4. Equilibrium fluctuations. The results formulated above are obtained under the condition of finite macroscopic total energy (3.5). By a duality argument they can be applied to obtain the following macroscopic behavior of the fluctuations when the system starts in an equilibrium measure $\mu_{\mathcal{E}_0,0,0}$. For the fluctuations of the energy mode we assume that $\gamma = \gamma_0 \epsilon^s$ for some $\gamma_0 > 0$ and $s \in [0, 1)$. Consider the energy fluctuation field

$$\tilde{\mathfrak{e}}_\epsilon(t, J) = \sqrt{\epsilon} \sum_x J(\epsilon x) \left[\mathfrak{e}_x \left(\frac{t}{\epsilon^\delta} \right) - \mathcal{E}_0 \right] \quad (3.25)$$

where δ is chosen as before: $\delta = (3 - s)/2$ in the unpinned case, and $\delta = 2 - s$ in the pinned case, and $J \in C_0^\infty(\mathbb{R})$. The covariance field is defined as

$$C_\epsilon^{(e)}(t, J_1, J_2) := \mathbb{E} [\tilde{\mathfrak{e}}_\epsilon(t, J_1) \tilde{\mathfrak{e}}_\epsilon(0, J_2)], \quad J_1, J_2 \in C_0^\infty(\mathbb{R}). \quad (3.26)$$

The following theorem is a direct corollary from Theorems 6.1 and 6.2, and Proposition 6.3.

Theorem 3.4. *For any functions $J_1, J_2 \in C_0^\infty(\mathbb{R})$ and $\phi \in L^1[0, +\infty)$ we have*

$$\lim_{\epsilon \rightarrow 0} \int_0^{+\infty} \phi(t) C_\epsilon^{(e)}(t, J_1, J_2) dt = \int_0^{+\infty} \phi(t) C^{(e)}(t, J_1, J_2) dt$$

where $C^{(e)}(t, J_1, J_2)$ satisfies the equation

$$\partial_t C^{(e)}(t, J_1, J_2) = C^{(e)}(t, \mathcal{A}J_1, J_2)$$

with the initial condition

$$C^{(e)}(0, J_1, J_2) = \mathcal{E}_0 \int_{\mathbb{R}} J_1(y) J_2(y) dy,$$

and $\mathcal{A} = -\hat{c}|\Delta_y|^{2/3}$ in the unpinned, or $\mathcal{A} = D\Delta_y$ in the pinned case, respectively. Coefficients \hat{c} and D are the same as in Theorems 3.2 and 3.3.

Concerning the phonon mode fluctuations (that of course are relevant only in the unpinned case) they are diffusive, as stated in the theorem below. Recall (3.8), and define the corresponding fluctuation field:

$$\mathcal{F}_\epsilon^{(\iota)}(t, J) = \sqrt{\epsilon} \sum_x J \left(\epsilon x - \iota \sqrt{\tau_1} \frac{t}{\epsilon} \right) \mathfrak{f}_x^{(\iota)} \left(\frac{t}{\epsilon^2} \right), \quad \iota = \pm 1 \quad (3.27)$$

and the covariance field

$$C_\epsilon^{(\iota, \iota')}(t, J_1, J_2) := \mathbb{E} \left[\mathcal{F}_\epsilon^{(\iota)}(t, J_1) \mathcal{F}_\epsilon^{(\iota')}(0, J_2) \right]$$

Assume also that in this case $\gamma > 0$ is independent of ϵ .

Theorem 3.5. *For any $J_1, J_2 \in C_0^\infty(\mathbb{R})$ and $t \geq 0$ we have*

$$\lim_{\epsilon \rightarrow 0} C_\epsilon^{(\iota, \iota')} (t, J_1, J_2) = \delta_{\iota, \iota'} C^{(\iota)} (t, J_1, J_2),$$

where $C^{(\iota)} (t, J_1, J_2)$ satisfies the equation

$$\partial_t C^{(\epsilon)} (t, J_1, J_2) = C^{(\epsilon)} (t, \bar{D} J_1'', J_2)$$

with the initial condition

$$C^{(\iota)} (0, J_1, J_2) = \mathcal{E}_0 \int_{\mathbb{R}} J_1(y) J_2(y) dy,$$

and $\bar{D} = 3\gamma/2$.

4. SOME BASIC NOTATION

The one dimensional torus \mathbb{T} considered here is the interval $[-1/2, 1/2]$ with identified endpoints. Let ℓ^2 be the space of all complex valued sequences (f_x) , $x \in \mathbb{Z}$, equipped with the norm $\|f\|_{\ell^2}^2 := \sum_x |f_x|^2$. Given $m \in \mathbb{R}$ we shall define the space h_m made of sequences (f_x) , $x \in \mathbb{Z}$, for which $\|f\|_{h_m}^2 := \sum_x (1+x^2)^m |f_x|^2 < +\infty$.

Given a sequence $(f_x) \in \ell^2$ define its Fourier transform as a function $\hat{f} : \mathbb{T} \rightarrow \mathbb{C}$ given by

$$\hat{f}(k) = \sum_x f_x e_x^*(k), \quad (4.1)$$

where z^* denotes the complex conjugate of a given $z \in \mathbb{C}$ and

$$e_x(k) := \exp\{2\pi i x k\}, \quad x \in \mathbb{Z}.$$

It is well known that \hat{f} belongs to $L^2(\mathbb{T})$ - the space of all complex valued functions equipped with the norm $\|\hat{f}\|_{L^2(\mathbb{T})} := \langle \hat{f}, \hat{f} \rangle_{L^2(\mathbb{T})}^{1/2}$, where

$$\langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{T})} := \int_{\mathbb{T}} \hat{f}(k) \hat{g}^*(k) dk.$$

Formula (4.1) determines an isometric isomorphism between h_m and $H^m(\mathbb{T})$ - the completion of $C^\infty(\mathbb{T})$ in the norm $\|\hat{f}\|_{H^m(\mathbb{T})}^2 := \|f\|_{h_m}^2$. We have $H^0(\mathbb{T}) = L^2(\mathbb{T})$.

For an arbitrary $J : \mathbb{T} \rightarrow \mathbb{C}$, $k \in \mathbb{T}$, $p \in \mathbb{R}$ and $\epsilon > 0$ we define

$$\delta_\epsilon J(p, k) := \frac{1}{\epsilon} \left[J \left(k + \frac{\epsilon p}{2} \right) - J \left(k - \frac{\epsilon p}{2} \right) \right], \quad (4.2)$$

$$\bar{J}(k, p) := \frac{1}{2} \left[J \left(k + \frac{p}{2} \right) + J \left(k - \frac{p}{2} \right) \right].$$

Given a set A and two functions $f, g : A \rightarrow \mathbb{R}$ we say that $f(x) \approx g(x)$, $x \in A$ if there exists $C > 1$ such that

$$\frac{f(x)}{C} \leq g(x) \leq Cf(x), \quad \forall x \in A.$$

We write $g(x) \preceq f(x)$, when only the upper bound on g is satisfied.

Denote by \mathcal{S} the set of functions $J : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{C}$ that are of C^∞ class and such that for any integers l, m, n we have

$$\sup_{x,k} (1+x^2)^n |\partial_x^l \partial_k^m J(x, k)| < +\infty.$$

For any $J \in \mathcal{S}$ we let

$$\hat{J}(p, k) := \int_{\mathbb{R}} e^{-2\pi i x p} J(x, k) dx$$

be its Fourier transform in the first variable.

Let $a \geq 1$. We introduce the norms

$$\|J\|_{\mathcal{A}_a} := \int_{\mathbb{R}} \left(\int_{\mathbb{T}} |\hat{J}(p, k)|^a dk \right)^{1/a} dp, \quad (4.3)$$

and

$$\|J\|_{\mathcal{A}} := \int_{\mathbb{R}} \sup_k |\hat{J}(p, k)| dp. \quad (4.4)$$

By \mathcal{A}_a and \mathcal{A} we denote the completion of \mathcal{S} in the respective norms.

4.1. Averaged Wigner transform. Suppose that (μ_ϵ) , $\epsilon \in (0, 1]$ is a family of Borel probability measures on ℓ^2 . We assume that

$$K_0 := \sup_{\epsilon \in (0, 1]} \frac{\epsilon}{2} \langle \|f\|_{\ell^2}^2 \rangle_{\mu_\epsilon} dk < +\infty, \quad (4.5)$$

$$\lim_{L \rightarrow +\infty} \sup_{\epsilon \in (0, 1]} \frac{\epsilon}{2} \langle \|\hat{f}_L\|_{L^2(\mathbb{T})}^2 \rangle_{\mu_\epsilon} = 0, \quad (4.6)$$

where $\langle \cdot \rangle_{\mu_\epsilon}$ denotes the expectation with respect to μ_ϵ and $\hat{f}_L(k) := \hat{f}(k) 1_{\{|\hat{f}(k)| \geq L\}}$.

Define $W_\epsilon^{(0)}, Y_\epsilon^{(0)} \in \mathcal{A}'$ as

$$\langle W_\epsilon^{(0)}, J \rangle := \frac{\epsilon}{2} \int_{\mathbb{R} \times \mathbb{T}} \left\langle \hat{f}^* \left(k - \frac{\epsilon p}{2} \right) \hat{f} \left(k + \frac{\epsilon p}{2} \right) \right\rangle_{\mu_\epsilon} \hat{J}^*(p, k) dp dk, \quad (4.7)$$

and

$$\langle Y_\epsilon^{(0)}, J \rangle := \frac{\epsilon}{2} \int_{\mathbb{R} \times \mathbb{T}} \left\langle \hat{f} \left(k + \frac{\epsilon p}{2} \right) \hat{f} \left(-k + \frac{\epsilon p}{2} \right) \right\rangle_{\mu_\epsilon} \hat{J}^*(p, k) dp dk \quad (4.8)$$

for any $J \in \mathcal{A}$. Note that from (4.6) and Plancherel's identity we conclude that

$$\limsup_{\epsilon \rightarrow 0^+} (\|Y_\epsilon^{(0)}\|_{\mathcal{A}'} + \|W_\epsilon^{(0)}\|_{\mathcal{A}'}) \leq K. \quad (4.9)$$

Functional $W_\epsilon^{(0)} \in \mathcal{A}'$ is called the *averaged Wigner transform*. We shall also refer to $Y_\epsilon^{(0)}$ as the *averaged anti-Wigner transform*. By Plancherel's identity we have

$$\langle W_\epsilon^{(0)}, J \rangle = \frac{\epsilon}{2} \sum_{x, x' \in \mathbb{Z}} \langle (f_{x'})^* f_x \rangle_\epsilon \int_{\mathbb{T}} e^{2\pi i(x'-x)k} J^* \left(\frac{\epsilon}{2}(x+x'), k \right) dk, \quad (4.10)$$

for any $J \in \mathcal{S}$.

As a consequence of (4.9), both $(W_\epsilon^{(0)})$ and $(Y_\epsilon^{(0)})$ are $*$ -weak (sequentially) compact in \mathcal{A}' , as $\epsilon \rightarrow 0^+$, i.e. for any sequence $\epsilon_n \rightarrow 0$ we can choose a subsequence $(W_{\epsilon'_n}^{(0)}, Y_{\epsilon'_n}^{(0)})$ whose each component is $*$ -weak convergent in \mathcal{A}' , see Section 4.1 of [5].

One can show, see Theorem B4 of [18], that if $(W_{\epsilon'_n}^{(0)})$ is $*$ -weak convergent then there exists a finite Borel measure $W_0(dx, dk)$ on $\mathbb{R} \times \mathbb{T}$ whose total mass does not exceed K and such that

$$\lim_{n \rightarrow +\infty} \langle W_{\epsilon'_n}^{(0)}, J \rangle = \int_{\mathbb{R} \times \mathbb{T}} J^*(x, k) W_0(dx, dk), \quad J \in \mathcal{A}.$$

Applied to functions $J(x, k) = J(x)$, constant in k , the Wigner distribution becomes:

$$\langle W_\epsilon^{(0)}, J \rangle = \frac{\epsilon}{2} \sum_x \langle |f_x|^2 \rangle_{\mu_\epsilon} J(\epsilon x). \quad (4.11)$$

4.2. Homogeneous random fields on \mathbb{Z} . Suppose that $\mathcal{E}(\cdot) : \mathbb{T} \rightarrow [0, +\infty)$ is a Borel measurable function such that

$$\sum_{x \in \mathbb{Z}} |\langle \mathcal{E}, e_x \rangle| < +\infty. \quad (4.12)$$

Let $\{\xi_y, y \in \mathbb{Z}\}$ be a sequence of i.i.d. complex Gaussian random variables such that $\mathbb{E}\xi_0 = 0$ and $\mathbb{E}|\xi_0|^2 = 1$. Define

$$\hat{\psi}(k) = \sum_{y \in \mathbb{Z}} \xi_y \mathcal{E}^{1/2}(k) e_y^*(k). \quad (4.13)$$

an Gaussian, random $H^{-m}(\mathbb{T})$ -valued element, where $m > 1/2$. Its covariance equals

$$\mathcal{C}(J_1, J_2) := \mathbb{E} \left[\langle J_1, \hat{\psi} \rangle \langle J_2, \hat{\psi} \rangle^* \right] = \int_{\mathbb{T}} \mathcal{E}(k) J_1(k) J_2^*(k) dk \quad (4.14)$$

for any $J_1, J_2 \in C^\infty(\mathbb{T})$. Then,

$$\psi_x := \int_{\mathbb{T}} e_x^*(k) \hat{\psi}(k) dk, \quad x \in \mathbb{Z},$$

is a complex Gaussian, stationary field whose covariance matrix equals

$$\langle \psi_x \rangle_{\mu_{\mathcal{E}_0}} = 0, \quad \langle \psi_x \psi_{x'} \rangle_{\mu_{\mathcal{E}_0}} = 0, \quad \langle \psi_x^* \psi_{x'} \rangle_{\mu_{\mathcal{E}_0}} = \int_{\mathbb{T}} e^{2\pi i(x-x')k} \mathcal{E}(k) dk.$$

Function $\mathcal{E}(k)$ is called a *spectral measure* of the field (ψ_x) . In the particular case when $\mathcal{E}(k) \equiv 2\mathcal{E}_0$ we denote by $\mu_{\mathcal{E}_0}$ the law of the respective field (ψ_x) . It is supported in h_{-m} , provided that $m > 1/2$. One can verify that (ψ_x) satisfies

$$\langle \psi_x \rangle_{\mu_{\mathcal{E}_0}} = 0, \quad \langle \psi_x \psi_{x'} \rangle_{\mu_{\mathcal{E}_0}} = 0, \quad \langle \psi_x^* \psi_{x'} \rangle_{\mu_{\mathcal{E}_0}} = 2\mathcal{E}_0 \delta_{x,x'} \quad (4.15)$$

for all $x, x' \in \mathbb{Z}$.

5. FINITE MACROSCOPIC ENERGY: INITIAL DATA IN L^2

We assume in this section that the family of initial distributions (μ_ϵ) satisfies conditions (4.5) and (4.6).

5.1. The wave function and its evolution. It is convenient to introduce the wave function that, adjusted to the macroscopic time, is given by

$$\psi_x^{(\epsilon)}(t) := (\tilde{\omega} * \mathbf{q}(\epsilon^{-\delta}t))_x + i\mathbf{p}_x(\epsilon^{-\delta}t), \quad (5.1)$$

where $(\mathbf{p}(t), \mathbf{q}(t))$ satisfies (2.5) and $\delta \in [0, 2]$ is to be chosen later. Function $(\tilde{\omega}_y)$, $y \in \mathbb{Z}$ is the inverse Fourier transform of the dispersion relation function $\omega(k) := \sqrt{\hat{\alpha}(k)}$. The Fourier transform of the wave function is given by

$$\hat{\psi}^{(\epsilon)}(t, k) = \omega(k) \hat{\mathbf{q}}\left(\frac{t}{\epsilon^\delta}, k\right) + i\hat{\mathbf{p}}\left(\frac{t}{\epsilon^\delta}, k\right). \quad (5.2)$$

Since (\mathbf{q}, \mathbf{p}) are real valued we have

$$(\hat{\psi}^{(\epsilon)})^*(t, -k) = \omega(k) \hat{\mathbf{q}}\left(\frac{t}{\epsilon^\delta}, k\right) - i\hat{\mathbf{p}}\left(\frac{t}{\epsilon^\delta}, k\right). \quad (5.3)$$

Recall that $R(k) = \hat{\beta}(k)/4$. From (2.1) we conclude that $(\hat{\psi}^{(\epsilon)}(t))$ is an $L^2(\mathbb{T})$ -valued, adapted process that is the (unique) solution of

the Itô stochastic differential equation

$$\begin{aligned}
d\hat{\psi}^{(\epsilon)}(t, k) &= \left\{ \frac{-i\omega(k)}{\epsilon^\delta} \hat{\psi}^{(\epsilon)}(t, k) - \frac{\gamma R(k)}{\epsilon^\delta} \left[\hat{\psi}^{(\epsilon)}(t, k) - (\hat{\psi}^{(\epsilon)})^*(t, -k) \right] \right\} dt \\
&+ \frac{i\gamma^{1/2}}{\epsilon^{\delta/2}} \int_{\mathbb{T}} r(k, k') \left[\hat{\psi}^{(\epsilon)}(t, k - k') - (\hat{\psi}^{(\epsilon)})^*(t, k' - k) \right] B(dt, dk'), \\
\hat{\psi}^{(\epsilon)}(0) &= \hat{\psi},
\end{aligned} \tag{5.4}$$

where $\hat{\psi} \in L^2(\mathbb{T})$ and

$$\begin{aligned}
r(k, k') &:= \mathfrak{s}(2k) + \mathfrak{s}(2(k - k')) + \mathfrak{s}(2(k' - 2k)) \\
&= 4\mathfrak{s}(k)\mathfrak{s}(k - k')\mathfrak{s}(2k - k') \\
&= 2\mathfrak{s}^2(k)\mathfrak{s}(2(k - k')) + 2\mathfrak{s}(2k)\mathfrak{s}^2(k - k'), \quad k, k' \in \mathbb{T}.
\end{aligned} \tag{5.5}$$

Process $B(dt, dk)$ is a cylindrical Wiener noise on $L^2(\mathbb{T})$ given by

$$B(dt, dk) = \sum_{x \in \mathbb{Z}} w_x(dt) e_x^*(k) dk,$$

where (w_x) are i.i.d. standard, 1-dimensional real Brownian motions, defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The solution of the stochastic equation (5.4) is understood in the mild sense, see the definition contained in Section 6.1 of [9]. Its existence and uniqueness follows from the results of *ibid.*, see in particular Theorem 7.4 of [9].

Random momentum exchange case. Analogous calculations, performed using (2.9) lead to the following evolution equation on the wave function

$$\begin{aligned}
d\hat{\psi}^{(\epsilon)}(t, k) &= \left\{ \frac{-i\omega(k)}{\epsilon^\delta} \hat{\psi}^{(\epsilon)}(t-, k) - \frac{\gamma R(k)}{\epsilon^\delta} \left[\hat{\psi}^{(\epsilon)}(t-, k) - (\hat{\psi}^{(\epsilon)})^*(t-, -k) \right] \right\} dt \\
&+ \frac{1}{2} \int_{\mathbb{T}} r(k, k') \left[\hat{\psi}^{(\epsilon)}(t-, k - k') - (\hat{\psi}^{(\epsilon)})^*(t-, k' - k) \right] \tilde{N}(dt, dk'),
\end{aligned} \tag{5.6}$$

where the initial data belongs to $L^2(\mathbb{T})$ and $\tilde{N}(dt, dk)$ is the cylindrical Levy noise on $L^2(\mathbb{T})$ given by

$$\tilde{N}(dt, dk) = \sum_{x \in \mathbb{Z}} \tilde{N}_{x, x+1}(dt) e_x^*(k) dk$$

and $(\tilde{N}_{x,x+1})$ are i.i.d., 1-dimensional compensated Poisson processes with intensities $3\gamma(2\epsilon^\delta)^{-1}$,

$$r(k, k') := (e^{-2\pi i k} - 1)(1 - e^{2\pi i(k-k')}), \quad k, k' \in \mathbb{T}, \quad (5.7)$$

$$R(k) = 3\mathfrak{s}^2(k), \quad k \in \mathbb{T}.$$

According to Theorem 9. 29 of [21] there exists a unique cadlag solution of (5.6) in $L^2(\mathbb{T})$.

5.2. Asymptotics of the Wigner transform. Assume that $\gamma = \gamma_0\epsilon^s$ for some $s \geq 0$. The noise in (2.5) is called weak (resp. strong) if $s > 0$ (resp. $s = 0$). Furthermore, suppose that the *-weak limit

$$* - \text{wlim}_{\epsilon \rightarrow 0^+} W_\epsilon^{(0)} = W_0, \quad (5.8)$$

exists in \mathcal{A}' . We suppose furthermore that $W_0(x, k)$ is a non-negative element of $L^1(\mathbb{R} \times \mathbb{T})$ and its Fourier transform

$$\widehat{W}_0(p, k) = \int_{\mathbb{T}} e^{-2\pi i p x} W_0(x, k) dx$$

satisfies

$$\|W_0\|_{\mathcal{A}'} = \sup_{p \in \mathbb{R}} \int_{\mathbb{T}} \widehat{W}_0(p, k) dk < +\infty. \quad (5.9)$$

Since the total energy of the system $\sum_{y \in \mathbb{Z}} |\psi_y(t)|^2$ is conserved in time, see Section 2 of [5], we have

$$\|\hat{\psi}^{(\epsilon)}(t)\|_{L^2(\mathbb{T})} = \|\hat{\psi}\|_{L^2(\mathbb{T})}, \quad t \geq 0, \quad \mathbb{P}_\epsilon \text{ a.s.} \quad (5.10)$$

Here $\mathbb{P}_\epsilon := \mu_\epsilon \otimes \mathbb{P}$. By \mathbb{E}_ϵ we denote the expectation with respect to \mathbb{P}_ϵ . Denote by $W_\epsilon(t)$ the Wigner transform of the law of $\psi^{(\epsilon)}(t)$ on ℓ^2 , i.e.

$$\langle W_\epsilon(t), J \rangle := \frac{\epsilon}{2} \int_{\mathbb{R} \times \mathbb{T}} \mathbb{E}_\epsilon \left[(\hat{\psi}^{(\epsilon)})^* \left(t, k - \frac{\epsilon p}{2} \right) \hat{\psi}^{(\epsilon)} \left(t, k + \frac{\epsilon p}{2} \right) \right] \hat{J}^*(p, k) dp dk. \quad (5.11)$$

From (4.6) and (5.10) we immediately conclude the following bound:

$$\|W_\epsilon(t)\|_{\mathcal{A}'} \leq K, \quad \forall \epsilon \in (0, 1], t \geq 0, \quad (5.12)$$

where K is the constant appearing in condition (4.6). As a direct consequence of the above estimate we infer that the family $(W_\epsilon(t))$, $\epsilon \in (0, 1]$ is *-weakly sequentially compact in any $L^\infty([0, T]; \mathcal{A}')$, where $T > 0$. From the results of [5], see Theorem 5 (and also calculations made in Section 7 below), it follows that for $s = \delta = 1$ we have

$$* - \text{wlim}_{\epsilon \rightarrow 0^+} W_\epsilon(t) = W(t). \quad (5.13)$$

The limit $W(t)$ belongs to $L^\infty([0, +\infty); L^1(\mathbb{R} \times \mathbb{T}))$ and is a unique weak solution of

$$\begin{aligned} & \int_0^t \int_{\mathbb{R} \times \mathbb{T}} \left(\partial_s + \frac{\omega'(k)}{2\pi} \partial_x + \gamma_0 \mathcal{L} \right) J(s, x, k) W(s, x, k) ds dx dk \\ &= \int_{\mathbb{R} \times \mathbb{T}} [J(t, x, k) W(t, x, k) - J(0, x, k) W_0(x, k)] dx dk, \end{aligned} \quad (5.14)$$

where $J(t, x, k)$ is a real valued function from $C_0^\infty([0, +\infty) \times \mathbb{R} \times \mathbb{T})$ and \mathcal{L} is the scattering operator, acting on the k -variable. It is defined by

$$\mathcal{L}w(k) := 2 \int_{\mathbb{T}} R(k, k') w(k') dk' - 2R(k)w(k), \quad w \in L^1(\mathbb{T}). \quad (5.15)$$

Here

$$R(k, k') := 16\mathfrak{s}^2(k)\mathfrak{s}^2(k')[\mathfrak{s}^2(k)\mathfrak{c}^2(k') + \mathfrak{c}^2(k)\mathfrak{s}^2(k')], \quad (5.16)$$

$$R(k) := \int_{\mathbb{T}} R(k, k') dk' = \frac{\hat{\beta}(k)}{4}, \quad k \in \mathbb{T}.$$

In what follows we assume that $s \in [0, 1)$. Our main result is that the exponent δ can be adjusted so that $\widehat{W}_\epsilon(t)$ is $*$ -weakly convergent. We consider separately the cases of pinned (when $\hat{\alpha}(0) > 0$) and unpinned chains (when $\hat{\alpha}(0) = 0$).

Remark. Similar calculations can be performed in the case of the random exchange model using (5.6). Then (5.13) remains in force with the limit $W(t, x, k)$ satisfying (5.14) with the scattering operator \mathcal{L} given by (5.15) with the kernel $R(k, k')$ and the total scattering cross-section $R(k)$ given by

$$R(k, k') = 6\mathfrak{s}^2(k)\mathfrak{s}^2(k'), \quad k, k' \in \mathbb{T} \quad (5.17)$$

and (5.7), respectively.

5.2.1. *Case of a pinning potential - diffusive transport of energy.* Suppose that

$$\hat{\alpha}(0) = \sum_x \alpha_x > 0. \quad (5.18)$$

From (2.7) we conclude that $R(k) \neq 0$ for $k \in \mathbb{T} \setminus \{0\}$ and

$$R(k) \approx \sin^2(\pi k), \quad k \in \mathbb{T}.$$

Since (5.18) together with the assumption $\hat{\alpha}''(0) > 0$ imply that

$$|\omega'(k)| \approx |\sin(\pi k)|, \quad k \in \mathbb{T}$$

we infer that in this case $\hat{\sigma}^2 < +\infty$, see (3.22). Let

$$\overline{W}_0(p) := \int_{\mathbb{T}} \widehat{W}_0(p, k) dk \quad (5.19)$$

and

$$W_0(x) := \int_{\mathbb{R}} e^{2\pi i p x} \overline{W}_0(p) dp. \quad (5.20)$$

Suppose that $\hat{c} > 0$. Let

$$W(t, x) := \int_{\mathbb{R}} e^{2\pi i p x} \widehat{W}(t, p) dp. \quad (5.21)$$

$$\widehat{W}(t, p) = \exp \left\{ -\frac{\hat{c} p^2 t}{2} \right\} \overline{W}_0(p) dp, \quad t \geq 0. \quad (5.22)$$

Theorem 5.1. *Assume that (5.8) and (5.18) hold. Then, for any $T > 0$ the Wigner transforms $W_\epsilon(t)$ converge in the *-weak sense in $L^\infty([0, T]; \mathcal{A}')$ to $W(t)$ given by (5.21) and (5.22). In addition, the following are true:*

i) (weak noise) if $1 > s > 0$ and $\delta = 2 - s$, then

$$\hat{c} = \frac{\hat{\sigma}^2}{\gamma_0}. \quad (5.23)$$

ii) (strong noise) if $s = 0$ and $\delta = 2$ then, the conclusion of part i) holds, provided

$$\hat{c} = \frac{\hat{\sigma}^2}{\gamma_0} + 8\gamma_0\pi^2. \quad (5.24)$$

5.2.2. *Case of a no pinning potential - 3/4 fractional superdiffusion.* Suppose that

$$\hat{\alpha}(0) = \sum_x \alpha_x = 0. \quad (5.25)$$

Recall that in this case the dispersion relation satisfies (2.4). Therefore,

$$\int_{\mathbb{T}} \frac{[\omega'(k)]^2 dk}{R(k)} = +\infty. \quad (5.26)$$

Define

$$\widehat{W}(t, p) = \exp \{ -\hat{c}|p|^{3/2}t \} \overline{W}_0(p), \quad t \geq 0, \quad (5.27)$$

with $\overline{W}_0(p)$ given by (5.19) and

$$\hat{c} := \frac{[\alpha''(0)]^{3/4}}{2^{9/4}(3\gamma_0)^{1/2}}. \quad (5.28)$$

Our result can be formulated as follows.

Theorem 5.2. *Assume that (5.8) and (5.25) hold. Then, the convergence assertion made in Theorem 5.1 still holds for any $s \in [0, 1)$ and $\delta = (3 - s)/2$. The limit $W(t)$ is given by (5.21) and (5.27).*

The proofs of the above two theorems are presented in Section 10.2.

Remark. The conclusions of both Theorems 5.1 and 5.2 hold also for the random momentum exchange model case, when the scattering kernel is given by (5.17). It will become clear, in the course of our proof of the aforementioned theorems, that our argument works (with some simplifications) in this case as well.

5.3. Energy modes. To ensure positivity of $\mathbf{e}_x(t)$ defined in (3.21) we assume that condition (2.3) holds. Note that

$$d\mathbf{e}_x(t) = \left[\sum_{z>0} \alpha(z) \nabla_z^* \mathfrak{J}_{x,z}^{(a,e)}(\mathbf{p}(t), \mathbf{q}(t)) + \gamma \nabla^* \mathfrak{J}_x^{(s,e)}(\mathbf{p}(t), \mathbf{q}(t)) \right] dt + d\mathcal{M}_x(t)$$

with

$$\begin{aligned} \mathfrak{J}_{x,z}^{(a,e)} &= \frac{1}{2} (\mathbf{p}_x + \mathbf{p}_{x+z}) \nabla_z \mathbf{q}_x, \\ \mathfrak{J}_x^{(s,e)} &= \frac{3}{2} (\mathbf{p}_x^2 - \mathbf{p}_{x+1}^2) + \frac{1}{2} (\mathbf{p}_{x-1}^2 - \mathbf{p}_{x+2}^2), \\ d\mathcal{M}_x(t) &:= \gamma^{1/2} \sum_{z=-1,0,1} \mathbf{p}_x(Y_{x+z} \mathbf{p}_x(t)) dw_{x+z}(t). \end{aligned}$$

A simple calculation, using the definition of the Wigner transform, see (4.10), shows that

$$\frac{\epsilon}{2} \sum_x J(\epsilon x) \mathbb{E}_\epsilon |\psi_x^{(\epsilon)}(t)|^2 = \langle W_\epsilon(t), J \rangle \quad (5.29)$$

for any $J \in C_0^\infty(\mathbb{R})$. Theorem 3.2 (resp. Theorem 3.3) is a consequence of (5.29), Theorem 5.2 (resp. Theorem 5.1) and the following result, proved in Section 13.1.

Proposition 5.3. *Assuming (4.5) and (4.6), we have*

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \sum_x J(\epsilon x) \mathbb{E}_\epsilon \left[\mathbf{e}_x \left(\frac{t}{\epsilon^\delta} \right) - \frac{1}{2} |\psi_x^{(\epsilon)}(t)|^2 \right] = 0 \quad (5.30)$$

for any $J \in C_0^\infty(\mathbb{R})$ and $t \geq 0$.

6. FLUCTUATIONS IN EQUILIBRIUM

In this section we assume that the system is in equilibrium, i.e. that (ψ_x) is a homogeneous, complex Gaussian random field whose covariance function is given by (4.15). As we have already mentioned, its law $\mu_{\mathcal{E}_0}$ is supported in h_{-m} for $m > 1/2$ and the Fourier transform $\hat{\psi}(k)$ belongs to $H^{-m}(\mathbb{T})$, $\mu_{\mathcal{E}_0}$ a.s.

Define $(\psi_x^{(\epsilon)}(t))$ as the field given by the Fourier coefficients of the solution $(\hat{\psi}^{(\epsilon)}(t, k))$ of the stochastic differential equation (5.4). It has been shown in [12], see Proposition 2.1, that there exists a unique solution of the equation, understood in the mild sense in $C([0, +\infty); H^{-m}(\mathbb{T}))$ for any $m > 1/2$, in case $\omega(0) > 0$ and for $m \in (1/2, 3/2)$, in the unpinned case. Furthermore, it can also be shown, see Section 5.1 of *ibid.*, that the law of $(\hat{\psi}(k))$ in $H^{-m}(\mathbb{T})$ is invariant in time under the Markov family determined by the equation.

6.1. Fluctuating Wigner distribution. For a given $J \in \mathcal{S}$ define the random Wigner transform as

$$\widetilde{\mathcal{W}}_\epsilon(\psi; J) := \frac{\sqrt{\epsilon}}{2} \sum_{x, x' \in \mathbb{Z}} (\psi_{x'}^* \psi_x - 2\delta_{x, x'} \mathcal{E}_0) \tilde{J}^* \left(\frac{\epsilon}{2}(x + x'), x' - x \right), \quad (6.1)$$

where $J \in \mathcal{S}$ and

$$\tilde{J}(y, x) = \int_{\mathbb{T}} e^{i2\pi x k} J^*(y, k) dk, \quad x \in \mathbb{Z}.$$

We will denote $\widetilde{\mathcal{W}}_\epsilon(t; J) := \widetilde{\mathcal{W}}_\epsilon(\psi^{(\epsilon)}(t); J)$. From the time stationarity of $\psi^{(\epsilon)}(t)$ and (4.15) we obtain

$$\mathbb{E} \widetilde{\mathcal{W}}_\epsilon(t; J) \equiv \mathbb{E} \widetilde{\mathcal{W}}_\epsilon(0; J) = 0.$$

Given $J_1, J_2 \in \mathcal{S}$ define also the covariance field

$$C_\epsilon(t; J_1, J_2) := \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(t; J_1) \widetilde{\mathcal{W}}_\epsilon(0; J_2)^* \right], \quad J_1, J_2 \in \mathcal{S}. \quad (6.2)$$

In the particular case when $t = 0$ we obtain

$$\begin{aligned} C_\epsilon(0; J_1, J_2) &= \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(0; J_1) \widetilde{\mathcal{W}}_\epsilon(0; J_2)^* \right] \\ &= \mathcal{E}_0^2 \epsilon \sum_{x, x'} \tilde{J}_1 \left(\frac{\epsilon(x + x')}{2}, x' - x \right) \tilde{J}_2^* \left(\frac{\epsilon(x + x')}{2}, x' - x \right) \\ &= \mathcal{E}_0^2 \epsilon \sum_x \int_{\mathbb{T}} \tilde{J}_1(\epsilon x, k) \tilde{J}_2^*(\epsilon x, k). \end{aligned} \quad (6.3)$$

The limit, as $\epsilon \rightarrow 0$, gives

$$\lim_{\epsilon \rightarrow 0^+} C_\epsilon(0; J_1, J_2) = \mathcal{E}_0^2 \int_{\mathbb{R} \times \mathbb{T}} J_1(y, k) J_2^*(y, k) dy dk, \quad (6.4)$$

for any $J_1, J_2 \in \mathcal{S}$. If the functions J_1, J_2 are constant in k , i.e. $J_i(x, k) = J_i(x)$, we have

$$C_\epsilon(0; J_1, J_2) = \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(0; J_1) \widetilde{\mathcal{W}}_\epsilon(0; J_2)^* \right] = \mathcal{E}_0^2 \epsilon \sum_x J_1(\epsilon x) J_2^*(\epsilon x)$$

and

$$\lim_{\epsilon \rightarrow 0^+} C_\epsilon(0; J_1, J_2) = \mathcal{E}_0^2 \int_{\mathbb{R}} J_1(y) \tilde{J}_2^*(y) dy. \quad (6.5)$$

6.2. Statements of the results.

6.2.1. *Case of a pinning potential.* Recall, see (5.19), that then

$$\bar{J}(p) := \int_{\mathbb{T}} \hat{J}(p, k) dk, \quad J \in \mathcal{S}.$$

Our result dealing with this situation can be formulated as follows.

Theorem 6.1. *Suppose that $\hat{\alpha}(0) > 0$ and $\delta = 2 - s$, $s \in [0, 1)$ in (5.4). Then,*

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0^+} \int_0^{+\infty} \phi(t) C_\epsilon(t; J_1, J_2) dt \\ &= \mathcal{E}_0^2 \int_0^{+\infty} \int_{\mathbb{R}} \exp \left\{ -\frac{\hat{c} p^2 t}{2} \right\} \phi(t) \bar{J}_1(p) \bar{J}_2(p) dt dp, \end{aligned} \quad (6.6)$$

for any $\phi \in L^1[0, +\infty)$, $J_1, J_2 \in \mathcal{S}$. Here

$$\hat{c} = \begin{cases} \frac{\hat{\sigma}^2}{\gamma_0}, & \text{if } 1 > s > 0, \\ \frac{\hat{\sigma}^2}{\gamma_0} + 8\gamma_0\pi^2, & \text{if } s = 0. \end{cases} \quad (6.7)$$

6.2.2. *Case of a no pinning potential.* Then, we have the following.

Theorem 6.2. *Assume that $\hat{\alpha}(0) = 0$ and $\delta = (3 - s)/2$, $s \in [0, 1)$. Then,*

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int_0^{+\infty} \phi(t) C_\epsilon(t; J_1, J_2) dt \\ = \mathcal{E}_0^2 \int_0^{+\infty} \int_{\mathbb{R}} \exp\{-\hat{c}|p|^{3/2}t\} \phi(t) \bar{J}_1(p) \bar{J}_2(p) dt dp, \end{aligned} \quad (6.8)$$

for any $\phi \in L^1[0, +\infty)$, $J_1, J_2 \in \mathcal{S}$. Here \hat{c} given by (5.23).

The proofs Theorems 6.1 and 6.2 are presented in Section 11.

6.2.3. *Energy fluctuations.* Applying the Wigner fluctuating field to a function $J(x)$ constant in k we obtain the fluctuation field

$$\widetilde{W}_\epsilon(t; J) = \sqrt{\epsilon} \sum_x \left(\frac{1}{2} |\psi_x^{(\epsilon)}(t)|^2 - \mathcal{E}_0 \right) J(\epsilon x). \quad (6.9)$$

Denote the empirical fluctuation of energy field by

$$\mathcal{E}_\epsilon(t; J) := \sqrt{\epsilon} \sum_x \left(\mathbf{e}_x \left(\frac{t}{\epsilon^\delta} \right) - \mathcal{E}_0 \right) J(\epsilon x), \quad J \in C_0^\infty(\mathbb{R}) \quad (6.10)$$

and the respective second mixed moment by

$$C_\epsilon^{(e)}(t; J_1, J_2) := \mathbb{E} [\mathcal{E}_\epsilon(t; J_1) \mathcal{E}_\epsilon(0; J_2)], \quad J_1, J_2 \in C_0^\infty(\mathbb{R}). \quad (6.11)$$

Our next result shows the fields defined by (6.9) and (6.10) are asymptotically equal.

Proposition 6.3. *For any $t \geq 0$ we have*

$$\lim_{\epsilon \rightarrow 0^+} \mathbb{E} \left[\mathcal{E}_\epsilon(t, J) - \widetilde{W}_\epsilon(t; J) \right]^2 = 0, \quad J \in C_0^\infty(\mathbb{R}). \quad (6.12)$$

We shall present the proof of the proposition in Section 13.2.

As a result the conclusions of Theorems 6.1 and 6.2 hold for $C_\epsilon^{(e)}(t; J_1, J_2)$ substituted in place of $C_\epsilon(t; J_1, J_2)$. Thus we obtain the assertions made in Theorem 3.4.

6.2.4. *Phonon modes.* Concerning the behavior of the phonon modes we consider their covariance matrix $\bar{C}(t, x) := [\bar{C}_{\iota, \iota'}(t, x)]$, given by

$$\bar{C}_{\iota, \iota'}(t, x) := \mathbb{E}[\mathfrak{f}_x^{(\iota')}(t)\mathfrak{f}_0^{(\iota)}(0)], \quad \iota, \iota' \in \{-, +\}.$$

Let $\hat{C}_{\iota, \iota'}(t, k)$ be its Fourier transform. Note that

$$\begin{aligned} \frac{d\hat{C}_{\iota, \iota'}}{dt}(t, k) &= \iota'(1 - e^{-2i\pi k})\sqrt{\tau_1}\hat{C}_{\iota, \iota'}(t, k) \\ &+ D_{\iota'}(1 - e^{2i\pi k})^2\hat{C}_{\iota, \iota'}(t, k) + O(\sin^3(\pi k)), \end{aligned} \quad (6.13)$$

where τ_1 is given in (3.4) and

$$D_{\pm} = \frac{3\gamma \pm \sqrt{\tau_1}}{2}. \quad (6.14)$$

Arguing as in the proof of Theorem 3.1, see Section 12, we obtain.

Theorem 6.4. *For any $J \in C_0^\infty(\mathbb{R})$ and $t \geq 0$ we have*

$$\lim_{\epsilon \rightarrow 0^+} \sqrt{\epsilon} \sum_x J(\epsilon x) \bar{C}_{\iota, \iota'}\left(\frac{t}{\epsilon}, x\right) = \frac{\mathcal{E}_0}{2}(1 + \iota')J(-\iota'\sqrt{\tau_1}t)$$

and

$$\lim_{\epsilon \rightarrow 0^+} \sqrt{\epsilon} \sum_x J\left(\epsilon x - \iota'\sqrt{\tau_1}\frac{t}{\epsilon}\right) \bar{C}_{\iota, \iota'}\left(\frac{t}{\epsilon^2}, x\right) = \frac{\mathcal{E}_0}{2}(1 + \iota') \int_{\mathbb{R}} P_t(x)J(x)dx,$$

$\iota, \iota' \in \{-, +\}$ and $P_t(x)$ is given by (3.11).

7. EVOLUTION OF THE WIGNER TRANSFORM

For a given $\epsilon > 0$ let $\hat{\psi}^{(\epsilon)}(t)$ be a solution of (5.4) with the initial condition $\hat{\psi}$ distributed according to a probability measure μ_ϵ on $L^2(\mathbb{T})$. The Fourier transform of the Wigner transform of $\hat{\psi}^{(\epsilon)}(t)$ is given by

$$\widehat{W}_\epsilon(t, p, k) := \frac{\epsilon}{2} \mathbb{E}_\epsilon \left[\left(\hat{\psi}^{(\epsilon)} \right)^* \left(t, k - \frac{\epsilon p}{2} \right) \hat{\psi}^{(\epsilon)} \left(t, k + \frac{\epsilon p}{2} \right) \right], \quad (7.1)$$

where, as we recall, \mathbb{E}_ϵ is the average with respect to the initial condition and the realization of the noise. In what follows we shall also need the functions

$$\begin{aligned} \widehat{Y}_\epsilon(t, p, k) &:= \frac{\epsilon}{2} \mathbb{E}_\epsilon \left[\hat{\psi}^{(\epsilon)} \left(t, -k + \frac{\epsilon p}{2} \right) \hat{\psi}^{(\epsilon)} \left(t, k + \frac{\epsilon p}{2} \right) \right], \\ \widehat{Y}_{\epsilon, -}(t, p, k) &:= \widehat{Y}_\epsilon^*(t, -p, k), \\ \widehat{W}_{\epsilon, -}(t, p, k) &:= \widehat{W}_\epsilon(t, p, -k). \end{aligned} \quad (7.2)$$

We shall also write $\widehat{W}_{\epsilon,+} = \widehat{W}_\epsilon$ and $\widehat{Y}_{\epsilon,+} = \widehat{Y}_\epsilon$. Define

$$R(k, k', p) := \frac{1}{2} \sum_{\sigma=\pm 1} r\left(k - \frac{p}{2}, k - \sigma k'\right) r\left(k + \frac{p}{2}, k - \sigma k'\right). \quad (7.3)$$

With the above definition we can write

$$\begin{aligned} \partial_t \widehat{W}_\epsilon(t, p, k) &= - \left[\frac{i}{\epsilon^{\delta-1}} \delta_\epsilon \omega(k; p) + \frac{2\gamma \bar{R}(k, \epsilon p)}{\epsilon^\delta} \right] \widehat{W}_\epsilon(t, p, k) \\ &+ \frac{\gamma}{\epsilon^\delta} \left\{ R\left(k - \frac{\epsilon p}{2}\right) \widehat{Y}_\epsilon(t, p, k) + R\left(k + \frac{\epsilon p}{2}\right) \widehat{Y}_{\epsilon,-}(t, p, k) \right\} \\ &+ \frac{4\gamma}{\epsilon^\delta} \int_{\mathbb{T}} r\left(k - \frac{\epsilon p}{2}, k'\right) r\left(k + \frac{\epsilon p}{2}, k'\right) \times \\ &\quad \times \mathbb{E} \left[(\hat{\mathbf{p}}^{(\epsilon)})^* \left(t, k - k' - \frac{\epsilon p}{2}\right) \hat{\mathbf{p}}^{(\epsilon)} \left(t, k - k' + \frac{\epsilon p}{2}\right) \right] dk', \end{aligned} \quad (7.4)$$

where $\delta_\epsilon \omega(k, p)$ and $\bar{R}(k, p)$ are defined according to (4.2) and

$$\hat{\mathbf{p}}^{(\epsilon)}(t, k) := \frac{1}{2i} \left[\hat{\psi}^{(\epsilon)}(t, k) - (\hat{\psi}^{(\epsilon)})^*(t, -k) \right]$$

is the Fourier transform of the momentum. Since the latter is real valued, its Fourier transform is complex even. The last term appearing on the right hand side of (7.4) can be replaced by

$$\begin{aligned} \frac{\gamma}{\epsilon^\delta} \int_{\mathbb{T}} R(k, k', \epsilon p) \mathbb{E}_\epsilon \left[(\hat{\psi}^{(\epsilon)})^* \left(t, k' - \frac{\epsilon p}{2}\right) \hat{\psi}^{(\epsilon)} \left(t, k' + \frac{\epsilon p}{2}\right) \right] dk' \\ = \frac{2\gamma}{\epsilon^\delta} (\mathcal{R}_{\epsilon p} \widehat{W}_\epsilon)(t, p, k) \end{aligned} \quad (7.5)$$

where

$$\mathcal{R}_p f(k) := \int_{\mathbb{T}} R(k, k', p) f(k') dk'. \quad (7.6)$$

Therefore,

$$\begin{aligned} \partial_t \widehat{W}_\epsilon(t, p, k) &= - \frac{i}{\epsilon^{\delta-1}} \delta_\epsilon \omega(k; p) \widehat{W}_\epsilon(t, p, k) + \frac{\gamma}{\epsilon^\delta} \mathcal{L}_{\epsilon p} \widehat{W}_\epsilon(t, p, k) \\ &- \frac{\gamma}{2\epsilon^\delta} \sum_{\sigma \in \{-, +\}} \mathcal{L}_{\sigma \epsilon p}^+ \widehat{Y}_{\epsilon, -\sigma}(t, p, k), \end{aligned} \quad (7.7)$$

where

$$\mathcal{L}_p f(k) := 2\mathcal{R}_p f(k) - 2\bar{R}(k, p) f(k), \quad (7.8)$$

$$\mathcal{L}_p^\pm f(k) := 2\mathcal{R}_p f(k) - 2R\left(k \pm \frac{p}{2}\right) f(k).$$

In addition,

$$\begin{aligned} \partial_t \widehat{Y}_\epsilon(t, p, k) &= -\frac{2i}{\epsilon^\delta} \bar{\omega}(k, \epsilon p) \widehat{Y}_\epsilon(t, p, k) - \frac{2\gamma}{\epsilon^\delta} \bar{R}(k, \epsilon p) \widehat{Y}_\epsilon(t, p, k) \\ &+ \frac{\gamma}{\epsilon^\delta} R\left(k - \frac{\epsilon p}{2}\right) \widehat{W}_\epsilon(t, p, k) + \frac{\gamma}{\epsilon^\delta} R\left(k + \frac{\epsilon p}{2}\right) \widehat{W}_{\epsilon,-}(t, p, k) + \frac{\gamma}{\epsilon^\delta} \mathcal{U}_\epsilon(t, p, k) \end{aligned} \quad (7.9)$$

where

$$\begin{aligned} \mathcal{U}_\epsilon(t, p, k) &:= 4 \int_{\mathbb{T}} r\left(-k + \frac{\epsilon p}{2}, k'\right) r\left(k + \frac{\epsilon p}{2}, -k'\right) \\ &\times \mathbb{E}_\epsilon \left\{ \left[\mathbf{p}^{(\epsilon)}\left(t, -k - k' + \frac{\epsilon p}{2}\right) \mathbf{p}^{(\epsilon)}\left(t, k + k' + \frac{\epsilon p}{2}\right) \right] \right\} dk. \end{aligned}$$

After straightforward calculations, similar to those performed in (7.4) – (7.7), we conclude that

$$\mathcal{U}_\epsilon(t, p, k) = \mathcal{R}_{\epsilon p} \left(\widehat{Y}_\epsilon + \widehat{Y}_{\epsilon,-} - \widehat{W}_\epsilon - \widehat{W}_{\epsilon,-} \right) (t, p, k).$$

Then,

$$\begin{aligned} \partial_t \widehat{Y}_\epsilon(t, p, k) &= -\frac{2i}{\epsilon^\delta} \bar{\omega}(k, \epsilon p) \widehat{Y}_\epsilon(t, p, k) + \frac{\gamma}{\epsilon^\delta} \mathcal{L}_{\epsilon p} \widehat{Y}_\epsilon(t, p, k) \\ &+ \frac{\gamma}{\epsilon^\delta} \mathcal{R}_{\epsilon p} (\widehat{Y}_{\epsilon,-} - \widehat{Y}_\epsilon)(t, p, k) - \frac{\gamma}{2\epsilon^\delta} \sum_{\sigma \in \{-, +\}} \mathcal{L}_{\sigma \epsilon p}^+ \widehat{W}_{\epsilon,-\sigma}(t, p, k). \end{aligned} \quad (7.10)$$

From (7.7) and (7.10) we conclude that for any fixed $p \in \mathbb{R}$ the evolution $(\widehat{W}_\epsilon(t, p, k), \widehat{Y}_\epsilon(t, p, k), \widehat{Y}_{\epsilon,-}(t, p, k), \widehat{W}_{\epsilon,-}(t, p, k))$ is governed by a linear equation with a generator that is a bounded operator in $(L^r(\mathbb{T}))^4$ for any $r \in [1, +\infty]$. In particular, under the assumption that the initial distribution of the wave functions satisfies (8.22) the components of $(\widehat{W}_\epsilon(t), \widehat{Y}_\epsilon(t), \widehat{Y}_{\epsilon,-}(t), \widehat{W}_{\epsilon,-}(t))$ belong to $C([0, +\infty); \mathcal{A}')$.

In the case of random momentum exchange the evolution of $\widehat{W}_\epsilon(t, p, k)$ and $\widehat{Y}_\epsilon(t, p, k)$ can be still described by (7.7) and (7.10) but then $R(k)$ is given by (5.7) and

$$R(k, k', p) := \frac{3}{2} \left(e^{2\pi i(k-p/2)} - 1 \right) \left(e^{-2\pi i(k+p/2)} - 1 \right) \mathfrak{s}^2(k').$$

8. AUXILIARY COMPUTATIONS

8.1. Some computations concerning the scattering kernel. Directly from (5.5) it follows that

$$r(-k, -k') = -r(k, k') \quad (8.1)$$

and

$$\begin{aligned} & r\left(k - \frac{p}{2}, k - k'\right) r\left(k + \frac{p}{2}, k - k'\right) \\ &= 16 \left[\mathfrak{s}^2(k) - \mathfrak{s}^2\left(\frac{p}{2}\right) \right] \mathfrak{s}^2(k') \left[\mathfrak{s}^2(k + k') - \mathfrak{s}^2\left(\frac{p}{2}\right) \right]. \end{aligned} \quad (8.2)$$

From (7.3) and (8.1) we have

$$R(\pm k, \pm k', \pm p) = R(k, k', p). \quad (8.3)$$

Equality (8.2) allows us to write the following expansion

$$R(k, k', p) = R(k, k') - \mathfrak{s}^2\left(\frac{p}{2}\right) R_1(k, k') + \mathfrak{s}^4\left(\frac{p}{2}\right) R_2(k, k'). \quad (8.4)$$

Here $R(k, k')$ is given by (5.16). It equals

$$R(k, k') = R(k, k', 0) = \frac{3}{4} \sum_{\iota \in \{-, +\}} \mathfrak{e}_\iota(k) \otimes \mathfrak{e}_{-\iota}(k') \quad (8.5)$$

and

$$R_1(k, k') = 4\mathfrak{f}_+(k) \otimes \mathfrak{f}_+(k') + \mathfrak{f}_+(k) \otimes \mathfrak{e}_-(k') + 3\mathfrak{f}_-(k) \otimes \mathfrak{e}_+(k'), \quad (8.6)$$

$$R_2(k, k') = 8\mathfrak{f}_+(k'),$$

with

$$\mathfrak{e}_+(k) := \frac{8}{3}\mathfrak{s}^4(k), \quad \mathfrak{e}_-(k) := 2\mathfrak{s}^2(2k) \quad (8.7)$$

and

$$\mathfrak{f}_+(k) := 2\mathfrak{s}^2(k), \quad \mathfrak{f}_-(k) := 2\mathfrak{c}^2(k).$$

Note that

$$\int_{\mathbb{T}} \mathfrak{e}_\iota(k) dk = 1, \quad \int_{\mathbb{T}} \mathfrak{f}_\iota(k) dk = 1, \quad \iota \in \{-, +\}. \quad (8.8)$$

As a result

$$\partial_p R(k, k', 0) = 0, \quad (8.9)$$

$$\partial_p^2 R(k, k', 0) = -\frac{\pi^2}{2} R_1(k, k').$$

From (8.5) and (8.8) we get

$$R(k) = \int_{\mathbb{T}} R(k, k') dk' = \frac{3}{4} \sum_{\iota \in \{-, +\}} \mathfrak{e}_\iota(k) = 2\mathfrak{s}^2(k)(1 + 2\mathfrak{c}^2(k)). \quad (8.10)$$

Using (2.7) we conclude

$$R'(k) = 2\pi(\mathfrak{s}(2k) + \mathfrak{s}(4k)) \quad (8.11)$$

and

$$R''(k) = 4\pi^2(4\mathfrak{c}^2(2k) + \mathfrak{c}(2k) - 2). \quad (8.12)$$

Let

$$R_\epsilon(k) := R(k) + \frac{(\epsilon p)^2}{8} R''(k). \quad (8.13)$$

Since $R''(0) = 12\pi^2 > 0$, see (8.12), we conclude that for a fixed $p \in \mathbb{R}$ there exist $\epsilon_0 > 0$ such that

$$R_\epsilon(k) \approx R(k) + \epsilon^2, \quad \forall k \in \mathbb{T}, \epsilon \in (0, \epsilon_0). \quad (8.14)$$

Lemma 8.1. *If $\omega(k) \approx |\sin(\pi k)|$ then, for a fixed $p \in \mathbb{R}$ there exists $\epsilon_0 > 0$ such that*

$$\bar{\omega}(k, \epsilon p) \approx R_\epsilon^{1/2}(k), \quad k \in \mathbb{T}, \epsilon \in (0, \epsilon_0). \quad (8.15)$$

If on the other hand $\omega(0) > 0$ we have

$$|\delta_\epsilon \omega(k, p)| \preceq R_\epsilon^{1/2}(k), \quad k \in \mathbb{T}, \epsilon \in (0, \epsilon_0). \quad (8.16)$$

Proof of (8.15). Using (2.4) we obtain

$$\begin{aligned} \bar{\omega}(k, \epsilon p) &\approx \left[\left| \sin \left(\pi \left(k - \frac{\epsilon p}{2} \right) \right) \right| + \left| \sin \left(\pi \left(k + \frac{\epsilon p}{2} \right) \right) \right| \right] \\ &\approx |\sin(\pi k)| + \epsilon, \quad \forall k \in \mathbb{T}, \epsilon \in (0, \epsilon_0). \end{aligned} \quad (8.17)$$

Estimate (8.15) follows from (2.7).

Proof of (8.16). Note that in case $\omega(0) > 0$ we have $\omega \in C^2(\mathbb{T})$. Since $\omega(k)$ is even we have $\omega'(0) = 0$, therefore

$$|\omega'(k)| \preceq |\sin(\pi k)|, \quad k \in \mathbb{T}. \quad (8.18)$$

Assume that $p \geq 0$. The case $p < 0$ can be handled in a similar fashion. We can write

$$\begin{aligned} \delta_\epsilon \omega(k, p) - \omega'(k)p &= \frac{1}{\epsilon} \int_{-\epsilon p/2}^{\epsilon p/2} [\omega'(k+h) - \omega'(k)] dh \\ &= \frac{1}{\epsilon} \int_0^{\epsilon p/2} [\omega'(k+h) - \omega'(k)] dh + \frac{1}{\epsilon} \int_0^{\epsilon p/2} [\omega'(k-h) - \omega'(k)] dh. \end{aligned} \quad (8.19)$$

The absolute value of the right hand side of (8.19) equals

$$\begin{aligned} & \frac{1}{\epsilon} \left| \int_0^{\epsilon p/2} dh \int_0^h [\omega''(k+h_1) - \omega''(k-h_1)] dh_1 \right| \\ & \leq \frac{p^2 \epsilon}{4} \sup_{|h_1| \leq \epsilon p/2} |\omega''(k+h_1) - \omega''(k-h_1)|. \end{aligned} \quad (8.20)$$

From (8.18)–(8.20) it follows that there exists $\epsilon_0 > 0$ such that

$$|\delta_\epsilon \omega(k, p)| \preceq |\sin(\pi k)| + \epsilon, \quad k \in \mathbb{T}, \epsilon \in (0, \epsilon_0). \quad (8.21)$$

Combining this with (2.7) and (8.10) we conclude (8.16). \square

8.2. Expansion of the dynamics. Since throughout this section we shall need square integrability of both \widehat{W}_ϵ and \widehat{Y}_ϵ in k we assume temporarily that the law μ_ϵ of the initial condition in (2.5) satisfies

$$K_1 := \limsup_{\epsilon \rightarrow 0^+} \frac{\epsilon}{2} \langle \|\widehat{f}\|_{L^4(\mathbb{T})}^2 \rangle_{\mu_\epsilon} < +\infty. \quad (8.22)$$

This assumption guarantees that

$$\sum_{\iota \in \{-, +\}} (\|\widehat{W}_{\epsilon, \iota}(0, p, \cdot)\|_{L^2(\mathbb{T})} + \|\widehat{Y}_{\epsilon, \iota}(0, p, \cdot)\|_{L^2(\mathbb{T})}) < +\infty. \quad (8.23)$$

Thanks to the existence and uniqueness result concerning the dynamics of $(\widehat{W}_\epsilon(t), \widehat{Y}_\epsilon(t), \widehat{Y}_{\epsilon, -}(t), \widehat{W}_{\epsilon, -}(t))$ in $(L^2(\mathbb{T}))^4$ we conclude that

$$\sup_{t \in [0, T]} \sum_{\iota \in \{-, +\}} (\|\widehat{W}_{\epsilon, \iota}(t, p, \cdot)\|_{L^2(\mathbb{T})} + \|\widehat{Y}_{\epsilon, \iota}(t, p, \cdot)\|_{L^2(\mathbb{T})}) < +\infty. \quad (8.24)$$

for any $T > 0$. Hypothesis (8.22) shall eventually be relinquished, see Theorem 9.2, using a density argument.

We expand the scattering kernel appearing on the right hand side of (7.7), (7.10) into the powers of ϵ , up to the second order. To abbreviate the notation we shall write

$$R_\epsilon := R(k) + \frac{(\epsilon p)^2}{8} R''(k), \quad \bar{\omega} := \bar{\omega}(k, \epsilon p), \quad \delta_\epsilon \omega := \delta_\epsilon \omega(k, p). \quad (8.25)$$

Then,

$$\begin{aligned} \partial_t \widehat{W}_\epsilon &= -\frac{i\delta_\epsilon \omega}{\epsilon^{\delta-1}} \widehat{W}_\epsilon + \left(\frac{\gamma}{\epsilon^\delta} \mathcal{L} + \frac{\gamma p^2}{2\epsilon^{\delta-2}} (\delta^2 \mathcal{L}) \right) \left[\widehat{W}_\epsilon - \frac{1}{2} (\widehat{Y}_\epsilon + \widehat{Y}_{\epsilon, -}) \right] \\ &+ \frac{\gamma R' p}{2\epsilon^{\delta-1}} (\widehat{Y}_{\epsilon, -} - \widehat{Y}_\epsilon) + \epsilon^{3-\delta} \mathfrak{R}_\epsilon^{(1)}, \end{aligned} \quad (8.26)$$

and

$$\begin{aligned} \partial_t \widehat{Y}_\epsilon &= -\frac{2i\bar{\omega}}{\epsilon^\delta} \widehat{Y}_\epsilon + \left(\frac{\gamma}{\epsilon^\delta} \mathcal{L} + \frac{\gamma p^2}{2\epsilon^{\delta-2}} (\delta^2 \mathcal{L}) \right) \left[\widehat{Y}_\epsilon - \frac{1}{2} (\widehat{W}_\epsilon + \widehat{W}_{\epsilon,-}) \right] \\ &+ \left(\frac{\gamma}{\epsilon^\delta} \mathcal{R} + \frac{\gamma p^2}{2\epsilon^{\delta-2}} (\delta^2 \mathcal{R}) \right) (\widehat{Y}_{\epsilon,-} - \widehat{Y}_\epsilon) + \frac{\gamma R' p}{2\epsilon^{\delta-1}} (\widehat{W}_{\epsilon,-} - \widehat{W}_\epsilon) + \epsilon^{3-\delta} \mathfrak{R}_\epsilon^{(2)}, \end{aligned} \quad (8.27)$$

where \mathcal{L} , $\mathcal{R}f$ are given by (7.8) and (7.6) and

$$\begin{aligned} (\delta^2 \mathcal{L})f(k) &:= 2 \int_{\mathbb{T}} \partial_p^2 R(k, k', 0) f(k') dk' - \frac{1}{2} R''(k) f(k) \\ &= -\pi^2 \int_{\mathbb{T}} R_1(k, k') f(k') dk' - \frac{1}{2} R''(k) f(k), \quad (8.28) \\ (\delta^2 \mathcal{R})f(k) &:= \int_{\mathbb{T}} \partial_p^2 R(k, k', 0) f(k') dk' = -\frac{\pi^2}{2} \int_{\mathbb{T}} R_1(k, k') f(k') dk'. \end{aligned}$$

Note that

$$(\delta^2 \mathcal{L})f_-(k) = ((\delta^2 \mathcal{L})f)(-k), \quad (8.29)$$

where $f_-(k) := f(-k)$.

In addition,

$$\mathfrak{R}_\epsilon^{(i)}(t, 0, k) \equiv 0, \quad i = 1, 2. \quad (8.30)$$

Adopting the convention $\widehat{W}_\epsilon = \widehat{W}_{\epsilon,+}$, $\widehat{Y}_\epsilon = \widehat{Y}_{\epsilon,+}$, we can write that for any $M > 0$

$$\|\mathfrak{R}_\epsilon^{(i)}(t, p, \cdot)\|_{L^2(\mathbb{T})} \preceq \sum_{\iota \in \{-, +\}} (\|\widehat{W}_{\epsilon, \iota}(t, p, \cdot)\|_{L^2(\mathbb{T})} + \|\widehat{Y}_{\epsilon, \iota}(t, p, \cdot)\|_{L^2(\mathbb{T})}) \quad (8.31)$$

for $i = 1, 2$ and all $t \geq 0$, $\epsilon \in (0, 1]$, $|p| \leq M$.

Define

$$\begin{aligned} \widehat{U}_{\epsilon,+}(t, p, k) &:= \frac{1}{2} (\widehat{Y}_\epsilon + \widehat{Y}_{\epsilon,-})(t, p, k), \quad (8.32) \\ \widehat{U}_{\epsilon,-}(t, p, k) &:= \frac{1}{2i} (\widehat{Y}_\epsilon - \widehat{Y}_{\epsilon,-})(t, p, k). \end{aligned}$$

From (8.26) and (8.27) we get

$$\begin{aligned}
\partial_t \widehat{W}_{\epsilon,+} &= -\frac{i\delta_\epsilon \omega}{\epsilon^{\delta-1}} \widehat{W}_{\epsilon,+} - \frac{i\gamma R' p}{\epsilon^{\delta-1}} \widehat{U}_{\epsilon,-} + \left(\frac{\gamma}{\epsilon^\delta} \mathcal{L} + \frac{\gamma p^2}{2\epsilon^{\delta-2}} (\delta^2 \mathcal{L}) \right) (\widehat{W}_{\epsilon,+} - U_{\epsilon,+}) + \epsilon^{3-\delta} \bar{\mathfrak{R}}_\epsilon^{(1)}, \\
\partial_t \widehat{U}_{\epsilon,+} &= \frac{2\bar{\omega}}{\epsilon^\delta} \widehat{U}_{\epsilon,-} + \left(\frac{\gamma}{\epsilon^\delta} \mathcal{L} + \frac{\gamma p^2}{2\epsilon^{\delta-2}} (\delta^2 \mathcal{L}) \right) \left[\widehat{U}_{\epsilon,+} - \frac{1}{2} (\widehat{W}_{\epsilon,+} + \widehat{W}_{\epsilon,-}) \right] + \epsilon^{3-\delta} \bar{\mathfrak{R}}_\epsilon^{(2)}, \\
\partial_t \widehat{U}_{\epsilon,-} &= -\frac{2\bar{\omega}}{\epsilon^\delta} \widehat{U}_{\epsilon,+} - \frac{2\gamma}{\epsilon^\delta} R_\epsilon \widehat{U}_{\epsilon,-} - \frac{i\gamma R' p}{2\epsilon^{\delta-1}} (\widehat{W}_{\epsilon,-} - \widehat{W}_{\epsilon,+}) + \epsilon^{3-\delta} \bar{\mathfrak{R}}_\epsilon^{(3)}, \\
\partial_t \widehat{W}_{\epsilon,-} &= \frac{i\delta_\epsilon \omega}{\epsilon^{\delta-1}} \widehat{W}_{\epsilon,-} + \frac{i\gamma R' p}{\epsilon^{\delta-1}} \widehat{U}_{\epsilon,-} + \left(\frac{\gamma}{\epsilon^\delta} \mathcal{L} + \frac{\gamma p^2}{2\epsilon^{\delta-2}} (\delta^2 \mathcal{L}) \right) (\widehat{W}_{\epsilon,-} - \widehat{U}_{\epsilon,+}) + \epsilon^{3-\delta} \bar{\mathfrak{R}}_\epsilon^{(4)},
\end{aligned} \tag{8.33}$$

From (8.31) we conclude that for any $M > 0$

$$\|\bar{\mathfrak{R}}_\epsilon^{(i)}(t, p, \cdot)\|_{L^2(\mathbb{T})} \preceq \sum_{\iota \in \{-, +\}} (\|\widehat{W}_{\epsilon,\iota}(t, p, \cdot)\|_{L^2(\mathbb{T})} + \|\widehat{U}_{\epsilon,\iota}(t, p, \cdot)\|_{L^2(\mathbb{T})}) \tag{8.34}$$

for $i = 1, 2, 3, 4$, $t \geq 0$ and $\epsilon \in (0, 1]$, $|p| \leq M$.

Let $\mathcal{D}(\phi) := \mathcal{D}(\phi, \phi)$, where

$$\begin{aligned}
\mathcal{D}(\phi, \psi) &:= \int_{\mathbb{T}} (-\mathcal{L}) \phi(k) \psi^*(k) dk \\
&= \int_{\mathbb{T}^2} R(k, k') [\phi(k) - \phi(k')] [\psi(k) - \psi(k')]^* dk dk',
\end{aligned}$$

and

$$\mathfrak{E}_\epsilon(t, p) := \sum_{\iota \in \{-, +\}} \left(\frac{1}{2} \|\widehat{W}_{\epsilon,\iota}(t, p)\|_{L^2(\mathbb{T})}^2 + \|\widehat{U}_{\epsilon,\iota}(t, p)\|_{L^2(\mathbb{T})}^2 \right)$$

for any $\phi, \psi \in L^2(\mathbb{T})$. Taking the scalar products of both sides of equations appearing in (8.33) against the respective $\widehat{W}_{\epsilon,\iota}$, $\widehat{U}_{\epsilon,\iota}$, $\iota \in \{-, +\}$

we obtain

$$\begin{aligned}
& \frac{1}{2}\mathfrak{E}_\epsilon(t, p) + \frac{\gamma}{\epsilon^\delta} \int_0^t \mathcal{D} \left((\widehat{W}_\epsilon - \widehat{U}_{\epsilon,+})(s, p) \right) ds \\
& + \frac{2\gamma}{\epsilon^\delta} \int_0^t ds \int_{\mathbb{T}} R(k) |\widehat{U}_{\epsilon,-}(s, p, k)|^2 dk \\
& + 2\epsilon^{1-\delta} \gamma p \int_0^t ds \int_{\mathbb{T}} R'(k) \text{Im} \left(\widehat{U}_{\epsilon,-}^* \widehat{W}_{\epsilon,+} \right) (s, p, k) dk \\
& = \frac{1}{2}\mathfrak{E}_\epsilon(0, p) + \epsilon^{2-\delta} \int_0^t \mathfrak{R}_\epsilon(s, p) ds,
\end{aligned} \tag{8.35}$$

where for any $p \in \mathbb{R}$ we have

$$\mathfrak{R}_\epsilon(t, p) \preceq \mathfrak{E}_\epsilon(t, p) \tag{8.36}$$

for $t \geq 0$ and $\epsilon \in (0, 1]$. Using Young's inequality and the fact that $(R'(k))^2 \preceq R(k)$ (see (8.11)) we conclude that there exists a constant $C > 0$ such that

$$\begin{aligned}
& \epsilon^{1-\delta} \gamma p \int_0^t ds \int_{\mathbb{T}} R'(k) \text{Im} \left(\widehat{U}_{\epsilon,-}^* \widehat{W}_{\epsilon,+} \right) (s, p, k) dk \\
& \geq -\frac{\gamma}{\epsilon^\delta} \int_0^t ds \int_{\mathbb{T}} R(k) |\widehat{U}_{\epsilon,-}(s, p, k)|^2 dk - C\epsilon^{2-\delta} \int_0^t \|\widehat{W}_{\epsilon,+}(s, p)\|_{L^2(\mathbb{T})}^2 ds,
\end{aligned} \tag{8.37}$$

for $t \geq 0$ and $\epsilon \in (0, 1]$. From the above, identity (8.35) and Gronwall's inequality we obtain the following.

Proposition 8.2. *For any $M > 0$ there exists $C_1 > 0$ such that*

$$\mathfrak{E}_\epsilon(t, p) \leq \mathfrak{E}_\epsilon(0, p) e^{C_1 \epsilon^{2-\delta} t}, \quad \forall \epsilon \in (0, 1], |p| \leq M, t \geq 0. \tag{8.38}$$

9. LAPLACE TRANSFORM OF SYSTEM (8.26)

Throughout most of this section, except Theorem 9.2, we assume that condition (8.22) holds. For any $\lambda > \lambda_0^{(\epsilon)} := C_1 \epsilon^{2-\delta}$ we let

$$\bar{w}_{\epsilon,t}(\lambda, p, k) := \int_0^{+\infty} e^{-\lambda t} \widehat{W}_{\epsilon,t}(t, p, k) dt \tag{9.1}$$

$$\bar{u}_{\epsilon,t}(\lambda, p, k) := \int_0^{+\infty} e^{-\lambda t} \widehat{U}_{\epsilon,t}(t, p, k) dt.$$

Thanks to Proposition 8.2 the above integrals are well defined as elements of $L^2(\mathbb{T})$. Moreover $\bar{r}_\epsilon^{(i)}(\lambda, p, k)$ - the Laplace transform of

$\bar{\mathfrak{R}}_\epsilon^{(i)}(t, p, k)$ satisfies: for any $\lambda > C_1 \epsilon^{2-\delta}$

$$\|\bar{r}_\epsilon^{(i)}(\lambda, p)\|_{L^2(\mathbb{T})} \preceq \sum_{\iota \in \{-, +\}} (\|\bar{w}_{\epsilon, \iota}(\lambda, p)\|_{L^2(\mathbb{T})} + \|\bar{u}_{\epsilon, \iota}(\lambda, p)\|_{L^2(\mathbb{T})}) \quad (9.2)$$

for $i = 1, 2, 3, 4$ and $\epsilon \in (0, 1]$. From Proposition 8.2 we conclude that for any $M > 0$, compact interval $I \subset (\lambda_0, +\infty)$

$$C_I := \sup_{\epsilon \in (0, 1]} \sup_{\lambda \in I, |p| \leq M} \left(\|\bar{w}_{\epsilon, \iota}(\lambda, p)\|_{L^2(\mathbb{T})} + \sum_{\iota \in \{-, +\}} \|\bar{u}_{\epsilon, \iota}(\lambda, p)\|_{L^2(\mathbb{T})} \right) < +\infty, \quad (9.3)$$

where

$$\lambda_0 := \begin{cases} 0, & \delta < 2 \\ C_1, & \delta = 2, \end{cases} \quad (9.4)$$

and C_1 is as in (8.38).

Taking the Laplace transform of both sides of equations of the system (8.33) we obtain

$$\lambda \bar{w}_{\epsilon, +} - \widehat{W}_{\epsilon, +}^{(0)} = -\frac{i\delta_\epsilon \omega}{\epsilon^{\delta-1}} \bar{w}_{\epsilon, +} - \frac{i\gamma R' p}{\epsilon^{\delta-1}} \bar{u}_{\epsilon, -} + \frac{\gamma}{\epsilon^\delta} L_\epsilon (\bar{w}_{\epsilon, +} - \bar{u}_{\epsilon, +}) + \epsilon^{3-\delta} \bar{r}_\epsilon^{(1)}, \quad (9.5)$$

$$\lambda \bar{u}_{\epsilon, +} - \widehat{U}_{\epsilon, +}^{(0)} = \frac{2\bar{\omega}}{\epsilon^\delta} \bar{u}_{\epsilon, -} + \frac{\gamma}{\epsilon^\delta} L_\epsilon \left[\bar{u}_{\epsilon, +} - \frac{1}{2} (\bar{w}_{\epsilon, +} + \bar{w}_{\epsilon, -}) \right] + \epsilon^{3-\delta} \bar{r}_\epsilon^{(2)},$$

$$\lambda \bar{u}_{\epsilon, -} - \widehat{U}_{\epsilon, -}^{(0)} = -\frac{2\bar{\omega}}{\epsilon^\delta} \bar{u}_{\epsilon, +} - \frac{2\gamma}{\epsilon^\delta} R_\epsilon \bar{u}_{\epsilon, -} - \frac{i\gamma R' p}{2\epsilon^{\delta-1}} (\bar{w}_{\epsilon, -} - \bar{w}_{\epsilon, +}) + \epsilon^{3-\delta} \bar{r}_\epsilon^{(3)}$$

$$\lambda \bar{w}_{\epsilon, -} - \widehat{W}_{\epsilon, -}^{(0)} = \frac{i\delta_\epsilon \omega}{\epsilon^{\delta-1}} \bar{w}_{\epsilon, -} + \frac{i\gamma R' p}{\epsilon^{\delta-1}} \bar{u}_{\epsilon, -} + \frac{\gamma}{\epsilon^\delta} L_\epsilon (\bar{w}_{\epsilon, -} - \bar{u}_{\epsilon, +}) + \epsilon^{3-\delta} \bar{r}_\epsilon^{(4)},$$

where

$$L_\epsilon := \mathcal{L} + \frac{(\epsilon p)^2}{2} (\delta^2 \mathcal{L})$$

and \mathcal{L} , $\delta^2 \mathcal{L}$ are given by (5.15) and (8.28), respectively. Taking the real parts of the scalar products in $L^2(\mathbb{T})$ of the respective equations of the above system with $(1/2)\bar{w}_{\epsilon, \pm}(\lambda, p, k)$, $\bar{u}_{\epsilon, \pm}(\lambda, p, k)$ and adding them

sideways we get

$$\begin{aligned}
& \lambda \left(\|\bar{w}_{\epsilon,+}(\lambda, p)\|_{L^2(\mathbb{T})}^2 + \sum_{\iota \in \{-, +\}} \|\bar{u}_{\epsilon,+}(\lambda, p)\|_{L^2(\mathbb{T})}^2 \right) \\
& + \frac{2\gamma}{\epsilon^\delta} \int_{\mathbb{T}} R_\epsilon(k) |\bar{u}_{\epsilon,-}(\lambda, p, k)|^2 dk + 2\epsilon^{1-\delta} \gamma p \int_{\mathbb{T}} R'(k) \text{Im}(\bar{u}_{\epsilon,-}^* \bar{w}_{\epsilon,+})(\lambda, p, k) dk \\
& + \frac{\gamma}{\epsilon^\delta} \mathcal{D}((\bar{w}_{\epsilon,+} - \bar{u}_{\epsilon,+})(\lambda, p)) = \text{Re} \int_{\mathbb{T}} \widehat{W}_{\epsilon,+}^{(0)}(p, k), \bar{w}_{\epsilon,+}(\lambda, p, k) dk \quad (9.6) \\
& + \sum_{\iota \in \{-, +\}} \text{Re} \int_{\mathbb{T}} \widehat{U}_{\epsilon,\iota}^{(0)}(p, k) \bar{u}_{\epsilon,\iota}(\lambda, p, k) dk + \bar{\mathfrak{R}}_\epsilon(\lambda, p)
\end{aligned}$$

and, given $M > 0$ and $I \subset (\lambda_0, +\infty)$ compact, we have

$$\bar{\mathfrak{R}}_\epsilon(\lambda, p) \preceq \|\bar{w}_{\epsilon,\iota}(\lambda, p)\|_{L^2(\mathbb{T})}^2 + \sum_{\iota \in \{-, +\}} \|\bar{u}_{\epsilon,\iota}(\lambda, p)\|_{L^2(\mathbb{T})}^2$$

for all $\epsilon \in (0, 1]$ and $\lambda \in I$, $|p| \leq M$. Using again Young's inequality, as in (8.37), together with (9.3) we conclude that for any $M > 0$ and a compact interval $I \subset (\lambda_0, +\infty)$

$$\sup_{\lambda \in I, |p| \leq M} \left[\int_{\mathbb{T}} R_\epsilon(k) |\bar{u}_{\epsilon,-}(\lambda, p, k)|^2 dk + \mathcal{D}((\bar{w}_\epsilon - \bar{u}_{\epsilon,+})(\lambda, p)) \right] \preceq \epsilon^{\delta-s},$$

for all $\epsilon \in (0, 1]$.

From the third equation of (9.5) we get

$$\bar{u}_{\epsilon,-} = \left(\lambda + \frac{2\gamma}{\epsilon^\delta} R_\epsilon \right)^{-1} \left\{ \widehat{U}_{\epsilon,-}(0) - \frac{2\bar{\omega}}{\epsilon^\delta} \bar{u}_{\epsilon,+} - \frac{i\gamma R' p}{2\epsilon^{\delta-1}} (\bar{w}_{\epsilon,-} - \bar{w}_\epsilon) + \epsilon^{3-\delta} \bar{r}_\epsilon^{(3)} \right\}.$$

In the unpinned case, i.e. $\omega(k) \approx |\sin(\pi k)|$, we obtain (due to (8.15)) from here that

$$\sup_{\lambda \in I, |p| \leq M} \int_{\mathbb{T}} |\bar{u}_{\epsilon,+}(\lambda, p, k)|^2 dk \preceq \epsilon^{\delta-s}. \quad (9.7)$$

for all $\epsilon \in (0, 1]$. On the other hand in the pinned case, i.e. $\omega(0) > 0$, we get

$$\sup_{\lambda \in I, |p| \leq M} \int_{\mathbb{T}} |\bar{u}_{\epsilon,+}(\lambda, p, k)|^2 \frac{dk}{R_\epsilon(k)} \preceq \epsilon^{\delta-s}.$$

Summarizing the above considerations we have shown the following.

Proposition 9.1. *For any $M > 0$ and a compact interval $I \subset (\lambda_0, +\infty)$*

$$\begin{aligned} \sup_{\lambda \in I, |p| \leq M} \left[\int_{\mathbb{T}} R_\epsilon(k) |\bar{u}_{\epsilon,-}(\lambda, p, k)|^2 dk \right. \\ \left. + \int_{\mathbb{T}} |\bar{u}_{\epsilon,+}(\lambda, p, k)|^2 \frac{\bar{\omega}(k, \epsilon p) dk}{R_\epsilon(k)} + \mathcal{D}(\bar{w}_\epsilon(\lambda, p)) \right] \preceq \epsilon^{\delta-s}, \end{aligned} \quad (9.8)$$

for all $\epsilon \in (0, 1]$.

Recall the definition (8.7) of the projectors $\mathbf{e}_\pm(k)$, and observe that are even function of k , then denote

$$w_\epsilon^{(\iota)}(\lambda, p) := \int w_{\epsilon,\pm}(\lambda, p, k) e_\iota(k) dk, \quad \iota \in \{-, +\}.$$

In the following result we have no need to assume (8.22).

Theorem 9.2. *Suppose that the initial laws (μ_ϵ) satisfy condition (4.6). Then, for any $M > 0$ and a compact interval $I \subset (\lambda_0, +\infty)$ we have*

$$\lim_{\epsilon \rightarrow 0^+} \sup_{\lambda \in I, |p| \leq M} \int_{\mathbb{T}} |\bar{w}_\epsilon(\lambda, p, k) - w_\epsilon^{(\pm)}(\lambda, p)| dk = 0, \quad (9.9)$$

and

$$\lim_{\epsilon \rightarrow 0^+} \sup_{\lambda \in I, |p| \leq M} \int_{\mathbb{T}} |\bar{u}_{\epsilon,\iota}(\lambda, p, k)| dk = 0, \quad \iota \in \{-, +\}. \quad (9.10)$$

Proof. For $\iota = -$, by (9.3), it is enough to prove that

$$\lim_{\epsilon \rightarrow 0^+} \sup_{\lambda \in I, |p| \leq M} \int_{\mathbb{T}} |\bar{w}_\epsilon(\lambda, p, k) - w_\epsilon^{(-)}(\lambda, p)| e_-(k) dk = 0. \quad (9.11)$$

We work first under the assumption that the stronger hypothesis (8.22) holds. Then

$$\begin{aligned} & \int_{\mathbb{T}} |\bar{w}_\epsilon(\lambda, p, k) - \bar{w}_\epsilon^{(-)}(\lambda, p)| \mathbf{e}_-(k) dk \\ & \leq \int_{\mathbb{T}^2} |\bar{w}_\epsilon(\lambda, p, k) - \bar{w}_\epsilon(\lambda, p, k')| \mathbf{e}_-(k) \mathbf{e}_-(k') dk dk' \\ & \leq \left\{ \int_{\mathbb{T}^2} R(k, k') |\bar{w}_\epsilon(\lambda, p, k) - w_\epsilon(\lambda, p, k')|^2 dk dk' \right\}^{1/2} \\ & \quad \times \left\{ \int_{\mathbb{T}^2} \frac{\mathbf{e}_-^2(k) \mathbf{e}_-^2(k')}{R(k, k')} dk dk' \right\}^{1/2} \preceq \epsilon^{\delta/2}. \end{aligned}$$

The last estimate follows from (9.8) and the fact that

$$\frac{\mathbf{e}_-^2(k) \mathbf{e}_-^2(k')}{R(k, k')} \preceq \frac{\mathbf{e}_-(k) \mathbf{e}_-(k')}{\mathbf{e}_-(k) + \mathbf{e}_-(k')} \preceq 1,$$

see (8.5). This together with (9.3) implies (9.11). Similar proof for $\iota = +$.

The proof of (9.10) is a consequence of (9.3) and (9.8).

In order to get rid of condition (8.22), observe that in the general case, when the laws (μ_ϵ) satisfy only (4.6), for any $\rho > 0$, we can find laws $(\tilde{\mu}_\epsilon)$, obtained by restricting μ_ϵ to the set $C_L := [\|\hat{\psi}\|_{L^\infty(\mathbb{T})} \leq L]$ for some sufficiently large L , so that (8.22) is satisfied and

$$\sup_{\epsilon \in (0,1]} \frac{\epsilon}{2} |\langle \|f\|_{\ell^2}^2 \rangle_{\mu_\epsilon} - \langle \|f\|_{\ell^2}^2 \rangle_{\tilde{\mu}_\epsilon}| < \rho.$$

Thanks to conservation of energy, see (5.10), we can conclude from the above that

$$\sup_{t \geq 0} \sup_{\epsilon \in (0,1]} (\|Y_\epsilon(t) - \tilde{Y}_\epsilon(t)\|_{\mathcal{A}'} + \|W_\epsilon(t) - \tilde{W}_\epsilon(t)\|_{\mathcal{A}'}) < \rho.$$

Here $W_\epsilon(t)$, $\tilde{W}_\epsilon(t)$ (resp. $Y_\epsilon(t)$, $\tilde{Y}_\epsilon(t)$) are the Wigner transforms (resp. anti-Wigner transforms) for (μ_ϵ) and $(\tilde{\mu}_\epsilon)$, respectively. Now using a standard density argument we can extend the conclusion of the theorem to the case the initial laws (μ_ϵ) satisfy only (4.6). \square

10. IDENTIFICATION OF THE LIMIT OF THE WIGNER TRANSFORM

Recall that $(W_\epsilon(t))$ is *-weakly sequentially compact in $L^\infty([0, T], \mathcal{A}')$ for any $T > 0$. Therefore for any $\epsilon_n \rightarrow 0$, as $n \rightarrow +\infty$, we can choose a subsequence, denoted in the same way, such that it *-weakly converges to some $W(\cdot) \in L^\infty([0, T], \mathcal{A}')$. In light of (5.12) we have

$$\|W(t)\|_{\mathcal{A}'} \leq K, \quad \forall t \geq 0, \quad (10.1)$$

with K the same as in (4.6). Therefore, we can define its Laplace-Fourier transform $w(\lambda, p, k)$ for any $\lambda > 0$. As a consequence of theorem 9.2, any limit $w(\lambda, p, k)$ obtained this way will be constant in k .

In the present section we show that for any $\lambda > \lambda_0$, see (9.4), and $p \in \mathbb{R}$ we have either

$$\left(\lambda + \frac{\hat{c}p^2}{2}\right) w(\lambda, p) = \int W_0(p, k) dk, \quad (10.2)$$

or

$$(\lambda + \hat{c}|p|^{3/2}) w(\lambda, p) = \int W_0(p, k) dk \quad (10.3)$$

depending on whether the potential is pinning or not. Coefficients \hat{c} are as in Theorems 5.1 and 5.2, respectively. Since the functions given by either (5.22) or (5.27) have the Laplace transforms that satisfy (10.2) and (10.3), respectively, we conclude from the uniqueness property of

the Laplace-Fourier transform that $(W_\epsilon(t))$ is indeed $*$ -weakly convergent, as $\epsilon \rightarrow 0+$, in $L^\infty([0, T], \mathcal{A}')$, for any $T > 0$. This would end the proof of the assertions made in Theorems 5.1 and 5.2.

10.1. Derivation of (10.2) and (10.3). Recall the definition

$$w_\epsilon^{(\iota)}(\lambda, p) := \int_{\mathbb{T}} \bar{w}_\epsilon(\lambda, p, k) \mathbf{e}_\iota(k) dk, \quad \iota \in \{-, +\}$$

In light of Theorem 9.2 it suffices only to show that any $*$ -weak limit of $w_\epsilon^{(+)}(\lambda, p)$ in $L^\infty(I \times \bar{B}_M)$, where $B_M := [p : |p| < M]$, satisfies either

$$\left(\lambda + \frac{\hat{c}p^2}{2} \right) w^{(+)}(\lambda, p) = \int_{\mathbb{T}} W_0(p, k) dk, \quad (10.4)$$

or

$$(\lambda + \hat{c}|p|^{3/2}) w^{(+)}(\lambda, p) = \int_{\mathbb{T}} W_0(p, k) dk, \quad (10.5)$$

depending on whether the potential is pinning or not.

From the first equation of the system (9.5) we get

$$D^{(\epsilon)} \bar{w}_\epsilon = \epsilon^\delta \widehat{W}_\epsilon^{(0)} + \frac{3}{2} \gamma \sum_{\iota \in \{-, +\}} \mathbf{e}_\iota w_\epsilon^{(-\iota)} + q_\epsilon, \quad (10.6)$$

where $q_\epsilon := \sum_{i=1}^4 q_\epsilon^{(i)}$,

$$D^{(\epsilon)}(p, k) := \epsilon^\delta \lambda + 2\gamma R_\epsilon(p, k) + i\epsilon \delta_\epsilon \omega(k), \quad (10.7)$$

and

$$\begin{aligned} q_\epsilon^{(1)} &:= -\frac{\gamma(\pi\epsilon p)^2}{2} \left[\mathbf{f}_+(k) \int_{\mathbb{T}} \bar{w}_\epsilon(\lambda, p, k') (4\mathbf{f}_+(k') + \mathbf{e}_-(k')) dk' \right. \\ &\quad \left. + 3\mathbf{f}_-(k) \int_{\mathbb{T}} \bar{w}_\epsilon(\lambda, p, k') \mathbf{e}_+(k') dk' \right] \\ q_\epsilon^{(2)} &:= -\gamma \mathcal{L} \bar{u}_{\epsilon,+}, \\ q_\epsilon^{(3)} &:= -i\epsilon \gamma R' p \bar{u}_{\epsilon,-}, \\ q_\epsilon^{(4)} &:= \frac{\gamma(\pi\epsilon p)^2}{2} (\delta^2 \mathcal{L}) \bar{u}_{\epsilon,+} + \epsilon^3 \bar{r}_\epsilon^{(1)}. \end{aligned} \quad (10.8)$$

In addition, $\bar{r}_\epsilon^{(1)}(\lambda, p, k)$ – the Laplace transforms of $\mathfrak{R}_\epsilon^{(1)}(t, p, k)$ appearing in (8.26) – satisfies the following estimate: for any compact set $I \subset (\lambda_0, +\infty)$ and $M > 0$

$$\limsup_{\epsilon \rightarrow 0+} \sup_{\lambda \in I, |p| \leq M} \|r_\epsilon^{(i)}(\lambda, p)\|_{L^1(\mathbb{T})} < +\infty, \quad i = 1, 2. \quad (10.9)$$

Computing \bar{w}_ϵ from (10.6) and then multiplying scalarly both sides of the resulting equation by $\gamma \mathbf{e}_\iota$, $\iota \in \{-, +\}$ we get the following system of equations

$$\begin{aligned} & \gamma w_\epsilon^{(\iota)} \int_{\mathbb{T}} \left(1 - \frac{3\gamma \mathbf{e}_- \mathbf{e}_+}{2D^{(\epsilon)}} \right) dk - \frac{3\gamma^2}{2} w_\epsilon^{(-\iota)} \int_{\mathbb{T}} \frac{\mathbf{e}_\iota^2}{D^{(\epsilon)}} dk \\ &= \gamma \epsilon^\delta \int_{\mathbb{T}} \frac{\mathbf{e}_\iota \widehat{W}_\epsilon^{(0)}}{D^{(\epsilon)}} dk + \gamma \int_{\mathbb{T}} \frac{\mathbf{e}_\iota q_\epsilon}{D^{(\epsilon)}} dk, \quad \iota \in \{-, +\}. \end{aligned}$$

Adding sideways the above equations corresponding to both values of ι and then dividing both sides of the resulting equation by ϵ^δ we obtain

$$\begin{aligned} a_w^{(\epsilon)} w_\epsilon^{(+)} - a_+^{(\epsilon)} (w_\epsilon^{(+)} - w_\epsilon^{(-)}) &= \frac{4\gamma}{3} \int_{\mathbb{T}} \frac{R(k) \widehat{W}_\epsilon^{(0)}(p, k)}{D^{(\epsilon)}(\lambda, p, k)} dk \\ &+ \frac{4\gamma}{3\epsilon^\delta} \int_{\mathbb{T}} \frac{R(k) q_\epsilon}{D^{(\epsilon)}(\lambda, p, k)} dk, \end{aligned} \quad (10.10)$$

where

$$\begin{aligned} a_w^{(\epsilon)}(\lambda, p) &:= \frac{4\gamma}{3\epsilon^\delta} \int_{\mathbb{T}} \left(1 - \frac{2\gamma R(k)}{D^{(\epsilon)}(\lambda, p, k)} \right) R(k) dk \\ a_+^{(\epsilon)}(\lambda, p) &:= \frac{\gamma}{\epsilon^\delta} \int_{\mathbb{T}} \left(1 - \frac{2\gamma R(k)}{D^{(\epsilon)}(\lambda, p, k)} \right) \mathbf{e}_+(k) dk. \end{aligned} \quad (10.11)$$

Let $\theta(\delta) = 0$, when $\delta < 2$ and $\theta(2) = 1$. In what follows we prove the following result, that obviously implies either (10.2) or (10.3), under an appropriate hypothesis on the respective dispersion relation.

Proposition 10.1. *For any $M > 0$ and a compact interval $I \subset (\lambda_0, +\infty)$ we have*

$$\lim_{\epsilon \rightarrow 0^+} \sup_{\lambda \in I, |p| \leq M} \left| \frac{4\gamma}{3} \int_{\mathbb{T}} \frac{R(k) \widehat{W}_\epsilon^{(0)}(p, k)}{D^{(\epsilon)}(\lambda, p, k)} dk - \frac{2}{3} \int_{\mathbb{T}} W_0(p, k) dk \right| = 0, \quad (10.12)$$

and

$$\lim_{\epsilon \rightarrow 0^+} \sup_{\lambda \in I, |p| \leq M} \left| a_+^{(\epsilon)}(\lambda, p) (w_\epsilon^{(+)}(\lambda, p) - w_\epsilon^{(-)}(\lambda, p)) \right| = 0. \quad (10.13)$$

Moreover, under the assumptions of Theorem 5.1, we have

$$\lim_{\epsilon \rightarrow 0^+} \sup_{\lambda \in I, |p| \leq M} \left| a_w^{(\epsilon)}(\lambda, p) - \frac{2\lambda}{3} - \frac{\hat{\sigma}^2 p^2}{3\gamma_0} \right| = 0, \quad (10.14)$$

where $\hat{\sigma}$ is given by (3.22), and

$$\lim_{\epsilon \rightarrow 0^+} \sup_{\lambda \in I, |p| \leq M} \left| \frac{4\gamma}{3\epsilon^\delta} \int_{\mathbb{T}} \frac{R q_\epsilon}{D^{(\epsilon)}} dk - \frac{8\theta(\delta)\gamma_0(\pi p)^2}{3} \bar{w}_\epsilon^{(+)} \right| = 0. \quad (10.15)$$

If, on the other hand, the assumptions of Theorem 5.2 hold then

$$\lim_{\epsilon \rightarrow 0^+} \sup_{\lambda \in I, |p| \leq M} \left| a_w^{(\epsilon)}(\lambda, p) - \frac{2\lambda}{3} - \frac{2\hat{c}|p|^{3/2}}{3} \right| = 0, \quad (10.16)$$

with \hat{c} is given by (5.23), and

$$\lim_{\epsilon \rightarrow 0^+} \sup_{\lambda \in I, |p| \leq M} \left| \frac{4\gamma}{3\epsilon^\delta} \int_{\mathbb{T}} \frac{Rq_\epsilon}{D^{(\epsilon)}} dk \right| = 0. \quad (10.17)$$

Equalities (10.2) and (10.3) then follow directly from the above proposition by taking the limit, as $\epsilon \rightarrow 0^+$, in (10.10).

10.2. Proof of Proposition 10.1.

Proof of (10.13). It is a simple consequence of the following.

Lemma 10.2. *Under the assumptions of Proposition 10.1 we have*

$$\sup_{\lambda \in I, |p| \leq M} |a_+^{(\epsilon)}(\lambda, p)| \preceq 1, \quad \epsilon \in (0, 1].$$

Proof. Since $k \mapsto \delta_\epsilon \omega(k; p)$ is an odd function for any fixed p and $\epsilon \in (0, 1]$, after a straightforward calculation, we conclude that

$$a_+^{(\epsilon)} = \gamma \int_{\mathbb{T}} \frac{\epsilon^\delta \lambda + 2\gamma R_\epsilon}{|D^{(\epsilon)}|^2} \left(\lambda + \gamma \epsilon^{2-\delta} \frac{p^2 R''}{4} \right) \mathbf{e}_+ dk.$$

The lemma then follows from the fact that $\gamma \mathbf{e}_+ \preceq \gamma R$ so the integrand in the expression above is bounded for $\epsilon \in (0, 1]$. \square

Proof of (10.12). Note that $2\gamma R/|D^{(\epsilon)}|$ is bounded and convergent, as $\epsilon \rightarrow 0^+$, to a function identically equal to 1. Using (5.8) we conclude (10.12).

Proof of (10.14) and (10.16). From (10.11) we get

$$a_w^{(\epsilon)} = \frac{4\gamma}{3} \int_{\mathbb{T}} \frac{R}{|D^{(\epsilon)}|^2} \left\{ (\epsilon^\delta \lambda + 2\gamma R_\epsilon) \left(\lambda + \frac{\gamma \epsilon^{2-\delta}}{4} p^2 R'' \right) + \epsilon^{2-\delta} (\delta_\epsilon \omega)^2 \right\} dk. \quad (10.18)$$

Diffusive scaling. Here $\omega(0) > 0$, $(\delta_\epsilon \omega)^2 \approx R_\epsilon$ (see (8.15)) and, as we recall $\delta = 2 - s$. Thus (10.14) follows.

Super-diffusive scaling. In this case $\omega(k) \approx |\sin(\pi k)|$ and $\delta = (3-s)/2$. From (10.18) we get

$$\lim_{\epsilon \rightarrow 0+} a_w^{(\epsilon)}(\lambda, p) = \frac{2\lambda}{3} + \lim_{\epsilon \rightarrow 0+} \frac{4\gamma\epsilon^{2-\delta}}{3} \int_{\mathbb{T}} \frac{R(\delta_\epsilon \omega)^2}{|D(\epsilon)|^2} dk. \quad (10.19)$$

We shall show that

$$\lim_{\epsilon \rightarrow 0+} \frac{4\gamma\epsilon^{2-\delta}}{3} \int_{\mathbb{T}} \frac{R(\delta_\epsilon \omega)^2}{|D(\epsilon)|^2} dk = \frac{2\hat{c}|p|^{3/2}}{3}, \quad (10.20)$$

uniformly in $\lambda \in I$ and $|p| \leq M$. Here \hat{c} is given by (5.23).

Assume that $p > 0$, as the consideration in the case $p < 0$ is analogous. Divide the domain of integration in the integral appearing in (10.20) into three sets $|k| \leq \epsilon^{\rho_1}$, $\epsilon^{\rho_1} \leq |k| \leq \epsilon^{\rho_2}$ and $\epsilon^{\rho_2} \leq |k|$ with $\rho_1 > \rho_2 > 0$ to be adjusted later on, and denote the expressions corresponding to the resulting integrals by $I_1^{(\epsilon)}$, $I_2^{(\epsilon)}$ and $I_3^{(\epsilon)}$.

The limit of $I_1^{(\epsilon)}$. Suppose that $\rho_1 \in (\delta - 1, 1)$. According to our assumptions in this case $\delta \leq 3/2$, so it is possible to find such ρ_1 . Using the fact that

$$|D(\epsilon)|^2 \succeq \epsilon |\delta_\epsilon \omega| (\gamma R_\epsilon + \epsilon^\delta),$$

therefore,

$$I_1^{(\epsilon)} \preceq \epsilon^{1-\delta} \int_0^{\epsilon^\rho} \frac{\gamma k^2 dk}{\gamma k^2 + \epsilon^\delta} \preceq \epsilon^{1+\rho_1-\delta} \rightarrow 0,$$

as $\epsilon \rightarrow 0+$.

The limit of $I_2^{(\epsilon)}$. Suppose also that $\rho_2 \in (0, 2 - \delta - s)$. We have

$$\rho_2 < 2 - \delta - s = \delta - 1 < \rho_1.$$

In integral appearing in $I_2^{(\epsilon)}$ we change variable according to

$$k' := \frac{\gamma_0^{1/2} k}{C_*^{1/2} \epsilon^{(1-s)/2}} \quad \text{and} \quad C_* := |\omega'(0)|p = \left(\frac{\hat{\alpha}''(0)}{2} \right)^{1/2} p.$$

Define

$$\begin{aligned} \tilde{R}(k) &:= \frac{\gamma_0}{\epsilon^{1-s} C_*} R \left(\frac{C_*^{1/2} \epsilon^{(1-s)/2} k}{\gamma_0^{1/2}} \right), \\ \tilde{R}_\epsilon(k) &:= \frac{\gamma_0}{\epsilon^{1-s} C_*} R_\epsilon \left(\frac{C_*^{1/2} \epsilon^{(1-s)/2} k}{\gamma_0^{1/2}} \right), \end{aligned}$$

and

$$\widehat{\delta_\epsilon \omega}(k, p) := \frac{1}{C_*} \delta_\epsilon \omega \left(\frac{C_*^{1/2} \epsilon^{(1-s)/2} k}{\gamma_0^{1/2}}, p \right).$$

Then,

$$I_2^{(\epsilon)} = \frac{(\hat{\alpha}''(0))^{3/4} p^{3/2}}{2^{3/4} \cdot 3\gamma_0^{1/2}} \int_{\mathcal{I}_\epsilon} \frac{\tilde{R}(\widehat{\delta_\epsilon \omega})^2 dk}{|\hat{D}^{(\epsilon)}|^2}, \quad (10.21)$$

where

$$|\hat{D}^{(\epsilon)}|^2 := \left(2\tilde{R}_\epsilon(k) + \frac{\epsilon^{\delta-1}\lambda}{C_*} \right)^2 + (\widehat{\delta_\epsilon \omega})^2$$

and

$$\mathcal{I}_\epsilon := \left[k : \frac{\gamma_0^{1/2} \epsilon^{\bar{\rho}_1}}{C_*^{1/2}} \leq |k| \leq \frac{\gamma_0^{1/2} \epsilon^{\bar{\rho}_2}}{C_*^{1/2}} \right],$$

with $\bar{\rho}_i := \rho_i - (1-s)/2$, $i = 1, 2$. Note that

$$\bar{\rho}_1 > \delta - 1 - \frac{1-s}{2} = 0.$$

and

$$\bar{\rho}_2 < 2 - \delta - s - \frac{1-s}{2} = 0.$$

From (8.10) we conclude that both $\tilde{R}(k)$ and $\tilde{R}_\epsilon(k)$ converge uniformly to $6\pi^2 k^2$ when $k \in \mathcal{I}_\epsilon$, $|p| \leq M$. Likewise, $\widehat{\delta_\epsilon \omega}(k, p)$ converges uniformly to 1 when $k \in \mathcal{I}_\epsilon$ and $\lambda \in I$, $|p| \leq M$. Since in addition

$$\frac{\tilde{R}(k)(\widehat{\delta_\epsilon \omega}(k))^2 1_{\mathcal{I}_\epsilon}(k)}{|\hat{D}^{(\epsilon)}(k)|^2} \preceq \frac{k^2 + 1}{k^4 + 1}, \quad k \in \mathbb{R}$$

we obtain

$$\lim_{\epsilon \rightarrow 0^+} I_2^{(\epsilon)} = \frac{\hat{\alpha}''(0)^{3/4} p^{3/2}}{3 \cdot 2^{3/4} \gamma_0^{1/2}} \int_{\mathbb{R}} \frac{24\pi^2 k^2 dk}{144\pi^4 k^4 + 1} \quad (10.22)$$

and the convergence is uniform in $\lambda \in I$ and $|p| \leq M$. Using the calculus of residua one can show that

$$\int_{-\infty}^{+\infty} \frac{k^2 dk}{k^4 + 1} = \frac{\pi}{\sqrt{2}},$$

therefore

$$\int_{\mathbb{R}} \frac{24\pi^2 k^2 dk}{144\pi^4 k^4 + 1} = \frac{1}{\sqrt{6}}.$$

Thus,

$$\lim_{\epsilon \rightarrow 0^+} I_2^{(\epsilon)} = \frac{(\hat{\alpha}''(0))^{3/4} p^{3/2}}{3^{3/2} \cdot 2^{5/4} \gamma_0^{1/2}}.$$

The limit of $I_3^{(\epsilon)}$. Then,

$$I_3^{(\epsilon)} \preceq \gamma \epsilon^{2-\delta} \int_{|k| \geq \epsilon^{\rho_2}} \frac{R(\delta_\epsilon \omega)^2}{|D^{(\epsilon)}|^2} dk \preceq \gamma \epsilon^{2-\delta} \int_{|k| \geq \epsilon^{\rho_2}} \frac{dk}{(\gamma k)^2} \preceq \epsilon^{2-\delta-s-\rho_2} \rightarrow 0,$$

as $\epsilon \rightarrow 0+$, uniformly in $\lambda \in I$ and $|p| \leq M$ (recall that $\rho_2 \in (0, 2 - \delta - s)$). It ends the proof of (10.20), thus finishing the proof of (10.16).

Proof of (10.15) and (10.17). Denote

$$Q_\epsilon^{(i)}(\lambda, p) := \frac{4\gamma}{3\epsilon^\delta} \int_{\mathbb{T}} \frac{Rq_\epsilon^{(i)}}{D^{(\epsilon)}} dk.$$

The results in question follow easily from our next result.

Lemma 10.3. *Under the assumptions of Proposition 10.1 we have*

$$\lim_{\epsilon \rightarrow 0+} \sup_{\lambda \in I, |p| \leq M} |Q_\epsilon^{(i)}(\lambda, p)| = 0, \quad i = 2, 3, 4. \quad (10.23)$$

Equality (10.23) holds also for $Q_\epsilon^{(1)}(\lambda, p)$ when $\delta < 2$. When $\omega(0) > 0$ and $\delta = 2$ we obtain

$$\lim_{\epsilon \rightarrow 0+} \sup_{\lambda \in I, |p| \leq M} \left| Q_\epsilon^{(1)}(\lambda, p) - \frac{8\gamma_0(\pi p)^2}{3} w_\epsilon^{(+)}(\lambda, p) \right| = 0 \quad (10.24)$$

Proof. Since $|\gamma R|/|D^{(\epsilon)}|$ is bounded the conclusion of the lemma for $i = 4$ is a simple consequence of (9.10). When $i = 3$, both $k \mapsto \delta_\epsilon \omega(k; p)$ and $k \mapsto \hat{R}'(k)$ are odd. Since $k \mapsto \bar{u}_{\epsilon,-}(\lambda, p, k)$ is even, we can write

$$Q_\epsilon^{(3)}(\lambda, p) = -\frac{4\epsilon^{2-\delta}\gamma^2}{3} \int_{\mathbb{T}} \frac{R(k)R'(k)p \delta_\epsilon \omega(p, k) \bar{u}_{\epsilon,-}(p, k)}{|D^{(\epsilon)}(\lambda, p, k)|^2} dk.$$

In case $\omega(0) > 0$ we use (8.16) and conclude that

$$|\hat{\beta}' p \delta_\epsilon \omega| \preceq R_\epsilon,$$

therefore

$$|Q_\epsilon^{(3)}(\lambda, p)| \preceq \epsilon^{2-\delta} \int_{\mathbb{T}} |\bar{u}_{\epsilon,-}| dk$$

and the lemma follows then by virtue of (9.10).

In the unpinned case we use bound

$$|D^{(\epsilon)}| \succeq \gamma R_\epsilon + \epsilon |\delta_\epsilon \omega| \quad (10.25)$$

together with Cauchy-Schwartz inequality and (9.8). We obtain

$$\begin{aligned}
|Q_\epsilon^{(3)}(\lambda, p)| &\leq \frac{4\epsilon^{2-\delta}\gamma^2|p|}{3} \left\{ \int_{\mathbb{T}} R|\bar{u}_{\epsilon,-}|^2 dk \right\}^{1/2} \left\{ \int_{\mathbb{T}} \frac{R(R'\delta_\epsilon\omega)^2}{|D(\epsilon)|^4} dk \right\}^{1/2} \\
&\leq \gamma^2\epsilon^{2-\delta}\epsilon^{(\delta-s)/2} \left\{ \int_{\mathbb{T}} \frac{R(R'\delta_\epsilon\omega)^2}{(\gamma R_\epsilon)^4 + \epsilon^4|\delta_\epsilon\omega|^4} dk \right\}^{1/2} \\
&\leq \gamma^2\epsilon^{2-(\delta+s)/2} \left\{ \int_{\mathbb{T}} \frac{R(R'\delta_\epsilon\omega)^2}{(\gamma R_\epsilon)^2\epsilon^2|\delta_\epsilon\omega|^2} dk \right\}^{1/2} \leq \gamma\epsilon^{1-(\delta+s)/2} \left\{ \int_{\mathbb{T}} \frac{R(R')^2}{R_\epsilon^2} dk \right\}^{1/2} \\
&\leq \epsilon^{(2+s-\delta)/2} \left\{ \int_0^1 \frac{k^3 dk}{k^4 + \epsilon^4} \right\}^{1/2} \leq \epsilon^{(2+s-\delta)/2} \log^{1/2} \left(\frac{1}{\epsilon} \right) \rightarrow 0,
\end{aligned}$$

as $\epsilon \rightarrow 0+$, uniformly in $\lambda \in I$ and $|p| \leq M$.

Finally,

$$\begin{aligned}
Q_\epsilon^{(2)}(\lambda, p) &= -\frac{4\gamma}{3\epsilon^\delta} \int_{\mathbb{T}} \frac{R\mathcal{L}\bar{u}_{\epsilon,+}}{D(\epsilon)} dk = -\frac{4\gamma}{3\epsilon^\delta} \int_{\mathbb{T}} \frac{(R - R_\epsilon)\mathcal{L}\bar{u}_{\epsilon,+}}{D(\epsilon)} dk \\
&\quad + \frac{4\gamma}{3\epsilon^\delta} \int_{\mathbb{T}} \frac{(\epsilon^\delta\lambda + i\epsilon\delta_\epsilon\omega)\mathcal{L}\bar{u}_{\epsilon,+}}{D(\epsilon)} dk - \frac{4\gamma}{3\epsilon^\delta} \int_{\mathbb{T}} \mathcal{L}\bar{u}_{\epsilon,+} dk.
\end{aligned} \tag{10.26}$$

Denote the terms appearing on the utmost right hand of (10.26) by I_ϵ , II_ϵ and III_ϵ , respectively. Since $\int_{\mathbb{T}} \mathcal{L}f dk = 0$ for any $f \in L^1(\mathbb{T})$ we have $III_\epsilon = 0$. In addition,

$$\begin{aligned}
I_\epsilon &= -\frac{\gamma\epsilon^{\delta-2}p^2}{2} \int_{\mathbb{T}} \frac{R''\mathcal{L}\bar{u}_{\epsilon,+}}{D(\epsilon)} dk \\
&= -\sum_{\iota \in \{-, +\}} \frac{3\gamma\epsilon^{\delta-2}p^2}{4} \int_{\mathbb{T}} \frac{R''\mathbf{e}_{-\iota}\bar{u}_{\epsilon,+}^{(\iota)}}{D(\epsilon)} dk + \gamma\epsilon^{\delta-2}p^2 \int_{\mathbb{T}} \frac{R''R}{D(\epsilon)} \bar{u}_{\epsilon,+} dk.
\end{aligned} \tag{10.27}$$

Here

$$\bar{u}_{\epsilon,+}^{(\iota)}(\lambda, p) := \langle \bar{u}_{\epsilon,+}(\lambda, p), \mathbf{e}_\iota \rangle_{L^2(\mathbb{T})}, \quad \iota \in \{-, +\}.$$

Since

$$\sup_{\epsilon \in (0,1]} \sup_{\lambda \in I, |p| \leq M} \sup_{k \in \mathbb{T}} \left| \frac{R''R}{D(\epsilon)} \right| < +\infty$$

(10.27) implies that

$$\lim_{\epsilon \rightarrow 0+} \sup_{\lambda \in I, |p| \leq M} |I_\epsilon| = 0,$$

thanks to (9.10).

Concerning term II_ϵ note that, thanks to the fact that $k \mapsto \delta_\epsilon \omega(k, p)$ is odd and $k \mapsto \mathcal{L}\bar{u}_{\epsilon,+}(\lambda, p, k)$ is even we have

$$\begin{aligned} II_\epsilon &= \frac{4\gamma}{3} \int_{\mathbb{T}} \frac{\mathcal{L}\bar{u}_{\epsilon,+}}{|D(\epsilon)|^2} [\lambda(\epsilon^\delta \lambda + 2\gamma R_\epsilon) + \epsilon^{2-\delta}(\delta_\epsilon \omega)^2] dk \\ &= 2\gamma \sum_{\iota \in \{-, +\}} \int_{\mathbb{T}} \frac{\mathbf{e}_\iota \bar{u}_{\epsilon,+}^{-\iota}}{|D(\epsilon)|^2} [\lambda(\epsilon^\delta \lambda + 2\gamma R_\epsilon) + \epsilon^{2-\delta}(\delta_\epsilon \omega)^2] dk \\ &\quad - \frac{8\gamma}{3} \int_{\mathbb{T}} \frac{R\bar{u}_{\epsilon,+}}{|D(\epsilon)|^2} [\lambda(\epsilon^\delta \lambda + 2\gamma R_\epsilon) + \epsilon^{2-\delta}(\delta_\epsilon \omega)^2] dk. \end{aligned} \tag{10.28}$$

We conclude therefore that

$$|II_\epsilon| \leq \gamma \epsilon^{2-\delta} \sum_{\iota \in \{-, +\}} |\bar{u}_{\epsilon,+}^{-\iota}| \int_{\mathbb{T}} \frac{\mathbf{e}_\iota (\delta_\epsilon \omega)^2}{|D(\epsilon)|^2} dk + \gamma \epsilon^{2-\delta} \int_{\mathbb{T}} \frac{R|\bar{u}_{\epsilon,+}|(\delta_\epsilon \omega)^2}{|D(\epsilon)|^2} dk. \tag{10.29}$$

Denote the terms appearing on the right hand side by $II_\epsilon^{(1)}$ and $II_\epsilon^{(2)}$, respectively.

When $\omega(0) > 0$ we use (8.16) and estimate

$$\frac{R(\delta_\epsilon \omega)^2}{|D(\epsilon)|^2} \leq 1.$$

Therefore, from (9.10), we get

$$\lim_{\epsilon \rightarrow 0+} \sup_{\lambda \in \mathcal{I}, |p| \leq M} |II_\epsilon| = 0.$$

In the unpinned case, we use (9.7) and (10.25), and get

$$\begin{aligned} |II_\epsilon^{(1)}| &\leq \gamma \epsilon^{2-\delta} \epsilon^{(\delta-s)/2} \int_{\mathbb{T}} \frac{R(\delta_\epsilon \omega)^2}{|D(\epsilon)|^2} dk \\ &\leq \gamma \epsilon^{2-(s+\delta)/2} \int_{\mathbb{T}} \frac{R(\delta_\epsilon \omega)^2}{\epsilon(\gamma R_\epsilon)|\delta_\epsilon \omega|} dk \leq \epsilon^{[2-(s+\delta)]/2} \rightarrow 0+, \end{aligned} \tag{10.30}$$

uniformly in $\lambda \in \mathcal{I}$ and $|p| \leq M$, as $\delta < (3-s)/2 < 2-s$ for $s \in (0, 1]$.

To estimate $|II_\epsilon^{(2)}|$ we use Cauchy-Schwartz inequality together with (9.7). Then

$$\begin{aligned} |II_\epsilon^{(2)}| &\leq \gamma \epsilon^{2-\delta} \left\{ \int_{\mathbb{T}} \frac{R^2(\delta_\epsilon \omega)^4}{|D(\epsilon)|^4} dk \right\}^{1/2} \left\{ \int_{\mathbb{T}} |\bar{u}_{\epsilon,+}|^2 dk \right\}^{1/2} \\ &\leq \gamma \epsilon^{2-(s+\delta)/2} \left\{ \int_{\mathbb{T}} \frac{R^2(\delta_\epsilon \omega)^4}{\epsilon^2(\gamma R_\epsilon \delta_\epsilon \omega)^2} dk \right\}^{1/2} \leq \epsilon^{[2-(s+\delta)]/2} \rightarrow 0+ \end{aligned} \tag{10.31}$$

in the same sense as in (10.30). This ends the proof of (10.23) for $i = 2$.

Concerning $Q_\epsilon^{(1)}(\lambda, p)$ we can write

$$Q_\epsilon^{(1)}(\lambda, p) := -\frac{2\gamma^2(\pi p)^2\epsilon^{2-\delta}}{3} \int_{\mathbb{T}} \frac{R}{D^{(\epsilon)}} [\mathbf{f}_+(k)\langle w_\epsilon(\lambda, p), 4\mathbf{f}_+ + \mathbf{e}_- \rangle_{L^2(\mathbb{T})} + 3\mathbf{f}_-(k)\langle w_\epsilon(\lambda, p), \mathbf{e}_+ \rangle_{L^2(\mathbb{T})}] dk.$$

Since $|R|/|D^{(\epsilon)}|$ is bounded the conclusion of the lemma follows easily for $\delta < 2$. If $\delta = 2$, then $\gamma \equiv \gamma_0$. We can use the Lebesgue dominated convergence theorem and obtain (10.24). \square

10.3. The dual dynamics. The equations (7.7) and (7.10) describing the dynamics of the vector

$$\widehat{W}_\epsilon(t, p, k) = (\widehat{W}_{\epsilon,+}(t, p, k), \widehat{Y}_{\epsilon,+}(t, p, k), \widehat{Y}_{\epsilon,-}(t, p, k), \widehat{W}_{\epsilon,-}(t, p, k))$$

can be written in the form

$$\frac{d}{dt} \widehat{W}_\epsilon(t, p, k) = L_\epsilon \widehat{W}_\epsilon(t, p, k) \quad (10.32)$$

where L_ϵ is some matrix operator. We now define the dual dynamics that runs on test functions. Suppose that

$$\widehat{\mathcal{J}}^{(\epsilon)}(t, p, k) = \begin{bmatrix} \widehat{J}_\epsilon^{w,+}(t) \\ \widehat{J}_\epsilon^{y,+}(t) \\ \widehat{J}_\epsilon^{y,-}(t) \\ \widehat{J}_\epsilon^{w,-}(t) \end{bmatrix},$$

is the solution of the system dual to (10.32), i.e.

$$\frac{d}{dt} \widehat{\mathcal{J}}^{(\epsilon)}(t) = L_\epsilon^* \widehat{\mathcal{J}}^{(\epsilon)}(t). \quad (10.33)$$

with given initial conditions that are Fourier transform of functions belonging to \mathcal{S} . The adjoint matrix L_ϵ^* is given explicitly by

$$L_\epsilon^* \widehat{\mathcal{J}} := \begin{bmatrix} L_w^{(\epsilon)} & L_-^{(\epsilon)} & L_+^{(\epsilon)} & 0 \\ L_-^{(\epsilon)} & L_y^{(\epsilon)} & R_y^{(\epsilon)} & L_+^{(\epsilon)} \\ L_-^{(\epsilon)} & R_y^{(\epsilon)} & \bar{L}_y^{(\epsilon)} & L_+^{(\epsilon)} \\ 0 & L_-^{(\epsilon)} & L_+^{(\epsilon)} & \bar{L}_w^{(\epsilon)} \end{bmatrix} \widehat{\mathcal{J}}, \quad (10.34)$$

with

$$\begin{aligned} L_w^{(\epsilon)} &:= \epsilon^{-\delta}(i\epsilon\delta_\epsilon\omega(k, p) + \gamma\mathcal{L}_{ep}^*), & \bar{L}_w^{(\epsilon)} &:= \epsilon^{-\delta}(-i\epsilon\delta_\epsilon\omega(k, p) + \gamma\mathcal{L}_{ep}^*), \\ L_y^{(\epsilon)} &:= \epsilon^{-\delta}[i\bar{\omega}(k, \epsilon p) + \gamma(\mathcal{L}_{ep}^* - \mathcal{R}_{ep}^*)], & \bar{L}_y^{(\epsilon)} &:= \epsilon^{-\delta}[-i\bar{\omega}(k, \epsilon p) + \gamma(\mathcal{L}_{ep}^* - \mathcal{R}_{ep}^*)], \\ L_\pm^{(\epsilon)} &:= -2^{-1}\gamma\epsilon^{-\delta}(\mathcal{L}_{ep}^\pm)^*, & R_y^{(\epsilon)} &:= \epsilon^{-\delta}\gamma\mathcal{R}_{ep}^*. \end{aligned}$$

The operators \mathcal{L}_p^* , $(\mathcal{L}_p^\pm)^*$ and \mathcal{R}_p^* are adjoints of \mathcal{L}_p , \mathcal{L}_p^\pm and \mathcal{R}_p (see (7.6) and (7.8)) with respect to the Lebesgue measure on \mathbb{T} . Given $M > 0$, $r_1, r_2 \in [1, +\infty)$ we introduce the norm

$$\|J\|_{\mathcal{A}_{r_2, M}^{r_1}} := \left\{ \int_{B_M} dp \left\{ \int_{\mathbb{T}} |\hat{J}(p, k)|^{r_1} dk \right\}^{r_2/r_1} \right\}^{1/r_2}, \quad (10.35)$$

and denote by $\mathcal{A}_{r_2, M}^{r_1}$ as the completion of \mathcal{S} under the above norm. Let $\mathcal{A}_{r_2, loc}^{r_1} := \bigcup_{M>0} \mathcal{A}_{r_2, M}^{r_1}$. The above definitions extend in an obvious way when either r_1 or r_2 equals ∞ .

Given $J \in \mathcal{S}$, define

$$\underline{J}(p) := \int_{\mathbb{T}} \hat{J}(p, k) dk = \int_{\mathbb{R} \times \mathbb{T}} e^{-2\pi i p x} J(x, k) dx dk$$

and

$$\varphi(p) := \begin{cases} \frac{\hat{c} p^2}{2}, & \text{in case } \hat{\alpha}(0) > 0, \\ \hat{c} |p|^{3/2}, & \text{in case } \hat{\alpha}(0) = 0. \end{cases}$$

Coefficient \hat{c} is determined either as in Theorem 5.1, when $\hat{\alpha}(0) > 0$, or Theorem 5.2, when $\hat{\alpha}(0) = 0$. We can repeat the argument made so far and conclude the following statement concerning the convergence of the dual dynamics.

Theorem 10.4. *Suppose that the initial data in (10.33) belongs to $\mathcal{A}_{\infty, loc}^2$. Then the following are true:*

i) *for any $T, M > 0$ we have*

$$\sup_{\epsilon \in (0, 1]} \sup_{t \in [0, T]} \|\mathcal{J}^{(\epsilon)}(t)\|_{\mathcal{A}_{\infty, M}^2} < +\infty,$$

ii) *suppose that $M > 0$ is such that $\hat{\mathcal{J}}^{(\epsilon)}(0, p, k) \equiv 0$ for $|p| \geq M$ and $k \in \mathbb{T}$. Then $\hat{\mathcal{J}}^{(\epsilon)}(t, p, k) \equiv 0$ for $t \geq 0$, $|p| \geq M$ and $k \in \mathbb{T}$,*

iii) *for any $T > 0$ and $g \in L^1([0, T], \mathcal{A}_{1, loc}^2)$ and $\hat{\mathcal{J}}^{(\epsilon)}(t, p, k)$ as in ii) we have*

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_0^T \int_{B_M \times \mathbb{T}} \hat{J}_\epsilon^{w,+}(t, p, k) \hat{g}^*(t, p, k) dt dp dk \\ &= \int_0^T \int_{\mathbb{R}} e^{-\varphi(p)t} \underline{J}^{w,+}(p) \underline{g}^*(t, p) dt dp. \end{aligned} \quad (10.36)$$

11. PROOFS OF THEOREMS 6.1 AND 6.2

11.1. **Evolution of the random Wigner transform.** To describe the evolution of the fluctuating Wigner transform $\widetilde{\mathcal{W}}_\epsilon(t; J)$, see (6.1), we shall also need the following quantities

$$\widetilde{\mathcal{Y}}_{\epsilon,+}(t; J) := \sqrt{\epsilon} \sum_{x,x' \in \mathbb{Z}} \psi_{x'}^{(\epsilon)}(t) \psi_x^{(\epsilon)}(t) \tilde{J}^* \left(\frac{\epsilon}{2}(x+x'), x'-x \right),$$

$$\widetilde{\mathcal{Y}}_{\epsilon,-}(t; J) := \sqrt{\epsilon} \sum_{x,x' \in \mathbb{Z}} \left(\psi_{x'}^{(\epsilon)}(t) \psi_x^{(\epsilon)}(t) \right)^* \tilde{J}^* \left(-\frac{\epsilon}{2}(x+x'), x'-x \right),$$

$$\widetilde{\mathcal{W}}_{\epsilon,-}(t; J) := \sqrt{\epsilon} \sum_{x,x' \in \mathbb{Z}} \left(\psi_{x'}^{(\epsilon)}(t) \left(\psi_x^{(\epsilon)}(t) \right)^* - \delta_{x,x'} \mathcal{E}_0 \right) \tilde{J}^* \left(-\frac{\epsilon}{2}(x+x'), x'-x \right).$$

We identify $\widetilde{\mathcal{W}}_{\epsilon,+}(t) = \widetilde{\mathcal{W}}_\epsilon(t)$. Given $\mathcal{J} = \{J^{w,+}, J^{y,+}, J^{y,-}, J^{w,-}\}$ with components that are Fourier transform of functions in \mathcal{S} , denote

$$\widetilde{\mathcal{W}}_\epsilon(t; \mathcal{J}) := \sum_{\iota \in \{-,+\}} \left(\widetilde{\mathcal{W}}_{\epsilon,\iota}(t; J^{w,\iota}) + \widetilde{\mathcal{Y}}_{\epsilon,\iota}(t; J^{y,\iota}) \right).$$

For $J_1, J_2 \in \mathcal{S}$ let

$$C_\epsilon^{w,\pm}(t; J_1, J_2) := \mathbb{E} \left[\widetilde{\mathcal{W}}_{\epsilon,\pm}(t; J_1) \widetilde{\mathcal{W}}_\epsilon(0; J_2) \right], \quad (11.1)$$

$$C_\epsilon^{y,\pm}(t; J_1, J_2) := \mathbb{E} \left[\widetilde{\mathcal{Y}}_{\epsilon,\pm}(t; J_1) \widetilde{\mathcal{W}}_\epsilon(0; J_2) \right].$$

Computing the time differential as in Section 7, we obtain

$$d\widetilde{\mathcal{W}}_\epsilon(t; J) = \epsilon^{-\delta} \left\{ \widetilde{\mathcal{W}}_\epsilon(t; (i\epsilon A + \gamma \mathcal{L}_\epsilon^*) J) - \frac{\gamma}{2} \sum_{\sigma \in \{-,+\}} \widetilde{\mathcal{Y}}_{\epsilon,\sigma}(t; \mathcal{L}_{\epsilon,-\sigma}^* J) \right. \quad (11.2)$$

$$\left. + \epsilon^{-1/2} \mathcal{E}_0 \sum_{n \in \mathbb{Z}} \int_{\mathbb{T}} \left(i\epsilon A \hat{J} + \gamma \mathcal{L}_\epsilon^* \hat{J} \right) \left(\frac{n}{\epsilon}, k \right) dk \right\} dt + d\mathcal{M}_t^{(\epsilon)}(J),$$

$$d\widetilde{\mathcal{Y}}_\epsilon(t; J) = \epsilon^{-\delta} \left\{ \widetilde{\mathcal{Y}}_\epsilon(t; (iB + \gamma \mathcal{L}_\epsilon^*) J) + \gamma \widetilde{\mathcal{Y}}_{\epsilon,-}(t; \mathcal{R}_\epsilon^* J) - \gamma \widetilde{\mathcal{Y}}_\epsilon(t; \mathcal{R}_\epsilon^* J) \right. \quad (11.3)$$

$$\left. - \frac{\gamma}{2} \sum_{\sigma \in \{-,+\}} \widetilde{\mathcal{W}}_{\epsilon,\sigma}(t; \mathcal{L}_{\epsilon,-\sigma}^* J) - \frac{\gamma \mathcal{E}_0}{2\epsilon^{1/2}} \sum_{n \in \mathbb{Z}} \int_{\mathbb{T}} \left(\mathcal{L}_{\epsilon,-\sigma}^* \hat{J} \right) \left(\frac{n}{\epsilon}, k \right) dk \right\} dt + d\mathcal{N}_t^{(\epsilon)}(J),$$

where $\mathcal{M}_t^{(\epsilon)}(J)$, $\mathcal{N}_t^{(\epsilon)}(J)$, are some square integrable, continuous trajectory martingales. Summarizing, suppose that test functions $J^{w,\pm}$ and

$J^{y,\pm}$ are such that their respective Fourier transforms in the x variables $\hat{J}^{w,\pm}$ and $\hat{J}^{y,\pm}$ belong to $C_c^\infty(\mathbb{R} \times \mathbb{T})$. Then, using (11.2) and (11.3) we obtain

$$\frac{d}{dt} \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(t; \mathcal{J}) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] = \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(t; L_\epsilon^* \mathcal{J}) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] \quad (11.4)$$

where L_ϵ^* is given by (10.34).

Suppose that $\mathcal{J}^{(\epsilon)}(t)$ is the solution of an equation (10.33). From part ii) of Theorem 10.4 we conclude that

$$\widehat{\mathcal{J}}^{(\epsilon)}(t, p, k) \equiv 0, \quad \forall t \geq 0, |p| \geq M, k \in \mathbb{T}, \quad (11.5)$$

provided that $M > 0$ is such that $\widehat{\mathcal{J}}^{(\epsilon)}(0, p, k) \equiv 0$ for all $|p| \geq M$, $k \in \mathbb{T}$.

Combining (11.4) with (10.33) we obtain

$$\begin{aligned} \frac{d}{ds} \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(s; \mathcal{J}^{(\epsilon)}(t-s)) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] &= \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(s; L_\epsilon^* \mathcal{J}^{(\epsilon)}(t-s)) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] \\ &+ \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon \left(s; \frac{d}{ds} \mathcal{J}^{(\epsilon)}(t-s) \right) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] \equiv 0 \end{aligned}$$

for all $s \geq 0$. Hence,

$$\mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(t; \mathcal{J}) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] = \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(0; \mathcal{J}^{(\epsilon)}(t)) \widetilde{\mathcal{W}}_\epsilon(0; J) \right]. \quad (11.6)$$

Suppose that the initial data satisfies the hypothesis of part ii) of Theorem 10.4 and that \hat{J} is compactly supported. According to (6.3) the right hand side of (11.6) equals

$$\begin{aligned} \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(0; J_\epsilon^{w,+}(t)) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] &= \mathcal{E}_0^2 \epsilon \sum_x \int_{\mathbb{T}} J_\epsilon^{w,+}(t, \epsilon x, k) J^*(\epsilon x, k) dk \\ &= \mathcal{E}_0^2 \sum_n \int_{\mathbb{R} \times \mathbb{T}} \hat{J}_\epsilon^{w,+} \left(t, \frac{n}{\epsilon} - p, k \right) \hat{J}^*(-p, k) dp dk \\ &= \mathcal{E}_0^2 \int_{\mathbb{R} \times \mathbb{T}} \hat{J}_\epsilon^{w,+}(t, p, k) \hat{J}^*(p, k) dp dk \end{aligned}$$

for sufficiently small ϵ , since the supports of $\hat{J}_\epsilon^{w,+}(t)$ and \hat{J} are both compact in p . Using (10.36) we conclude that for any compactly supported $\phi \in L^1[0, +\infty)$ (11.5) we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0^+} \int_0^{+\infty} \phi(t) \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(t; J_1) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] dt \\ &= \lim_{\epsilon \rightarrow 0^+} \int_0^{+\infty} \phi(t) \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(0; J_\epsilon^{w,+}(t)) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] dt \\ &= \mathcal{E}_0^2 \int_0^{+\infty} dt \int_{\mathbb{R}} \phi(t) \exp \left\{ -\frac{\hat{c} p^2 t}{2} \right\} \underline{\mathbb{J}}_1(p) \underline{\mathbb{J}}(p) dp, \end{aligned}$$

with \hat{c} given by (5.23) when $\delta < 2$, or (5.24) when $\delta = 2$ in case of pinning potential, or

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0^+} \int_0^{+\infty} \phi(t) \mathbb{E} \left[\widetilde{\mathcal{W}}_\epsilon(t; J_1) \widetilde{\mathcal{W}}_\epsilon(0; J) \right] dt \\ &= \mathcal{E}_0^2 \int_0^{+\infty} dt \int_{\mathbb{R}} \phi(t) \exp \left\{ -\hat{c} |p|^{3/2} t \right\} \underline{\mathbb{J}}_1(p) \underline{\mathbb{J}}(p) dp, \end{aligned}$$

with \hat{c} given by (5.28) in the unpinned case. Generalization to arbitrary $J_1, J \in \mathcal{S}$ and $\phi \in L^1[0, +\infty)$ is standard and can be done via an approximation argument, thanks to the fact that $(\widetilde{\mathcal{W}}_\epsilon(t; J))$ is stationary for any $J \in \mathcal{S}$.

Remark: Observe that the proof does not really use time stationarity of the initial distribution, in fact it follows that for any initial homogeneous distribution with energy density given by some $\mathcal{E}(k)$ such that $\int_{\mathbb{T}} \mathcal{E}(k) dk = 2\mathcal{E}_0$, we have the same result. On the other hand we does use the stationarity in order to prove the equivalence of the energy distribution (6.12).

12. PROOF OF THEOREM 3.1

From the definition of the phonon modes, see (3.8), we conclude that

$$\begin{aligned} d\hat{\mathfrak{f}}^{(\pm)}(t, k) &= \left\{ \pm (1 - e^{-2i\pi k}) \sqrt{\tau_1} \hat{\mathfrak{f}}^{(\pm)}(t, k) \right. \\ &\quad \left. + D_\pm (1 - e^{2i\pi k}) {}^2\hat{\mathfrak{f}}^{(\pm)}(t, k) + O(\sin^3(\pi k)) \right\} dt + dM_t(k), \end{aligned} \quad (12.1)$$

where

$$dM_t(k) := 2i\gamma^{1/2} \int_{\mathbb{T}} r(k, k') \hat{\mathfrak{p}}(t, k - k') B(dt, dk'), \quad (12.2)$$

and D_{\pm} are given in (6.14). Let

$$\hat{\mathbf{f}}_{\epsilon}^{(\pm)}(t, k) := \epsilon \mathbb{E}_{\epsilon} \hat{\mathbf{f}}^{(\pm)}\left(\frac{t}{\epsilon}, \epsilon k\right).$$

It satisfies

$$\frac{d}{dt} \hat{\mathbf{f}}_{\epsilon}^{(\pm)}(t, k) = \pm \frac{1}{\epsilon} (1 - e^{-2i\pi\epsilon k}) \hat{\mathbf{f}}_{\epsilon}^{(\pm)}(t, k) + O(\epsilon). \quad (12.3)$$

An elementary stability theory for solutions of ordinary differential equations guarantees that

$$\hat{\mathbf{f}}_{\epsilon}^{(\pm)}(t, k) = e^{\pm 2i\pi kt} (1 + O(\epsilon)) \hat{\mathbf{f}}^{(\pm)}(0, \epsilon k). \quad (12.4)$$

After a straightforward calculation we obtain that the left hand side of (3.13) equals

$$\epsilon \int_{\mathbb{R}} \hat{J}(p) \mathbb{E}_{\epsilon} \hat{\mathbf{f}}^{(\pm)}\left(\frac{t}{\epsilon}, -\epsilon p\right) dp = \epsilon \int_{\mathbb{R}} \hat{J}(p) e^{\mp 2i\pi p t} \mathbb{E}_{\epsilon} \hat{\mathbf{f}}^{(\pm)}(0, -\epsilon p) dp,$$

which tends to the expression on the right hand side of (3.13), as $\epsilon \rightarrow 0+$ for any $J \in C_0^{\infty}(\mathbb{R})$.

To prove the diffusive approximation (up to time scale $\sim 1/\epsilon^2$) it suffices to consider test functions $J(y)$ such that $\hat{J} \in C_0^{\infty}(\mathbb{R})$. Let

$$\hat{\mathbf{v}}_{\epsilon}^{(\iota)}(t, k) := \epsilon \mathbb{E}_{\epsilon} \hat{\mathbf{f}}^{(\iota)}\left(\frac{t}{\epsilon^2}, \epsilon k\right) \exp\left\{-\iota(1 - e^{-2i\pi\epsilon k}) \frac{\sqrt{\tau_1} t}{\epsilon^2}\right\}, \quad \iota \in \{-, +\}.$$

It satisfies

$$\frac{d\hat{\mathbf{v}}_{\epsilon}^{(\iota)}}{dt}(t, k) = \frac{D_{\iota}}{\epsilon^2} (1 - e^{2i\pi\epsilon k})^2 \hat{\mathbf{v}}_{\epsilon}^{(\iota)}(t, k) + O(\epsilon). \quad (12.5)$$

Using again the elementary stability theory of ordinary differential equations we conclude that for any $K, T > 0$

$$\lim_{\epsilon \rightarrow 0+} \sup_{t \in [0, T], |k| \leq K} \left| \hat{\mathbf{v}}_{\epsilon}^{(\iota)}(t, k) - \epsilon \mathbb{E}_{\epsilon} \hat{\mathbf{f}}^{(\iota)}(0, \epsilon k) e^{-4\pi^2 t D_{\iota} |k|^2} \right| = 0, \quad \iota \in \{-, +\}.$$

For any J as indicated above we can write that the expression under the limit on the left hand side of (3.14) equals

$$\begin{aligned} & \epsilon \int_{\mathbb{R}} \hat{J}(p) \exp\left\{-\iota 2\pi i p \sqrt{\tau_1} \frac{t}{\epsilon}\right\} \mathbb{E}_{\epsilon} \hat{\mathbf{f}}^{(\iota)}\left(\frac{t}{\epsilon^2}, -\epsilon p\right) dp \\ &= \int_{\mathbb{R}} \hat{J}(p) \exp\left\{\iota \sqrt{\tau_1} (1 - e^{-2\pi i \epsilon p} - 2\pi i \epsilon p) \frac{t}{\epsilon^2}\right\} \hat{\mathbf{v}}_{\epsilon}^{(\iota)}(t, -p) dp \\ &= \int_{\mathbb{R}} \hat{J}(p) \exp\{2\iota \sqrt{\tau_1} \pi^2 p^2 t + O(\epsilon)\} \hat{\mathbf{v}}_{\epsilon}^{(\iota)}(t, -p) dp \end{aligned}$$

and the latest expression tends to

$$\int_{\mathbb{R}} \hat{J}(p) \hat{f}_0^{(\iota)}(-p) e^{-4\pi^2 t \bar{D} |p|^2} dp$$

as $\epsilon \rightarrow 0+$, with $\hat{f}_0^{(\iota)}(p)$ the Fourier transform of $\tilde{f}_0^{(\iota)}(y)$, given by (3.12), which ends the proof of (3.14).

13. EQUIVALENCE OF ENERGY FUNCTIONALS

13.1. Proof of Proposition 5.3.

13.1.1. *The case of a pinned potential.* The left hand side of (5.30) equals $\lim_{\epsilon \rightarrow 0+} (I_\epsilon + II_\epsilon)$, where

$$\begin{aligned} I_\epsilon &:= \frac{\epsilon}{4} \sum_x J(\epsilon x) \mathbb{E}_\epsilon \left(\hat{\alpha}(0) [\mathbf{q}_x^{(\epsilon)}(t)]^2 - \sum_y \alpha_{x-y} [\mathbf{q}_y^{(\epsilon)}(t)]^2 \right) \\ &= \frac{\epsilon}{4} \sum_x \mathbb{E}_\epsilon [\mathbf{q}_x^{(\epsilon)}(t)]^2 \int_{\mathbb{R}} e^{2\pi i \epsilon x p} \hat{J}(p) [\hat{\alpha}(0) - \hat{\alpha}(-\epsilon p)] dp \end{aligned}$$

and

$$\begin{aligned} II_\epsilon &:= \frac{\epsilon}{2} \sum_x J(\epsilon x) \mathbb{E}_\epsilon \left[\mathbf{q}_x^{(\epsilon)}(t) (\alpha * \mathbf{q}^{(\epsilon)}(t))_x - (\tilde{\omega} * \mathbf{q}^{(\epsilon)}(t))_x^2 \right] \\ &= \frac{\epsilon}{2} \sum_{y, y'} \mathbb{E}_\epsilon (\mathbf{q}_y^{(\epsilon)}(t) \mathbf{q}_{y'}^{(\epsilon)}(t)) \sum_x \int_{\mathbb{R}} e^{2\pi i \epsilon x p} \hat{J}(p) [\tilde{\omega}_{x-y}(0) - \tilde{\omega}_{x-y}(-\epsilon p)] \tilde{\omega}_{x-y'} dp. \end{aligned}$$

Here we have adopted the notation $\tilde{\omega}_x(p) := e^{-2\pi i x p} \tilde{\omega}_x$. Since in the case of a pinned chain we have

$$\sup_{\epsilon \in (0,1]} \epsilon \mathbb{E}_\epsilon [\mathbf{q}_x^{(\epsilon)}(t)]^2 < +\infty$$

it is clear that for any $J \in C_0^\infty(\mathbb{R})$ we have

$$\lim_{\epsilon \rightarrow 0+} I_\epsilon = 0 \quad \text{and} \quad \lim_{\epsilon \rightarrow 0+} II_\epsilon = 0. \quad (13.1)$$

13.1.2. *The case of an unpinned potential.* Recall that then $\hat{\alpha}(0) = 0$. Let (μ_ϵ) , $\epsilon \in (0, 1]$, be a family of probability distributions on ℓ^2 such that

$$\sup_\epsilon \epsilon \left\langle \sum_x |\psi_x|^2 \right\rangle_{\mu_\epsilon} < +\infty, \quad (13.2)$$

see condition (3.16). In addition we assume that at time $t = 0$ it also satisfies the bound

$$\sup_\epsilon \int_\Pi dk \left[\epsilon \left\langle |\hat{\psi}(k)|^2 \right\rangle_{\mu_\epsilon} \right]^2 < +\infty. \quad (13.3)$$

We prove (5.30) only for $t = 0$. Our argument shows that the equivalence of energy density functionals is a consequence of the above two bounds. By virtue of the conservation of energy property of the dynamics and Proposition 8.2 bounds (13.5) and (13.6) persist in time, so our proof shows that in fact (5.30) holds for any subsequent time $t \geq 0$. Finally, to remove assumption (13.6) we can use condition (4.6) and the truncation procedure described at the end of Section 9.

Since in the unpinned case $\mathfrak{q}_0 = 0$ we have

$$\mathfrak{q}_x = \int_{\mathbb{T}} (e^{2\pi i k x} - 1) \hat{\mathfrak{q}}(k) dk, \quad (13.4)$$

where

$$\hat{\mathfrak{q}}(k) := \frac{\hat{\psi}(k) + \hat{\psi}^*(-k)}{2\omega(k)}.$$

As a consequence of (13.2) and (13.3) we obtain

$$\sup_\epsilon \epsilon \left\langle \sum_x (\tilde{\omega} * \mathfrak{q})_x^2 \right\rangle_{\mu_\epsilon} < +\infty \quad (13.5)$$

and

$$\sup_\epsilon \int_{\mathbb{T}} \left[\epsilon \left\langle |\omega(k) \hat{\mathfrak{q}}(k)|^2 \right\rangle_{\mu_\epsilon} \right]^2 dk < +\infty. \quad (13.6)$$

Define

$$\delta \hat{\alpha}(k, k') := \hat{\alpha}(k + k') - \hat{\alpha}(k) - \hat{\alpha}(k'). \quad (13.7)$$

Lemma 13.1. *We have*

$$\left| \frac{\delta \hat{\alpha}(k, k')}{\omega(k)\omega(k')} \right| \preceq 1, \quad k, k' \in \mathbb{T}. \quad (13.8)$$

Proof. Observe that

$$\hat{\alpha}(k) = -2 \sum_x \alpha_x \mathfrak{s}^2(xk).$$

Therefore (13.8) is a consequence of the following elementary trigonometric inequality

$$|\mathfrak{s}^2(\alpha + \beta) - \mathfrak{s}^2(\alpha) - \mathfrak{s}^2(\beta)| \leq |\mathfrak{s}(\alpha)\mathfrak{s}(\beta)|, \quad \alpha, \beta \in \mathbb{R}$$

and the assumption a1) made on the decay of (α_x) . \square

Let

$$\phi_x := |\psi_x|^2 - 2\mathbf{e}_x = \frac{1}{2} \sum_y \alpha_{x-y} (\mathbf{q}_x - \mathbf{q}_y)^2 + (\tilde{\omega} * \mathbf{q})_x^2 \quad (13.9)$$

Obviously Proposition 5.3 is a consequence of the following.

Lemma 13.2. *For any $J \in C_0^\infty(\mathbb{R})$*

$$\lim_{\epsilon \rightarrow 0} \epsilon \sum_x J(\epsilon x) \langle \phi_x \rangle_{\mu_\epsilon} = 0. \quad (13.10)$$

Proof of Lemma 13.2. Using (13.4) we can write ϕ_x in Fourier transform coordinates as

$$\phi_x = \iint_{\mathbb{T}^2} \hat{F}(k, k') \omega(k) \hat{\mathbf{q}}(k) \omega(k') \hat{\mathbf{q}}(k') e^{i2\pi(k+k')x} dk dk' \quad (13.11)$$

where

$$\hat{F}(k, k') := \frac{\hat{\alpha}(k+k') - \hat{\alpha}(k) - \hat{\alpha}(k')}{2\omega(k)\omega(k')} + 1. \quad (13.12)$$

Note that $F(k, -k) = 0$. Moreover, according to Lemma 13.1 it is bounded. Observe that under the condition (13.5), the integral in (13.11) is well defined. Furthermore we can write:

$$\epsilon \sum_x J(\epsilon x) \langle \phi_x \rangle_{\nu_\epsilon} = \int_{\mathbb{R}} \hat{J}(p) Z_\epsilon(p) dp,$$

where

$$Z_\epsilon(p) := \epsilon \int_{\mathbb{T}} \hat{F}(k, -k - \epsilon p) \langle \omega(k) \hat{\mathbf{q}}(k) \omega(-k - \epsilon p) \hat{\mathbf{q}}(-k - \epsilon p) \rangle_{\mu_\epsilon} dp dk.$$

Then by Schwarz inequality and symmetry with respect to k and $k + \epsilon p$, we obtain

$$\begin{aligned} |Z_\epsilon(p)| &\leq \int_{\mathbb{T}} dk |\hat{F}(k, -k - \epsilon p)| \epsilon \langle \omega(k)^2 |\hat{\mathbf{q}}(k)|^2 \rangle_{\mu_\epsilon} \\ &\leq \left(\int_{\mathbb{T}} dk |\hat{F}(k, -k - \epsilon p)|^2 \right)^{1/2} \left(\int_{\mathbb{T}} dk \left[\epsilon \langle \omega(k)^2 |\hat{\mathbf{q}}(k)|^2 \rangle_{\mu_\epsilon} \right]^2 \right)^{1/2} \\ &\leq C \left(\int_{\mathbb{T}} dk |\hat{F}(k, -k - \epsilon p)|^2 \right)^{1/2} \xrightarrow{\epsilon \rightarrow 0} 0 \end{aligned}$$

and since $Z_\epsilon(p)$ is bounded, the result follows upon an application of the Lebesgue dominated convergence theorem. \square

13.2. Proof of Proposition 6.3. Obviously, stationarity implies that the limit in (6.12) does not depend on t and we show its validity for $t = 0$. To prove (6.12) it suffices to show that

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \mathbb{E} \left[\sum_x \phi_x J(\epsilon x) \right]^2 = 0, \quad (13.13)$$

where $\phi_x = \sum_{i=1}^3 \phi_x^{(i)}$ and

$$\begin{aligned} \phi_x^{(1)} &:= \frac{1}{2} \sum_y \alpha_{x-y} (\mathbf{q}_x - \mathbf{q}_y)^2, \\ \phi_x^{(2)} &:= (\tilde{\omega} * \mathbf{q})_x^2, \quad \phi_x^{(3)} := -\hat{\alpha}(0) \mathbf{q}_x^2. \end{aligned}$$

13.2.1. The case of an unpinned chain. We assume that $\hat{\alpha}(0) = 0$, therefore $\phi_x^{(3)} = 0$. Then, the field (\mathbf{q}_x) is Gaussian given by

$$\mathbf{q}_x = \sqrt{\mathcal{E}_0} \int_{\mathbb{T}} (e^{2\pi i k x} - 1) \frac{\hat{w}(dk)}{\omega(k)}, \quad (13.14)$$

where $\hat{w}(dk)$ is a complex even, Gaussian white noise in $L^2(\mathbb{T})$, i.e.

$$\mathbb{E}[\hat{w}(dk) \hat{w}^*(dk')] = \delta(k - k') dk dk', \quad \hat{w}^*(dk) = \hat{w}(-dk).$$

As a result

$$\mathbb{E}(\mathbf{q}_x \mathbf{q}_y) = \mathcal{E}_0 \int_{\mathbb{T}} [e^{2\pi i k x} - 1] [e^{2\pi i k y} - 1]^* \frac{dk}{\alpha(k)} \quad (13.15)$$

Note that $\mathbb{E} \phi_x = 0$ for all $x \in \mathbb{Z}$. Indeed, we have

$$\begin{aligned} \mathbb{E} \phi_x^{(1)} &= \frac{1}{2} \sum_y \alpha_{x-y} \mathbb{E}(\mathbf{q}_x - \mathbf{q}_y)^2 = \frac{1}{2} \sum_y \alpha_y \mathbb{E} \mathbf{q}_y^2 \\ &= \frac{\mathcal{E}_0}{2} \sum_y \alpha_y \int_{\mathbb{T}} [2 - (e^{2\pi i k y} + e^{-2\pi i k y})] \frac{dk}{\alpha(k)} \\ &= -\mathcal{E}_0 \int_{\mathbb{T}} \frac{\alpha(k) dk}{\alpha(k)} = -\mathcal{E}_0. \end{aligned}$$

In addition,

$$\begin{aligned} \mathbb{E} \phi_x^{(2)} &= \sum_{y, y'} \tilde{\omega}_{x-y} \tilde{\omega}_{x-y'} \mathbb{E}(\mathbf{q}_y \mathbf{q}_{y'}) \\ &= \mathcal{E}_0 \sum_{y, y'} \tilde{\omega}_{x-y} \tilde{\omega}_{x-y'} \int_{\mathbb{T}} [e^{2\pi i k y} - 1] [e^{2\pi i k y'} - 1]^* \frac{dk}{\alpha(k)} \\ &= \mathcal{E}_0 \int_{\mathbb{T}} \left(\sum_{y, y'} \tilde{\omega}_{x-y} \tilde{\omega}_{x-y'} e^{2\pi i k (y-y')} \right) \frac{dk}{\alpha(k)} = \mathcal{E}_0 \int_{\mathbb{T}} \omega^2(k) \frac{dk}{\alpha(k)} = \mathcal{E}_0. \end{aligned}$$

In our next step we calculate

$$r_x := \mathbb{E}(\phi_x \phi_0) = \sum_{i,i'=1}^2 r_x^{(i,i')}, \quad (13.16)$$

where $r_x^{(i,i')} := \mathbb{E}(\tilde{\phi}_x^{(i)} \tilde{\phi}_0^{(i')})$ and

$$\tilde{\phi}_x^{(1)} := \phi_x^{(1)} + \mathcal{E}_0, \quad \tilde{\phi}_x^{(2)} := \phi_x^{(2)} - \mathcal{E}_0.$$

We have

$$\begin{aligned} r_x^{(1,1)} &= \frac{1}{2} \sum_{y_1, y_2} \alpha_{x-y_1} \alpha_{y_2} \{ \mathbb{E}[(\mathbf{q}_x - \mathbf{q}_{y_1}) \mathbf{q}_{y_2}] \}^2 \\ &= \frac{\mathcal{E}_0^2}{2} \int_{\mathbb{T}^2} [\delta \hat{\alpha}(k, k')]^2 \frac{e^{2\pi i(k+k')x} dk dk'}{\hat{\alpha}(k) \hat{\alpha}(k')}, \\ r_x^{(2,1)} = r_x^{(1,2)} &= \sum_y \alpha_{x-y} \{ \mathbb{E}[(\mathbf{q}_x - \mathbf{q}_y)(\tilde{\omega} * \mathbf{q})_0] \}^2 \\ &= \mathcal{E}_0^2 \int_{\mathbb{T}^2} \delta \hat{\alpha}(k, k') \frac{e^{2\pi i(k+k')x} dk dk'}{\omega(k) \omega(k')}, \\ r_x^{(2,2)} &= 2 \{ \mathbb{E}[(\tilde{\omega} * \mathbf{q})_0 (\tilde{\omega} * \mathbf{q})_x] \}^2 = 2\mathcal{E}_0^2 \delta_{x,0}. \end{aligned}$$

Here $\delta \hat{\alpha}(k, k')$ is given by (13.7). Therefore,

$$\begin{aligned} \epsilon \mathbb{E} \left[\sum_x \phi_x J(\epsilon x) \right]^2 &= \epsilon \sum_{x, x'} J(\epsilon x) J(\epsilon x') r_{x-x'} \\ &= \frac{\epsilon \mathcal{E}_0^2}{2} \sum_{x, x'} \int_{\mathbb{R}^2 \times \mathbb{T}^2} e^{2\pi i(k+k')(x-x')} e^{2\epsilon \pi i(xp+x'p')} \hat{J}(p) \hat{J}(p') F(k, k') dp dp' dk dk', \end{aligned} \quad (13.17)$$

with

$$F(k, k') := \frac{[\delta \hat{\alpha}(k, k') + 2\omega(k)\omega(k')]^2}{\hat{\alpha}(k)\hat{\alpha}(k')}.$$

Observe that $F(-k, k) = 0$. Summing first over x and then over x' we obtain that the utmost right hand side of (13.17) equals

$$\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \hat{J}(p) \hat{J}\left(\frac{n}{\epsilon} - p\right) \left(\int_{\mathbb{T}} F(-k - \epsilon p, k) dk \right) dp. \quad (13.18)$$

Therefore (13.13) (thus also the conclusion of the proposition) is a consequence of the Lebesgue dominated convergence theorem and Lemma 13.1.

13.2.2. *The pinned case.* Then,

$$\mathbf{q}_x = \sqrt{\mathcal{E}_0} \int_{\mathbb{T}} \frac{e^{2\pi i k x} \hat{w}(dk)}{\omega(k)}, \quad (13.19)$$

We have $\mathbb{E}\phi_x = 0$ with

$$\begin{aligned} \mathbb{E}\phi_x^{(1)} &= \mathcal{E}_0 \int_{\mathbb{T}} \frac{[\hat{\alpha}(0) - \hat{\alpha}(k)] dk}{\hat{\alpha}(k)}, \\ \mathbb{E}\phi_x^{(2)} &= \mathcal{E}_0, \quad \mathbb{E}\phi_x^{(3)} = -\mathcal{E}_0 \int_{\mathbb{T}} \frac{\hat{\alpha}(0) dk}{\hat{\alpha}(k)}. \end{aligned}$$

We let

$$r_x := \mathbb{E}(\phi_x \phi_0) = \sum_{i,i'=1}^3 r_x^{(i,i')}, \quad (13.20)$$

where $r_x^{(i,i')} := \mathbb{E}(\tilde{\phi}_x^{(i)} \tilde{\phi}_0^{(i')})$ and

$$\tilde{\phi}_x^{(i)} := \phi_x^{(i)} - \mathbb{E}\phi_x^{(i)}, \quad i = 1, 2, 3.$$

We have

$$\begin{aligned} r_x^{(1,1)} &= \frac{1}{2} \sum_{y_1, y_2} \alpha_{x-y_1} \alpha_{y_2} \{ \mathbb{E}[(\mathbf{q}_x - \mathbf{q}_{y_1})(\mathbf{q}_{y_2} - \mathbf{q}_0)] \}^2 \\ &= \frac{\mathcal{E}_0^2}{2} \int_{\mathbb{T}^2} [\delta \hat{\alpha}(k, k')]^2 \frac{e^{2\pi i(k+k')x} dk dk'}{\hat{\alpha}(k) \hat{\alpha}(k')}, \end{aligned}$$

where $\delta \hat{\alpha}(k, k') := \hat{\alpha}(0) + \hat{\alpha}(k+k') - \hat{\alpha}(k) - \hat{\alpha}(k')$ and

$$\begin{aligned} r_x^{(2,1)} &= r_x^{(1,2)} = \sum_y \alpha_{x-y} \{ \mathbb{E}[(\mathbf{q}_x - \mathbf{q}_y)(\tilde{\omega} * \mathbf{q})_0] \}^2 \\ &= \mathcal{E}_0^2 \int_{\mathbb{T}^2} \delta \hat{\alpha}(k, k') \frac{e^{2\pi i(k+k')x} dk dk'}{\omega(k) \omega(k')}, \end{aligned}$$

$$r_x^{(2,2)} = 2 \{ \mathbb{E}[(\tilde{\omega} * \mathbf{q})_0 (\tilde{\omega} * \mathbf{q})_x] \}^2 = 2\mathcal{E}_0^2 \delta_{x,0},$$

$$\begin{aligned} r_x^{(3,1)} &= r_x^{(1,3)} = - \sum_y \alpha_{x-y} \hat{\alpha}(0) \{ \mathbb{E}[(\mathbf{q}_x - \mathbf{q}_y) \mathbf{q}_0] \}^2 \\ &= -\mathcal{E}_0^2 \int_{\mathbb{T}^2} \hat{\alpha}(0) \delta \hat{\alpha}(k, k') \frac{e^{2\pi i(k+k')x} dk dk'}{\hat{\alpha}(k) \hat{\alpha}(k')}, \end{aligned}$$

$$\begin{aligned} r_x^{(3,2)} &= r_x^{(2,3)} = -2\hat{\alpha}(0) \{ \mathbb{E}[(\tilde{\omega} * \mathbf{q})_x \mathbf{q}_0] \}^2 \\ &= -2\mathcal{E}_0^2 \int_{\mathbb{T}^2} \hat{\alpha}(0) \frac{e^{2\pi i(k+k')x} dk dk'}{\omega(k) \omega(k')}, \end{aligned}$$

and

$$r_x^{(3,3)} = 2\hat{\alpha}^2(0) \{E[\mathbf{q}_x \mathbf{q}_0]\}^2 = 2\mathcal{E}_0^2 \int_{\mathbb{T}^2} \hat{\alpha}^2(0) \frac{e^{2\pi i(k+k')x} dk dk'}{\hat{\alpha}(k)\hat{\alpha}(k')}.$$

Therefore, we can write (13.17) with

$$F(k, k') := \frac{1}{\hat{\alpha}(k)\hat{\alpha}(k')} \{-2\hat{\alpha}(0) + 2\omega(k)\omega(k') + \delta\hat{\alpha}(k, k')\}^2.$$

We have $F(-k, k) = 0$. Repeating the argument made in the unpinned case, this time easier since we do not have to bother about possible singularities of $F(-k - \epsilon p, k)$ in the vicinity of 0 we conclude the assertion of the proposition for pinned chains.

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